A QUANTIZATION OF CONJUGACY CLASSES OF MATRICES

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Abstract. We construct a generator system of the annihilator of the generalized Verma module of $\mathfrak{gl}(n,\mathbb{C})$ induced from any character of any parabolic subalgebra as an analogue of minors and elementary divisors. The generator system has a quantization parameter ϵ and it generates the defining ideal of the conjugacy class of square matrices at the classical limit $\epsilon = 0$.

1. Introduction

Let A be an element of the space $M(n,\mathbb{C})$ of square matrices of size n with components in $\mathbb{C}$. Then the conjugacy class containing A is the algebraic variety $V_A = \bigcup_{g\in G} \mathrm{Ad}(g)A$ by denoting $G = GL(n,\mathbb{C})$ and $\mathrm{Ad}(g)A = gAg^{-1}$. Under the G-action on $M(n,\mathbb{C})$, we will study a quantization of V_A interpreted as follows:

For the defining equations of V_A or the G-invariant defining ideal I_A of the closure of V_A in the ring of polynomial functions on $M(n,\mathbb{C})$, we will associate left invariant differential operators on G or an ideal J_A of the ring of the left invariant differential operators on G, which we call a *quantization* of I_A. The Lie algebra $\mathfrak{g}$ of $GL(n,\mathbb{C})$ is identified with $M(n,\mathbb{C})$ and we identify the left invariant differential operators on G with the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Then our quantization of V_A is a $U(\mathfrak{g})$-homomorphism of $U(\mathfrak{g})/J_A$ to a suitable $U(\mathfrak{g})$-module M. Note that the quantization of V_A becomes a representation space of a real form $G_{\mathbb{R}}$ of G if M is a function space on a homogeneous space of $G_{\mathbb{R}}$ or a space of sections of a $G_{\mathbb{R}}$-homogeneous vector bundle.

$$
\begin{array}{ccc}
V_A = \bigcup_{g\in G} \mathrm{Ad}(g)A & \longrightarrow & G\text{-invariant defining ideal of } V_A \\
\vdots & & \Big\downarrow {\scriptstyle \text{quantization}} \\
\text{Representations of } U(\mathfrak{g}) \text{ or } G_{\mathbb{R}} & \longleftarrow & \text{Ideal of } U(\mathfrak{g})
\end{array}
$$

The main purpose of this note is a unified explicit construction of the ideals I_A and J_A together with a study of certain properties of the ideals. Applications of the results in this paper to some representation spaces of the real form $G_{\mathbb{R}}$ will be studied in other papers since their arguments are valid for the general real reductive Lie groups. But one of the applications will be briefly explained in Example 3.3.

In §2 we introduce a homogenized universal enveloping algebra $U^\epsilon(\mathfrak{g})$ to study our quantization together with "the classical limit"($\epsilon = 0$). We construct generators of J_A from the generalized Capelli operators introduced by [15] which can be considered as quantizations of minors and we show in Theorem 2.9 that they generate the annihilator of a generalized Verma module induced from a character of a parabolic subalgebra of $\mathfrak{g}$. In fact, we give an explicit generator system of the annihilator of every generalized Verma module of $\mathfrak{gl}(n,\mathbb{C})$ of the scalar type. When

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$\epsilon = 0$ and moreover A is a nilpotent matrix, the corresponding result is Tanisaki's conjecture in [17], which is solved by Weyman [18]. In particular, if $\epsilon = 0$ and A is a regular nilpotent matrix, the result is due to Kostant [Ko].

In §3 we examine how the annihilator determines the gap between the generalized Verma module and the usual Verma module, which is important for applications. For example, Theorem 3.1 assures that the theorem on boundary value problems for symmetric spaces studied in [15, Theorem 5.1] is improved by the generator system defined in this note (cf. Example 3.3 and [16, §5]).

A similar construction of the annihilator using quantized Pfaffian in the case when $\mathfrak{g} = \mathfrak{o}(n)$ is studied by [12].

On the other hand, we can also quantize the minimal polynomial of V_A from which we can construct another generator system of the annihilator. This is valid for any reductive Lie algebra and is studied in [16] and [13].

There are other papers examining the generators of annihilator of a generalized Verma module induced from a character of a parabolic subalgebra. In particular [6], [7], [8] etc. study generators of the annihilator which span the adjoint representation of $\mathfrak{g}$. But their generators are less explicit and there are some assumptions on the character.

2. Elementary divisors

The Lie algebra $\mathfrak{g}$ of $G = GL(n,\mathbb{C})$ is identified with $M(n,\mathbb{C})$ and also with the space of left G-invariant holomorphic vector fields on G. Then $\mathfrak{g}$ is spanned by E_{ij} for $1 \le i \le n$ and $1 \le j \le n$ where E_{ij} is the fundamental matrix unit whose (p,q)-component equals $\delta_{i,p}\delta_{j,q}$ and

$$E_{ij} = \sum_{\nu=1}^{n} x_{\nu i}\frac{\partial}{\partial x_{\nu j}} \tag{2.1}$$

with the coordinate $\bigl(x_{ij}\bigr) \in G$. Then $\mathfrak{g}$ is naturally a $(\mathfrak{g}, G)$-module.

Using the non-degenerate symmetric bilinear form $\langle X, Y\rangle = \operatorname{Trace}(XY)$ on $M(n,\mathbb{C}) \times M(n,\mathbb{C})$, we identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. The dual basis $\{E_{ij}^*\}$ of $\{E_{ij}\}$ is given by $E_{ij}^* = E_{ji}$. For simplicity, we will denote $E_i = E_{ii}$ and $e_i = E_{ii}^*$.

Definition 2.1. The *homogenized universal enveloping algebra* $U^\epsilon(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined by

$$U^\epsilon(\mathfrak{g}) = \left(\sum_{k=0}^{\infty} \bigotimes^{k} \mathfrak{g}\right) \big/ \bigl\langle X\otimes Y - Y\otimes X - \epsilon[X,Y];\ X,\ Y \in \mathfrak{g}\bigr\rangle \tag{2.2}$$

and the subalgebra formed by the G-invariants in $U^\epsilon(\mathfrak{g})$ is denoted by $U^\epsilon(\mathfrak{g})^G$. Here ϵ is a complex number (or an element commuting with $\mathfrak{g}$) and the denominator is the span as a two-sided ideal of the tensor algebra of $\mathfrak{g}$ which equals the numerator.

Note that $U^\epsilon(\mathfrak{g})$ is naturally a $(\mathfrak{g}, G)$-module whose structure is induced from the map $\operatorname{Ad}(g)$ of $\mathfrak{g}$. $U^1(\mathfrak{g})$ and $U^0(\mathfrak{g})$ are the universal enveloping algebra $U(\mathfrak{g})$ and the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$, respectively. If $\epsilon \ne 0$, the map defined by $E_{ij} \mapsto \epsilon E_{ij}$ gives an algebra isomorphism of $U^\epsilon(\mathfrak{g})$ onto $U(\mathfrak{g})$.

The residue class of the element $X_1 \otimes X_2 \otimes \cdots \otimes X_m$ ($X_j \in \mathfrak{g}$) in $U^\epsilon(\mathfrak{g})$ will be denoted by $X_1X_2\cdots X_m$ and the image of $\sum_{k=0}^m \otimes^k \mathfrak{g}$ in $U^\epsilon(\mathfrak{g})$ is denoted by $U^\epsilon(\mathfrak{g})^{(m)}$.

For an ordered partition $\{n'_1,\dots,n'_L\}$ of a positive integer n into L positive integers put

$$\begin{cases} n_j &= n'_1+\cdots+n'_j \quad (1\le j\le L), \quad n_0=0,\\ \Theta &= \{n_1,n_2,\dots,n_L\},\\ \iota_\Theta(\nu) &= j \quad \text{if } n_{j-1}<\nu\le n_j \quad (1\le\nu\le n). \end{cases} \tag{2.3}$$

The ordered partition of n is expressed by the set Θ of strictly increasing positive integers ending at n. Define Lie subalgebras $\mathfrak{n}_\Theta$, $\bar{\mathfrak{n}}_\Theta$ and $\mathfrak{m}_\Theta$ by the span of E_{ij} with $\iota_\Theta(i)>\iota_\Theta(j)$, $\iota_\Theta(i)<\iota_\Theta(j)$ and $\iota_\Theta(i)=\iota_\Theta(j)$, respectively, and put $\mathfrak{p}_\Theta=\mathfrak{m}_\Theta+\mathfrak{n}_\Theta$. We denote $\mathfrak{m}^k_\Theta=\sum_{\iota_\Theta(i)=\iota_\Theta(j)=k}\mathbb{C}E_{ij}$, $\mathfrak{n}=\sum_{1\le j<i\le n}\mathbb{C}E_{ij}$, $\bar{\mathfrak{n}}=\sum_{1\le i<j\le n}\mathbb{C}E_{ij}$, $\mathfrak{a}=\sum_{j=1}^n\mathbb{C}E_i$ and $\mathfrak{p}=\mathfrak{a}+\mathfrak{n}$. Then $\mathfrak{m}_\Theta=\mathfrak{m}^1_\Theta\oplus\cdots\oplus\mathfrak{m}^L_\Theta$ and $\mathfrak{p}_\Theta$ is a *parabolic subalgebra* containing the *Borel subalgebra* $\mathfrak{p}$. We remark that $\mathfrak{p}_\Theta=\{X\in\mathfrak{g};\ \langle X,Y\rangle=0\ (\forall Y\in\mathfrak{n}_\Theta)\}$.

Fix $\lambda=(\lambda_1,\dots,\lambda_L)\in\mathbb{C}$ and define a closed affine subset of $\mathfrak{p}$:

$$\begin{aligned} A_{\Theta,\lambda} &= \sum_{j=1}^n \lambda_{\iota_\Theta(j)}E_j+\mathfrak{n}_\Theta\\ &= \left\{\begin{pmatrix} \lambda_1 I_{n'_1} & & & & 0\\ A_{21} & \lambda_2 I_{n'_2} & & & \\ A_{31} & A_{32} & \lambda_3 I_{n'_3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ A_{L1} & A_{L2} & A_{L3} & \cdots & \lambda_L I_{n'_L}\end{pmatrix};\ A_{ij}\in M(n'_i,n'_j;\mathbb{C})\right\}. \end{aligned} \tag{2.4}$$

Here I_m denotes the identity matrix of size m and $M(k,\ell;\mathbb{C})$ denotes the space of matrices of size $k\times\ell$ with components in $\mathbb{C}$.

Remark 2.2. It is easy to see that the generic element of $A_{\Theta,\lambda}$ has the Jordan canonical form

$$\bigoplus_{\mu\in\mathbb{C},\ 1\le k\le n} J(\#\{i;\ \lambda_i=\mu \text{ and } n_i\ge k\},\mu) \tag{2.5}$$

$$\textit{with}\quad J(m,\mu)=\begin{pmatrix}\mu & & & 0\\ 1 & \mu & & \\ & \ddots & \ddots & \\ & & 1 & \mu\end{pmatrix}\in M(m,\mathbb{C})$$

and any Jordan canonical form is obtained in this way with a suitable choice of Θ and λ.

If $\epsilon=0$, for $f\in U^0(\mathfrak{g})=S(\mathfrak{g})$ we have

$$\begin{aligned} f\Bigl(\bigcup_{g\in G}\mathrm{Ad}(g)A_{\Theta,\lambda}\Bigr)=0 &\Longleftrightarrow \bigl(\mathrm{Ad}(g)f\bigr)(A_{\Theta,\lambda})=0 \qquad (\forall g\in G)\\ &\Longleftrightarrow \mathrm{Ad}(g)f\in J^\epsilon_\Theta(\lambda) \qquad (\forall g\in G)\\ &\Longleftrightarrow f\in \mathrm{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr) \end{aligned}$$

with $\epsilon=0$, where

$$\begin{aligned} J^\epsilon_\Theta(\lambda) &= \sum_{X\in\mathfrak{p}_\Theta} U^\epsilon(\mathfrak{g})(X-\lambda_\Theta(X)),\\ M^\epsilon_\Theta(\lambda) &= U^\epsilon(\mathfrak{g})/J^\epsilon_\Theta(\lambda),\\ \mathrm{Ann}\bigl(M^\epsilon_\Theta(\lambda)\bigr) &= \{D\in U^\epsilon(\mathfrak{g});\ DM^\epsilon_\Theta(\lambda)=0\},\\ \mathrm{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr) &= \{D\in U^\epsilon(\mathfrak{g});\ \mathrm{Ad}(g)D\in\mathrm{Ann}\bigl(M^\epsilon_\Theta(\lambda)\bigr)\ (\forall g\in G)\} \end{aligned} \tag{2.6}$$

and the character λ_Θ of $\mathfrak{p}_\Theta$ is defined by

$$\lambda_\Theta(Y+\sum_{k=1}^L X_k)=\sum_{k=1}^L \lambda_k \operatorname{Trace}(X_k) \quad \text{for } X_k\in\mathfrak{m}_\Theta^k \text{ and } Y\in\mathfrak{n}_\Theta. \tag{2.7}$$

When $\epsilon=1$, $M_\Theta(\lambda)=M_\Theta^1(\lambda)$ is a *generalized Verma module* induced from the character λ_Θ of $\mathfrak{m}_\Theta$, which is a quotient of the *Verma module*

$$M(\lambda_\Theta)=U(\mathfrak{g})/J(\lambda_\Theta) \tag{2.8}$$

with

$$J^\epsilon(\lambda_\Theta)=\sum_{X\in\mathfrak{p}} U^\epsilon(\mathfrak{g})\bigl(X-\lambda_\Theta(X)\bigr) \text{ and } J(\lambda_\Theta)=J^1(\lambda_\Theta). \tag{2.9}$$

In general we will omit the superfix ϵ when $\epsilon=1$.

Proposition 2.3. *Under the definition* (2.6)

$$\operatorname{Ann}_G\left(M_\Theta^\epsilon(\lambda)\right)=\operatorname{Ann}\left(M_\Theta^\epsilon(\lambda)\right) \quad \textit{if } \epsilon\neq 0, \tag{2.10}$$

$$\operatorname{Ann}_G\left(M_\Theta^\epsilon(\lambda)\right)=\bigcap_{g\in G}\operatorname{Ad}(g)J_\Theta^\epsilon(\lambda). \tag{2.11}$$

Proof. We may assume $\epsilon\neq 0$ to prove the proposition.

Let $D\in\operatorname{Ann}\bigl(M_\Theta^\epsilon(\lambda)\bigr)$. Then for $X\in\mathfrak{g}$ and $v\in M_\Theta^\epsilon(\lambda)$, $(XD-DX)v=X(Dv)-D(Xv)=0$ and therefore $XD-DX\in\operatorname{Ann}\bigl(M_\Theta^\epsilon(\lambda)\bigr)$. Since $XD-DX=\epsilon\operatorname{ad}(X)D$ in $U^\epsilon(\mathfrak{g})$, $\operatorname{ad}(X)D\in\operatorname{Ann}\bigl(M_\Theta^\epsilon(\lambda)\bigr)$ and therefore $\operatorname{Ad}(g)D\in\operatorname{Ann}\bigl(M_\Theta^\epsilon(\lambda)\bigr)$ for $g\in G$.

Put $I=\bigcap_{g\in G}\operatorname{Ad}(g)J_\Theta^\epsilon(\lambda)$. Since $\operatorname{Ann}(M_\Theta^\epsilon(\lambda))\subset J_\Theta^\epsilon(\lambda)$, $\operatorname{Ann}_G\bigl(M_\Theta^\epsilon(\lambda)\bigr)\subset I$. For $P\in U^\epsilon(\mathfrak{g})$, $IP=PI\equiv 0 \mod J_\Theta^\epsilon(\lambda)$ because I is a two-sided ideal of $U^\epsilon(\mathfrak{g})$, which means $I\subset\operatorname{Ann}\bigl(M_\Theta^\epsilon(\lambda)\bigr)$ and therefore $I\subset\operatorname{Ann}_G\bigl(M_\Theta^\epsilon(\lambda)\bigr)$. ☐

Definition 2.4. Define the polynomials and integers

$$\begin{cases} d_m^\epsilon(x)=d_m^\epsilon(x;\Theta,\lambda)=\prod_{j=1}^L\bigl(x-\lambda_j-n_{j-1}\epsilon\bigr)^{(n'_j+m-n)},\\ d_m=d_m(\Theta)=\deg_x d_m^\epsilon(x;\Theta,\lambda)=\sum_{j=1}^L\max\{n'_j+m-n,0\},\\ e_m^\epsilon(x)=e_m^\epsilon(x;\Theta,\lambda)=d_m^\epsilon(x)/d_{m-1}^\epsilon(x),\\ q^\epsilon(x)=q^\epsilon(x;\Theta,\lambda)=\prod_{j=1}^L\bigl(x-\lambda_j-n_{j-1}\epsilon\bigr)\end{cases} \tag{2.12}$$

for $m=1,\ldots,m$ by putting

$$z^{(\ell)}=\begin{cases} z\bigl(z-\epsilon\bigr)\cdots\bigl(z-(\ell-1)\epsilon\bigr) & \text{if } \ell>0,\\ 1 & \text{if } \ell\leq 0\end{cases} \tag{2.13}$$

and call $d_n^\epsilon(x)$, $q^\epsilon(x)$ and $\{e_m^\epsilon(x);\,1\leq m\leq n\}$ the *characteristic polynomial*, the *minimal polynomial* and the *elementary divisors* of $M_\Theta^\epsilon(\lambda)$, respectively.

Remark 2.5. i) The set $\{e_m^\epsilon(x);\,1\leq m\leq n\}$ recovers $\{d_m^\epsilon(x);\,1\leq m\leq n\}$. Note that $e_m^\epsilon(x)\in\mathbb{C}[x]e_{m-1}^\epsilon(x)\cap\mathbb{C}[x]e_{m-1}^\epsilon(x-\epsilon)$.

ii) For the generic element A of $J_\Theta^0(\lambda)$, the greatest common divisor of m-minors of the matrix xI_n-A equals $d_m^0(x)$ and therefore when $\epsilon=0$, the above definition coincides with that in the linear algebra.

iii) The meaning of the minimal polynomial for $\epsilon\neq 0$ will be clear in [16].

Now we introduce quantized *minors*.

Definition 2.6. For set of indices $I = \{i_1, \dots, i_m\}$ and $J = \{j_1, \dots, j_m\}$ with i_μ, $j_\nu \in \{1, \dots, n\}$, define a *generalized Capelli operator* (cf. [15])

$$\det{}^\epsilon(x; E_{IJ}) = \det\Big(\big(x + (\nu - m)\epsilon\big)\delta_{i_\mu j_\nu} - E_{i_\mu j_\nu}\Big)_{\substack{1\le\mu\le m\\ 1\le\nu\le m}} \tag{2.14}$$

in $U^\epsilon(\mathfrak{g})[x]$ by the column determinant:

$$\det\Big(A_{\mu\nu}\Big)_{\substack{1\le\mu\le m\\ 1\le\nu\le m}} = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}(\sigma) A_{\sigma(1)1}A_{\sigma(2)2}\cdots A_{\sigma(m)m}. \tag{2.15}$$

Proposition 2.7. *The Capelli operators satisfy*

$$\det{}^\epsilon(x; E_{\sigma(I)\sigma'(J)}) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')\det{}^\epsilon(x; E_{IJ}) \quad \text{for } \sigma,\ \sigma' \in \mathfrak{S}_m, \tag{2.16}$$

$$\operatorname{ad}(E_{ij})\det{}^\epsilon(x; E_{IJ}) = D_1 - D_2 \tag{2.17}$$

where

$$\sigma(I) = \{i_{\sigma(1)}, \dots, i_{\sigma(m)}\}, \quad \sigma'(J) = \{j_{\sigma'(1)}, \dots, j_{\sigma'(m)}\},$$

$$D_1 = \begin{cases} \det{}^\epsilon(x; E_{\{i_1,\dots,i_{\mu-1},j,i_{\mu+1},\dots,i_m\}J}) & \text{if there exists only one } i_\mu \text{ with } i_\mu = j,\\ 0 & \text{otherwise,}\end{cases}$$

$$D_2 = \begin{cases} \det{}^\epsilon(x; E_{I\{j_1,\dots,j_{\nu-1},i,j_{\nu+1},\dots,j_m\}}) & \text{if there exists only one } j_\nu \text{ with } j_\nu = i,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. When $\epsilon = 1$, (2.16) and (2.17) are proved by [15, Lemma 2.2 and Proposition 2.4]. Combining this with the definition of $U^\epsilon(\mathfrak{g})$, we have the proposition. □

Definition 2.8. Under Definition 2.4 and Definition 2.6, put

$$\det{}^\epsilon(x; E_{IJ}) = h_{IJ}(x)d_m^\epsilon(x) + r_{IJ}^{d_m-1}x^{d_m-1} + \cdots + r_{IJ}^1 x + r_{IJ}^0 \tag{2.18}$$

in $U^\epsilon(\mathfrak{g})[x]$ with $h_{IJ}[x] \in U^\epsilon(\mathfrak{g})[x]$ and $r_{IJ}^j \in U^\epsilon(\mathfrak{g})^{(m-j)}$ for $j = 0, \dots, d_m - 1$ and define the two-sided ideal of $U^\epsilon(\mathfrak{g})$:

$$I_\Theta^\epsilon(\lambda) = \sum_{m=1}^{n}\sum_{\#I=\#J=m}\sum_{j=0}^{d_m-1} U^\epsilon(\mathfrak{g})r_{IJ}^j \tag{2.19}$$

Note that if $m \le n - \max\{n'_1, \dots, n'_L\}$ the summand equals 0 because $d_m = 0$. Moreover note that r_{IJ}^j with $\#I = \#J = n$ and $0 \le j < n$ are in $U^\epsilon(\mathfrak{g})^G$ by Proposition 2.7. In particular, if $\Theta = \{1, 2, \dots, n\}$, then $\mathfrak{p}_\Theta = \mathfrak{p}$ and $I_\Theta^\epsilon(\lambda)$ is generated by suitable n elements in $U^\epsilon(\mathfrak{g})^G$.

Now we can state the main result in this section and we call r_{IJ}^j *quantized Tanisaki generators* of $\operatorname{Ann}_G\big(M_\Theta^\epsilon(\lambda)\big)$. In the case when $\epsilon = \lambda = 0$, $d_m^0(x; \Theta, 0) = x^{d_m}$ and the generators r_{IJ}^j are introduced by [17].

Theorem 2.9. *Under the notation* (2.6) *and* (2.19)

$$\operatorname{Ann}_G\big(M_\Theta^\epsilon(\lambda)\big) = I_\Theta^\epsilon(\lambda).$$

If all the roots of $d_n^\epsilon(x) = 0$ are simple, which is equivalent to say that the infinitesimal character of $M_\Theta^\epsilon(\lambda)$ is regular (cf. Remark 2.15)*, then*

$$\operatorname{Ann}_G\big(M_\Theta^\epsilon(\lambda)\big) = \sum_{k=1}^{L}\sum_{\#I=\#J=n+1-n'_k} U^\epsilon(\mathfrak{g})D_{IJ}^\epsilon(\lambda_k + n_{k-1}\epsilon). \tag{2.20}$$

Here for $I = \{i_1, \dots, i_m\}$ and $J = \{j_1, \dots, j_m\}$ we put

$$D_{IJ}^\epsilon(x) = (-1)^m\det{}^\epsilon(x; E_{IJ}) = \det\Big(E_{i_\mu j_\nu} - (x + (\nu - m)\epsilon)\delta_{i_\mu j_\nu})\Big)_{\substack{1\le\mu\le m\\ 1\le\nu\le m}}. \tag{2.21}$$

If all the roots of $d^\epsilon_{n-1}(x)=0$ *are simple,* (2.20) *holds modulo the ideal generated by* $\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)\cap U^\epsilon(\mathfrak{g})^G$.

When $\epsilon=0$, (2.20) *holds if* $\lambda_i\neq\lambda_j$ *for* $1\le i<j\le L$ *and the last statement above holds if* $\lambda_i\neq\lambda_j$ *for* $1\le i<j\le L$ *satisfying* $n'_i>1$ *and* $n'_j>1$.

Remark 2.10. Let $\{\lambda'_1,\dots,\lambda'_k\}$ be the set of the roots of $d^\epsilon_m(x)=0$ and let m_k be the multiplicity of the root λ'_k. Here $d_m=m_1+\cdots+m_k$ and $\lambda'_\mu\neq\lambda'_\nu$ if $1\le\mu<\nu\le k$. Then

$$\sum_{j=0}^{d_m-1}\mathbb{C}r^j_{IJ}=\sum_{i=1}^{k}\sum_{j=1}^{m_i}\mathbb{C}\Bigl(\frac{d^{j-1}}{dx^{j-1}}D^\epsilon_{IJ}(x)\Bigr)\Big|_{x=\lambda'_i} \tag{2.22}$$

for $\#I=\#J=m$.

The remaining part in this section will be devoted to the proof of this theorem until Remark 2.17. First we will examine the image of our minors under the Harish-Chandra homomorphism.

Define the map ω of $U^\epsilon(\mathfrak{g})$ to $S(\mathfrak{a})=U^\epsilon(\mathfrak{a})$ by

$$D-\omega(D)\in U^\epsilon(\mathfrak{g})\mathfrak{n}+\bar{\mathfrak{n}}U^\epsilon(\bar{\mathfrak{n}}+\mathfrak{a}). \tag{2.23}$$

Fix $I=\{i_1,\dots,i_m\}$ and $J=\{j_1,\cdots,j_m\}$ with $1\le i_1<i_2<\cdots<i_m\le n$ and $1\le j_1<j_2<\cdots<j_m\le n$. Then [15, Corollary 2.11] in the case $\epsilon=1$ shows

$$\omega\bigl(D^\epsilon_{IJ}(x)\bigr)=\begin{cases}0 & \text{if } I\neq J,\\ \prod\limits_{\nu=1}^{m}\bigl(E_{i_\nu}-x+(\nu-1)\epsilon\bigr) & \text{if } I=J\end{cases} \tag{2.24}$$

under the notation in Theorem 2.9. Introducing the algebra isomorphism

$$\begin{gathered}\bar{\ }\colon S(\mathfrak{a})\to S(\mathfrak{a})\\ \text{with }\ \bar{E}_j=E_j-\bigl(-\tfrac{n-1}{2}+(j-1)\bigr)\epsilon\ \ \text{for } j=1,\dots,n\end{gathered} \tag{2.25}$$

(cf. Remark 2.15), put

$$\bar{\omega}(P)=\overline{\omega(P)}. \tag{2.26}$$

Then $\bar\omega$ defines the Harish-Chandra isomorphism of $U^\epsilon(\mathfrak{g})^G$ onto the algebra $S(\mathfrak{a})^W$ of $\mathfrak{S}_n$-invariants in $S(\mathfrak{a})$. Here we note that if $I=\{i_1<i_2<\cdots<i_m\}$,

$$\bar\omega\bigl(D^\epsilon_{II}(x)\bigr)=\prod_{\nu=1}^{m}\bigl(E_{i_\nu}-x+(\tfrac{n-1}{2}+\nu-i_\nu)\epsilon\bigr). \tag{2.27}$$

Since $D^\epsilon_{\{1,\dots,n\}\{1,\dots,n\}}(x)\in U^\epsilon(\mathfrak{g})^G[x]$ (cf. Proposition 2.7), it is clear that the coefficients of $D^\epsilon_{\{1,\dots,n\}\{1,\dots,n\}}(x)$ as a polynomial of x generate the algebra $U^\epsilon(\mathfrak{g})^G$.

Lemma 2.11. *Let* $\mathfrak{g}=\bar{\mathfrak{n}}\oplus\mathfrak{a}\oplus\mathfrak{n}$ *be a triangular decomposition of a reductive Lie algebra* $\mathfrak{g}$ *over* $\mathbb{C}$. *Here* $\mathfrak{n}$ *and* $\bar{\mathfrak{n}}$ *are nilpotent subalgebras of* $\mathfrak{g}$ *and* $\mathfrak{a}$ *is a Cartan subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{p}=\mathfrak{a}\oplus\mathfrak{n}$ *is a Borel subalgebra of* $\mathfrak{g}$. *For an element* D *of the universal enveloping algebra* $U(\mathfrak{g})$ *of* $\mathfrak{g}$, *we define* $\omega(D)\in S(\mathfrak{a})$ *so that*

$$D-\omega(D)\in U(\mathfrak{g})\mathfrak{n}+\bar{\mathfrak{n}}U(\bar{\mathfrak{n}}+\mathfrak{a}). \tag{2.28}$$

For a subspace V *of* $U(\mathfrak{g})$ *put*

$$\langle\omega(V)\rangle_{S(\mathfrak{a})}=\sum_{p\in\omega(V)}S(\mathfrak{a})p. \tag{2.29}$$

Then if $\operatorname{ad}(\mathfrak{g})V\subset V$, *we have*

$$\omega(PDQ)\in\langle\omega(V)\rangle_{S(\mathfrak{a})}\quad\text{for any } P,\ Q\in U(\mathfrak{g}) \text{ and any } D\in V. \tag{2.30}$$

Proof. Let $\{X_1,\ldots,X_N\}$, $\{Y_1,\ldots,Y_N\}$ and $\{H_1,\ldots,H_M\}$ be the basis of $\mathfrak{n}$, $\bar{\mathfrak{n}}$ and $\mathfrak{a}$, respectively. Then

$$\{Y^\alpha H^\beta X^\gamma = Y_1^{\alpha_1}\cdots Y_N^{\alpha_N}H_1^{\beta_1}\cdots H_M^{\beta_M}X_1^{\gamma_1}\cdots X_N^{\gamma_N};\, \alpha\in\mathbb{N}^N,\ \beta\in\mathbb{N}^M,\ \gamma\in\mathbb{N}^N\}$$

with $\mathbb{N}=\{0,1,2,\ldots\}$ is a Poincare-Birkhoff-Witt basis of $U(\mathfrak{g})$.

Let $D\in V$. The assumption implies $PDQ\in U(\mathfrak{g})V$ and therefore we may assume $Q=1$ in (2.30). Since $XD=\mathrm{ad}(X)D+DX\in V+U(\mathfrak{g})\mathfrak{n}$ for $X\in\mathfrak{n}$, we have $X^\gamma D\in V+U(\mathfrak{g})\mathfrak{n}$. On the other hand, $Y^\alpha H^\beta D - Y^\alpha H^\beta\omega(D)\in Y^\alpha H^\beta(\bar{\mathfrak{n}}U(\bar{\mathfrak{n}}+\mathfrak{a})+U(\mathfrak{g})\mathfrak{n})\subset\bar{\mathfrak{n}}U(\bar{\mathfrak{n}}+\mathfrak{a})+U(\mathfrak{g})\mathfrak{n}$ and therefore $\omega(Y^\alpha H^\beta D)=H^\beta\omega(D)$ if $\alpha=0$ and $\omega(Y^\alpha H^\beta D)=0$ otherwise. Hence $\omega(Y^\alpha H^\beta X^\gamma D)\in\langle\omega(V)\rangle_{S(\mathfrak{a})}$ and $\omega(PD)\in\langle\omega(V)\rangle_{S(\mathfrak{a})}$ for $P\in U(\mathfrak{g})$. □

Lemma 2.12. *Under the notation in Lemma 2.11, fix $H_\Theta\in\mathfrak{a}$ so that the condition $\mathrm{ad}(H_\Theta)Y=c_YY$ with $c_Y\in\mathbb{C}$ and $Y\in\mathfrak{n}\setminus\{0\}$ means $c_Y\geq 0$. Suppose $\mathrm{ad}(H_\Theta)\mathfrak{n}\neq\{0\}$. Let m_Θ be the centralizer of H_Θ in $\mathfrak{g}$ and let $\mathfrak{n}_\Theta$ and $\bar{\mathfrak{n}}_\Theta$ be subspaces spanned by the elements Y in $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively, satisfying $\mathrm{ad}(H_\Theta)Y=c_YY$ with $c_Y\neq 0$. Then $\mathfrak{p}_\Theta=\mathfrak{m}_\Theta\oplus\mathfrak{n}_\Theta$ be a Levi decomposition of a parabolic subalgebra $\mathfrak{p}_\Theta$ containing $\mathfrak{p}$. Let $\mathfrak{a}_\Theta$ denote the center of $\mathfrak{m}_\Theta$. For an element λ of the dual $\mathfrak{a}_\Theta^*$ of $\mathfrak{a}_\Theta$ we define a character λ_Θ of $\mathfrak{p}_\Theta$ so that $\lambda_\Theta(\mathfrak{n}_\Theta+[\mathfrak{m}_\Theta,\mathfrak{m}_\Theta])=0$ and $\lambda_\Theta(H)=\lambda(H)$ for $H\in\mathfrak{a}_\Theta$. Suppose there exist $D_1(\lambda),\ldots,D_m(\lambda)$ in $U(\mathfrak{g})[\lambda]$ so that*

$$\mathrm{ad}(X)D_k(\lambda)\in\sum_{j=1}^m U(\mathfrak{g})[\lambda]D_j(\lambda)\quad\text{for } X\in\mathfrak{g}\text{ and } k=1,\ldots,m, \tag{2.31}$$

$$D_k(\lambda)\in\sum_{X\in\mathfrak{p}_\Theta}U(\mathfrak{g})[\lambda]\bigl(X-\lambda_\Theta(X)\bigr)+\bar{\mathfrak{n}}U(\mathfrak{g})[\lambda]\quad\text{for } k=1,\ldots,m. \tag{2.32}$$

Then $D_k(\lambda)\in\sum_{X\in\mathfrak{p}_\Theta}U(\mathfrak{g})[\lambda]\bigl(X-\lambda_\Theta(X)\bigr)$ and therefore $D_k(\lambda)\in\mathrm{Ann}(M_\Theta(\lambda))$ for $k=1,\ldots,m$ under the same notation as in the case $\mathfrak{g}=\mathfrak{gl}(n,\mathbb{C})$.

Proof. Retain the notation in the proof of Lemma 2.11. We may assume $\{Y_1,\ldots,Y_{N'}\}$ is a basis of $\bar{\mathfrak{n}}_\Theta$ for a suitable N'. We note that for $D\in U(\mathfrak{g})[\lambda]$

$$D\equiv\sum_{\alpha\in\mathbb{N}^{N'}}c_\alpha(D;\lambda)Y^\alpha\mod\sum_{X\in\mathfrak{p}_\Theta}U(\mathfrak{g})[\lambda]\bigl(X-\lambda_\Theta(X)\bigr). \tag{2.33}$$

Here $c_\alpha(D;\lambda)\in\mathbb{C}[\lambda]$ are uniquely determined by D because of the decomposition $U(\mathfrak{g})=U(\bar{\mathfrak{n}}_\Theta)\oplus U(\mathfrak{g})\mathfrak{p}_\Theta$.

Put $I=\sum_{k=1}^m U(\mathfrak{g})D_k(\lambda)U(\mathfrak{g})$ and $I_\lambda=\sum_{H\in\mathfrak{a}}S(\mathfrak{a})[\lambda]\bigl(H-\lambda(H)\bigr)$ and suppose $D\in I$. Then (2.32) implies $\omega\bigl(D_k(\lambda)\bigr)\in I_\lambda$ for $k=1,\ldots,m$ and therefore $\omega(PD_k(\lambda)Q)\in I_\lambda$ for $P,\ Q\in U(\mathfrak{g})$ by Lemma 2.11 which implies $c_0(D;\lambda)=\omega(D)(\lambda)=0$. Hence $IM_\Theta(\lambda)$ is a proper $\mathfrak{g}$-submodule of $M_\Theta(\lambda)$ for any fixed $\lambda\in\mathfrak{a}_\Theta^*$.

Since $M_\Theta(\lambda)$ is an irreducible $\mathfrak{g}$-module for a generic λ, $IM_\Theta(\lambda)=0$ for a generic λ. Hence $c_\alpha(D;\lambda)=0$ for $\alpha\in\mathbb{N}^{N'}$ and $D\in I$ and therefore $IM_\Theta(\lambda)=0$ for any λ. □

The following remark is clear from the argument in the proof of Lemma 2.12.

Remark 2.13. i) Let ℓ be a positive integer and let $r(\lambda,\epsilon)$ be a polynomial function of $(\lambda,\epsilon)\in\mathbb{C}^{\ell+1}$ valued in $U^\epsilon(\mathfrak{g})$. If $r(\lambda,\epsilon)\in\mathrm{Ann}_G\left(M_\Theta^\epsilon(\lambda)\right)$ for generic (λ,ϵ), then $r(\lambda,\epsilon)\in\mathrm{Ann}_G\left(M_\Theta^\epsilon(\lambda)\right)$ for any (λ,ϵ).

ii) Let p be a suitable polynomial map of $\mathbb{C}^\ell$ to $\mathfrak{a}_\Theta^*$. Replacing $D_k(\lambda)$, $U(\mathfrak{g})[\lambda]$ and λ by $D_k(\mu)$, $U(\mathfrak{g})[\mu]$ and $p(\mu)$, respectively, in Lemma 2.12, we have the same conclusion if $M_\Theta(p(\mu))$ is irreducible for generic $\mu\in\mathbb{C}^\ell$.

Remark 2.14. Use the notation in Lemma 2.11. Let $\lambda \in \mathfrak{a}^*$ and consider the Verma module $M(\lambda) = U(\mathfrak{g})/\big(U(\mathfrak{g})\mathfrak{n} + \sum_{H\in\mathfrak{a}} U(\mathfrak{g})(H - \lambda(H))\big)$. Then

$$P_\lambda = \{D \in U(\mathfrak{g});\ \omega(D)(\lambda) = \omega\big(\operatorname{ad}(X)D\big)(\lambda) = 0\ (\forall X \in \mathfrak{g})\} \tag{2.34}$$

is the annihilator $\operatorname{Ann}\big(L(\lambda)\big)$ of the unique irreducible quotient $L(\lambda)$ of $M(\lambda)$. Here we identify $S(\mathfrak{a})$ with the space of polynomial functions of $\mathfrak{a}^*$. This may be also considered to be a *quantization* of the conjugacy class of semisimple matrices.

Proof. Lemma 2.11 proves that P_λ is a two-sided ideal of $U(\mathfrak{g})$. Since the assumption means that the projection of $P_\lambda L(\lambda)$ into the highest weight space of $L(\lambda)$ vanishes, $P_\lambda L(\lambda) = 0$ because of the irreducibility of $L(\lambda)$. On the other hand, $Dv = 0$ for the highest weight vector v of $L(\lambda)$ implies $\omega(D)(\lambda) = 0$. Since $\operatorname{Ann}\big(L(\lambda)\big)$ is a two-sided ideal of $U(\mathfrak{g})$, we have $\operatorname{Ann}\big(L(\lambda)\big) \subset P_\lambda$. □

Remark 2.15. Define $\rho \in \mathfrak{a}^*$ by $\rho(X) = \frac{1}{2}\operatorname{Trace}\operatorname{ad}(H)|_{\mathfrak{n}}$ and $w.\lambda = w(\lambda+\rho) - \rho$ for the element w of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{a})$. Then the infinitesimal character of the highest weight module $M(\lambda)$ is parametrized by $W.\lambda$. We say that the infinitesimal character is *regular* if $w.\lambda \neq \lambda$ for any $w \in W$ satisfying $w \neq e$.

If $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, then

$$\rho = \big(-\tfrac{n-1}{2} + (1-1)\big)\,e_1 + \cdots + \big(-\tfrac{n-1}{2} + (n-1)\big)\,e_n, \tag{2.35}$$

$W \simeq \mathfrak{S}_n$ and

$$w\Big(\sum_{j=1}^n \mu_j e_j\Big) = \sum_{j=1}^n \mu_j e_{w(j)} = \sum_{j=1}^n \mu_{w^{-1}(j)} e_j \quad \text{for } (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n \text{ and } w \in W.$$

In $U^\epsilon(\mathfrak{g})$, ρ changes into $\rho^\epsilon = \epsilon\rho$ and the infinitesimal character of $M^\epsilon_\Theta(\lambda)$ equals that of $M^\epsilon(\lambda_\Theta)$. Hence the infinitesimal character is regular if and only if all the roots of $d^\epsilon_n(x) = 0$ are simple because the set of roots is $\{\bar\lambda_\nu + \frac{n-1}{2}\epsilon;\ \nu = 1, \ldots, n\}$ by putting

$$\lambda_\Theta + \rho^\epsilon = \bar\lambda_1 e_1 + \cdots + \bar\lambda_n e_n. \tag{2.36}$$

Here we note that

$$\bar\lambda_\nu = \lambda_k + \big(-\frac{n-1}{2} + (\nu - 1)\big)\epsilon \quad \text{if} \quad n_{k-1} < \nu \leq n_k. \tag{2.37}$$

Lemma 2.16. *Let $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_{m-1}\}$ be sets of positive numbers with $m > 0$, $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_{m-1}$. Then there exists a positive integer $\mu \leq m$ such that $\#\{j \in J;\ j < i_\mu\} = \mu - 1$ and $i_\mu \notin J$.*

Proof. Suppose $m > 1$ since the lemma is clear when $m = 1$. If $j_{m-1} < i_m$, we can put $\mu = m$. If $j_{m-1} \geq i_m$, we can reduce to the case when $I = \{i_1, \ldots, i_{m-1}\}$ and $J = \{j_1, \ldots, j_{m-2}\}$. □

Retain the notation in Theorem 2.9. Fix k with $1 \leq k \leq L$ and put $m = n+1-n'_k$ and $J = \{1, 2, \ldots, n\} \setminus \{n_{k-1}+1, n_{k-1}+2, \ldots, n_k\}$. Note that $\#J = m - 1$. For $I = \{i_1, \ldots, i_m\}$ with $1 \leq i_1 < \cdots < i_m \leq n$, choose an integer μ as in Lemma 2.16. Then $n_{k-1} < i_\mu \leq n_k$ and $\#\{1, 2, \ldots, n_{k-1}\} = \mu - 1$, from which we have $\mu = n_{k-1} + 1$ and $\lambda(E_{i_\mu}) - (\lambda_k + n_{k-1}\epsilon) + (\mu - 1)\epsilon = 0$ and therefore (2.24) and (2.16) show

$$\omega\big(D^\epsilon_{IJ}(\lambda_k + n_{k-1}\epsilon)\big) \in \sum_{H\in\mathfrak{a}} S(\mathfrak{a})\big(H - \lambda(H)\big) \quad \text{if } \#I = \#J = n + 1 - n'_k. \tag{2.38}$$

Denoting

$$J(m, x) = \sum_{\#I=\#J=m} \mathbb{C} D^\epsilon_{IJ}(x), \tag{2.39}$$

the basis of $J(n+1-n'_k, \lambda_k + n_{k-1}\epsilon)$ satisfies the assumption in Lemma 2.12 and then

$$J(n+1-n'_k, \lambda_k + n_{k-1}\epsilon) \subset \operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr) \quad \text{for } k = 1, \ldots, L \tag{2.40}$$

in the case when $\epsilon = 1$. But this holds for any ϵ because of Remark 2.13 i) with the isomorphism between $U(\mathfrak{g})$ and $U^\epsilon(\mathfrak{g})$.

Now the Laplace expansions of $D^\epsilon_{IJ}(x)$ with respect to the first and the last columns show (cf. [15, Proposition 2.6 i)])

$$J(m+1, \lambda) + J(m+1, \lambda + \epsilon) \subset U^\epsilon(\mathfrak{g})J(m, \lambda) \quad \text{if } m < n \tag{2.41}$$

and therefore

$$J(n+1-n'_k+j, \lambda_k + (n_{k-1}+i)\epsilon) \in \operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr) \quad \text{for } 0 \le i \le j \le n'_k - 1. \tag{2.42}$$

Hence if $c \in \mathbb{C}$ satisfies $d^\epsilon_m(c;\lambda) = 0$, then $\det^\epsilon_m(c; E_{IJ}) \in I^\epsilon_\Theta(\lambda)'$ for $\#I = \#J = m$ under the notation

$$I^\epsilon_\Theta(\lambda)' = \sum_{k=1}^{L} U^\epsilon(\mathfrak{g})J(n+1-n'_k, \lambda_k + n_{k-1}\epsilon). \tag{2.43}$$

We have proved

$$I^\epsilon_\Theta(\lambda)' \subset I^\epsilon_\Theta(\lambda) \ \text{ and } \ I^\epsilon_\Theta(\lambda)' \subset \operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr). \tag{2.44}$$

Moreover we have $I^\epsilon_\Theta(\lambda)' = I^\epsilon_\Theta(\lambda)$ if all the roots of $d^\epsilon_m(x;\lambda) = 0$ are simple for $m = 1, \ldots, n$ (cf. Remark 2.10). Hence it follows from Remark 2.13 i) that

$$I^\epsilon_\Theta(\lambda) \subset \operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr). \tag{2.45}$$

Note that the elements r^j_{IJ} for $\#I = \#J = n$ in (2.18) are contained in $J^\epsilon(\lambda_\Theta)$ because they are in the center $U^\epsilon(\mathfrak{g})^G$ of $U^\epsilon(\mathfrak{g})$ (cf. (2.11)).

Thus we have only to show $I^\epsilon_\Theta(\lambda) \supset \operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)$ to complete the proof of Theorem 2.9. We can prove this for generic λ with $\epsilon \neq 0$ using the result in the next section (cf. [16]) or Theorem 2.22 but here we reduce it to the claim

$$I^0_\Theta(0) = \operatorname{Ann}_G\bigl(M^0_\Theta(0)\bigr). \tag{2.46}$$

For $\epsilon = \lambda = 0$, this is conjectured by [17] and is proved by [18]. In this case $r^j_{IJ} \in S(\mathfrak{g})$ are of homogeneous polynomials of $\mathfrak{g}^*$ with degree $\#I - j$. Here we note that $\det{}^\epsilon(x; E_{IJ})$ is homogeneous of degree $\#I$ with respect to $(x, \mathfrak{g}, \epsilon, \lambda)$, which is well-defined under any choice of Poincare-Birkhoff-Witt basis because of the definition of the homogenized universal enveloping algebra.

Let $S(\mathfrak{g})_m$ be the space of homogeneous elements of $S(\mathfrak{g})$ with degree m. Then $U^\epsilon(\mathfrak{g})^{(m)}/U^\epsilon(\mathfrak{g})^{(m-1)} \simeq S(\mathfrak{g})_m$ and for $D \in U^\epsilon(\mathfrak{g})^{(m)}$, we denote by $\sigma_m(D)$ the corresponding element in $S(\mathfrak{g})_m$. Note that $\sigma_{\#I-j}(r^j_{IJ})$ in (2.18) does not depend on λ and ϵ. Hence

$$I^0_\Theta(0) = \sum_{m=n+1-\max\{n'_1,\ldots,n'_L\}}^{n} \ \sum_{\#I=\#J=m} \ \sum_{j=0}^{d_m-1} S(\mathfrak{g})\sigma_{m-j}(r^j_{IJ}). \tag{2.47}$$

Put $R^\epsilon(\lambda)^{(m)} = \operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr) \cap U^\epsilon(\mathfrak{g})^{(m)}$ and $D \in R^\epsilon(\lambda)^{(m)} \setminus R^\epsilon(\lambda)^{(m-1)}$. We will prove $D \in I^\epsilon_\Theta(\lambda)$ by the induction on m. Since (2.11) implies $\operatorname{Ad}(g)D \equiv 0 \mod U^\epsilon(\mathfrak{g})^{(m-1)}\mathfrak{p}_\Theta + U^\epsilon(\mathfrak{g})^{(m-1)}$, we have

$$\sigma_m(D)\bigl(\operatorname{Ad}(g)\mathfrak{n}_\Theta\bigr) = 0 \quad (\forall g \in G) \tag{2.48}$$

and $\sigma_m(D) \in I^0_\Theta(0)$. Hence it follows from (2.46) and (2.47) that there exist homogeneous elements $p^j_{IJ} \in S(\mathfrak{g})$ satisfying $\sigma_m(D) = \sum p^j_{IJ}\sigma_{\#I-j}(r^j_{IJ})$. Here r^j_{IJ} are generators of $I^\epsilon_\Theta(\lambda)$ appeared in (2.18) and $\deg(p^j_{IJ}) + \#I - j = m$ if $p^j_{IJ} \neq 0$. Let $P^j_{IJ} \in U^\epsilon(\mathfrak{g})^{(m-\#I+j)}$ with $\sigma_{m-\#I+j}(P^j_{IJ}) = p^j_{IJ}$ and put $D' = \sum P^j_{IJ}D^j_{IJ}$. Then

$D' \in I^\epsilon_\Theta(\lambda)$ and $D - D' \in R^\epsilon(\lambda)^{(m-1)}$ and therefore we have $D - D' \in I^\epsilon_\Theta(\lambda)$ by the hypothesis of the induction. Thus we have completed the proof of Theorem 2.9. □

Remark 2.17. The procedure to deform λ to 0 under the classical limit $\epsilon = 0$ is studied by [3].

In the proof of Theorem 2.9 we have shown the following, which is proved by [2] together with the fact that it is not valid for a generalized Verma module of a general semisimple Lie algebra induced from a character of a parabolic subalgebra.

Corollary 2.18. *The graded ring* $\operatorname{gr}\bigl(\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)\bigr) = \bigoplus_{m=0}^{\infty}\bigl(\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr) \cap U^\epsilon(\mathfrak{g})^{(m)}\bigr)/\bigl(\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr) \cap U^\epsilon(\mathfrak{g})^{(m-1)}\bigr)$ *equals the defining ideal of the closure of the nilpotent conjugacy class of the generic element* $A_{\Theta,0}$ *of the form* (2.4)*. In particular it is a prime ideal and does not depend on* (λ, ϵ)*.*

Corollary 2.19. *The following two conditions are equivalent.*

$$\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr) \supset \operatorname{Ann}_G\bigl(M^\epsilon_{\Theta'}(\lambda')\bigr). \tag{2.49}$$

$$d^\epsilon_m(x;\Theta,\lambda) \in \mathbb{C}[x] d^\epsilon_m(x;\Theta',\lambda') \quad \text{for } m = 1,\ldots,n. \tag{2.50}$$

Proof. It follows from Theorem 2.9 that the latter condition implies the former. Hence suppose the first condition. Let $f_m(x)$ be the least common multiple of $d^\epsilon_m(x;\Theta,\lambda)$ and $d^\epsilon_m(x;\Theta',\lambda')$. Then if $\#I = \#J = m$, $\det^\epsilon(x;E_{IJ}) \in U^\epsilon(\mathfrak{g}) f_m(x)$ mod $\mathbb{C}[x] \otimes \operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)$. Applying σ_m to this relation as in the proof of Theorem 2.9, we have $\det^0(x;E_{IJ}) \in S(\mathfrak{g}) x^{\deg(f_m)}$ mod $\mathbb{C}[x] \otimes \operatorname{Ann}_G\bigl(M^0_\Theta(0)\bigr)$ because of the homogeneity with respect to $(x, \mathfrak{g}, \epsilon, \lambda)$. Let $A_{\Theta,0}$ be the generic element of the form (2.4) and let J_Θ be the maximal ideal of $S(\mathfrak{g})$ corresponding to $A_{\Theta,0}$. Considering under modulo J_Θ, we can conclude that all the m-minors of the matrix $\bigl(x - A_{\Theta,0}\bigr)$ are in $\mathbb{C}[x] x^{\deg(f_m)}$. On the other hand, $x^{d_m(\Theta)}$ is the greatest common divisors of m-minors of $\bigl(x - A_{\Theta,0}\bigr)$ and therefore $\deg f_m(x) \le d_m(\Theta) = \deg d^\epsilon_m(x;\Theta,\lambda)$ and we have the latter condition. □

Remark 2.20. If $\epsilon = 0$, Corollary 2.19 gives the closure relation in the conjugacy classes of the matrices.

Remark 2.21. i) The following theorem is a part of a conjecture proposed by [14] for the general symmetric pair. The case in this note corresponds to the pair $(GL(n,\mathbb{C}), U(n))$.

ii) When $\operatorname{Ann}_G(M_\Theta(\lambda))$ is realized as a system of differential equations (cf. Example 3.3) on a Riemannian symmetric space of the non-compact type, the following theorem describes the characteristic exponents of the system along the boundary and hence the boundary value of the solutions of the system vanishes with respect to some exponents.

iii) In the case of the classical limit $\epsilon = \lambda = 0$, the following theorem is obtained by [4] and [17].

Theorem 2.22. *Let* W_Θ *be the Weyl group of* $\mathfrak{m}_\Theta$ *and let* $W = W(\Theta)W_\Theta$ *be the decomposition of* $W = \mathfrak{S}_n$ *so that* $W(\Theta)$ *is the set of the representatives of* W/W_Θ *with the minimal length. Then the common zeros of* $\omega\bigl(\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)\bigr)$ *coincides with the set* $\{w.\lambda_\Theta;\ w \in W(\Theta)\}$ *counting their multiplicities.*

In particular, the space $S(\mathfrak{a})/\omega\bigl(\operatorname{Ann}_G\bigl(M^0_\Theta(\lambda)\bigr)\bigr)$ *is naturally a representation space of* W *which is isomorphic to* $\operatorname{Ind}^W_{W_\Theta}\operatorname{id}$.

Proof. Under the notation (2.36)

$$\bar\lambda_\nu = \lambda_{\iota_\Theta(\nu)} - \tfrac{n-1}{2} + (\nu - 1) \quad \text{for } \nu = 1,\ldots,n$$

and

$$\bar\omega(D^\epsilon_{II})(\lambda_k+n_{k-1}\epsilon)=\prod_{\mu=1}^{m}\bigl(E_{i_\mu}-\lambda_k+(\tfrac{n-1}{2}-n_{k-1}+\mu-i_\mu)\epsilon\bigr).$$

Fix k with $1\le k\le L$ and $w\in W(\Theta)$. Put $m=n+1-n'_k$, $K=\{n_{k-1}+1,\dots,n_k\}$, $K^c=\{1,\dots,n\}\setminus K$ and $J=w(K^c)$. For $I=\{i_1,\dots,i_m\}$ with $1\le i_1<\cdots<i_m\le n$, choose μ as in Lemma 2.16 and put $\ell=w^{-1}(i_\mu)$. Then $\ell\in K$ and $\{\nu\in K^c;\,w(\nu)<i_\mu\}=\mu-1$, which implies $\#\{\nu\in K;\,w(\nu)<i_\mu\}=i_\mu-\mu$. On the other hand, since the condition $n_{k-1}<\nu<\nu'\le n_k$ means $w(\nu)<w(\nu')$, we have $\{\nu\in K;\,w(\nu)<i_\mu\}=\{n_{k-1}+1,n_{k-1}+2,\dots,\ell-1\}$. Hence $\ell-n_{k-1}-1=i_\mu-\mu$ and

$$\bar\lambda_\ell-\lambda_k+(\tfrac{n-1}{2}-n_{k-1}+\mu-i_\mu)\epsilon=\bigl(\ell-1-n_{k-1}+\mu-i_\mu\bigr)\epsilon=0.$$

Since $\bar\lambda_\ell$ is the i_μ-th component of $(\bar\lambda_{w^{-1}(1)},\dots,\bar\lambda_{w^{-1}(n)})$, we can conclude that $\bar\omega(D_{II})(\lambda_k+n_{k-1}\epsilon)$ vanishes at $w(\lambda_\Theta+\rho^\epsilon)$, which is equivalent to the condition that $\omega(D_{II})(\lambda_k+n_{k-1}\epsilon)$ vanishes at $w.\lambda_\Theta$. Hence if λ is generic, $\omega\bigl(I^\epsilon_\Theta(\lambda)\bigr)$ vanishes at $w.\lambda_\Theta$ for $w\in W(\Theta)$ and therefore for any $\lambda\in\mathbb{C}^L$ because of the continuity. In particular, $\dim S(\mathfrak a)/\omega\bigl(I^\epsilon_\Theta(\lambda)\bigr)\ge\#W(\Theta)$ for generic λ and therefore for any λ by the same reason.

Since $\omega\bigl(I^\epsilon_\Theta(\lambda)\bigr)$ are generated by homogeneous polynomials of $(\mathfrak a,\lambda,\epsilon)$ and [17, Theorem 1] shows $\dim S(\mathfrak a)/\omega\bigl(I^0_\Theta(0)\bigr)=\#W(\Theta)$, we have $\dim S(\mathfrak a)/\omega\bigl(I^\epsilon_\Theta(\lambda)\bigr)\le\#W(\Theta)$. Thus we can conclude $\dim S(\mathfrak a)/\omega\bigl(I^\epsilon_\Theta(\lambda)\bigr)=\#W(\Theta)$ and the theorem follows from this. In fact, the last claim is clear because $I^0_\Theta(\lambda)$ is W-invariant. □

3. Generalized Verma modules

In this section we examine the necessary and sufficient condition on $\lambda\in\mathbb{C}^L$ so that

$$J^\epsilon_\Theta(\lambda)=\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)+J^\epsilon(\lambda_\Theta)\tag{3.1}$$

under the notation (2.6) and (2.9). Note that it is clear by the definition that $J^\epsilon_\Theta(\lambda)\supset\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)+J^\epsilon(\lambda_\Theta)$ and

$$\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)=\operatorname{Ann}_G\bigl(U^\epsilon(\mathfrak g)/(\operatorname{Ann}_G\bigl(M^\epsilon_\Theta(\lambda)\bigr)+J^\epsilon(\lambda_\Theta))\bigr).\tag{3.2}$$

In general it is proved by [1] and [7] that for $\mu\in\mathfrak a^*$ the map

$$\begin{aligned}&\{I;\ I\text{ is the two sided ideal of }U(\mathfrak g)\text{ with }I\supset\operatorname{Ann}\bigl(M(\mu)\bigr)\}\\&\qquad\ni I\mapsto I+J(\mu)\in\{J;\ J\text{ is the left ideal of }U(\mathfrak g)\text{ with }J\supset J(\mu)\}\end{aligned}\tag{3.3}$$

is injective if μ is dominant:

$$2\frac{\langle\mu+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle}\notin\{-1,-2,\dots\}\quad\text{for any positive root }\alpha\text{ for the pair }(\mathfrak n,\mathfrak a).\tag{3.4}$$

Moreover the map is surjective if μ is regular, that is,

$$\langle\mu+\rho,\alpha\rangle\ne0\quad\text{for any root }\alpha\text{ for the pair }(\mathfrak n,\mathfrak a)\tag{3.5}$$

and dominant. Hence in our case when $\epsilon\ne0$, (3.1) is valid if $\lambda_\Theta+\rho^\epsilon$ is regular and dominant, which is equivalent to

$$\bar\lambda_j-\bar\lambda_i\notin\{0,-\epsilon,-2\epsilon,\dots\}\quad\text{for }1\le i<j\le n.\tag{3.6}$$

For $\mu\in\mathfrak a^*$ and $D\in U^\epsilon(\mathfrak g)$ let $\gamma(\mu;D)$ denote the unique element in $U^\epsilon(\bar{\mathfrak n})$ with $D\equiv\gamma(\mu;D)\mod J^\epsilon(\mu)$. For a basis $\{R_j\}$ of an $\operatorname{ad}(\mathfrak g)$-invariant subspace V of $U^\epsilon(\mathfrak g)$ we note that

$$\gamma(\mu;\sum P_jR_j)\in\sum U^\epsilon(\bar{\mathfrak n})\gamma(\mu;R_j)\quad\text{for }P_j\in U^\epsilon(\mathfrak g).\tag{3.7}$$

Let R_- denote the set of weights of $U^\epsilon(\bar{\mathfrak{n}})$ with respect to $\mathfrak{a}$. Then

$$R_- = \{\sum_{i=1}^n m_i e_i;\ m_i \in \mathbb{Z},\ \sum m_i = 0 \ \text{ and } \ m_1 \ge m_2 \ge \cdots \ge m_n\} \setminus \{0\}.$$

Suppose $R_j \in U^\epsilon(\mathfrak{g})$ are weight vectors and $U^\epsilon(\mathfrak{g})V + J^\epsilon(\mu) \ne U^\epsilon(\mathfrak{g})$. Since $\gamma(\mu; R_j)$ has the weight which equals that of R_j, $\gamma(\mu; R_j) = 0$ if the weight of R_j is not in R_-. Moreover since E_{ii+1} has a maximal weight $e_i - e_{i+1}$ in R_- for any integer i with $1 \le i < n$,

$$E_{ii+1} \in U^\epsilon(\mathfrak{g})V + J^\epsilon(\bar\lambda) \ \Leftrightarrow\ \mathbb{C}E_{ii+1} = \sum_{\text{the weight of } R_j = e_i - e_{i+1}} \mathbb{C}\gamma(\mu; R_j). \tag{3.8}$$

The key to studying the condition for (3.1) is the following argument used in [15, proof of Theorem 5.1]:

Fix positive integers k, $\bar\imath$ and $\bar\jmath$ satisfying $1 \le k \le L$ and $n_{k-1} < \bar\imath < \bar\jmath \le n_k$. Let $I = \{i_m, \dots, i_1\}$ and $J = \{j_m, \dots, j_1\}$ be a set of positive numbers such that

$$\begin{gathered} 1 \le i_1 < i_2 < \cdots < i_m \le n, \\ i_\nu = j_\nu \quad \text{if } \nu \ne \ell, \\ i_\ell = \bar\imath < j_\ell = \bar\jmath < i_{\ell+1} \end{gathered} \tag{3.9}$$

with a suitable $1 \le \ell \le m$. Define non-negative integers

$$\begin{cases} m' &= n - m, \\ a'_j &= n'_j - \#\{\nu;\ n_{j-1} < i_\nu \le n_j\}, \\ a_j &= n_j - \#\{\nu;\ i_\nu \le n_j\} = a'_1 + \cdots + a'_j,\ a_0 = 0, \\ b &= \#\{\nu;\ n_{k-1} < i_\nu < \bar\imath\}, \\ b' &= \#\{\nu;\ \bar\jmath < i_\nu \le n_k\}. \end{cases} \tag{3.10}$$

Then

$$\begin{gathered} 1 \le a_L = m' \le n-2,\ 1 \le a'_k = n'_k - b - b' - 1, \\ 0 \le a'_j \le n'_j - \delta_{kj},\ 0 \le b \le \bar\imath - n_{k-1} + 1,\ 0 \le b' \le n_k - \bar\jmath \end{gathered} \tag{3.11}$$

and we have

$$\begin{aligned} \det{}^\epsilon(x; E_{IJ}) &\equiv \prod_{\nu=\ell+1}^m (x - E_{i_\nu} - (\nu-1)\epsilon) \cdot E_{\bar\imath\bar\jmath} \\ &\qquad \cdot \prod_{\nu=1}^{\ell-1} (x - E_{i_\nu} - (\nu-1)\epsilon) \mod U^\epsilon(\mathfrak{g})\mathfrak{n} \\ &\equiv \frac{\prod_{j=1}^L p^j_{IJ}(x)}{s_{IJ}(x)} E_{\bar\imath\bar\jmath} \mod J^\epsilon(\lambda_\Theta) \end{aligned} \tag{3.12}$$

by putting

$$\begin{cases} p^j_{IJ}(x) = \bigl(x - \lambda_j - (n_{j-1} - a_{j-1})\epsilon\bigr)^{(n'_j - a'_j)}, \\ s_{IJ}(x) = x - \lambda_k - (n_{k-1} - a_{k-1} + b)\epsilon. \end{cases} \tag{3.13}$$

Hence it follows from from (2.18) that

$$\sum_{i=0}^{d_m-1} \mathbb{C}r^i_{IJ} \equiv \begin{cases} \mathbb{C}E_{\bar\imath\bar\jmath} & \mod J^\epsilon(\lambda_\Theta) \quad \text{if } \prod_{j=1}^L p^j_{IJ}(x) \notin \mathbb{C}[x]s_{IJ}(x)d^\epsilon_m(x), \\ 0 & \mod J^\epsilon(\lambda_\Theta) \quad \text{otherwise}. \end{cases} \tag{3.14}$$

Since $(n'_j - a'_j - a_{j-1}) - (n'_j - m') = m' - a_j \ge m' - a_L \ge 0$, we can define polynomials

$$\bar p^j_{IJ}(x) = \frac{p^j_{IJ}(x)}{(x - \lambda_j - n_{j-1}\epsilon)^{(n'_j - m')}}.$$

Then the condition $\prod_{j=1}^L p_{IJ}^j(x) \in \mathbb{C}[x]s_{IJ}(x)d_m^\epsilon(x)$ is equivalent to the existence of j with

$$\bar{p}_{IJ}^j(x) \in \mathbb{C}[x]s_{IJ}(x). \tag{3.15}$$

If $\epsilon \neq 0$, the condition (3.15) is equivalent to the condition that ν is an integer satisfying

$$0 \le \nu \le n_j' - a_j' - 1 \quad \text{and} \quad \bigl(\nu < a_{j-1} \quad \text{or} \quad \nu \ge a_{j-1} + n_j' - m'\bigr) \tag{3.16}$$

by denoting

$$\lambda_k + (n_{k-1} - a_{k-1} + b)\epsilon = \lambda_j + (n_{j-1} - a_{j-1} + \nu)\epsilon. \tag{3.17}$$

If $\epsilon = 0$, it is equivalent to

$$\lambda_j = \lambda_k \quad \text{and} \quad a_j' < m'. \tag{3.18}$$

Put $I = \{n, n-1, \ldots, n_k+1, \bar{i}, n_{k-1}, n_{k-1}-1, \ldots, 1\}$ and $J = \{n, n-1, \ldots, n_k + 1, \bar{j}, n_{k-1}, n_{k-1} - 1, \ldots, 1\}$. Then

$$m' = n_k' - 1,\ b = b' = 0,\ a_k' = n_k' - 1,\ a_j' = 0 \text{ and } n_j' - a_j' - 1 = n_j' - 1 \text{ if } j \neq k.$$

Suppose (3.15) holds. Then $j \neq k$ because $\bar{p}_{IJ}^k(x) = 1$. Since

$$\begin{cases} a_{j-1} - 1 = -1 < 0 \ \text{ and } \ a_{j-1} + n_j' - m' = n_j' - n_k' + 1 & \text{if } j < k, \\ a_{j-1} - 1 = n_k' - 2 \ \text{ and } \ a_{j-1} + n_j' - m' = n_j' > n_j' - a_j' - 1 & \text{if } j > k, \end{cases}$$

the condition (3.16) is equivalent to

$$\begin{cases} \max\{0, n_j' - n_k' + 1\} \le \nu' \le n_j' - 1 & \text{if } j < k, \\ 1 - n_k' \le \nu' \le \min\{n_j' - n_k', -1\} & \text{if } k < j \end{cases}$$

with

$$\nu' = (\nu - a_{j-1}) - (b - a_{k-1}) = \begin{cases} \nu & \text{if } j < k, \\ \nu - n_k' + 1 & \text{if } k < j. \end{cases}$$

Hence (3.15) is equivalent to the condition (cf. Remark 2.15)

$$\begin{aligned} &\Lambda_k \cap \Lambda_j \neq \emptyset,\ \Lambda_k \not\subset \Lambda_j \text{ and } \Bigl(\mu \in \Lambda_j,\ \mu' \in \Lambda_k \setminus \Lambda_j \Rightarrow (\mu' - \mu)(k - j) > 0\Bigr) \\ &\quad \text{with } \Lambda_i := \{\bar{\lambda}_\nu;\ n_{i-1} < \nu \le n_i\} = \{\lambda_i + \bigl((\nu-1) - \tfrac{n-1}{2}\bigr)\epsilon;\ n_{i-1} < \nu \le n_i\} \\ &\qquad\qquad\qquad \text{if } \epsilon \neq 0, \\ &\lambda_j = \lambda_k \text{ and } n_k' > 1 \quad \text{if } \epsilon = 0. \end{aligned} \tag{3.19}$$

Thus we have the following theorem.

Theorem 3.1. i) *Fix k with $1 \le k \le L$. Recall $\mathfrak{m}_\Theta^k = \sum_{\substack{n_{k-1} < i \le n_k \\ n_{k-1} < j \le n_k}} \mathbb{C}E_{ij}$. Then*

$$\operatorname{Ann}_G\bigl(M_\Theta^\epsilon(\lambda)\bigr) + J^\epsilon(\lambda_\Theta) \supset \mathfrak{m}_\Theta^k \cap \bar{\mathfrak{n}} \tag{3.20}$$

if and only if (3.19) *does not hold for $j = 1, \ldots, L$.*

ii) *The equality* (3.1) *is valid if and only if* (3.19) *does not hold for $j = 1, \ldots, L$ and $k = 1, \ldots, L$, which is equivalent to the condition*

$$\begin{cases} \Lambda_i \cap \Lambda_j = \emptyset \ \textit{or} \ \Lambda_i = \Lambda_j \ \textit{or} \ \min \bar{\Lambda}_i > \min \bar{\Lambda}_j \ \textit{or} \ \max \bar{\Lambda}_i > \max \bar{\Lambda}_j & \textit{if } \ \epsilon \neq 0, \\ \lambda_i \neq \lambda_j \ \textit{or} \ n_i' = n_j' = 1 & \textit{if } \ \epsilon = 0, \end{cases} \tag{3.21}$$
$$\textit{for } \ 1 \le i < j \le L.$$

Here $\bar{\Lambda}_i = \{\operatorname{Re}\mu;\ \mu \in \Lambda_i\}$ etc. and Λ_i is given in (3.19)*. In particular* (3.1) *is valid if the infinitesimal character of $M_\Theta^\epsilon(\lambda)$ is regular.*

Proof. We have only to prove that (3.20) is not valid if (3.19) holds for a suitable j. Suppose there exists $j = j_o$ which satisfies (3.19). Fix such j_o and continue the argument just before the theorem. Put $\bar{j} = \bar{i} + 1$ and suppose (3.15) does not valid for $j = k$. Then if $\epsilon \neq 0$, $\nu = b$ in (3.17) and since $0 \le b \le n'_k - a'_k - 1$ and (3.16) is not valid with $j = k$, we have

$$a_{k-1} \le b < a_{k-1} + n'_k - m' \quad \text{and} \quad m' < n'_k \quad \text{if } \epsilon \neq 0. \tag{3.22}$$

On the other hand, if $\epsilon = 0$, we have $a'_k = m'$ because $a'_k \le a_L = m'$.

First consider the case when $j_o < k$. Put $\ell = \lambda_k + n_{k-1} - \lambda_{j_o} - n_{j_o-1}$. If $\epsilon \neq 0$, it follows from (3.19) that

$$0 \le \ell < n'_{j_o} \quad \text{and} \quad \ell + n'_k > n'_{j_o}.$$

Put $\bar{i} = n_{k-1} + n'_{j_o} - \ell$ and $j = j_o$ in (3.17). Note that $n_{k-1} < \bar{i} < \bar{j} = \bar{i} + 1 \le n_k$ and $\nu = \ell + b - a_{k-1} + a_{j_o-1}$. Then we have $\nu = \ell + (b - a_{k-1}) + a_{j_o-1} \ge 0$, $(n'_{j_o} - a'_{j_o} - 1) - \nu = (\bar{i} - n_{k-1} - b - 1) + (a_{k-1} - a_{j_o-1}) \ge 0$ and $\nu - (a_{j_o-1} + n'_{j_o} - m') = \ell + b - a_{k-1} - n'_{j_o} + m' = m' - (\bar{i} - n_{k-1} - b - 1) - a_{k-1} \ge m' - a_k \ge 0$ in (3.16), which implies $\bar{p}^{j_0}_{IJ}(x) \in \mathbb{C}[x]s_{IJ}(x)$. We have this relation also in the case when $\epsilon = 0$ because $\deg \bar{p}^{j_0}_{IJ}(x) = n'_{j_o} - a'_{j_o} - (n'_{j_o} - m') = m' - a'_{j_o} \ge m' - (m' - a'_k) = a'_k > 0$. Thus we can conclude $r^j_{IJ} \equiv 0 \mod J^\epsilon(\lambda_\Theta)$ if the weight of r^j_{IJ} is $e_{\bar{i}} - e_{\bar{i}+1}$. Note that the weight of $r^j_{\{i_1,\dots,i_m\}\{j_1,\dots,j_m\}}$ is $\sum_{\nu=1}^m e_{i_\nu} - e_{j_\nu}$. Hence $E_{\bar{i}\bar{j}} \notin \mathrm{Ann}_G\left(M^\epsilon_\Theta(\lambda)\right) + J^\epsilon(\lambda_\Theta)$ because of (3.8).

Lastly consider the case when $k < j_o$. If $\epsilon = 0$, the same argument as in the case when $j_o < k$ works. Therefore we may assume $\epsilon \neq 0$. Put $\ell = \lambda_{j_o} + n_{j_o-1} - \lambda_k - n_{k-1}$. It follows from (3.19) that

$$1 \le \ell < n'_k \quad \text{and} \quad n'_k \le \ell + n'_{j_o}.$$

Put $\bar{i} = n_{k-1} + \ell$. Then similarly we have $n_{k-1} < \bar{i} < \bar{j} = \bar{i} + 1 \le n_k$, $\nu = a_{j_o-1} - a_{k-1} + b - \ell = a_{j_o-1} - a_{k-1} + (n'_k - a'_k - b' - 1) - \ell = (a_{j_o-1} - a_k) + (n_k - \bar{j} - b') \ge 0$, $(n'_{j_o} - a'_{j_o} - 1) - \nu = n'_{j_o} - a'_{j_o} - 1 - (a_{j_o-1} - a_{k-1} + b - \ell) = (\ell + n'_{j_o} - n'_k) + (a_{k-1} + n'_k - m' - b - 1) + (m' - a_{j_o}) \ge 0$ and $a_{j_o-1} - \nu = a_{k-1} - b + \ell = a_{k-1} + (\bar{i} - n_{k-1} - b - 1) + 1 > 0$ in (3.16). Hence $\bar{p}^{j_0}_{IJ}(x) \in \mathbb{C}[x]s_{IJ}(x)$ and thus $E_{\bar{i}\bar{j}} \notin \mathrm{Ann}_G\left(M^\epsilon_\Theta(\lambda)\right) + J^\epsilon(\lambda_\Theta)$ as in the previous case. □

Example 3.2. Suppose $n = 3$, $\Theta = \{2, 3\}$ and $\lambda = (\lambda_1, \lambda_2)$. Then

$$\begin{gathered}
d^\epsilon_1(x) = 1,\ d^\epsilon_2(x) = x - \lambda_1,\ d^\epsilon_3(x) = (x - \lambda_1)(x - \lambda_1 - \epsilon)(x - \lambda_2 - 2\epsilon),\\
J^\epsilon(\lambda_\Theta) = \sum_{3 \ge i > j \ge 1} U(\mathfrak{g})E_{ij} + U(\mathfrak{g})(E_1 - \lambda_1) + U(\mathfrak{g})(E_2 - \lambda_1) + U(\mathfrak{g})(E_3 - \lambda_2),\\
J^\epsilon_\Theta(\lambda) = J^\epsilon(\lambda_\Theta) + U^\epsilon(\mathfrak{g})E_{12}.
\end{gathered}$$

Since

$$\begin{aligned}
D^\epsilon_{IJ}(x) = {}&(E_{i_1j_1} - (x - \epsilon)\delta_{i_1j_1})(E_{i_2j_2} - x\delta_{i_2j_2})\\
&- (E_{i_2j_1} - (x - \epsilon)\delta_{i_2j_1})(E_{i_1j_2} - x\delta_{i_1j_2})
\end{aligned}$$

for $I=\{i_1>i_2\}$ and $J=\{j_1>j_2\}$, we have

$$
(3.23)\quad
\begin{cases}
D^{\epsilon}_{\{21\}\{21\}}(\lambda_1)=(E_2-\lambda_1+\epsilon)(E_1-\lambda_1)-E_{12}E_{21}\equiv 0,\\
D^{\epsilon}_{\{32\}\{32\}}(\lambda_1)=(E_3-\lambda_1+\epsilon)(E_2-\lambda_1)-E_{23}E_{32}\equiv 0,\\
D^{\epsilon}_{\{31\}\{31\}}(\lambda_1)=(E_3-\lambda_1+\epsilon)(E_1-\lambda_1)-E_{13}E_{31}\equiv 0,\\
D^{\epsilon}_{\{21\}\{32\}}(\lambda_1)=E_{23}E_{12}-E_{13}(E_2-\lambda_1)\equiv E_{23}E_{12},\\
D^{\epsilon}_{\{21\}\{31\}}(\lambda_1)=E_{23}(E_1-\lambda_1)-E_{13}E_{21}\equiv 0,\\
D^{\epsilon}_{\{32\}\{21\}}(\lambda_1)=E_{32}E_{21}-(E_2-\lambda_1+\epsilon)E_{31}\equiv 0,\\
D^{\epsilon}_{\{32\}\{31\}}(\lambda_1)=(E_3-\lambda_1+\epsilon)E_{21}-E_{23}E_{31}\equiv 0,\\
D^{\epsilon}_{\{31\}\{21\}}(\lambda_1)=E_{32}(E_1-\lambda_1)-E_{12}E_{31}\equiv 0,\\
D^{\epsilon}_{\{31\}\{32\}}(\lambda_1)=(E_3-\lambda_1+\epsilon)E_{12}-E_{13}E_{32}\equiv (\lambda_2-\lambda_1+\epsilon)E_{12}.
\end{cases}
$$

Here the above $\equiv$ is considered under modulo $J^{\epsilon}(\lambda_\Theta)$. Note that

$$
(3.24)\quad \operatorname{Ann}_G\bigl(M(\Theta^{\epsilon}(\lambda)\bigr)=\sum_{\substack{3\ge i_1>i_2\ge 1\\ 3\ge j_1>j_2\ge 1}} U^{\epsilon}(\mathfrak{g})D^{\epsilon}_{\{i_1i_2\}\{j_1j_2\}}(\lambda_1)
+\sum_{D\in U^{\epsilon}(\mathfrak{g})^G} U^{\epsilon}(\mathfrak{g})\bigl(D-\omega(D)(\lambda_\Theta)\bigr).
$$

Hence if $\lambda_1\ne\lambda_2+\epsilon$ which is equivalent to (3.21), we have (3.1).

Suppose $\lambda_1=\lambda_2+\epsilon$. Then since $\operatorname{ad}(\mathfrak{p})(E_{32}E_{12})\subset J^{\epsilon}(\lambda_\Theta)$, we have

$$
(3.25)\quad
\begin{aligned}
J^{\epsilon}_{\Theta}(\lambda)&=U^{\epsilon}(\bar{\mathfrak{n}})E_{12}\oplus J^{\epsilon}(\lambda_\Theta)\\
&\supsetneqq \operatorname{Ann}_G\bigl(M_\Theta(\lambda)\bigr)+J^{\epsilon}(\lambda_\Theta)=U^{\epsilon}(\bar{\mathfrak{n}})E_{23}E_{12}\oplus J^{\epsilon}(\lambda_\Theta)\supsetneqq J^{\epsilon}(\lambda_\Theta).
\end{aligned}
$$

If $\epsilon\ne 0$, the above inclusion relation gives a Jordan-Hörder sequence of $M^{\epsilon}(\lambda_\Theta)$ and

$$
(3.26)\quad J^{\epsilon}_{\Theta}(\lambda)/\bigl(\operatorname{Ann}_G\bigl(M^{\epsilon}_{\Theta}(\lambda)\bigr)+J^{\epsilon}(\lambda_\Theta)\bigr)\simeq M^{\epsilon}_{\Theta'}(\lambda')
$$

with $\Theta'=\{1,3\}$ and $\lambda'=(\lambda_1+\epsilon,\lambda_1-\epsilon)$. Note that $\rho^{\epsilon}=(-\epsilon,0,\epsilon)$, $\lambda_\Theta+\rho^{\epsilon}=(\lambda_1-\epsilon,\lambda_1,\lambda_1)$, $\lambda'_{\Theta'}-\lambda_\Theta=\epsilon(e_1-e_2)$, $(1,2).\lambda_\Theta=\lambda'_{\Theta'}$ and $\operatorname{Ann}_G\bigl(M^{\epsilon}_{\Theta}(\lambda)\bigr)=\operatorname{Ann}_G\bigl(M^{\epsilon}_{\Theta'}(\lambda')\bigr)$ under the notation in Remark 2.15. Here $\operatorname{Ann}\bigl(M_\Theta(\lambda)\bigr)$ is the unique two-sided proper ideal of $U(\mathfrak{g})$ which is larger than $U(\mathfrak{g})\bigl(J(\lambda_\Theta)\cap U(\mathfrak{g})^G\bigr)$.

Example 3.3. Let G be a real form of $GL(n,\mathbb{C})$, let K be a maximal compact subgroup of G and let P be a parabolic subgroup of G with the Langlands decomposition $P=MAN$. Fix a minimal parabolic subgroup P_o of G contained in P. Let $\mathfrak{a}^*$ be the dual space of the Lie algebra $\mathfrak{a}$ of A. Define

$$
\begin{aligned}
\chi_\lambda\,:\, P\ni man&\mapsto\chi_\lambda(man)=a^{\lambda}\in\mathbb{C}\quad(m\in M,\ a\in A,\ n\in N),\\
\mathcal{B}(G/P;L_\lambda)&=\{f\in\mathcal{B}(G);\ f(xp)=\chi_\lambda(p)f(x)\quad(\forall p\in P)\},\\
\mathcal{B}(G/P_o;L_\lambda)&=\{f\in\mathcal{B}(G);\ f(xp)=\chi_\lambda(p)f(x)\quad(\forall p\in P_o)\}.
\end{aligned}
$$

for $\lambda\in\mathfrak{a}^*$. Here $\mathcal{B}(G)$ is the space of hyperfunctions on G. Let $\mathfrak{p}_\Theta$ be the complexification of the Lie algebra of P. Then the totality of the elements of $U(\mathfrak{g})$ which kill all the elements of $\mathcal{B}(G/P_o;L_\lambda)$ equals $J_\Theta(\lambda)$. Here we note that $U(\mathfrak{g})$ is identified with the ring of left invariant differential operators on G.

Note that (3.1) implies that $f\in\mathcal{B}(G/P_o;L_\lambda)$ belongs to $B(G/P;L_\lambda)$ if and only if f is killed by $\operatorname{Ann}_G(M_\Theta(\lambda))$.

The Poisson transformation $\mathcal{P}_\lambda$ of the space $\mathcal{B}(G/P_o;L_\lambda)$ is defined by

$$
\mathcal{P}_\lambda\,:\, B(G/P_o;L_\lambda)\ni f\mapsto(\mathcal{P}_\lambda(f))(x)=\int_K f(xk)dk\in\mathcal{B}(G/K).
$$

Here dk is the normalized Haar measure on K.

Suppose $\mathcal{P}_\lambda$ is injective. It is known by Helgason [5] that this is valid for generic λ including $\lambda = 0$. Then [10] shows that the image is characterized by a simultaneous eigenspace of the ring of invariant differential operators on the symmetric space G/K. In this case (3.1) assures that the image of the Poisson transform of $\mathcal{B}(G/P; L_\lambda)$ is $\{f \in \mathcal{B}(G/K);\ Pf = 0 \quad \bigl(\forall P \in \mathrm{Ann}_G(M_\Theta(\lambda))\bigr)\}$.

Johnson [6] studies this problem when $\lambda = 0$ and $P \neq P_0$. Here we have solved this problem for generic λ including $\lambda = 0$. More precise argument and similar applications are given in [16, §5].

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