A classification of roots of symmetric Kac-Moody root systems and its application

Kazuki Hiroe* and Toshio Oshima†

Dedicated to Professor Michio Jimbo on the occasion of his 60th birthday

Abstract

We study Weyl group orbits in symmetric Kac-Moody root systems and show a finiteness of orbits of roots with a fixed index. We apply this result to the study of the Euler transform of linear ordinary differential equations on the Riemann sphere whose singular points are regular singular or unramified irregular singular points. The Euler transform induces a transformation on spectral types of the differential equations and it keeps their indices of rigidity. Then as a generalization of the result in [10], we show a finiteness of Euler transform orbits of spectral types with a fixed index of rigidity.

1 Introduction

Recall the definition of symmetric Kac-Moody root systems (precise definition of terminology appearing below can be found in the latter section, see §2.1). For a finite index set I, define a lattice $Q := \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ with a basis $\{\alpha_i \mid i \in I\}$ and consider a symmetric bilinear form on Q which satisfies

$$\begin{aligned}&\langle\alpha_i,\alpha_i\rangle = 2,\\ &\langle\alpha_i,\alpha_j\rangle = \langle\alpha_j,\alpha_i\rangle \in \mathbb{Z}_{\leq 0} \quad (i,\, j \in I \text{ and } i \neq j).\end{aligned}$$

The Weyl group W acting on Q is generated by simple reflections, $\sigma_i(\beta) := \beta - \langle\beta,\alpha_i\rangle\alpha_i$ for $\beta \in Q$ and $i \in I$.

Then a certain subset of Q, called the set of roots, is defined by

$$\Delta := \bigcup_{i\in I} W\alpha_i \sqcup WF \sqcup -WF.$$

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan; E-mail:kazuki@kurims.kyoto-u.ac.jp

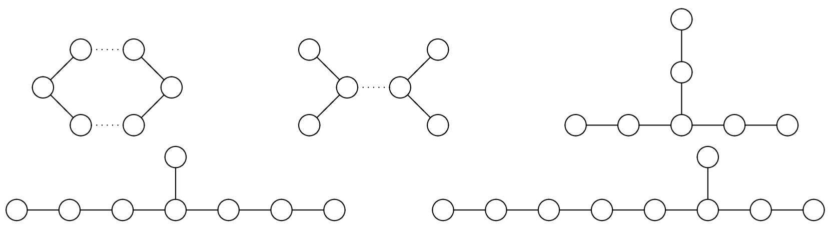
†Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan; E-mail:oshima@ms.u-tokyo.ac.jp

Here $F := \{\alpha \in Q^+ \setminus \{0\} \mid \operatorname{supp}\alpha \text{ is connected and } \langle\alpha, \alpha_i\rangle \le 0 \text{ for all } i \in I\}$ with $Q^+ := \bigoplus_{i\in I} \mathbb{Z}_{\ge 0}\alpha_i$. In particular we call $\Delta_{\mathrm{re}} := \bigcup_{i\in I} W\alpha_i$ the set of real roots and $\Delta_{\mathrm{im}} := WF \sqcup -WF$ the set of imaginary roots. If $\alpha \in \Delta$ is in Q^+, it is called a positive root. Moreover we call elements in F basic positive imaginary roots or shortly basic roots. Then we call the triple $(I, \langle\ ,\ \rangle, \Delta)$ or shortly $(I, \langle\ ,\ \rangle)$ the symmetric Kac-Moody root system.

A symmetric Kac-Moody root system $(\langle\ ,\ \rangle_1,\ I_1)$ is a subsystem of a symmetric Kac-Moody root system $(\langle\ ,\ \rangle_2,\ I_2)$ if there is an injective map ϕ of I_1 to I_2 such that $\langle\alpha_i, \alpha_j\rangle_1 = \langle\alpha_{\phi(i)}, \alpha_{\phi(j)}\rangle_2$ for $i,\ j \in I_1$ and in this case the root of $(\langle\ ,\ \rangle_1,\ I_1)$ is naturally identified with a root of $(\langle\ ,\ \rangle_2,\ I_2)$. Thus we can define the universal symmetric Kac-Moody root system by the inductive limit of symmetric Kac-Moody root systems under the injective maps defining subsystems.

One of our main aim is to classify the orbits of roots under the action of the Weyl group in the universal symmetric Kac-Moody root system. Since the real roots form a single orbit of the Weyl group, it is sufficient to classify the orbits contained in the set of positive imaginary roots, i.e., elements in $\Delta^+_{\mathrm{im}} = \Delta_{\mathrm{im}} \cap Q^+ = WF$. Thus what we need to do is to classify elements in F, i.e., basic roots.

For an element α in a root lattice, the index of α is defined by $\operatorname{idx}\alpha := \langle\alpha, \alpha\rangle$. The classification of basic roots with index 0 is known as follows. Dynkin diagrams of supports of them are classified by the following 5 cases.



Moreover for each diagram, there exists a unique indivisible root and any basic roots are scalar multiples of one of these indivisible roots. Here $\alpha = \sum_{i\in I} m_i\alpha_i$ is indivisible if the greatest common divisor of coefficients m_i is 1.

Hence in this case, the classification of Weyl group orbits of imaginary roots is obtained by the classification of indivisible basic roots which correspond to the above finite cases.

One of the main results in this paper is to show a finiteness of basic roots with a general index. For this purpose we introduce the shape of an element in a root lattice. Fix a root lattice $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ and $\alpha = \sum_{i\in I} m_i\alpha_i \in Q$. For the Dynkin diagram of the support of α, we attach each coefficient m_i of α to the vertex corresponding to α_i, then we obtain the diagram with the coefficients, which we call the shape of α.

We say $i_1, \ldots, i_k \in \{i \in I \mid m_i \neq 0\}$ is a constant connected sequence of

α if $m_{i_1} = \cdots = m_{i_k}$ and the Dynkin diagram of $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ is

$$\circ\!-\!\!-\!\circ\cdots\circ\!-\!\!-\!\circ.$$

Theorem 1.1 (see Corollary 2.3)**.** *If a basic root $\alpha = \sum_{i \in I} m_i \alpha_i$ contains a constant connected sequence $i_1, \ldots, i_k$ of I satisfying $k > 2$ and $\langle \alpha, \alpha_{i_\nu} \rangle = 0$ for $\nu = 2, \ldots, k-1$, then the shape obtained from that of α by shrinking or extending the length of the sequence corresponds to a basic root with the same index. Expressing such a sequence by $\overset{m}{\circ}\cdots\overset{m}{\circ}$, we have shapes of roots which may contain such expressions. We call these shapes reduced shapes.*

Then the basic roots with a fixed nonzero index are classified by a finite number of reduced shapes. The indivisible basic roots with index 0 are also classified by a finite number of reduced shapes.

Moreover proceeding further from the classification of basic roots with index 0 seen above, we give the complete list of shapes of basic roots with index -2 in §2.4.

Another aim of this paper is to give a classification of orbits of linear ordinary differential equations under the action of the Euler transform as an application of our classification of basic roots.

Consider a Fuchsian system of ordinary differential equations on the Riemann sphere of the form $\frac{d}{dx}Y(x) = \sum_{i=0}^{p} \frac{A_i}{x - c_i} Y(x)$ where $A_i\,(i = 0, \ldots, p)$ are $n \times n$ matrices with coefficients in $\mathbb{C}$ and $Y(x)$ is a $\mathbb{C}^n$-valued function. For this system, W. Crawley-Boevey [2] constructs a representation of a quiver, more precisely, a deformed preprojective algebra, with a star-shaped quiver. His result shows that for an irreducible Fuchsian system, the dimension vector of the corresponding representation of the quiver is a positive root in the Kac-Moody root system of its quiver. Then the index of rigidity of the Fuchsian system equals the index of the root and reflection functors on representations of the quiver are obtained by algebraic transformations on Fuchsian systems, the Euler transform and the addition. Thus to study orbits of irreducible Fuchsian systems under the actions of the Euler transform and the additions, we can apply the classification of Weyl group orbits of the roots.

In [10, 11] the corresponding results for Fuchsian single differential equations together with the analysis of their global solutions, namely, integral representations of the solutions and the connection problem etc., are studied.

In [7], we consider a generalization of the result of Crawley-Boevey to ordinary differential equations whose singular points are regular singular or unramified irregular singular points. As in the case of the Fuchsian equations, there exists a Kac-Moody root system attached to a differential equation such that its spectral type corresponds to an element in the root lattice (see Theorem 3.14, Theorem 3.15 and Definition 3.20). Here a spectral type is a tuple of integers representing multiplicities of characteristic exponents of local formal solutions of a differential equation where we ignore integer differences of characteristic exponents (see §3.1.2 for the precise definition).

Thus for spectral types it shall be defined an analogy of basic roots, called basic pairs (see Definition 3.16). Then we shall consider a classification of basic pairs in §3.3 as an application of that of basic roots.

Combining this result with Theorem 1.1, we show the following theorem which generalizes the result of the second author [9, 10] in the Fuchsian case.

Theorem 1.2 (see Theorem 3.7)**.** *Fix an integer $r \geq 0$ and consider linear differential equations with index of rigidity $-r$ on the Riemann sphere whose singular points are regular singular or unramified irregular singular points.*

Then we have the finiteness of orbits of spectral types of the differential equations under the actions of the Euler transform and the addition. Namely, if $r > 0$, there exist only a finite number of orbits and if $r = 0$, there exist a finite number of orbits of indivisible spectral types.

Finally in §3.3.2 and §3.3.3, we classify basic pairs with indices of rigidity 0 and -2. This gives classifications of Euler transform orbits of differential equations with these indices of rigidity. When all singular points are regular singular points, these classifications are given by V. Kostov [6] and the second author [9, 10], respectively.

2 A classification of basic roots

2.1 Symmetric Kac-Moody root systems

Let $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be a $\mathbb{Z}$-lattice with the basis $\{\alpha_i \mid i \in I\}$ where I is a finite set of indices. The set of positive elements in Q is written by $Q^+ := Q \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. Fix a symmetric $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle$ on Q satisfying

$$\langle \alpha_i, \alpha_i \rangle = 2 \quad (i \in I),$$
$$\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle \in \mathbb{Z}_{\leq 0} \quad (i,\, j \in I \text{ and } i \neq j).$$

We call this lattice Q with the bilinear form $\langle\ ,\ \rangle$ the *symmetric Kac-Moody root lattice*.

For an element $\alpha \in Q$, we define an even integer

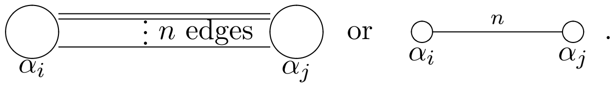
$$\operatorname{idx} \alpha := \langle \alpha, \alpha \rangle,$$

which we call the *index* of α. For each α_i $(i \in I)$, we can define a $\mathbb{Z}$-endomorphism of Q by

$$\sigma_i(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i \quad (\beta \in Q),$$

which is called the *simple reflection* with respect to α_i. The transformation group W on Q generated by all these σ_i $(i \in I)$ is called the *Weyl group*.

For this lattice Q, we associate a diagram which consists of edges and vertices as follows. Regard the elements in $\Pi := \{\alpha_i \mid i \in I\}$ as vertices. Connect two vertices α_i, $\alpha_j \in \Pi$ by n edges if $\langle \alpha_i, \alpha_j \rangle = -n$ with a positive integer n. We express this by



We call the diagram constructed as above the *Dynkin diagram* of Q.

Let $\alpha = \sum_{i \in I} m_i \alpha_i \in Q$ with $m_i \in \mathbb{Z}$. The *support* of α is $\operatorname{supp} \alpha := \{\alpha_i \mid m_i \neq 0\}$. We say the support of α is *connected* if for any two distinct elements α_i, $\alpha_j \in \operatorname{supp} \alpha$, there exists a sequence $\alpha_i = \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r} = \alpha_j$ of elements of $\operatorname{supp} \alpha$ such that $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$ for $k = 1, \ldots, r-1$. We define that α is *indivisible* if the greatest common divisor of $\{m_i \mid i \in I\}$ equals 1.

Recall the root system of Q. Each element α_i $(i \in I)$ of the basis of Q is called the *simple root*. The *real roots* are the elements of

$$\Delta_{\mathrm{re}} := \bigcup_{i \in I} W \alpha_i,$$

i.e., a real root belongs to the Weyl group orbit of a simple root α_i. Define the fundamental subset of Q,

$$F := \{\alpha \in Q^+ \setminus \{0\} \mid \operatorname{supp} \alpha \text{ is connected and } \langle \alpha, \alpha_i \rangle \leq 0 \text{ for all } i \in I\}.$$

Then the imaginary roots are the elements of

$$\Delta_{\mathrm{im}} := WF \sqcup -WF.$$

Here $WF = \{w\alpha \mid w \in W, \alpha \in F\}$ and $-WF = \{-\alpha \mid \alpha \in WF\}$. The root is the element of $\Delta := \Delta_{\mathrm{re}} \sqcup \Delta_{\mathrm{im}}$. The root in $\Delta^+ := \Delta \cap Q^+$ and that in F are called *positive* and *basic*, respectively.

In general the symmetric Kac-Moody root system determined by the pair $\langle\ ,\ \rangle$ and I shall be denoted by $(\langle\ ,\ \rangle,\ I)$. A symmetric Kac-Moody root system $(\langle\ ,\ \rangle_1,\ I_1)$ is a subsystem of a symmetric Kac-Moody root system $(\langle\ ,\ \rangle_3,\ I_3)$ if there is a map ϕ of I_1 to I_3 such that $\langle \alpha_i, \alpha_j \rangle_1 = \langle \alpha_{\phi(i)}, \alpha_{\phi(j)} \rangle_3$ for i, $j \in I_1$ and in this case the root of $(\langle\ ,\ \rangle_1,\ I_1)$ is naturally identified with a root of $(\langle\ ,\ \rangle_3,\ I_3)$.

We define a root α of $(\langle\ ,\ \rangle_1,\ I_1)$ and a root α' of $(\langle\ ,\ \rangle_2,\ I_2)$ are in a same Weyl group orbit in a universal symmetric Kac-Moody root system if there exists a symmetric Kac-Moody root system $(\langle\ ,\ \rangle_3,\ I_3)$ such that $(\langle\ ,\ \rangle_1,\ I_1)$ and $(\langle\ ,\ \rangle_2,\ I_2)$ are subsystems of $(\langle\ ,\ \rangle_3,\ I_3)$ and moreover α and α' are in the same orbit under the action of the Weyl group of $(\langle\ ,\ \rangle_3,\ I_3)$. Namely, the universal symmetric Kac-Moody root system is defined by the inductive limit of symmetric Kac-Moody root systems under the injective maps defining subsystems.

Our purpose is to classify the Weyl group orbits in the universal symmetric Kac-Moody root system. Since the real roots form a single Weyl group orbit, it is sufficient to classify the orbits contained in the set of positive imaginary roots.

For an element $\alpha = \sum_{i\in I} m_i\alpha_i \in Q$, we consider the diagram of $\operatorname{supp}\alpha$, that is, we restrict the Dynkin diagram of Π to $\operatorname{supp}\alpha$. Then we attach each coefficient m_i of α to the vertex corresponding to α_i and obtain the diagram of the support of α with the coefficients. We call this diagram with coefficients the *shape* of α.

For example, if $\alpha = m_1\alpha_{i_1} + m_2\alpha_{i_2} + m_3\alpha_{i_3} \in Q$ with the diagram of the support $\underset{\alpha_{i_1}}{\bigcirc}\!-\!\underset{\alpha_{i_2}}{\bigcirc}\!-\!\underset{\alpha_{i_3}}{\bigcirc}$, the diagram with coefficients is $\overset{m_1}{\underset{\alpha_{i_1}}{\bigcirc}}\!-\!\overset{m_2}{\underset{\alpha_{i_2}}{\bigcirc}}\!-\!\overset{m_3}{\underset{\alpha_{i_3}}{\bigcirc}}$.

Note that each Weyl group orbit contained in the set of positive imaginary roots has a unique representative in F and therefore the orbits containing positive imaginary roots are classified by the shapes of the basic roots in the orbits.

2.2 Basic roots with a fixed index

First we examine some properties of the shapes of basic roots.

Fix an indivisible basic root

$$\alpha = \sum_{i\in I} m_i\alpha_i \quad (m_i \in \mathbb{Z}_{\geq 0}) \tag{1}$$

in this section and define subsets of I

$$\begin{cases} \bar{I} = \{i \in I \mid m_i > 0\}, \\ I_0 = \{i \in \bar{I} \mid \langle\alpha, \alpha_i\rangle = 0\}, \\ I_1 = \bar{I} \setminus I_0. \end{cases} \tag{2}$$

Lemma 2.1. *Let $\{i_1, \ldots, i_k\} \subset J$ for a subset J of $\bar{I}$ such that $i_\nu \neq i_{\nu'}$ for $1 \leq \nu < \nu' \leq k$ and $\langle\alpha_{i_\nu}, \alpha_{i_{\nu+1}}\rangle \neq 0$ for $\nu = 1, \ldots, k-1$. Then we call that $i_1, \ldots, i_k$ is a connected sequence of length k in J. Moreover if $m_{i_1} = m_{i,2} = \cdots = m_{i,k}$, we call $i_1, \ldots, i_k$ is a constant connected sequence.*

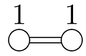
i) *Suppose i_1, i_2 is a connected sequence in $\bar{I}$ with $i_2 \in I_0$. Then*

$$m_{i_1} \leq 2m_{i_2} \tag{3}$$

and if $m_{i_1} = 2m_{i_2}$,

$$\langle\alpha_{i_1}, \alpha_{i_2}\rangle = -1 \tag{4}$$

and $\langle\alpha_{i_2}, \alpha_\nu\rangle = 0$ for $\nu \in \bar{I} \setminus \{i_1, i_2\}$.

Furthermore if $i_1 \in I_0$, then (4) is valid or the shape of α is 

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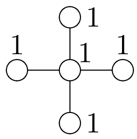
ii) *Fix $i_0 \in I_0$ and put $J_{i_0} = \{i \in I_0 \mid \langle \alpha_i, \alpha_{i_0} \rangle < 0\}$. Then $\# J_{i_0} \le 4$ and the equality holds if and only if the shape of α is* 

. *If $\# J_{i_0} = 3$, then $m_i < m_{i_0}$ for $i \in J_{i_0}$ or $\{m_i \mid i \in J_{i_0}\} = \{m_{i_0}, \frac{1}{2} m_{i_0}, \frac{1}{2} m_{i_0}\}$.*

iii) *Let $i_1, \ldots, i_k$ be a connected sequence in I with $k \ge 3$. Suppose $i_\nu \in I_0$ for $\nu = 2, \ldots, k-1$ and $m_{i_1} \ge m_{i_2}$. Then*

$$m_{i_1} - m_{i_k} \ge (k-1)(m_{i_1} - m_{i_2}).$$

If $m_{i_1} - m_{i_k} = (k-1)(m_{i_1} - m_{i_2})$, then

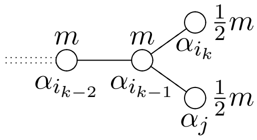
$$\langle \alpha_i, \alpha_{i_\nu} \rangle = \begin{cases} -1 & (i = i_{\nu-1} \text{ or } i_{\nu+1} \text{ and } 1 < \nu < k), \\ 0 & (i \in \bar{I} \setminus \{i_{\nu-1},\ i_\nu,\ i_{\nu+1}\} \text{ and } 1 < \nu < k). \end{cases}$$

If $m_{i_1} - m_{i_{k-1}} = (k-2)(m_{i_1} - m_{i_2})$ and $m_{i_1} - m_{i_k} > (k-1)(m_{i_1} - m_{i_2})$, then there exists $j \in \bar{I}$ such that

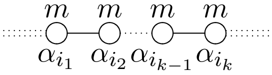
$$\langle \alpha_{i_{k-1}}, \alpha_j \rangle < 0 \text{ and } j \in I_1 \tag{5}$$

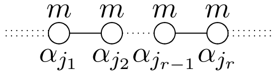
or

$$\begin{cases} m_{i_1} = m_{i_2},\ i_k \ne j,\ m_{i_{k-1}} = 2m_{i_k} = 2m_j \\ \text{and } \ \langle \alpha_{k-1}, \alpha_j \rangle = \langle \alpha_{k-1}, \alpha_{i_k} \rangle = -1. \end{cases} \tag{6}$$



Suppose $m_{i_1} = m_{i_2} = \cdots = m_{i_k}$. Then $\{j \in \bar{I} \mid \langle \alpha_{i_\nu}, \alpha_j \rangle < 0\} = \{i_{\nu-1}, i_{\nu+1}\}$ for $\nu = 2, \ldots, k-1$. Moreover suppose $\langle \alpha_{i_1}, \alpha_{i_k} \rangle = 0$. Fix $r \in \mathbb{Z}_{>0}$, put $m = m_{i_1}$ and introduce new simple roots $\alpha_{j_1}, \ldots, \alpha_{j_r}$ and put $I' = \big(\bar{I} \cup \{j_1, \ldots, j_r\}\big) \setminus \{i_1, \ldots, i_k\}$. Then the element $\alpha' = \sum_{i \in I'} m_i \alpha_i$ with $m_{j_\nu} = m$ $(1 \le \nu \le r)$ is also a basic root such that $r = 1$ or $j_1, \ldots, j_r$ is a constant connected sequence satisfying $\langle \alpha, \alpha_{j_\nu} \rangle = 0$ for $\nu = 2, \ldots, r-1$ and $\operatorname{idx} \alpha = \operatorname{idx} \alpha'$. *Here $\langle \alpha_j, \alpha_{j_1} \rangle = \langle \alpha_j, \alpha_{i_1} \rangle + \delta_{r,1} \langle \alpha_j, \alpha_{i_k} \rangle$ for $j \in \bar{I} \setminus \{i_1, \ldots, i_k\}$ etc.*

$\alpha :$ 

$\rightarrow$ $\alpha' :$ 

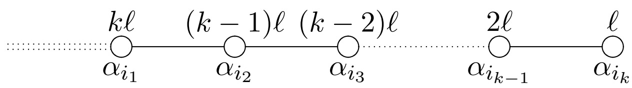
$(r = 1, 2, \ldots)$

iv) *Suppose that i_1, i_2 is a connected sequence in $\bar{I}$ with $i_2 \in I_0$ and $\ell := m_{i_1} - m_{i_2} \ge 0$. Then there exists a connected sequence $i_1, i_2, \ldots, i_k$ in $\bar{I}$ such that*

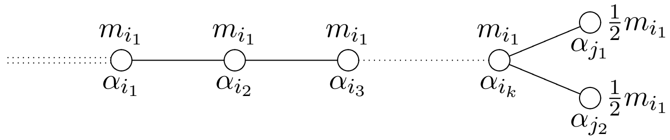
$$\langle \alpha_{i_\nu}, \alpha_i \rangle = 0 \quad (i \in \bar{I} \setminus \{i_{\nu-1}, i_\nu, i_{\nu+1}\},\ \nu = 2, \ldots, k-1)$$

and one of the following is valid.

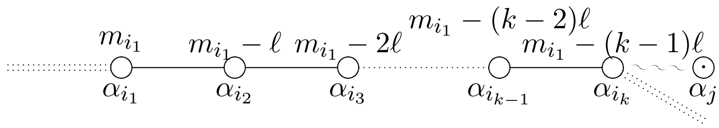
(a) $\ell > 0$, $k\ell = m_{i_1}$, $m_{i_\nu} = m_{i_1} - (\nu - 1)\ell$ $(0 \le \nu \le k)$, *and* $\langle \alpha_{i_k}, \alpha_i \rangle = 0$ $(i \in \bar{I} \setminus \{i_{k-1},\, i_k\})$: (7)



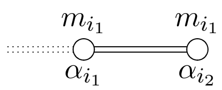
(b) $\ell = 0$, $m_{i_1} = \cdots = m_{i_{k-1}} = m_{i,k}$ *and there exist* $j_\nu \in I_0$ *for* $\nu = 1, 2$ *such that* $2m_{j_\nu} = m_{i_1}$, $\langle \alpha_{j_\nu}, \alpha_i \rangle = 0$ $(i \in \bar{I} \setminus \{i_k,\ j_\nu\})$, $\langle \alpha_{j_\nu}, \alpha_{i_k} \rangle = -1$, $\langle \alpha_{i_k}, \alpha_i \rangle = 0$ $(i \in \bar{I} \setminus \{i_{k-1},\, i_k,\, j_1,\, j_2\})$: (8)



(c) $k\ell < m_{i_1}$, $m_{i_\nu} = m_{i_1} - (\nu - 1)\ell$ *for* $\nu = 1, \ldots, k$ *and there exists* $j \in I_1$ *with* $\langle \alpha_{i_k}, \alpha_j \rangle < 0$: (9)



(d) $k = 2$, $l = 0$, $m_{i_1} = m_{i_2}$, $\langle \alpha_{i_1}, \alpha_{i_2} \rangle = -2$, $\langle \alpha_{i_2}, \alpha_i \rangle = 0$ $(i \in \bar{I} \setminus \{i_1,\, i_2\})$

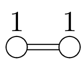


(10)

Proof. i) Since $\langle \alpha_{i_1}, \alpha_{i_2} \rangle \le -1$ and $\langle \alpha_\nu, \alpha_{i_2} \rangle \in \mathbb{Z}_{\le 0}$ for $\nu \in \bar{I} \setminus \{i_1, i_2\}$ and

$$2m_{i_2} + m_{i_1}\langle \alpha_{i_1}, \alpha_{i_2} \rangle + \sum_{\nu \in I \setminus \{i_1, i_2\}} m_\nu \langle \alpha_\nu, \alpha_{i_2} \rangle = \langle \alpha, \alpha_{i_2} \rangle = 0,$$

we have $m_{i_1} \le 2m_{i_2}$ and the condition $m_{i_1} = 2m_{i_2}$ implies $\langle \alpha_{i_1}, \alpha_{i_2} \rangle = -1$ and $\langle \alpha_\nu, \alpha_{i_2} \rangle = 0$ for $\nu \in \bar{I} \setminus \{i_1, i_2\}$.

Suppose $i_1 \in I_0$ and $\langle \alpha_{i_1}, \alpha_{i_2} \rangle < -1$. Then we have $m_{i_2} \le m_{i_1}$. In the same way we have $m_{i_1} \le m_{i_2}$ and hence $m_{i_1} = m_{i_2}$ and $\langle \alpha_{i_1}, \alpha_{i_2} \rangle = -2$ and the shape of α is 

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ii) We may assume $\# J_{i_0} > 2$. Since the claim i) shows $\langle \alpha_{i_0}, \alpha_\nu \rangle = -1$ and $m_{i_0} \le 2m_\nu$ for $\nu \in J_{i_0}$ and the condition $i_0 \in I_0$ implies $2m_{i_0} - \sum_{\nu \in J_{i_0}} m_\nu \ge 0$, we have $\# J_{i_0} \le 4$. If $\# J_{i_0} = 4$, $2m_\nu = m_{i_0}$ for $\nu \in J_{i_0}$ and the shape of α is given in the claim.

Suppose $\# J_{i_0} = 3$. Put $J_{i_0} = \{i_1, i_2, i_3\}$ with $m_{i_1} \ge m_{i_2} \ge m_{i_3}$. Then $m_{i_1} \le 2m_{i_0} - m_{i_2} - m_{i_3} \le 2m_{i_0} - \frac{1}{2}m_{i_0} - \frac{1}{2}m_{i_0} = m_{i_0}$. If $m_{i_1} = m_{i_0}$, $m_{i_2} = m_{i_3} = \frac{1}{2}m_{i_0}$.

iii) We may assume $k = 3$. Since $m_{i_1} \geq m_{i_2}$ and

$$0 = \langle \alpha, \alpha_{i_2} \rangle = 2m_{i_2} + m_{i_1} \langle \alpha_{i_1}, \alpha_{i_2} \rangle + m_{i_3} \langle \alpha_{i_3}, \alpha_{i_2} \rangle + \sum_{\nu \in I \setminus \{i_1, i_2, i_3\}} m_\nu \langle \alpha_\nu, \alpha_{i_2} \rangle,$$

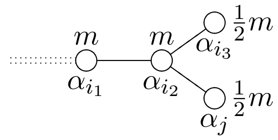
we have $\langle \alpha_{i_1}, \alpha_{i_2} \rangle = -1$ and $2m_{i_2} \geq m_{i_1} + m_{i_3}$, which means $m_{i_1} - m_{i_3} \geq 2(m_{i_1} - m_{i_2})$. Moreover the condition $m_{i_1} - m_{i_3} = 2(m_{i_1} - m_{i_2})$ implies $\langle \alpha_{i_2}, \alpha_{i_3} \rangle = -1$ and $\langle \alpha_\nu, \alpha_{i_2} \rangle = 0$ for $\nu \in \bar{I} \setminus \{i_1, i_2, i_3\}$.

Suppose $2m_{i_2} > m_{i_1} + m_{i_3}$ and $i_3 \in I_0$. Then the claim i) shows $\langle \alpha_{i_2}, \alpha_{i_3} \rangle = -1$ and there exists $j \in \bar{I} \setminus \{i_1, i_2\}$ satisfying $\langle \alpha_{i_2}, \alpha_j \rangle < 0$. Suppose $j \in I_0$. Then $2m_j \geq m_{i_2}$, $2m_{i_3} \geq m_{i_2}$ and $m_{i_2} - m_{i_3} - m_j \geq 2m_{i_2} - m_{i_1} - m_{i_3} - m_j \geq 0$ and therefore $2m_j = 2m_{i_3} = m_{i_1} = m_{i_2}$ and

$$\langle \alpha_{i_2}, \alpha_\nu \rangle = \begin{cases} -1 & (\nu = i_1,\, i_3,\, j), \\ 0 & (\nu \in \bar{I} \setminus \{i_1,\, i_2,\, i_3,\, j\}), \end{cases}$$

$$\langle \alpha_j, \alpha_\nu \rangle = 0 \quad (\nu \in \bar{I} \setminus \{i_2,\, j\}),$$

$$\langle \alpha_{i_3}, \alpha_\nu \rangle = 0 \quad (\nu \in \bar{I}\{i_2,\, i_3\}).$$



Thus we have iii) since the last claim in iii) is clear.

iv) The claims easily follow from iii). □

Now we give one of our main results in this paper.

Theorem 2.2. *Fix integers $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{>0}$. Let α be the basic root satisfying the following conditions:*

1. $\operatorname{idx} \alpha = -N$.

2. $N \neq 0$ *or* α *is indivisible.*

3. α *has no constant connected sequence in* I_0 *whose length is larger than* M.

Then there are only finite shapes which can be the shape of α.

Proof. Since the basic roots with index 0 are well-known as are given in the next section, we may assume $N > 0$. We shall use the notation in the previous lemma. Since

$$N = -\sum_{i \in I_1} m_i \langle \alpha, \alpha_i \rangle \geq \sum_{i \in I_1} m_i, \tag{11}$$

we have

$$m_i \leq N \text{ for } i \in I_1 \text{ and } \# I_1 \leq N. \tag{12}$$

Let $i \in I_1$ and $j \in I_0$ and suppose $\langle \alpha_i, \alpha_j \rangle < 0$. Then

$$N = -\sum_{\nu \in I_1} m_\nu \langle \alpha, \alpha_\nu \rangle \geq -m_i \langle \alpha, \alpha_i \rangle \geq m_i m_j |\langle \alpha_j, \alpha_i \rangle| - 2m_i^2$$

and therefore

$$m_j \le m_i^{-1}N + 2m_i \le 3N \quad\text{and}\quad |\langle \alpha_i, \alpha_j \rangle| \le 3N. \tag{13}$$

Since $N = -\sum_{\nu\in I_1} m_\nu \langle \alpha, \alpha_\nu \rangle = -\sum_{i\in I}\sum_{\nu\in I_1} m_i m_\nu \langle \alpha_i, \alpha_\nu\rangle - 2\sum_{\nu\in I_1} m_\nu^2$, we have

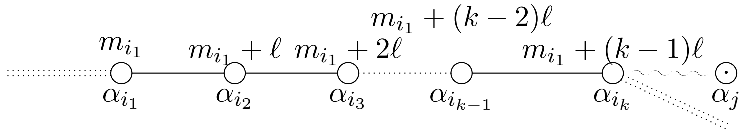
$$\sum_{i\in I_0}\sum_{\nu\in I_1} |\langle \alpha_i, \alpha_\nu\rangle| \le N + 2\sum_{\nu\in I_1} m_\nu^2 \le N + 2N^2 \le 3N^2 \tag{14}$$

and therefore $\#\partial I_0 \le 3N^2$ by denoting $\partial I_0 := \{i \in I_0 \mid \sum_{\nu\in I_1}\langle \alpha_i, \alpha_\nu\rangle \ne 0\}$.

Fix $i_1 \in \partial I_0$. Suppose $J_{i_1} := \{j \in I_0 \mid \langle \alpha_j, \alpha_{i_i}\rangle < 0\} \ne \emptyset$. Note that $\#J_{i_1} \le 3$. Fix $i_2 \in J_{i_1}$ and put

$$\begin{aligned} J(i_1, i_2) = \{i \in I_0 \mid \exists\ &\text{connected sequence } i_1, i_2, \ldots, i_k = i \text{ in } I_0 \\ &\text{with } \ i_\nu \notin \partial I_0 \ (1 < \nu < k)\}. \end{aligned}$$

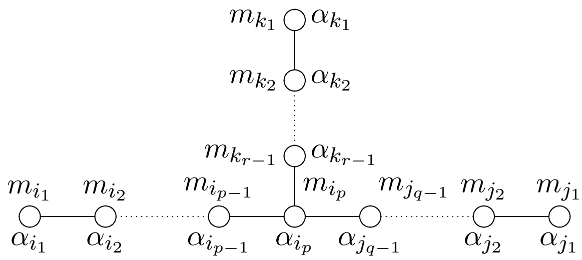
Then the Dynkin diagram of $J(i_1, i_2)$ equals that in (7) or (8) or (9) or



(15)

$\ell \in \mathbb{Z}_{>0},\ \alpha_j \in I_1,\ J(i_1, i_2) = \{i_1, \ldots, i_k\},\ i_2, \ldots, i_{k-1} \in I_0 \setminus \partial I_0$

or



(16)

$m_{i_\nu} = m_{i_p} - (p-\nu)\ell_i,\ m_{j_\nu} = m_{i_p} - (q-\nu)\ell_j,\ m_{k_\nu} = m_{i_p} - (q-\nu)\ell_k,$
$\ell_i,\ \ell_j,\ \ell_k \in \mathbb{Z}_{>0},\ i_2, \ldots, i_p,\ j_2, \ldots, j_{q-1}, k_2, \ldots, k_{r-1} \in I_0 \setminus \partial I_0,$
$\{i_1, j_1, k_1\} \cap \partial I_0 \ne \emptyset,\ p \ge 2,\ q \ge 2,\ r \ge 2.$

If the Dynkin diagram is not of the form (16), we have

$$\#J(i_1, i_2) \le 3N + M \quad\text{and}\quad m_i \le 3N \quad (i \in J(j_1, j_2)) \tag{17}$$

by the estimate (13).

Hence we assume the Dynkin diagram is of the form (16). We may assume $\ell_i \le \ell_j \le \ell_k$ without loss of generality. Since $i_p \in I_0$, we have $2m_{i_p}^2 = m_{i_p}(m_{i_p} - \ell_i) + m_{i_p}(m_{i_p} - \ell_j) + m_{i_p}(m_{i_p} - \ell_k)$ and therefore

$$\ell_i + \ell_j + \ell_k = m_{i_p}.$$

Hence $3\ell_k \geq m_{i_p}$ and $r \leq 3$.

If $k_1 \in \partial I_0$, $r = 2$ and $m_{k_1} \leq 3N$ and we have

$$m_{i_p} < 6N,\ \#J(i_1, i_2) < 12N \text{ and } m_i < 6N \text{ for } i \in J(i_1, i_2) \tag{18}$$

because $p < 6N$ and $q < 6N$.

Suppose $k_1 \in I_0 \setminus \partial I_0$. Then $r = 2$ or $r = 3$.

If $r = 3$, $\ell_i = \ell_j = \ell_k = \frac{1}{3}m_{i_p}$ and we may assume $i_1 \in \partial I_0$ and we have the same claim (18).

Suppose $r = 2$. Then $\ell_k = \frac{1}{2}m_{i_p}$ and $\frac{1}{2}m_{i_p} > \ell_j \geq \frac{1}{4}m_{i_p}$. If $j_1 \in I_0 \setminus \partial I_0$, $4\ell_j = 4\ell_i = m_{i_p}$ or $3\ell_j = 6\ell_i = m_{i_p}$ and therefore $p \leq 5$ and

$$\#J(i_1, i_2) < 9 \ \text{ and } \ m_i \leq 15N \quad (i \in J(i_1, i_2)). \tag{19}$$

If $j_1 \in \partial I_0$, $q \leq 3$ and $m_p \leq 9N$ and therefore

$$\#J(i_1, i_2) \leq 9N + 2 + 1 = 9N + 3 \ \text{ and } \ m_i \leq 9N \quad (i \in J(i_1, i_2)). \tag{20}$$

Since $\#\{(i_1, i_2) \mid i_1 \in \partial I_0,\ i_2 \in I_0,\ \langle \alpha_{i_1}, \alpha_{i_2} \rangle < 0\} \leq 3 \cdot \#\partial I_0 \leq 9N^2$, we have

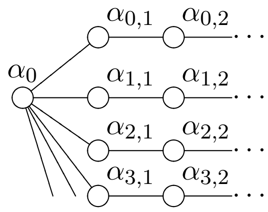
$$\#I \leq 9N^2 \cdot (12N + M) + \#\partial I_0 + \#I_1 \leq 108N^3 + 9MN^2 + 3N^2 + N,$$
$$m_i \leq 15N \quad (i \in I) \ \text{ and } \ |\langle \alpha_i, \alpha_j \rangle| \leq 3N \quad (i,\, j \in I).$$

These estimates imply the theorem. $\square$

The proof of Theorem 2.2 assures the following finiteness of the shapes.

Corollary 2.3. *If a basic root $\alpha = \sum_{i \in I} m_i \alpha_i$ contains a constant connected sequence $i_1, \ldots, i_k$ of I such that $k \geq 2$ and $\langle \alpha, \alpha_{i_\nu} \rangle = 0$ for $\nu = 2, \ldots, k-1$, then the shape obtained from that of α by shrinking or extending the length of the sequence corresponds to a basic root with the same index. Expressing such a sequence by* $\overset{m}{\bigcirc}\cdots\overset{m}{\bigcirc}$ *, we have shapes of roots which may contain such expressions. We call these shapes reduced shapes. Then the basic roots with a fixed nonzero index are classified by a finite number of reduced shapes. Also the indivisible basic roots with index 0 are classified by a finite number of reduced shapes.*

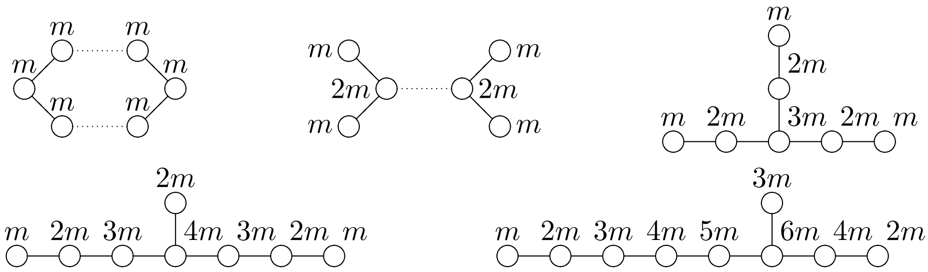
Remark 2.4. *The Dynkin diagram of the form*

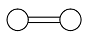
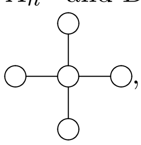
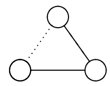
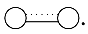


is called star-shaped. The basic roots whose shapes have star-shaped Dynkin diagrams are studied and the finiteness of such basic roots with a fixed index is proved in [9]. *The number of such shapes with index* 0, -2, -4, -6,… *equals* 4, 13, 36, 67, …, *respectively, and the list of them is given in* [10].

2.3 Basic roots with index 0

Theorem 2.2 assures that in the universal symmetric Kac-Moody root system there are only a finite number of Weyl group orbits with a fixed index. The basic roots with index 0 are well-known and we list their shapes as follows.



These are diagrams obtained by attaching coefficients to the Dynkin diagrams called *Euclidean diagrams*, which are denoted by $A_n^{(1)}$ $(n \geq 1)$, $D_n^{(1)}$ $(n \geq 4)$, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$, respectively. Here m are positive integers and $A_n^{(1)}$ and $D_n^{(1)}$ have $n+1$ vertices. Moreover $A_1^{(1)}$ and $D_4^{(1)}$ mean  and , respectively. Hence $A_n^{(1)}$ for $n \geq 1$ shall be written by  or .

2.4 Basic roots with index -2

In this section, we shall give a classification of the basic roots whose indices are -2. Suppose that $\alpha = \sum_{i \in I} m_i \alpha_i \in Q$ is basic and $\operatorname{idx} \alpha = -2$. Retain the notation in §2.2 and put $N_i = -\langle \alpha, \alpha_i \rangle \geq 0$. Then $\langle \alpha, \alpha \rangle = -\sum_{i \in I} m_i N_i$ and $I_1 = \{i \in I \mid N_i > 0,\ m_i > 0\}$.

Lemma 2.5. *Let $\alpha \in Q$ be as above. Put*

$$\operatorname{Ed}(\alpha_i) := -\sum_{j \in \bar{I} \setminus \{i\}} \langle \alpha_j, \alpha_i \rangle, \tag{21}$$

which equals the number of edges spread out from α_i. Then we have the following.

i) *The cardinality $\# I_1$ is* 1 *or* 2*.*

ii) *If $I_1 = \{i\}$, there are two cases.*

Case 1: $m_i = 2$, $N_i = 1$ *and* $\operatorname{Ed}(\alpha_i) \leq 5$.

Case 2: $m_i = 1$, $N_i = 2$ *and* $\operatorname{Ed}(\alpha_i) \leq 4$.

iii) *If $I_1 = \{i, i'\}$, then $m_i = m_{i'} = N_i = N_{i'} = 1$,* $\operatorname{Ed}(\alpha_i) \leq 3$ *and* $\operatorname{Ed}(\alpha_{i'}) \leq 3$.

Proof. Since $2 = \sum_{i \in I} m_i N_i = \sum_{j \in I_1} m_j N_j$, we have $\# I_1 = 1$ or 2. Then $(m_i, N_i) = (1, 2)$ or $(2, 1)$ if $I_1 = \{i\}$ and $m_i = m_{i'} = N_i = N_{i'} = 1$ if $I_1 = \{i, i'\}$. The remaining assertions follow from $N_i = \sum_{j \in \bar{I} \setminus \{i\}} m_j \langle \alpha_j, \alpha_i \rangle - 2m_i \geq \mathrm{Ed}\,(\alpha_i) - 2m_i$. $\square$

From this lemma and Lemma 2.1, the basic roots with index -2 are classified by the following cases.

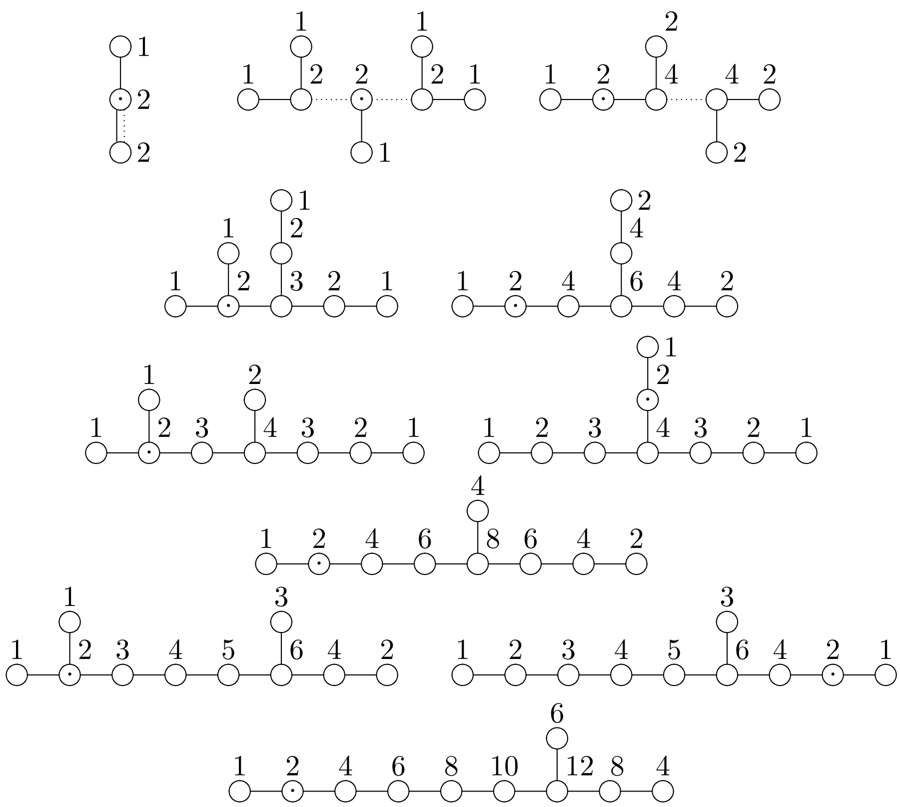
<u>Case 1</u>: $I_1 = \{i\}$, $m_i = 2$ and $\mathrm{Ed}\,(\alpha_i) \leq 5$
Since

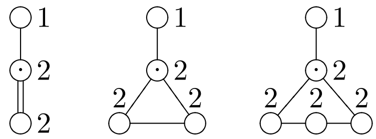
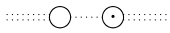
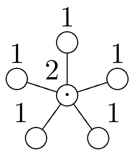
$$\sum_{\nu \in \bar{I} \setminus \{i\}} m_\nu |\langle \alpha_i, \alpha_\nu \rangle| = 5,$$

one of the following Case 1.1, Case 1.2 or Case 1.3 is valid.

<u>Case 1.1</u>: There exists α_k such that $m_k = 1$ and $\langle \alpha_i, \alpha_k \rangle < 0$.

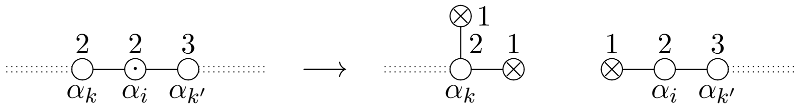
It follows from Lemma 2.1 i) that $\langle \alpha_j, \alpha_k \rangle = 0$ for $j \in \bar{I} \setminus \{i,\, k\}$ and $\langle \alpha_i, \alpha_k \rangle = -1$. Then the element $\alpha' = \alpha - \alpha_k \in Q^+$ satisfies $\langle \alpha', \alpha_i \rangle = 0$ for $i \in \bar{I} \setminus \{k\}$ and $\mathrm{supp}\, \alpha'$ is connected. Hence the diagram of $\{\alpha_i \mid i \in \bar{I} \setminus \{k\}\}$ is one of the Euclidean diagrams $A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ given in the previous section and we have the list of the shapes with indicating α_i by dotted circles.



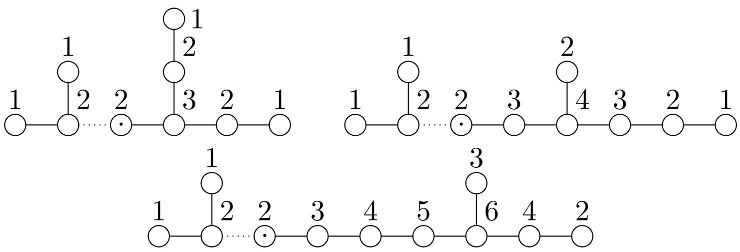
Note that the first shape represents  $\cdots$, etc., and the part  in the diagram above can be  as a special case. Hence  is a special case of the second shape.

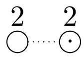
Case 1.2: There exist α_k and $\alpha_{k'}$ such that $(m_k, m_{k'}) = (2, 3)$ and $\langle \alpha_i, \alpha_k \rangle = \langle \alpha_i, \alpha_{k'} \rangle = -1$.

Then cutting the shape of the basic root between α_k and α_i and adding three vertices, we have one or two Euclidean diagrams with coefficients corresponding to some basic roots of index 0:



Here each $\otimes$ represents a new vertex. It follows from the shapes given in the previous section that the corresponding diagrams are $D_n^{(1)}$ and $E_k^{(1)}$ ($k = 6$, 7, 8) and we have the following list of shapes:



Replacing  by  in the above shapes, we may regard three shapes in Case 1.1 as special cases in Case 1.2.

Case 1.3: There uniquely exists α_k with $k \in \bar{I}$ such that $\langle \alpha_i, \alpha_k \rangle < 0$. Put

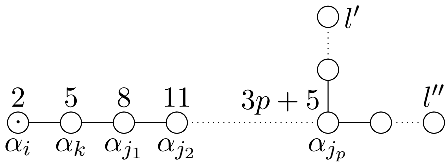
$$\{j \in \bar{I} \mid \langle \alpha_k, \alpha_j \rangle < 0\} = \{i, l_1, \ldots, l_r\}$$

with suitable r. Note that $m_i = 2$, $m_k = 5$, $k \in I_0$, $\ell_\nu \in I_0$ and $\langle \alpha_{l_\nu}, \alpha_k \rangle = -1$ for $\nu = 1, \ldots, r$. Since $\sum_{j \in \bar{I}} m_j \langle \alpha_k, \alpha_j \rangle = 0$, we have

$$m_{l_1} + \cdots + m_{l_r} = 2m_k - m_i = 2 \times 5 - 2 = 8$$

and Lemma 2.1 iv) shows $m_{l_\nu} \geq 4$ and $m_{l_\nu} \neq 5$ for $\nu = 1, \ldots . r$. Hence $\{m_{l_1}, \ldots, m_{l_r}\} = \{4, 4\}$ or $\{8\}$.

Suppose $\{m_{l_1}, \ldots, m_{l_r}\} = \{8\}$. Then the shape of α is



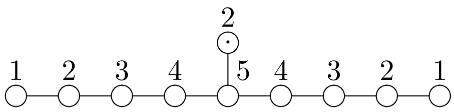
$(p \geq 1,\ j_1 = l_1)$

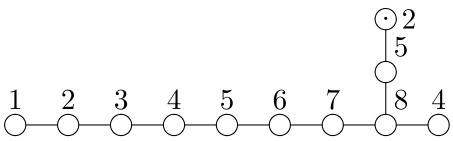
and there exist positive integers p' and p'' such that $p'l' = p''l'' = 3p+5$. The condition $j_p \in I_0$ shows

$$\begin{aligned}
2(3p+5) &= (3p+2) + (3p+5)\frac{p'-1}{p'} + (3p+5)\frac{p''-1}{p''}, \\
2 &= \frac{3p+2}{3p+5} + \frac{p'-1}{p'} + \frac{p''-1}{p''}, \\
1 &= \frac{3}{3p+5} + \frac{1}{p'} + \frac{1}{p''}.
\end{aligned}$$

Since $\{1 - \frac{1}{p'} - \frac{1}{p''} \mid p',\, p'' \in \mathbb{Z}_{>0}\} \cap (0,1) \subset [\frac{1}{6}, 1)$, it follows that $3p+5 = 8,\ 11,\ 14,\ 17$ and $1 - \frac{3}{3p+5} = \frac{5}{8},\ \frac{8}{11},\ \frac{11}{14},\ \frac{14}{17}$. Then we can conclude $p = 1$ and $\{p', p''\} = \{2, 8\}$, which corresponds to $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$.

Hence α is one of the following.





Case 2 : $I_1 = \{i\}$, $m_i = 1$ and $\mathrm{Ed}\,(\alpha_i) \leq 4$.

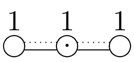
Since

$$\sum_{\nu \in \bar{I} \setminus \{i\}} m_\nu |\langle \alpha_i, \alpha_\nu \rangle| = 4,$$

one of Case 2.1, ..., Case 2.4 is valid.

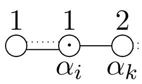
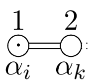
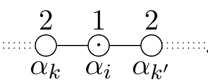
Case 2.1: The condition $\langle \alpha_\nu, \alpha_i \rangle \neq 0$ implies $m_\nu \leq 1$.

Then it follows from Lemma 2.1 that the shape of α is the following:

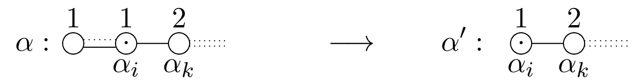


Hence the diagrams is obtained by connecting Euclidean diagrams $A_n^{(1)}$ $(n \geq 1)$ and $A_{n'}^{(1)}$ $(n' \geq 1)$ at the common vertex α_i.

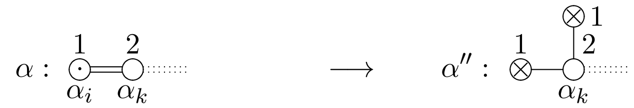
Case 2.2: There exists α_k such that $\langle \alpha_k, \alpha_i \rangle \neq 0$ and $m_k = 2$.

Then the shape of α is  or  or .

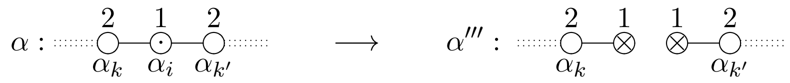
Modify these diagrams with coefficients as follows.



(22)



(23)



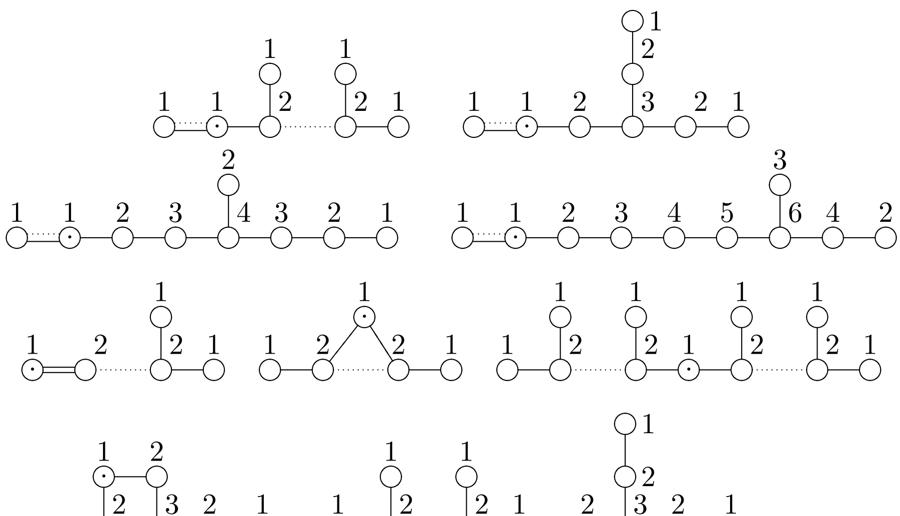
(24)

Here we do not modify the parts ⋯⋯ in the above. Then α', α'' and α''' are elements of F with the given shapes and their indices are 0.

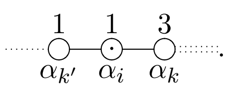
The element $\alpha' \in F$ is a basic root with the diagram $D_n^{(1)}$ or $E_7^{(1)}$ or $E_8^{(1)}$ and in this case α corresponds to the first four shapes in the list below.

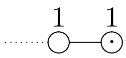
The element $\alpha'' \in F$ is an indivisible basic root with the diagram $D_n^{(1)}$ and we have the fifth shape in the list below.

The shape of $\alpha''' \in F$ is that of an indivisible basic root with the diagram $D_n^{(1)}$ or $E_6^{(1)}$ or $E_7^{(1)}$ or a disjoint union of the shapes $\mathcal{D}_1$ and $\mathcal{D}_2$ of indivisible basic roots with the diagrams $\mathcal{D}_\nu \in \{D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}\}$ for $\nu = 1$ and 2. In each case it is easy to write the shape of α and therefore we only give some examples in the list below.



Case 2.3: There exists α_k such that $\langle \alpha_k, \alpha_i \rangle \neq 0$ and $m_k = 3$.

Then α corresponds to . However applying Lemma 2.1

iv) to the part 

of this shape, we can conclude that such α does not exist.

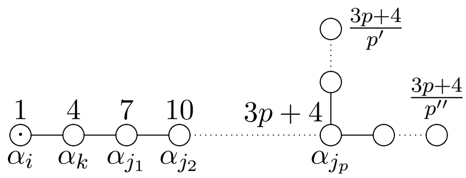
Case 2.4: There exists α_k such that $\langle \alpha_k, \alpha_i \rangle \neq 0$ and $m_k = 4$.

In the same way as in Case 1.3 we have

$$\{j \in \bar{I} \mid \langle \alpha_k, \alpha_j \rangle < 0\} = \{i, l_1, \dots, l_r\},$$
$$m_{l_1} + \cdots + m_{l_r} = 2m_k - m_i = 7,$$
$$m_{l_\nu} \geq 2, \quad m_{l_\nu} \neq 4 \quad (1 \leq \nu \leq r).$$

Hence $\{m_{\ell_1}, \dots, m_{\ell_r}\} = \{3, 2, 2\}$ or $\{5, 2\}$ or $\{4, 3\}$ or $\{7\}$.

Suppose $\{m_{\ell_1}, \dots, m_{\ell_r}\} = \{7\}$. Then α corresponds to

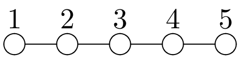


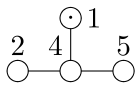
$(p \geq 1,\ j_1 = l_1)$

and we have

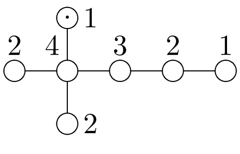
$$2(3p+4) = 3p + 1 + (3p+4)\frac{p'-1}{p'} + (3p+4)\frac{p''-1}{p''},$$
$$1 = \frac{3}{3p+4} + \frac{1}{p'} + \frac{1}{p''}.$$

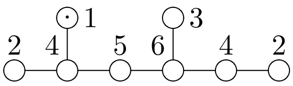
Here $\frac{3}{3p+4}$ should be $\frac{3}{7}$, $\frac{3}{10}$, $\frac{3}{13}$ or $\frac{3}{16}$. It is easy to see that $p = 2$ together with $\{p', p''\} = \{2, 5\}$ is the unique solution of the above equation.

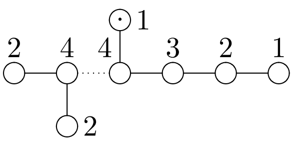
If $\{m_{\ell_1}, \dots, m_{\ell_r}\} = \{5, 2\}$, the shape of α is obtained by replacing a part 

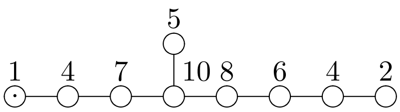
of the shape of a basic root with index 0 by 

. Hence α corresponds to one of the following:









<u>Case 3</u> : $I_1 = \{i,\, i'\}$, $m_i = m_{i'} = 1$, $\mathrm{Ed}\,(\alpha_i) \leq 3$ and $\mathrm{Ed}\,(\alpha_{i'}) \leq 3$.

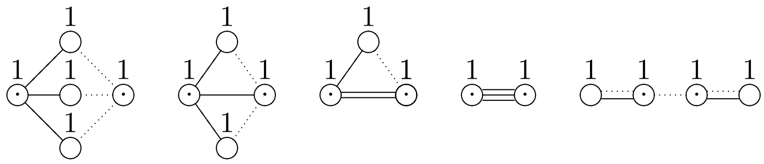
Since

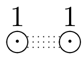
$$\begin{cases} \sum_{\nu\in\bar{I}\setminus\{i\}} m_\nu|\langle\alpha_i,\alpha_\nu\rangle| + \sum_{\nu\in\bar{I}\setminus\{i,i'\}} m_\nu|\langle\alpha_{i'},\alpha_\nu\rangle| = 6, \\ \sum_{\nu\in\bar{I}\setminus\{i\}} m_\nu|\langle\alpha_i,\alpha_\nu\rangle| \geq 3,\ \sum_{\nu\in\bar{I}\setminus\{i'\}} m_\nu|\langle\alpha_{i'},\alpha_\nu\rangle| \geq 3, \end{cases}$$

one of Case 3.1,..., Case 3.4 is valid.

Case 3.1: $\langle\alpha_k,\alpha_i\rangle = \langle\alpha_k,\alpha_{i'}\rangle = 0$ if $m_k > 1$.

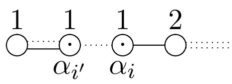
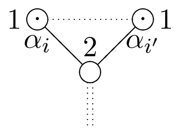
It is easy to see that the shape of α is one of the following:



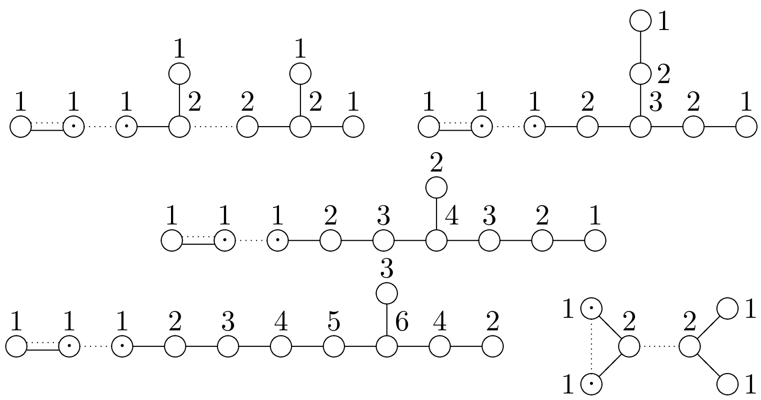
The first four shapes may be expressed by  .

Case 3.2: There uniquely exists α_k such that $\langle\alpha_k,\alpha_i\rangle \neq 0$, $\langle\alpha_k,\alpha_{i'}\rangle \neq 0$ and $m_k = 2$.

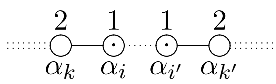
Then α is

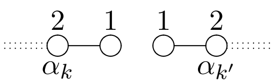
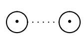
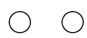
 or  .

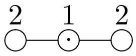
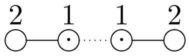
Hence α has one of the following shapes:

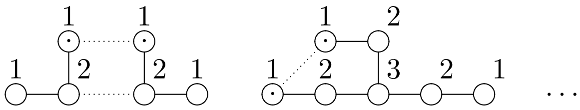


Case 3.3: There are different elements α_k and $\alpha_{k'}$ such that $\langle\alpha_k,\alpha_i\rangle \neq 0$ and $\langle\alpha_{k'},\alpha_{i'}\rangle \neq 0$ and $m_k = m_{k'} = 2$.

Then the shape of α is  and therefore the shape

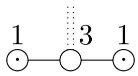
 obtained by replacing  by  are one or two

of the shapes of basic roots of index 0. Hence the list of the shapes of α is obtained by replacing  by  in the shapes classified in Case 2.2. For example we have



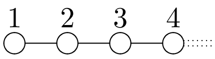
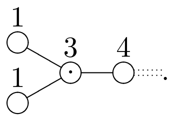
Consequently the shapes in Case 2.2 may be regarded as special cases of those in Case 3.3 except for the first five shapes listed there.

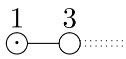
Case 3.4: There uniquely exists a pair α_k and $\alpha_{k'}$ such that $\langle\alpha_k,\alpha_i\rangle \neq 0$, $\langle\alpha_{k'},\alpha_{i'}\rangle \neq 0$ with $m_k = m_{k'} = 3$.

Suppose $k = k'$. Then the shape of α is . Then as in Case 1.3 we have

$$\{j \in \bar{I} \mid \langle\alpha_k,\alpha_j\rangle < 0\} = \{i, i', l_1, \ldots, l_r\},$$
$$m_{l_1} + \cdots + m_{l_r} = 2m_k - m_i - m_{i'} = 4.$$

Since $m_{l_\nu} \geq 2$, we have $\{m_{l_1},\ldots,m_{l_r}\} = \{2,2\}$ or $\{4\}$. If $\{m_{l_1},\ldots,m_{l_r}\} = \{2,2\}$, it corresponds to the first shape in the list below .

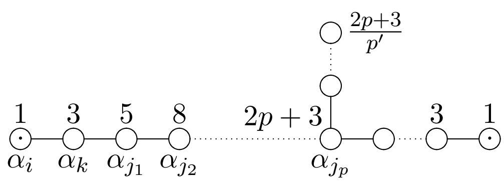
If $\{m_{l_1},\ldots,m_{l_r}\} = \{4\}$, the shape of α is obtained by replacing a part  of the shape of a basic root with index 0 by . It corresponds to the second and the third shape in the below.

Suppose $k \neq k'$. Then the shape of α contains  twice and we have

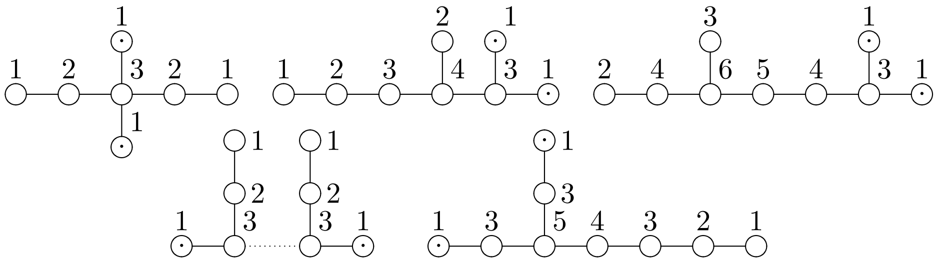
$$\{j \in \bar{I} \mid \langle\alpha_k,\alpha_j\rangle < 0\} = \{i, l_1, \ldots, l_r\},$$
$$m_{l_1} + \cdots + m_{l_r} = 2m_k - m_i = 5$$

and $m_{l_\nu} \geq 2$ for $\nu = 1,\ldots,r$. Hence $\{m_{l_1},\ldots,m_{l_r}\} = \{3,2\}$ or $\{5\}$. If $\{m_{l_1},\ldots,m_{l_r}\} = \{3,2\}$, Lemma 2.1 iv) assures that the shape of α is the forth shape in the below.

Suppose $\{m_{l_1},\ldots,m_{l_r}\} = \{5\}$. The shape of α is



with $p \geq 1$ and $j_1 = l_1$. Then $2(2p+3) = 2 + 2 + (2p+3)\frac{p'-1}{p'}$, which means $1 = \frac{4}{2p+3} + \frac{1}{p'}$ and we have $(p,p') = (1,5)$. The shape of α is the last one in below. Thus we see the following list:



3 Spectral types of differential equations

In this section, we consider linear differential equations on the Riemann sphere whose singular points are regular singular or unramified irregular singular points. For these differential equations, we define spectral types as tuples of integers representing multiplicities of characteristic exponents of local formal solutions where we ignore integer differences of characteristic exponents. We shall classify orbits of spectral types under algebraic transformations on differential equations, called the Euler transform and the addition and show the finiteness of orbits with a fixed index of rigidity, where we note that the index does not change under the transformations.

First we explain that spectral types can be seen as elements of a certain $\mathbb{Z}$-lattice L which has a group action defined by these transformations. Moreover we shall see that there exists a Kac-Moody root lattice Q_L and the lattice L can be seen as a quotient lattice of Q_L. Then the group action on L coincides with the Weyl group action on Q_L and an analogy of the root system for L shall be defined. As in the previous section, we study the classification of basic roots of L, in particular we show the finiteness of basic roots with a fixed index and give lists of basic roots with index 0 and -2.

3.1 Differential equations and spectral types

The detail of this section can be found in [7]. Let K be an algebraically closed field of characteristic zero. Let $W[x] = K[x][\partial]$ be the ring of differential operators with polynomial coefficients and $W(x) = K(x)[\partial]$ the ring of differential operators with coefficients in $K(x)$. Moreover $W((x))$ denotes the ring of differential operators with coefficients in $K((x))$, the quotient field of the ring of formal power series $K[[x]]$.

3.1.1 Local structures

In this section we review the local structure of elements in $W(x)$. We fix an element in $W(x)$, $P = \sum_{i=0}^{n} a_i(x)\partial^i$ $(a_n(x) \neq 0)$. Here the non-negative integer n is called the *rank* and written by $\operatorname{rank} P$. For $c \in K$ and a monomial

$(x-c)^a\partial^b$, we introduce the *weight*

$$\mathrm{wt}_c((x-c)^a\partial^b) := a-b.$$

The weight of $P \in W(x) \subset W((x-c))$ is defined by

$$\mathrm{wt}_c(P) := \min\bigl\{\mathrm{wt}_c((x-c)^i\partial^j) \mid P = \sum_{i,j} a_{i,j}(x-c)^i\partial^j,\ a_{i,j} \neq 0\bigr\}.$$

For $f(x) \in K((x-c))$, weight $\mathrm{wt}_c(f(x))$ is defined by regarding $f(x)$ as an element in $W((x-c))$.

For an integer k, the k-homogeneous part of $P \in W((x-c))$ is

$$P_{(k)} := \sum_{i-j=k} a_{i,j}(x-c)^i\partial^j$$

if $P = \sum_{i,j} a_{i,j}(x-c)^i\partial^j$ with $a_{i,j} \in K$.

Similarly we can define wt_∞ by

$$\mathrm{wt}_\infty(x^a\partial^b) = b-a.$$

The *singular points* of P are poles of $\frac{a_i(x)}{a_n(x)}$ $(i = 1, \ldots, n)$. We also say that ∞ is a singular point of P if

$$P^{(\infty)} := \sum_{i=0}^{n} a_i(\frac{1}{x})(-x^2\partial)^i$$

has a singular point at 0. Suppose that c $(\neq \infty)$ is a singular point of P. The $\mathrm{wt}_c(P)$-homogeneous part of P equals

$$\sum_{i-j=\mathrm{wt}_c(P)} a_{i,j}(x-c)^i\partial^j$$

and then the *characteristic polynomial* of P at c is defined by

$$\mathrm{C}_c(P)(t) := \sum_{i-j=\mathrm{wt}_c(P)} a_{i,j}t(t-1)\cdots(t-j+1).$$

If $\deg_{K[t]} \mathrm{C}_c(P)(t) = \operatorname{rank} P$, we say that c is a *regular singular point* of P. Otherwise, c is an *irregular singular point* of P. For the point ∞, we can define characteristic polynomials, regular and irregular singular points as well as the above replacing $x-c$ by $\frac{1}{x}$.

Suppose that c is an irregular singular point of P. For simplicity of notation, we put $c = 0$. There exists an algebraic extension $K((x^{\frac{1}{q}}))$ of $K((x))$ for a positive integer q and we denote the ring of differential operators with coefficients in $K((x^{\frac{1}{q}}))$ by $W_q((x))$. Then we can decompose the left-$W_q((x))$-module $W_q((x))/W_q((x))P$ as follows.

Definition 3.1 (Local decomposition (see [8] for example)). *For $P \in W(x)$ with an irregular singular point c, there exists the algebraic extension $K(((x-c)^{\frac{1}{q}}))$ of $K((x-c))$, distinct polynomials w_i of $(x-c)^{-\frac{1}{q}}$ with no constant terms and $P_i(t) \in K(((x-c)^{\frac{1}{q}}))[t]$ for $1 \leq i \leq r$ such that we have the following.*

i) *Each $P_i(\vartheta_c)$ has a regular singular point at c.*

ii) *We can write P as the least left common multiple of*

$$\{P_1(\vartheta_c - w_1), \ldots, P_r(\vartheta_c - w_r)\}.$$

Namely there exist $R_i \in W_q((x-c))$ such that

$$P = R_i P_i(\vartheta_c - w_i) \text{ for } i = 1, \ldots, r.$$

Here $\vartheta_c = (x-c)\partial$ and for $Q(t) = \sum_{\nu \geq 0} q_\nu(x) t^\nu \in K(((x-c)^{\frac{1}{q}}))[t]$ and $w \in K(((x-c)^{\frac{1}{q}})))$, we put

$$Q(\vartheta_c - w) = \sum_{\nu \geq 0} q_\nu(x)(\vartheta_c - w)^\nu.$$

iii) *We have the decomposition*

$$W_q((x-c))/W_q((x-c))P \simeq \bigoplus_{i=1}^{r} W_q((x-c))/W_q((x-c))P_i(\vartheta_c - w_i)$$

as $W_q((x-c))$-modules.

We call the decomposition in iii) *the* local decomposition *of P at c. Moreover we call $P_i(\vartheta_c - w_i) \in W_q((x-c))$* local factors *and w_i the* exponential factors *of $P_i(\vartheta_c - w_i)$ for $1 \leq i \leq r$.*

If the local decomposition at c is obtained in $W_1((x-c)) = W((x-c))$, we say that c is an unramified irregular singular point*. Otherwise, c is called a* ramified irregular singular point*.*

We introduce the notion of spectral data. Let $P \in W((x))$. We regard the left $W((x))$-module $M_P = W((x))/W((x))P$ as the $K((x))$-vector space of $\dim M_P = \operatorname{rank} P$. For a basis $\{u_1, \ldots, u_n\}$ of M_P as $K((x))$-vector space, we can represent the action of $\vartheta = x\partial$ by the matrix as follows. For $u \in M_P$, there exists $A = (a_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} \in M\big(n, K((x))\big)$ such that

$$\vartheta u_i = \sum_{j=1}^{n} a_{ij} u_j.$$

Moreover if 0 is a regular singular point of P, there exists a basis such that we can take $A \in M(n,K)$. We call this matrix $A \in M(n,K)$ a *local matrix* of P at 0. For any other regular singular point $c \in K$ and ∞, we can define a local matrix in the same way.

Definition 3.2 (Spectral data). *Fix $m_1, \ldots, m_s \in \mathbb{Z}_{>0}$ and $\lambda_1, \ldots, \lambda_s \in K$ which satisfy*

$$\lambda_i - \lambda_j \notin \mathbb{Z} \quad (i \neq j).$$

We say $P \in W(x)$ has the spectral data

$$\{(\lambda_1, \ldots, \lambda_s); (m_1, \ldots, m_s)\}$$

at c if P has a regular singular point at c and satisfies the following.

i) *The characteristic polynomial is*

$$\mathrm{C}_c(P)(t) = C \prod_{i=1}^{s} \prod_{j=0}^{m_i-1} \bigl(t - (\lambda_i + j)\bigr)$$

for a constant C.

ii) *A local matrix of P is a semisimple matrix.*

Here we note that condition ii) does not depend on the choice of local matrices.

3.1.2 Spectral types and the Euler transform

Fix $P \in W(x)$ satisfying the assumption below.

Assumption 3.3. *We assume that $P \in W(x)$ satisfies the following.*

i) *All singular points of P, written by $c_0 = \infty, c_1, \ldots, c_p \in K$, are regular singular or unramified irregular singular points.*

ii) *Denote the set of local factors of P at c_i by*

$$\{P_{i,1}(\vartheta_{c_i} - w_{i,1}), \ldots, P_{i,k_i}(\vartheta_{c_i} - w_{i,k_i})\}.$$

Then there exist positive integers $m_{i,j,s}$ and $\lambda_{i,j,s} \in K$ for $i = 0, \ldots, p$, $j = 1, \ldots, k_i$, $s = 1, \ldots, l_{i,j}$ such that $\lambda_{i,j,s} - \lambda_{i,j,s'} \notin \mathbb{Z}$ if $s \neq s'$ and $P_{i,j}(\vartheta)$ have spectral data

$$\left\{(\lambda_{i,j,1}, \ldots, \lambda_{i,j,l_{i,j}}); (m_{i,j,1}, \ldots, m_{i,j,l_{i,j}})\right\},$$

respectively. Here $w_{i,j}$ are the exponential factors of the corresponding local factors.

Put

$$\lambda(P) = \bigl((\lambda_{i,j,1}, \ldots, \lambda_{i,j,l_{i,j}})\bigr)_{\substack{0 \leq i \leq p \\ 1 \leq j \leq k_i}},$$
$$\mathbf{m}(P) = \bigl((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}})\bigr)_{\substack{0 \leq i \leq p \\ 1 \leq j \leq k_i}}.$$

The *index of rigidity* is defined by

$$
\begin{aligned}
\operatorname{idx} P := -\sum_{i=0}^{p}\sum_{1\le j\ne j'\le k_i} d_i(j,j')\Big(\sum_{s=1}^{l_{i,j}} m_{i,j,s}\Big)\Big(\sum_{s'=1}^{l_{i,j'}} m_{i,j',s'}\Big)\\
+\sum_{i=0}^{p}\sum_{j=1}^{k_i}\sum_{s=1}^{l_{i,j}} m_{i,j,s}^2-(p-1)(\operatorname{rank} P)^2
\end{aligned}
\tag{25}
$$

where $d_i(j,j') = -\mathrm{wt}_{c_i}(w_{i,j}-w_{i,j'})$ for $i=0,\dots,p$ and $j,\, j'=1,\dots,k_i$. Here we notice that these $d_i(j,j')$ satisfy

$$
\begin{aligned}
d_i(j,j') = 0 \text{ if and only if } j=j'\\
d_i(j,j') = d_i(j,j'),\\
d_i(j_1,j_2) \le \max\big\{d_i(j_1,j_3),\, d_i(j_2,j_3)\big\}
\end{aligned}
\tag{26}
$$

for all $i=0,\dots,p$ and $j,\, j',\, j_1,\, j_2,\, j_3 \in \{1,\dots,k_i\}$.

Remark 3.4. *The index of rigidity is defined by N. Katz in* [4] *and can be computed from local structures of differential equations (see* Proposition 3.1 *in* [1] *for example). One can check that our definition of the index of rigidity coincides with the original one.*

Remark 3.5. *Suppose $P\in W(x)$ satisfies* Assumption 3.3 *and put $Z_i := \bigoplus_{j=1}^{k_i}\mathbb{Z}^{l_{i,j}}$. If $p>0$ and there exists $i_0\in\{0,\dots,p\}$ such that $k_{i_0}=1$ and $l_{i_0,1}=1$, then c_{i_0} is not a singular point of* $\mathrm{Ad}\,(e^{-w_{i_0,1}})\mathrm{Ad}\,((x-c_{i_0})^{-\lambda_{i_0,1,1}})P$*. Here the operator* $\mathrm{Ad}\,(f(x))$ *is defined in Definition 3.8. Hence in this case, we identify $\mathbf{m}(P)$ and $\mathrm{pr}_{\{0,\dots,p\}\setminus\{i_0\}}\mathbf{m}(P)$. Here $\mathrm{pr}_{\{0,\dots,p\}\setminus\{i_0\}}\colon \bigoplus_{i=0}^{p} Z_i \to \bigoplus_{i\in\{0,\dots,p\}\setminus\{i_0\}} Z_i$ is the natural projection.*

Thus for $\mathbf{m}(P)$, we assume $k_i\cdot l_{i,k_i}>1$ for all $i=0,\dots,p$ if $p>0$.

Definition 3.6 (Spectral type)**.** *Choose arbitrary integers $p\in\mathbb{Z}_{\ge 0}$, $k_i\in\mathbb{Z}_{>0}$ $(i=0,\dots,p)$ and $l_{i,j}\in\mathbb{Z}_{>0}$ $(i=0,\dots,p,\ j=1,\dots,k_i)$. Fix integers $d_i(j,j')\in\mathbb{Z}_{\ge 0}$ satisfying the relation* (26) *and take a tuple of positive integers*

$$
\mathbf{m} = \big((m_{i,j,1},\dots,m_{i,j,l_{i,j}})\big)_{\substack{0\le i\le p\\ 1\le j\le k_i}} \in \bigoplus_{i=0}^{p}\bigoplus_{j=1}^{k_i}\mathbb{Z}_{\ge 0}^{l_{i,j}}.
$$

Then we call $\mathbf{m}$ with the integers $(d_i(j,j'))_{\substack{0\le i\le p\\ 1\le j,j'\le k_i}}$ a spectral type.

The spectral type of $P\in W(x)$ satisfying Assumption 3.3 *is defined by $\mathbf{m}=\mathbf{m}(P)$ and $d_i(j,j') = -\mathrm{wt}_{c_i}(w_{i,j}-w_{i,j'})$. A spectral type is called irreducible if there exists an irreducible operator $P\in W(x)$ with the spectral type which satisfies* Assumption 3.3.

In the remaining of this paper, we investigate orbits of spectral types under the action of the twisted Euler transform which is defined below. The following is one of our main theorem which tells us that the finiteness of Euler transform orbits of spectral types with a fixed index of rigidity.

Theorem 3.7. *Fix an integer* $r \geq 0$. *If* $r > 0$, *there exist only a finite number of orbits of irreducible spectral types with index of rigidity* $-r$ *under the action of twisted Euler transforms.*

Moreover there exist a finite number of orbits of indivisible irreducible spectral types with index of rigidity 0 *under the action of twisted Euler transforms.*

Here we say that a spectral type $\mathbf{m} = \big((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}})\big)_{\substack{0\leq i\leq p\\ 1\leq j\leq k_i}}$ *with integers is indivisible if the greatest common divisor of* $\{m_{i,j,s} \mid i = 0, \ldots, p,\ j = 1, \ldots, k_i,\ s = 1, \ldots, l_{i,j}\}$ *is* 1.

This theorem follows from Theorem 3.15 and Theorem 3.24 which appear in the latter sections.

We give a brief review of algebraic transformations on $W[x]$ and $W(x)$.

Definition 3.8 (Addition)**.** *For* $f(x) \in K(x)$, *define*

$$\begin{array}{rccc} \mathrm{Ad}\,(e^{\int f(x)\,dx}): & W(x) & \longrightarrow & W(x) \\ & x & \longmapsto & x \\ & \partial & \longmapsto & \partial - f(x) \end{array}.$$

In particular,

$$\begin{array}{rccc} \mathrm{Ad}\,((x-c)^{\lambda}): & W(x) & \longrightarrow & W(x) \\ & x & \longmapsto & x \\ & \partial & \longmapsto & \partial - \frac{\lambda}{x-c} \end{array}$$

for $c, \lambda \in K$ *is called the addition at* c *with the parameter* λ.

Definition 3.9 (Fourier-Laplace transform)**.** *The Fourier-Laplace transform is the* K*-algebra automorphism of* $W[x]$,

$$\begin{array}{rccc} \mathcal{L}: & W[x] & \longrightarrow & W[x] \\ & x & \longmapsto & -\partial \\ & \partial & \longmapsto & x \end{array}.$$

Definition 3.10 (Primitive component)**.** *We say that* $P = \sum_{i=0}^{n} a_i(x)\partial^i \in W[x]$ *is primitive if*

i) $\gcd_{K[x]}\{a_i(x) \mid i = 0, \ldots, n\} = 1$,

ii) *the highest term* $a_n(x)$ *is monic.*

For $P \in W(x)$, *there exist* $f(x) \in K(x)$ *and the primitive element* $\tilde{P} \in W[x]$, *and then we can decompose* P *by*

$$P = f(x)\tilde{P},$$

uniquely.

We denote the primitive element by $\mathrm{Prim}(P)$ *and call this the primitive component of* P.

Definition 3.11 (Euler transform)**.** *The Euler transform of* $P \in W(x)$ *with the parameter* λ *is*

$$E(\lambda)P := \mathcal{L} \circ \mathrm{Prim} \circ \mathrm{Ad}\,(x^{\lambda}) \circ \mathcal{L}^{-1} \circ \mathrm{Prim}(P) \in W[x].$$

For $P \in W(x)$ satisfying Assumption 3.3, we consider following special Euler transforms.

Definition 3.12 (Twisted Euler transform)**.** *Let* $P \in W(x)$ *satisfying* Assumption 3.3*. Define* $\mathcal{J} := \bigoplus_{i=0}^{p}\{1, \ldots, k_i\}$*. Then for* $\hat{j} = (j_0, \ldots, j_p) \in \mathcal{J}$*, the twisted Euler transform* $E(\hat{j})P$ *is*

$$\begin{aligned} E(\hat{j})P := &\prod_{i=0}^{p} \mathrm{Ad}\left(e^{w_{i,j_i}}\right) \prod_{i=1}^{p} \mathrm{Ad}\left((x-c_i)^{\lambda_{i,j_i,1}}\right) \\ &\circ E(1-\lambda(P;\hat{j})) \prod_{i=1}^{p} \mathrm{Ad}\left((x-c_i)^{-\lambda_{i,j_i,1}}\right) \prod_{i=0}^{p} \mathrm{Ad}\left(e^{-w_{i,j_i}}\right) P \end{aligned}$$

where

$$\lambda(P;\hat{j}) := \sum_{i=0}^{p} \lambda_{i,j_i,1}.$$

The following theorem gives explicit changes of spectral types induced by the twisted Euler transform.

Theorem 3.13 (Theorem 3.2 in [7])**.** *Let* $P \in W(x)$ *satisfying* Assumption 3.3*. Choose* $\hat{j} = (j_0, \ldots, j_p) \in \mathcal{J}$ *and suppose* $\lambda(P)$ *is generic* (*see* [7, Theorem 2.18])*.*

Then $E(\hat{j})P \in W(x)$ *also satisfies conditions in* Assumption 3.3*. If the spectral type of* $P_{\hat{j}} = E(\hat{j})P$ *is* $\mathbf{m}(P_{\hat{j}}) = \left((\tilde{m}_{i,j,1}, \ldots, \tilde{m}_{i,j,l_{i,j}})\right)_{\substack{0 \le i \le p \\ 1 \le j \le k_i}}$ *with* $(\tilde{d}_i(j,j'))_{\substack{0 \le i \le p \\ 1 \le j,j' \le k_i}}$*, then we have*

$$\begin{aligned} \tilde{m}_{i,j,1} &= m_{i,j,1} + d(\hat{j}) && \text{if } j = j_i, \\ \tilde{m}_{i,j,s} &= m_{i,j,s} && \text{otherwise}, \\ \tilde{d}_i(j,j') &= d_i(j,j') \end{aligned}$$

where

$$\begin{aligned} d(\hat{j}) = &\sum_{i=1}^{p}\sum_{j=1}^{k_i}\left(-\mathrm{wt}_{c_i}(w_{i,j}-w_{i,j_i})+1\right)\sum_{s=1}^{l_{i,j}} m_{i,j,s} \\ &+ \sum_{j=1}^{k_0}\left(-\mathrm{wt}_{c_0}(w_{0,j}-w_{0,j_0})-1\right)\sum_{s=1}^{l_{0,j}} m_{0,j,s} - \sum_{i=0}^{p} m_{i,j_i,1}. \end{aligned}$$

3.2 The Lattice of spectral types and the root system

Theorem 3.13 shows that twisted Euler transforms $E(\hat{j})$ $(\hat{j} \in \mathcal{J})$ induce transformations of the spectral type $\mathbf{m}(P)$ of $P \in W(x)$ satisfying Assumption 3.3. From these transformations we shall construct a transformation group on a certain lattice where $\mathbf{m}(P)$ can be seen as an element in this lattice. Moreover we shall see this lattice with the transformation group is a quotient lattice of a Kac-Moody root lattice.

3.2.1 The Lattice of spectral types

Choose arbitrary integers $p \in \mathbb{Z}_{\geq 0}$, $k_i \in \mathbb{Z}_{>0}$ $(i = 0, \dots, p)$ and $l_{i,j} \in \mathbb{Z}_{>0}$ $(i = 0, \dots, p,\ j = 1, \dots, k_i)$. Fix integers $d_i(j, j') \in \mathbb{Z}_{\geq 0}$ satisfying the relation (26).

Then we consider the following $\mathbb{Z}$-lattice

$$L := \Big\{ \big((m_{i,j,1}, \dots, m_{i,j,l_{i,j}}) \big)_{\substack{0 \leq i \leq p \\ 1 \leq j \leq k_i}} \in \bigoplus_{i=0}^{p} \bigoplus_{j=1}^{k_i} \mathbb{Z}^{l_{i,j}} \;\Big|\; \sum_{j=1}^{k_0} \sum_{s=1}^{l_{0,j}} m_{0,j,s} = \cdots = \sum_{j=1}^{k_p} \sum_{s=1}^{l_{p,j}} m_{p,j,s} \Big\}.$$

We denote the set of positive elements in L by

$$L^+ := L \cap \bigoplus_{i=0}^{p} \bigoplus_{j=1}^{k_i} \mathbb{Z}_{\geq 0}^{l_{i,j}}$$

and define the *rank* of $\mathbf{m} = \big((m_{i,j,1}, \dots, m_{i,j,l_{i,j}}) \big)_{\substack{0 \leq i \leq p \\ 1 \leq j \leq k_i}} \in L$ by

$$\operatorname{rank} \mathbf{m} := \sum_{j=1}^{k_i} \sum_{s=1}^{l_{i,j}} m_{i,j,s}$$

for any $i = 0, \dots, p$. Note that the definition of $\operatorname{rank} \mathbf{m}$ is independent of the choice of $i = 0, \dots, p$.

Then we define transformations on L as an analogy of the transformation of spectral types given in Theorem 3.13. Namely, for each $\hat{j} = (j_0, j_1, \dots, j_p) \in \mathcal{J} := \bigoplus_{i=0}^{p} \{1, \dots, k_i\}$, we define the lattice transformation on L,

$$\begin{array}{cccc} \sigma(\hat{j})\colon & L & \longrightarrow & L \\ & \mathbf{m} = \big((m_{i,j,1}, \dots, a_{i,j,l_{i,j}}) \big)_{\substack{0 \leq i \leq p \\ 1 \leq j \leq k_i}} & \longmapsto & \big((\tilde{m}_{i,j,1}, \dots, \tilde{m}_{i,j,l_{i,j}}) \big)_{\substack{0 \leq i \leq p \\ 1 \leq j \leq k_i}} \end{array},$$

where

$$\begin{aligned} \tilde{m}_{i,j,1} &:= m_{i,j,1} + d(\mathbf{m}; \hat{j}) & &\text{if } (i,j) = (i, j_i), \\ \tilde{m}_{i,j,s} &:= m_{i,j,s} & &\text{otherwise} \end{aligned}$$

and

$$d(\mathbf{m};\hat{j}) := \sum_{i'=1}^{p}\sum_{j'=1}^{k_{i'}}\bigl(d_{i'}(j',j_{i'})+1\bigr)\sum_{s=1}^{l_{i',j'}} m_{i',j',s} + \sum_{j'=1}^{k_0}\bigl(d_0(j',j_0)-1\bigr)\sum_{s=1}^{l_{0,j'}} m_{0,j',s} - \sum_{i'=0}^{p} m_{i',j_{i'},1}.$$

In addition, for $i_0 = 0,\dots,p$, $j_0 = 1,\dots,k_{i_0}$, $s_0 = 1,\dots,l_{i_0,j_0}-1$, we also define permutations on L,

$$\begin{array}{llll} \sigma(i_0,j_0,s_0)\colon L(P) & \longrightarrow & L(P) & \\ m_{i_0,j_0,s_0} & \longmapsto & m_{i_0,j_0,s_0+1} & \\ m_{i_0,j_0,s_0+1} & \longmapsto & m_{i_0,j_0,s_0} & \\ m_{i,j,s} & \longmapsto & m_{i,j,s} & (i,j,s)\neq(i_0,j_0,s_0),\ (i_0,j_0,s_0+1). \end{array}$$

Then L has the action of the group $\overline{W}$ generated by these $\sigma(\hat{j}), \sigma(i,j,s)$, i.e.,

$$\overline{W} := \langle \sigma(\hat{j}), \sigma(i,j,s) \mid \hat{j}\in\mathcal{J},\ i=0,\dots,p,\ j=1,\dots,k_i,\ s=1,\dots,l_{i,j}-1\rangle.$$

We call L with $\overline{W}$ action the *lattice of spectral types* and denote it by $(L,\overline{W})$ or shortly by L.

3.2.2 The lattice of spectral types as a quotient lattice

We shall explain that the lattice of spectral types $(L,\overline{W})$ can be seen as a quotient lattice of the Kac-Moody root lattice Q_L with the index set

$$I := \mathcal{J} \sqcup \{(i,j,s) \mid i=0,\dots,p,\ j=1,\dots,k_i,\ s=1,\dots,l_{i,j}-1\}$$

and the basis $\{\alpha_t \mid t\in I\}$. Namely, $Q_L := \bigoplus_{t\in I}\mathbb{Z}\alpha_t$. We define the symmetric bilinear form $\langle\ ,\ \rangle$ on Q_L,

$$\langle\alpha_{\hat{j}},\alpha_{\hat{j}'}\rangle := 2 - \sum_{\substack{0\le i\le p\\ j_i\neq j_i'}} (d_i(j_i,j_i')+1),$$

$$\langle\alpha_{\hat{j}},\alpha_{(i,j,s)}\rangle := \begin{cases} -1 & \text{if } j_i = j \text{ and } s=1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\langle\alpha_{(i,j,s)},\alpha_{(i',j',s')}\rangle := \begin{cases} 2 & \text{if } (i,j,s)=(i',j',s'),\\ -1 & \text{if } (i,j)=(i',j') \text{ and } |s-s'|=1,\\ 0 & \text{otherwise.}\end{cases}$$

Here $\hat{j} = (j_0, \ldots, j_p)$, $\hat{j}' = (j'_0, \ldots, j'_p) \in \mathcal{J}$. Let $W_L := \langle \sigma_t \mid t \in I \rangle$ be the Weyl group of Q_L. Then we have the surjection $\Phi\colon Q_L \to L$ by which W_L action on Q_L coincides with the $\overline{W}$ action on L.

Theorem 3.14 (Theorem 3.3 in [7]). *Define the $\mathbb{Z}$-module homomorphism*

$$\Phi\colon Q_L \longrightarrow L$$

as follows. For

$$\alpha = \sum_{\hat{j} \in \mathcal{J}} m_{\hat{j}} \alpha_{\hat{j}} + \sum_{i=0}^{p} \sum_{j=1}^{k_i} \sum_{s=1}^{l_{i,j}-1} m_{(i,j,s)} \alpha_{(i,j,s)} \in Q_L,$$

the image $\Phi(\alpha) = \big((\bar{m}_{i,j,1}, \ldots, \bar{m}_{i,j,l_{i,j}})\big)_{\substack{0 \le i \le p \\ 1 \le j \le k_i}}$ *is given by*

$$\begin{aligned} \bar{m}_{i,j,1} &= \sum_{\{\hat{j} \in \mathcal{J} \mid j_i = j\}} m_{\hat{j}} - m_{(i,j,1)}, \\ \bar{m}_{i,j,s} &= m_{(i,j,s-1)} - m_{(i,j,s)} \quad \textit{for} \;\; 2 \le s \le l_{i,j}. \end{aligned}$$

Here we put $m_{(i,j,l_{i,j})} = 0$. Then we have the following.

i) *The map Φ is surjective.*

ii) *Φ is injective if and only if $\#\{i \in \{0, 1, \ldots, p\} \mid k_i > 1\} \le 1$.*

iii) *The Weyl group action on Q_L corresponds to the action of $\overline{W}$ on L. Namely, we have*

$$\begin{aligned} \Phi(\sigma_{\hat{j}} \alpha) &= \sigma(\hat{j}) \Phi(\alpha) & (\alpha \in Q_L), \\ \Phi(\sigma_{(i,j,s)} \alpha) &= \sigma(i,j,s) \Phi(\alpha) & (\alpha \in Q_L). \end{aligned}$$

iv) *If $\alpha \in \operatorname{Ker} \Phi$, then $\langle \alpha, \beta \rangle = 0$ for any $\beta \in Q_L$.*

v) *Let $\mathbf{m} \in L$. Then we have*

$$\langle \alpha, \alpha_{\hat{j}} \rangle = -d(\mathbf{m}; \hat{j}) \quad (\alpha \in \Phi^{-1}(\mathbf{m}),\ \hat{j} \in \mathcal{J})$$

vi) *For $\alpha \in \Phi^{-1}(\mathbf{m})$, we have*

$$\begin{aligned} \langle \alpha, \alpha \rangle = &- \sum_{i=0}^{p} \sum_{1 \le j \ne j' \le k_i} d_i(j, j') \Big(\sum_{s=1}^{l_{i,j}} m_{i,j,s} \Big) \Big(\sum_{s'=1}^{l_{i,j'}} m_{i,j',s'} \Big) \\ &+ \sum_{i=0}^{p} \sum_{j=1}^{k_i} \sum_{s=1}^{l_{i,j}} m_{i,j,s}^2 - (p-1)(\operatorname{rank} \mathbf{m})^2. \end{aligned}$$

Form vi) in this theorem, we define the *index of rigidity* of $\mathbf{m} \in L$ by

$$\operatorname{idx} \mathbf{m} := \operatorname{idx} \alpha = \langle \alpha, \alpha \rangle$$

for $\alpha \in \Phi^{-1}(\mathbf{m})$.

3.2.3 Φ-root system

We shall define the Φ-root system of $(L, \overline{W})$ which is an analogue of the root system of Q_L.

First consider the following subset of L,

$$\Delta^{\Phi}_{\mathrm{re}} := \bigcup_{\hat{j} \in \mathcal{J}} \overline{W} \Phi(\alpha_{\hat{j}}),$$

i.e., the union of $\overline{W}$-orbits of $\Phi(\alpha_{\hat{j}})$, which is called the set of Φ-*real roots*. We also consider the subset

$$F^{\Phi} := \left\{ \mathbf{m} \in L^+ \setminus \{0\} \;\middle|\; \substack{m_{i,j,1} \geq m_{i,j,2} \geq \cdots \geq m_{i,j,l_{i,j}},\, d(\mathbf{m};\hat{j}) \geq 0 \\ \text{for all } i=0,\ldots,p,\, j=1,\ldots,k_i,\, \hat{j} \in \mathcal{J}, \\ \overline{W}\mathbf{m} \subset L^+} \right\}.$$

Then the set of Φ-*imaginary roots* is

$$\Delta^{\Phi}_{\mathrm{im}} := \overline{W} F^{\Phi} \cup -\overline{W} F^{\Phi}.$$

We call

$$\Delta^{\Phi} := \Delta^{\Phi}_{\mathrm{re}} \cup \Delta^{\Phi}_{\mathrm{im}}$$

the set of Φ-*roots*.

3.2.4 Spectral types of differential equations and root systems

We explain that for $P \in W(x)$ satisfying Assumption 3.3, the spectral type of P can be seen as an element in the lattice of spectral types $(L, \overline{W})$.

Suppose $P \in W(x)$ satisfies Assumption 3.3. If we put

$$d_i(j, j') = -\mathrm{wt}_{c_i}(w_{i,j} - w_{i,j'})$$

for $i = 0, \ldots, p$ and j, $j' = 1, \ldots, k_i$, then $d_i(j, j')$ satisfy the relations (26).

Thus we can define the lattice of spectral types $(L, \overline{W})$ and see $\mathbf{m}(P) \in L$. Then the index of rigidity of P equals that of $\mathbf{m}(P) \in L$, namely, $\mathrm{idx}\, P = \mathrm{idx}\, \mathbf{m}(P)$. Also $\mathrm{rank}\, P = \mathrm{rank}\, \mathbf{m}(P)$ as well.

Theorem 3.13 shows that the spectral types of $P_{\hat{j}} = E(\hat{j})P$ ($\hat{j} \in \mathcal{J}$) are obtained by the transformation $\sigma(\hat{j})$ on L, i.e.,

$$\mathbf{m}(P_{\hat{j}}) = \sigma(\hat{j})\mathbf{m}(P).$$

Hence we can associate an element in Δ^{Φ} to P as follows.

Theorem 3.15 (Theorem 3.11 in [7]). *Suppose $\lambda(P)$ is generic* (*see* [7, Definition 3.8]). *If P is irreducible in $W(x)$, then we have the following.*

i) $\mathbf{m}(P) \in \Delta^{\Phi}$.

ii) *If* $\mathrm{idx}\, \mathbf{m}(P) > 0$, *then* $\mathrm{idx}\, \mathbf{m}(P) = 2$.

iii) *We have*

$$\mathbf{m}(P) \in \begin{cases} \Delta^{\Phi}_{re} & \text{if } \mathrm{idx}\, \mathbf{m}(P) = 2, \\ \Delta^{\Phi}_{im} & \text{if } \mathrm{idx}\, \mathbf{m}(P) \leq 0. \end{cases}$$

3.3 A classification of basic pairs

At the end of §3.2, we see that the spectral type of the irreducible operator $P \in W(x)$ satisfying Assumption 3.3 corresponds to an element in $\Delta^{\Phi+} = \Delta^{\Phi} \cap L^{+}$. By the definition of Δ^{Φ}, any element in $\Delta^{\Phi+}$ can be reduced to an element in $\{\Phi(\alpha_{\hat{j}}) \mid \hat{j} \in \mathcal{J}\} \sqcup F^{\Phi}$ by $\overline{W}$ action. This means that $\mathbf{m}(P)$ can be reduced to an element in $\{\Phi(\alpha_{\hat{j}}) \mid \hat{j} \in \mathcal{J}\} \sqcup F^{\Phi}$ by the Euler transform.

Thus to see Euler transform orbits of spectral types, it suffices to see elements in $F^{\Phi} \sqcup \{\Phi(\alpha_{\hat{j}}) \mid \hat{j} \in \mathcal{J}\}$.

The differential operator corresponding to an element in $\{\Phi(\alpha_{\hat{j}}) \mid \hat{j} \in \mathcal{J}\}$ is an obvious operator of the first order. Hence we study F^{Φ}.

Definition 3.16 (Basic pair)**.** *Let $(L, \overline{W})$ be a lattice of spectral types with $\overline{W}$-action. Denote the corresponding Kac-Moody root lattice by Q_L and the surjection by $\Phi\colon Q_L \to L$ defined as in §3.2.2. We also define the subset $F^{\Phi} \subset L$ as in §3.2.3.*

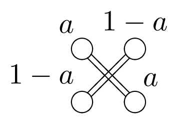
Choose an element $\mathbf{m} = \big((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}})\big)_{\substack{0\le i\le p\\ 1\le j\le k_i}} \in F^{\Phi}$ and suppose $m_{i,j,s} \neq 0$ for all $i = 0, \ldots, p,\ j = 1, \ldots, k_i$ and $s = 1, \ldots, l_{i,j}$.

Then we call $(\mathbf{m}, L, \overline{W})$ the basic triple. We usually omit $\overline{W}$ and call $(\mathbf{m}, L)$ the basic pair.

We define the shape of a basic pair $(\mathbf{m}, L)$.

Definition 3.17 (Shape of a basic pair)**.** *Let $(\mathbf{m}, L)$ be a basic pair. The shape of $(\mathbf{m}, L)$ is the set of shapes of elements in $\Phi^{-1}(\mathbf{m}) \subset Q_L$.*

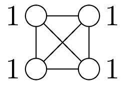
Example 3.18. *For example, suppose $p = 1$, $k_0 = k_1 = 2$, $l_{i,j} = 1$ $(i = 0, 1$ and $j = 1, 2)$, $d_0(1, 2) = d_1(1, 2) = 1$. Consider $\mathbf{m} = ((m_{i,j,1}))_{\substack{0\le i\le 1\\ 1\le j\le 2}}$ such that $m_{i,j,1} = 1$ for all i, j. Then $(\mathbf{m}, L)$ is a basic pair and its shape is*



$(a \in \mathbb{Z})$, (27)

where we simply denote $\{x_a \mid a \in \mathbb{Z}\}$ by x_a $(a \in \mathbb{Z})$.

Suppose $p = 0$, $k_0 = 4$, $d_0(i, j) = 2$ for $1 \le i < j \le 4$ and $l_{0,\nu} = 2$ for $1 \le \nu \le 4$. If $m_{0,j,1} = 1$ for $1 \le j \le 4$, the shape of $(\mathbf{m}, L)$ equals



. (28)

Now we prepare the following lemma to have an element in $\Phi^{-1}(\mathbf{m}) \cap Q_L^{+}$.

Lemma 3.19. *Let* $\big(m_{i,j}\big)_{\substack{0\le i\le p\\ 1\le j\le k_i}}$ *be a tuple of* $p+1$ *partitions of a positive integer* n*, namely,* n*,* p*,* $m_{i,j}$ *and* k_i *are positive integers satisfying*

$$m_{i,1}+\cdots+m_{i,k_i}=n \quad (j=0,\ldots,p).$$

Then there exist non-negative integers $\tilde m_{\nu_0,\ldots,\nu_p}$ *for* $1\le\nu_i\le k$ *and* $0\le i\le p$ *such that*

$$\sum_{\substack{0\le j\le p\\ j\ne i}}\sum_{\nu_j=1}^{k_j}\tilde m_{\nu_0,\nu_1,\ldots,\nu_p}=m_{i,\nu_i}\quad (0\le i\le p,\ 1\le\nu_i\le k_i),$$

$$\tilde m_{j_0,\ldots,j_p}\cdot\tilde m_{j'_0,\ldots,j'_p}\ne 0\ \Rightarrow\ \begin{cases} j_\nu\le j'_\nu & (0\le\nu\le p)\\ or & \\ j_\nu\ge j'_\nu & (0\le\nu\le p),\end{cases}$$

$$\tilde m_{1,\ldots,1}\cdot\tilde m_{k_0,\ldots,k_p}\ne 0.$$

Proof. Put

$$\tilde m_{\nu_0,\ldots,\nu_p}=\#\{k\in\{1,2,\ldots,n\}\mid \\ m_{j,1}+\cdots+m_{j,\nu_j-1}<k\le m_{j,1}+\cdots+m_{j,\nu_j}\text{ for } j=0,\ldots,p\}.$$

Then the lemma is clear. Here we note that $\tilde m_{1,\ldots,1}=\min\{m_{0,1},\ldots,m_{p,1}\}$ and $\tilde m_{k_0,\ldots,k_p}=\min\{m_{0,k_0},\ldots,m_{p,k_p}\}$. $\square$

Definition 3.20. *Fix* $\mathbf{m}=\big((m_{i,j,1},\ldots,m_{i,j,l_{i,j}})\big)_{\substack{0\le i\le p\\ 1\le j\le k_i}}\in L^+$*. Put* $n=\operatorname{rank}\mathbf{m}$ *and* $m_{i,j}=\sum_{j=1}^{l_{i,j}}m_{i,j,s}$*. Applying* Lemma 3.19 *to* $\mathbf{m}$ *and putting* $m_{\hat\jmath}=\tilde m_{\nu_0,\ldots,\nu_p}\quad(\hat\jmath=(\nu_0,\ldots,\nu_p))$ *and* $m_{(i,j,s)}=\sum_{t=s+1}^{l_{i,j}}m_{i,j,t}$*, we define*

$$\alpha(\mathbf{m}):=\sum_{\hat\jmath\in\mathcal J}m_{\hat\jmath}\alpha_{\hat\jmath}+\sum_{i=0}^{p}\sum_{j=1}^{k_i}\sum_{s=1}^{l_{i,j}-1}m_{(i,j,s)}\alpha_{(i,j,s)}\in\Phi^{-1}(\mathbf{m})\cap Q_L^+.$$

The following lemma gives some properties of $\alpha(\mathbf{m})$.

Lemma 3.21. *Retain the notation in* Definition 3.20.

Let I *be the index set of the basis of* Q_L *:*

$$I=\mathcal J\sqcup\{(i,j,s)\mid i=0,\ldots,p,\ j=1,\ldots,k_i,\ s=1,\ldots,l_{i,j}-1\}.$$

Put $\mathcal C_{\mathbf m}=\operatorname{supp}\alpha(\mathbf m)$ *and define*

$$\bar I:=\{i\in I\mid \alpha_i\in\mathcal C_{\mathbf m}\},\ I_0:=\{t\in\bar I\mid\langle\alpha(\mathbf m),\alpha_t\rangle=0\},\ I_1:=\bar I\setminus I_0.$$

Assume $k_0\ge k_1\ge\cdots\ge k_{N-1}>k_N=\cdots=k_p=1$*. Here* N *is a non-negative integer. Put* $\hat\jmath_0=(1,\ldots,1)\in\mathcal J$*,* $\hat\jmath_1=(k_0,\ldots,k_p)\in\mathcal J$*.*

i) *The element* $\alpha(\mathbf{m})$ *is indivisible if* $\mathbf{m}$ *is indivisible.*

ii) *We have* $m_{\hat{j}_0} > 0$, $m_{\hat{j}_1} > 0$ *and*

$$\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}_1} \rangle \le 2 - 2N,$$

$$\max\{k_0, \ldots, k_p\} \le \#(\bar{I} \cap \mathcal{J}) \le 1 + \sum_{i=0}^{p} (k_i - 1),$$

$$\sum_{\hat{j} \in \bar{I} \cap \mathcal{J}} m_{\hat{j}} = \operatorname{rank} \mathbf{m}.$$

iii) *The Dynkin diagram of a subset of* $\mathcal{C}_{\mathbf{m}}$ *is never equal to* $D_n^{(1)}$ *with* $n > 4$.

Preceding to the proof of Lemma 3.21, we remark the following.

Lemma 3.22. *Let* $\hat{j}_\nu = (j_{\nu,0}, \ldots, j_{\nu,p}) \in \mathcal{J}$ *for* $\nu = 1, 2, \ldots$. *Then we have*

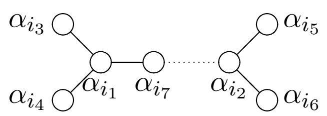
$$\langle \alpha_{\hat{j}_1}, \alpha_{\hat{j}_2} \rangle \le 2 - 2\#\{i \in \{0, 1, \ldots, p\} \mid j_{1,i} \neq j_{2,i}\}, \tag{29}$$

$$\langle \alpha_{\hat{j}_1}, \alpha_{\hat{j}_2} \rangle = \langle \alpha_{\hat{j}_1}, \alpha_{\hat{j}_3} \rangle = 0 \ \Rightarrow\ \langle \alpha_{\hat{j}_2}, \alpha_{\hat{j}_3} \rangle \neq -1. \tag{30}$$

Proof. Definition 3.20 directly shows (29). Suppose $-1 \le \langle \alpha_{\hat{j}_\nu}, \alpha_{\hat{j}_{\nu'}} \rangle \le 0$ for $1 \le \nu < \nu' \le 3$. Then there exists $l \in \mathbb{Z}_{\ge 0}$ such that $j_{1,i} = j_{\nu,i}$ for $i \in \{0, \ldots, p\} \setminus \{l\}$ and therefore (30) follows from the relation (26). $\square$

Proof of Lemma 3.21. The claims i) and ii) follow from Definition 3.20, Lemma 3.19 and (29).

iii) Suppose the Dynkin diagram of a subset of $\mathcal{C}_{\mathbf{m}}$ is $D_n^{(1)}$ with $n > 4$:



.

Define $c_{\mu,\nu} = \langle \alpha_{i_\mu}, \alpha_{i_\nu} \rangle$. For $i_\nu \in I$ put $i_\nu = (j_{\nu,0}, \ldots, j_{\nu,p})$ if $i_\nu \in \mathcal{J}$ and put $i_\nu = (k_\nu, j_\nu, s_\nu)$ otherwise. The proof of Lemma 3.22 shows that there exists l with $0 \le l \le N$ such that $j_{\nu,i} = j_{\nu',i}$ if $i \neq l$ and $i_\nu, i_{\nu'} \in \mathcal{J}$.

Suppose $i_1 \in \mathcal{J}$ and $i_2 \in \mathcal{J}$. Then (30) shows $\#(\{i_3, i_4, i_5, i_6\} \cap \mathcal{J}) \le 1$ and there exists $i_\nu \notin \mathcal{J}$ such that $i_\nu = (k_\nu, j_\nu, 1)$ with $k_\nu \neq l$ and $j_\nu = j_{1,k_\nu}$. Then $c_{\nu,1} = c_{\nu,2} = -1$, which contradicts to the Dynkin diagram.

Suppose $i_1 \notin \mathcal{J}$ and $i_2 \in \mathcal{J}$. Then $\{i_3, i_4\} \cap \mathcal{J} \neq \emptyset$. We may assume $i_3 \in \mathcal{J}$ and then the claim i) shows $i_5 \notin \mathcal{J}$ and $i_6 \notin \mathcal{J}$, The same argument as above shows $c_{3,5} = -1$ or $c_{3,6} = -1$, which leads a contradiction.

Lastly suppose $i_1 = (k_1, j_1, 1) \notin \mathcal{J}$ and $i_2 \notin \mathcal{J}$. We may assume $i_3 \in \mathcal{J}$ and $i_5 \in \mathcal{J}$. Then there exists $\alpha_{i_7} \in \mathcal{J}$ such that $i_7 \neq i_3$, $i_7 \neq i_4$ and $c_{1,7} = -1$. Since $c_{1,7} = c_{1,3} = -1$, we have $k_1 \neq l$, $j_1 = j_{3,k_1}$ and $c_{1,5} = -1$, which leads a contradiction. $\square$

We shall show some properties of $\alpha(\mathbf{m})$ when $(\mathbf{m}, L)$ is basic.

Lemma 3.23. *Retain the notation and the assumption in* Lemma 3.21. *Suppose* $(\mathbf{m}, L)$ *is basic.*

i) $\mathcal{C}_{\mathbf{m}}$ *is connected.*

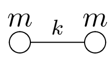
ii) *Put* $\alpha' = \sum_{i\in\bar I'} m_i\alpha_i$ *for a proper subset* $\bar I' \subsetneqq \bar I$. *Then*

$$\operatorname{idx}\alpha' > \operatorname{idx}\alpha(\mathbf{m}).$$

iii) *We have*

$$\langle\alpha_{i_1},\alpha_{i_2}\rangle \ge \tfrac12\operatorname{idx}\mathbf{m} - 2 \quad \text{for} \quad i_1,\, i_2 \in \bar I \tag{31}$$

and the equality holds if and only if the shape of $\alpha(\mathbf{m})$ *is*



$$(m = 1 \quad \text{if} \quad k \ne 2) \tag{32}$$

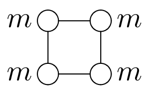
with $k = \frac12 - \operatorname{idx}\mathbf{m}$.

iv) *We have*

$$N \le 2 + \tfrac14|\operatorname{idx}\mathbf{m}| \tag{33}$$

and the equality holds if and only if the shape of $\alpha(\mathbf{m})$ *is the one in* (32) *with* $k = 2N-2$.

v) *Suppose* $(\mathbf{m}, L)$ *is basic. Let* $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_K}$ *be a constant connected sequence in* I_0 $(k > 1)$. *Then* $K \le 4$ *and* $1 \le N \le 2$.

If $K = 4$, *then* $N = 1$ *and the shape of* $\alpha(\mathbf{m})$ *is*  .

Suppose $N = 2$ *and* $K = 3$. *Then* $i_2 = \hat\jmath_2$ *or* $i_2 = \hat\jmath_3$ *by denoting*

$$\hat\jmath_2 := (1, k_1, 1, \dots, 1) \in \mathcal{J} \quad \text{and} \quad \hat\jmath_3 := (k_0, 1, 1, \dots, 1) \in \mathcal{J}.$$

Moreover $i_1 \notin \mathcal{J}$ *and* $i_3 \notin \mathcal{J}$.

Proof. i) We say that two elements α and α' in $\mathcal{C}_{\mathbf{m}}$ are connected in $\mathcal{C}_{\mathbf{m}}$ if they belong to a connected component of the Dynkin diagram of $\mathcal{C}_{\mathbf{m}}$. Note that $\alpha_{\hat\jmath_0}$ and $\alpha_{\hat\jmath_1}$ are connected in $\mathcal{C}_{\mathbf{m}}$.

Fix $\alpha_{(i,j,s)}$ with $1 \le s \le l_{i,j} - 1$. Then there exists $\hat\jmath = (j_0, \dots, j_p)$ such that $\alpha_{\hat\jmath} \in \mathcal{C}_{\mathbf{m}}$ and $j_i = j$. Then $\alpha_{(i,j,s)}$ and $\alpha_{\hat\jmath}$ are connected in $\mathcal{C}_{\mathbf{m}}$.

Let $\hat\jmath \in \mathcal{J}$ with $\alpha_{\hat\jmath} \in \mathcal{C}_{\mathbf{m}}$. If $N \ge 3$, then $\langle\alpha_{\hat\jmath_0},\alpha_{\hat\jmath}\rangle \ne 0$ or $\langle\alpha_{\hat\jmath_1},\alpha_{\hat\jmath}\rangle \ne 0$, which means $\alpha_{\hat\jmath}$ and $\alpha_{\hat\jmath_0}$ are connected in $\mathcal{C}_{\mathbf{m}}$ and therefore $\mathcal{C}_{\mathbf{m}}$ is connected.

Hence we assume $N = 2$ and $\langle\alpha_{\hat\jmath_0},\alpha_{\hat\jmath}\rangle = \langle\alpha_{\hat\jmath_1},\alpha_{\hat\jmath}\rangle = 0$. Then $\hat\jmath = \hat\jmath_2$ or $\hat\jmath = \hat\jmath_3$.

Suppose $\hat\jmath = \hat\jmath_2$. Then $\alpha_{\hat\jmath_3} \notin \mathcal{C}_{\mathbf{m}}$, which follows from Lemma 3.19. Since $\alpha(\mathbf{m})$ is basic, there exists $\alpha \in \mathcal{C}_{\mathbf{m}}$ satisfying $\langle\alpha,\alpha_{\hat\jmath_2}\rangle < 0$.

Suppose $\alpha = \alpha_{\hat{j}'}$ with $\hat{j}' \in \mathcal{J}$, $\hat{j}' = (1, j_1, \ldots, 1)$ or $\hat{j}' = (j_0, k_1, 1, \ldots, 1)$. Here $1 < j_1 < k_1$ and $1 < j_0 < k_0$, respectively. If $\hat{j}' = (1, j_1, \ldots, 1)$, α and $\alpha_{\hat{j}_1}$ are connected in $\mathcal{C}_{\mathbf{m}}$. We have the same conclusion when $\hat{j}' = (j_0, k_1, 1, \ldots, 1)$.

Suppose $\alpha = \alpha_{(i,j,s)}$. Then $s = 1$. If $i \geq 2$ or $i = 0$, $j = 1$ and $\langle \alpha_{(i,1,1)}, \alpha_{\hat{j}_0} \rangle < 0$. If $i = 1$, $j = k_1$ and $\langle \alpha_{(i,1,1)}, \alpha_{\hat{j}_1} \rangle < 0$. Hence α and $\alpha_{\hat{j}_1}$ are connected in $\mathcal{C}_{\mathbf{m}}$ and so are $\alpha_{\hat{j}}$ and $\alpha_{\hat{j}_1}$.

In the same way as above we have the same conclusion when $\hat{j} = \hat{j}_3$. Thus we have the claim.

If $N = 0$, $\#\mathcal{J} = 1$ and the Dynkin diagram of $\mathcal{C}_{\mathbf{m}}$ is star-shaped and hence connected.

Lastly assume $N = 1$. If there exists $i \in \{2, \ldots, p\}$ such that $l_{i,1} > 1$, then $\langle \alpha_{\hat{j}}, \alpha_{(i,1,1)} \rangle = -1$ for any $\hat{j} \in \bar{I} \cap \mathcal{J}$ and therefore the Dynkin diagram of $\mathcal{C}_{\mathbf{m}}$ is connected. Hence we assume $l_{i,1} = 1$ for all $i \in \{2, \ldots, p\}$. If $\langle \alpha_{\hat{j}_0}, \alpha_j \rangle = 0$ for any $\hat{j} \in (\bar{I} \cap \mathcal{J}) \setminus \{\hat{j}_0\}$, then $\langle \alpha(\mathbf{m}), \alpha_{\hat{j}_0} \rangle = 2m_{\hat{j}} - m_{(0,1,1)} > 0$, which contradicts to the fact that $\alpha(\mathbf{m})$ is basic. Hence there exists $\hat{j} \in \bar{I} \cap \mathcal{J}$ satisfying $\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}} \rangle < 0$. Then the relation (26) assures $\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}'} \rangle < 0$ or $\langle \alpha_{\hat{j}}, \alpha_{\hat{j}'} \rangle < 0$ for any $\hat{j}' \in (\bar{I} \cap \mathcal{J}) \setminus \{\hat{j}_0, \hat{j}\}$, which proves that $\alpha_{\hat{j}_0}$ and $\alpha_{\hat{j}'}$ are connected in $\mathcal{C}_{\mathbf{m}}$ and moreover $\mathcal{C}_{\mathbf{m}}$ is connected.

ii) The claim easily follows from the definition of the index and the connectedness of the Dynkin diagram of $\mathcal{C}_{\mathbf{m}}$.

iii) We may assume $\langle \alpha_{i_1}, \alpha_{i_2} \rangle \leq -2$. If $m_{i_1} < m_{i_2}$, we have

$$\begin{aligned}
\operatorname{idx} \mathbf{m} &\leq \langle \alpha(\mathbf{m}), m_{i_1} \alpha_{i_1} \rangle \\
&\leq m_{i_1}^2 + m_{i_1} m_{i_2} \langle \alpha_{i_1}, \alpha_{i_2} \rangle \\
&\leq m_{i_1}^2 (1 + \langle \alpha_{i_1}, \alpha_{i_2} \rangle) + m_{i_1} \langle \alpha_{i_1}, \alpha_{i_2} \rangle \\
&\leq m_{i_1}^2 (1 + \langle \alpha_{i_1}, \alpha_{i_2} \rangle) + m_{i_1}^2 \langle \alpha_{i_1}, \alpha_{i_2} \rangle, \\
\langle \alpha_{i_1}, \alpha_{i_2} \rangle &\geq \frac{\operatorname{idx} \mathbf{m}}{2m_{i_1}^2} - \frac{1}{2}.
\end{aligned}$$

If $m_{i_1} = m_{i_2}$, we have

$$\begin{aligned}
\operatorname{idx} \mathbf{m} &\leq \langle \alpha(\mathbf{m}), m_{i_1} \alpha_{i_1} \rangle + \langle \alpha(\mathbf{m}), m_{i_2} \alpha_{i_2} \rangle \\
&\leq 2m_{i_1}^2 + 2m_{i_1} m_{i_2} \langle \alpha_{i_1}, \alpha_{i_2} \rangle + 2m_{i_2}^2, \\
\langle \alpha_{i_1}, \alpha_{i_2} \rangle &\geq \frac{\operatorname{idx} \mathbf{m}}{2m_{i_1} m_{i_2}} - 2.
\end{aligned}$$

Hence we have $\langle \alpha_{i_1}, \alpha_{i_2} \rangle \geq \frac{1}{2} \operatorname{idx} \mathbf{m} - 2$ and the equality implies $m_{i_1} = m_{i_2}$ and moreover $m_{i,1} = 1$ if $\operatorname{idx} \mathbf{m} \neq 0$. It follows from the claim ii) that the equality implies that the shape of $\alpha(\mathbf{m})$ is the one in (32) with $k = 2 - \frac{1}{2} \operatorname{idx} \mathbf{m}$.

iv) Since $\langle \alpha_{\hat{j}_0}, \alpha_{\hat{j}_1} \rangle \leq 2 - 2N$, the claim in iii) implies that in iv).

v) Put $A=\bar I\cap\mathcal J$ and $B=\bar I\setminus A$.

Let $(i,j,s)\in B$ with $s\ge 2$. Then $m_{(i,j,s-1)}>m_{(i,j,s)}>m_{i,j,s+1}\ge 0$ and $\{i\in I\mid \langle\alpha_{(i,j,s)},\alpha_i\rangle<0\}\subset\{(i,j,s-1),(i,j,s+1)\}$. Hence $i_\nu\ne(i,j,s)$ for $\nu=1,\ldots,K$.

Note that $\langle\alpha_{(i,j,1)},\alpha_{(i',j',1)}\rangle=0$ for two different elements $(i,j,1)$ and $(i',j',1)$ of B. Hence there exists no constant connected sequence in B.

If $N=0$, then $\#\mathcal J=1$ and it is clear that there is no constant connected sequence.

Suppose $N\ge 3$. Then $\alpha_{\hat j_\nu}\notin I_0$ for $\nu=0$ and 1 and moreover $\langle\alpha_{\hat j},\alpha_{\hat j_0}\rangle\le -2$ or $\langle\alpha_{\hat j},\alpha_{\hat j_1}\rangle\le -2$ for any $\hat j\in A$. Hence there exists no constant connected sequence in I_0.

Suppose $N=2$ and $K=3$. If $\hat j\in A\setminus\{\hat j_0,\hat j_1,\hat j_2,\hat j_3\}$, then $\langle\alpha_{\hat j},\alpha_{\hat j_0}\rangle\le -2$ or $\langle\alpha_{\hat j},\alpha_{\hat j_1}\rangle\le -2$. Hence $i_2=\hat j_2$ or $i_2=\hat j_3$. Since $\{\hat j_2,\hat j_3\}\not\subset A$, the length of the constant connected sequence in I_0 is not larger than 3 and the corresponding claim in iii) is valid.

Lastly suppose $N=1$. Suppose $K=2$ and $i_2\in B$. Then $i_1=(j_1,1,\ldots,1)\in A$ and $i_2=(0,j_1,1)$ or $i_2=(i,1,1)$ with $i>0$. If $i_2=(0,j_1,1)$, then $m_{i_1}>m_{i_2}$, which implies $i_2=(i,1,1)$ and $\langle\alpha_{\hat j},\alpha_{i_2}\rangle<0$ for any $\hat j\in A$.

Fix a constant connected sequence in I_0. The number M of the elements α_i in the sequence with $i\in B$ is not larger than 2. If $M>0$, the number of the elements α_j with $j\in A$ in the sequence is not larger than 2 and therefore $K\le 4$. If $M>0$ and $K=4$, then $M=2$ and the shape of $\alpha(\mathbf m)$ is the one given in v).

Suppose $M=0$ and $K\ge 4$. Put $i_\nu=(j_\nu,1,\ldots,1)\in A$ for $\nu=1,\ldots,K$. Since $\langle\alpha_{i_1},\alpha_{i_3}\rangle=0$ we have $\langle\alpha_{i_3},\alpha_{i_4}\rangle=0$ if $\langle\alpha_{i_1},\alpha_{i_4}\rangle=0$. Hence $K=4$ and $\langle\alpha_{i_1},\alpha_{i_4}\rangle<0$ and the condition $i_1\in I_0$ shows the claim v). □

3.3.1 The finiteness of basic pairs

We show the finiteness theorem which is an analogue of Theorem 2.2.

We say that a basic pair $(\mathbf m,L)$ is *indivisible* if the greatest common divisor of $\{m_{i,j,s}\mid i=0,\ldots,p,\ j=1,\ldots,k_i,\ s=1,\ldots,l_{i,j}\}$ is 1 for $\mathbf m=\big((m_{i,j,1},\ldots,m_{i,j,l_{i,j}})\big)_{\substack{0\le i\le p\\1\le j\le k_i}}$.

We also say that a basic pair $(\mathbf m,L)$ is *reduced* when we have $l_{i,1}>1$ for all $i=0,\ldots,p$ satisfying $k_i=1$ (cf. Remark 3.5).

Theorem 3.24 (Corollary of Theorem 2.2)**.** *Fix an integer $r\ge 0$. If $r>0$, then there exist only a finite number of reduced basic pairs $(\mathbf m,L)$ with* $\operatorname{idx}\mathbf m=-r$. *Moreover there exist only a finite number of reduced indivisible basic pairs $(\mathbf m,L)$ with* $\operatorname{idx}\mathbf m=0$.

Proof. Theorem 3.24 and Lemma 3.23 assure that there are only finite possibilities of shapes of $\alpha(\mathbf m)$. Hence there exists a positive integer n_r such that rank $\mathbf{m} \le n_r$. Hence the theorem is reduced to the following lemma. $\square$

Lemma 3.25. *Fix integers $n > 0$ and r. Then there exist a finite number of reduced basic pairs $(\mathbf{m}, L)$ satisfying* $\operatorname{rank}\mathbf{m} \le n$ *and* $\operatorname{idx}\mathbf{m} \ge -r$.

Proof. Let $(\mathbf{m}, L)$ be a reduced basic pair satisfying the assumption. Since $\sum_{u=1}^{v} e_u^2 - \bigl(\sum_{u=1}^{v} e_u\bigr)^2 \le -2$ if $v \ge 2$ and $e_u \in \mathbb{Z}_{>0}$ for $u = 1, \ldots, v$, we have

$$\begin{aligned}
\operatorname{idx}\mathbf{m} &+ \sum_{i=0}^{p} \sum_{1\le j\ne j'\le k_i} d_i(j,j')\Bigl(\sum_{s=1}^{l_{i,j}} m_{i,j,s}\Bigr)\Bigl(\sum_{s'=1}^{l_{i,j'}} m_{i,j',s'}\Bigr)\\
&= \sum_{i=0}^{p}\Bigl(\sum_{j=1}^{k_i}\sum_{s=1}^{l_{i,j}} m_{i,j,s}^2 - (\operatorname{rank}\mathbf{m})^2\Bigr) + 2(\operatorname{rank}\mathbf{m})^2\\
&\le -2(p+1) + 2(\operatorname{rank}\mathbf{m})^2
\end{aligned}$$

by putting $\mathbf{m} = \bigl((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}})\bigr)_{\substack{0\le i\le p\\ 1\le j\le k_i}}$, which implies

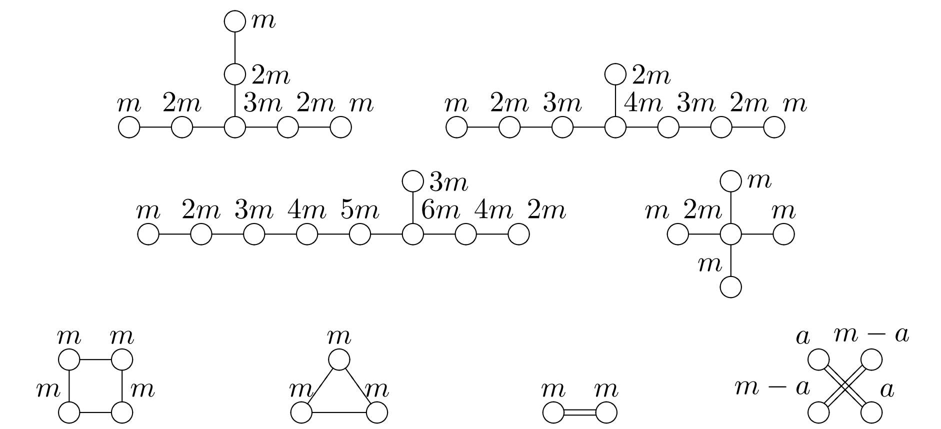
$$\begin{aligned}
2(p+1) &\le r + 2n^2,\\
d_i(j,j') &\le 2n^2 - 2(p+1) + r\\
&\le 2n^2 + r - 2 \quad (0 \le i \le p,\ 1 \le j < j' \le k_i).
\end{aligned}$$

This shows the lemma. $\square$

3.3.2 The classification of basic pairs with idx 0

We shall give lists of shapes of basic pairs of index 0 and -2. First we consider basic pairs of index 0.

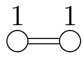
Theorem 3.26. *If a basic pair $(\mathbf{m}, L)$ satisfies $\operatorname{idx}\mathbf{m} = 0$, then its shape is one of the following.*



$(m)(m), mm, mm$ $((m))((m)), mm$ $(((m)))(((m)))$ $(a \in \mathbb{Z})$

$((m)(m))((m)(m))$ $((m))((m))((m))$ $(m)(m), (m)(m)$

Here m are arbitrary elements in $\mathbb{Z}_{>0}$. We simply write sets $\{x_a \mid a \in \mathbb{Z}\}$ and $\{x\}$ by x_a $(a \in \mathbb{Z})$ and x, respectively. The sequences of integers written under the shapes except for star-shaped ones stand for the corresponding basic pairs $(\mathbf{m}, L)$.

Proof. Retain the notation in the proof of Lemma 3.23. We may assume $\mathbf{m}$ is indivisible. If $N = 2$, the shape of $\alpha(\mathbf{m})$ is . Then $\operatorname{rank} \mathbf{m} = 2$ and the shape of $(\mathbf{m}, L)$ is the last shape in the above list with $m = 1$.

Then we may assume $N \leq 1$ and the shape of $(\mathbf{m}, L)$ corresponds to the shape of $\alpha(\mathbf{m})$. Hence the claim in §2.3, Lemma 3.21 and Lemma 3.23 show the theorem. $\square$

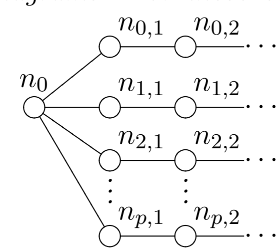
Remark 3.27. *We shall explain the notation expressing $(\mathbf{m}, L)$ in* Theorem 3.26*. The number of parentheses ( ) represents the number $d_i(j, j')$. For instance, if $(\mathbf{m}, L)$ is written by*

$$\cdots m_{i,j,1} m_{i,j,2} \ldots m_{i,j,l_{i,j}}))((m_{i,j',1} m_{i,j',2} \cdots ,$$

then we can see the double parenthesis (( )) between $m_{i,j,1} \ldots$, and $m_{i,j',1} \ldots$. This means $d_i(j, j') = 2$. Let us see an example. Consider a basic pair $(\mathbf{m}, L)$ where $p = 1$, $(k_0, k_1) = (2, 3)$, $(l_{0,1}, l_{0,2}, l_{1,1}, l_{1,2}, l_{1,3}) = (1, 2, 1, 1, 2)$ and $(d_0(1, 2), d_1(1, 2), d_1(2, 3), d_1(1, 3)) = (1, 1, 2, 2)$.

Then $\mathbf{m} = ((m_{i,j,1}, \ldots, m_{i,j,l_{i,j}}))_{\substack{0 \leq i \leq p \\ 1 \leq j \leq k_i}}$ is written by

$$(m_{0,1,1})(m_{0,2,1} m_{0,2,2}),\ ((m_{1,1,1})(m_{1,2,1}))((m_{1,3,1} m_{1,3,2})).$$

Remark 3.28. *In the above list of shapes, we omit the corresponding $(\mathbf{m}, L)$ for star-shaped diagrams. For these cases $(\mathbf{m}, L)$ are obtained as follows. Consider a shape*  *and put $m_{(i,1)} = n_0 - n_{i,1}$, $m_{(i,j+1)} = n_{i,j} - n_{i,j+1}$, $m_{(i,0)} = \sum_{\substack{0 \leq k \leq p \\ k \neq i}} n_{k,1} - n_0$ and $m_{(0)} = \sum_{i=0}^{p} n_{i,1} - n_0$. Then the shape corresponds to the following 5 types of $(\mathbf{m}, L)$ with $0 \leq i \leq p$.*

$$m_{(0,1)} m_{(0,2)} \ldots,\ m_{(1,1)} m_{(1,2)} \ldots,\ \ldots,\ m_{(p,1)} m_{(p,2)} \ldots,$$
$$m_{(0)} n_0,\ (m_{(0,2)} m_{(0,3)} \ldots) \ldots (m_{(p,2)} m_{(p,3)} \ldots),$$
$$m_{(i,0)} m_{(i,1)} \ldots,\ (m_{(0,2)} m_{(0,3)} \ldots) \ldots (m_{(i-1,2)} \ldots)(m_{(i+1,2)} \ldots) \ldots,$$
$$((m_{(i,1)} m_{(i,2)} \ldots))((m_{(0,2)} m_{(0,3)} \ldots) \ldots (m_{(i-2,2)} \ldots)(m_{(i+1,2)} \ldots) \ldots),$$
$$((n_0))((m_{(0,2)} m_{(0,3)} \ldots) \ldots (m_{(p,2)} m_{(p,3)} \ldots)).$$

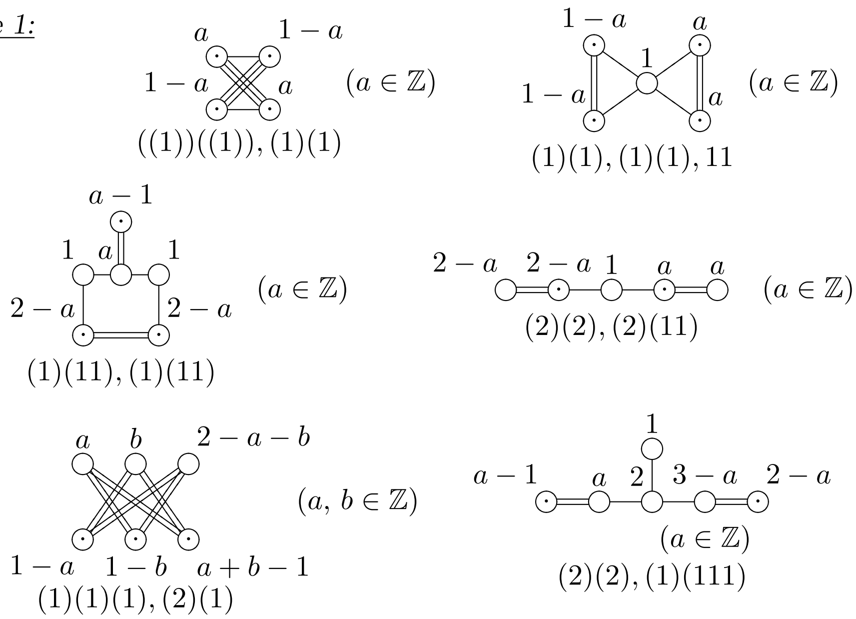
In [12], K. Takemura obtains a part of the classification in Theorem 3.26 under some conditions (see Proposition 4.3 in [12]).

3.3.3 The classification of basic pairs with idx -2

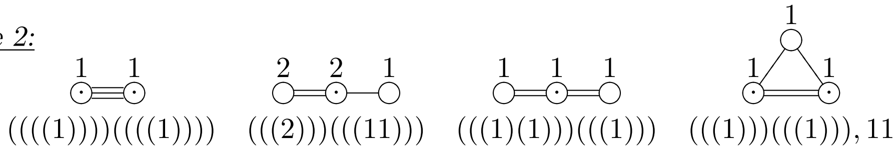
We shall give a classification of basic pairs of idx -2.

Theorem 3.29. *Let $(\mathbf{m}, L)$ be a basic pair with $\operatorname{idx}\mathbf{m} = -2$. Then its shape is one of the following.*

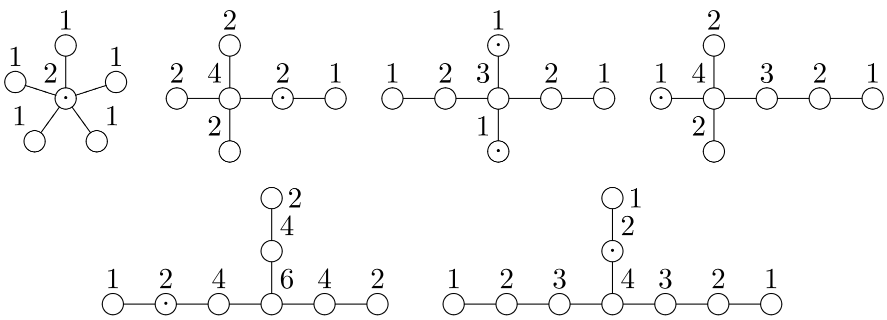
<u>*Case 1:*</u>

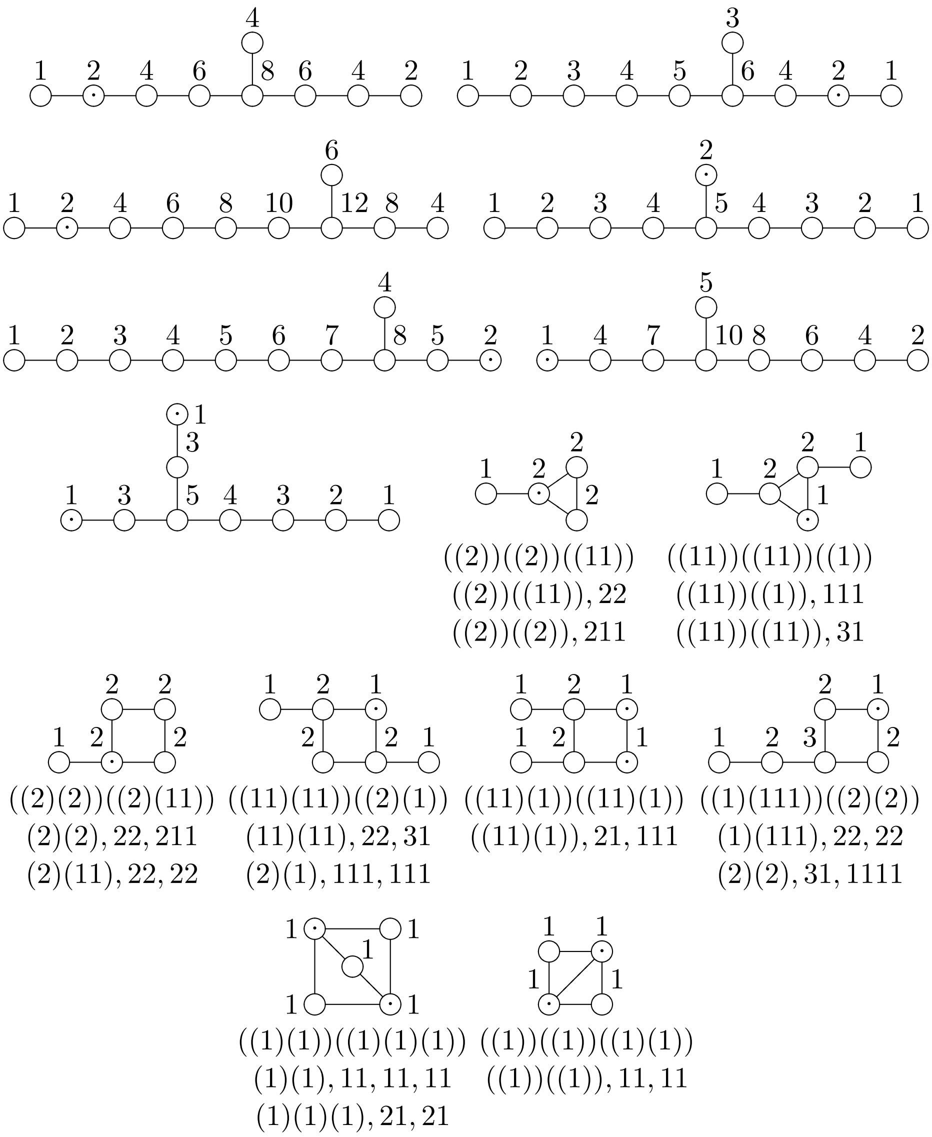


<u>*Case 2:*</u>



<u>*Case 3:*</u>



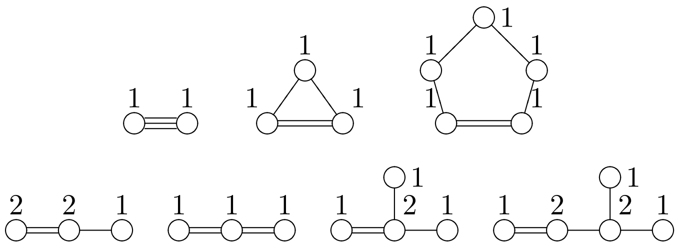


Here we simply denote the sets $\{x_a \mid a \in \mathbb{Z}\}$ and $\{y\}$ by x_a $(a \in \mathbb{Z})$ and y, respectively. The sequences of integers written under the shapes except for star-shaped ones stand for the corresponding basic pairs $(\mathbf{m}, L)$.

Retain the notation in the previous section. To prove the theorem we may assume $k_0 \geq k_1 \geq \cdots \geq k_{N-1} > k_N = \cdots = k_p = 1$ and $l_{N,1} \geq l_{N+1,1} \geq \cdots \geq l_{p,1} > 1$. Note that Lemma 3.23 iv) assures $N \leq 2$.

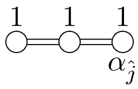
Then the proof of the theorem deduced to the following three lemmas.

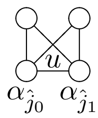
Lemma 3.30. *Suppose $N = 2$. Then the shape of $\alpha(\mathbf{m})$ is one of the following.*



Moreover the shape of $(\mathbf{m}, L)$ *is one of the shapes in* <u>*Case 1*</u> *in* Theorem 3.29.

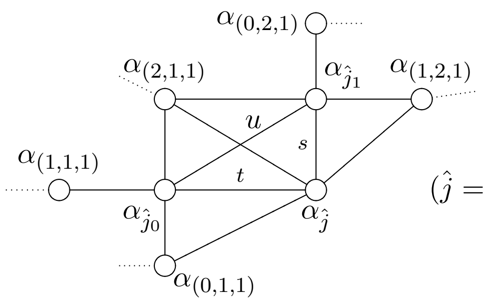
Proof. Use the notation in Lemma 3.23.

First suppose $k_0 \geq 3$. Then there exists $\hat{\jmath} = (2, l, 1 \ldots, 1) \in \mathcal{J} \cap \bar{I}$. If $l \neq 1$, $\langle \alpha_{\hat{\jmath}}, \alpha_{\hat{\jmath}_1} \rangle \leq -2$. If $l \neq k_1$, $\langle \alpha_{\hat{\jmath}}, \alpha_{\hat{\jmath}_0} \rangle \leq -2$. Since $\langle \alpha_{\hat{\jmath}_0}, \alpha_{\hat{\jmath}_1} \rangle \leq -2$, the lists in §2.4 show that the shape of $\alpha(\mathbf{m})$ equals $E)$  .

If $p \geq 3$, the Dynkin diagram of $\{\alpha_{\hat{\jmath}_0}, \alpha_{\hat{\jmath}_1}, \alpha_{(2,1,1)}, \alpha_{(3,1,1)}\}$ equals 

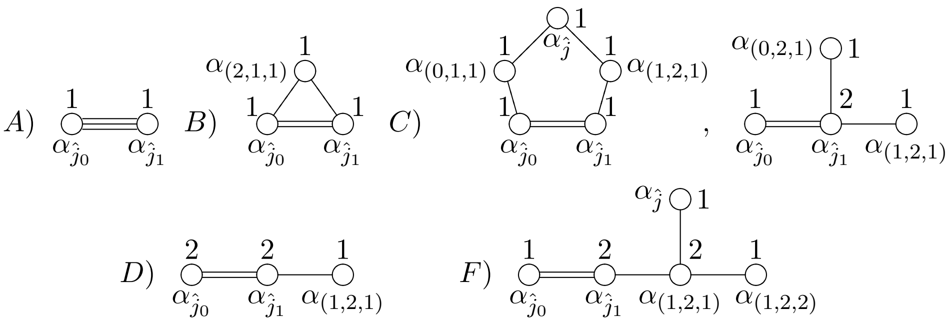
with $u = -\langle \alpha_{\hat{\jmath}_0}, \alpha_{\hat{\jmath}_1} \rangle \geq 2$, which contradicts to the lists in §2.4.

Next suppose $k_0 = k_1 = 2$ and $p \leq 2$. Then $\#(\bar{I} \cap \mathcal{J}) \leq 3$ and the support of $\alpha(\mathbf{m})$ is a subset of the set of simple roots whose Dynkin diagram is



$(\hat{\jmath} = (1, 2, 1, \ldots, 1)),$

where $s, t \geq 0$, $u = s + t + 2 \geq 2$ and $\hat{\jmath} = (1, 2, 1 \ldots, 1)$ or $\hat{\jmath} = (2, 1, 1, \ldots, 1)$. Here the Dynkin diagram in the case $\hat{\jmath} = (2, 1, 1 \ldots, 1)$ is similar as above and hence we assume $\hat{\jmath} = (1, 2, 1 \ldots, 1)$. Then the lists in §2.4 tell us that the shape of $\alpha(\mathbf{m})$ is one of the following.



Here $s = t = 0$ when $m_{\hat{j}} > 0$ and the simple roots indicated in the shape are examples corresponding to the shapes.

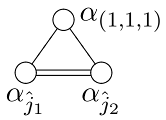
Since $\alpha \in Q_L$ and $\Phi(\alpha) = \mathbf{m}$, $\mathbf{m}$ is uniquely determined from $\alpha(\mathbf{m})$ for fixed L. Then if we write the shapes of $(\mathbf{m}, L)$ from the shapes $A)$, $B)$, $C)$, $D)$, $E)$ and $F)$, then we have the shapes in <u>*Case 1*</u> in Theorem 3.29, respectively. Here we note that the shapes of $\alpha(\mathbf{m})$ labeled by $C)$ correspond to a single shape of $(\mathbf{m}, L)$, which is the third one in <u>*Case 1*</u>. $\square$

Next consider the case $N = 1$. We notice that Φ is injective in this case. Hence the shape of $(\mathbf{m}, L)$ consists only of the shape of $\alpha(\mathbf{m})$. Put $\mathcal{J}_0 := \{1, \ldots, k_0\}$ for simplicity.

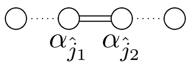
Lemma 3.31. *Retain the notation above. If* $\max\{d_0(j, j') \mid j,\ j' \in \mathcal{J}_0\} \geq 3$, *the shape of* $\alpha(\mathbf{m})$ *is one of the shapes in* <u>*Case 2*</u> *in* Theorem 3.29.

Proof. Lemma 3.23 proves $\max\{d_0(j, j') \mid j,\ j' \in \mathcal{J}_0\} \leq 4$ and the equality means that the shape of $\alpha(\mathbf{m})$ is the first one in <u>*Case 2*</u>.

Suppose $\max\{d_0(j, j') \mid j,\ j' \in \mathcal{J}_0\} = 3$. We may assume $d_0(1, 2) = 3$. Put $\hat{j}_\nu = (\nu, 1, \ldots, 1)$.

If $p \geq 1$, the Dynkin diagram of $\{\alpha_{\hat{j}},\ \alpha_{\hat{j}'},\ \alpha_{(1,1,1)}\}$ equals 

and the lists in §2.4 show that the shape of $\alpha(\mathbf{m})$ is the last one in <u>*Case 2*</u>.

If $k_0 = 2$, the shape of $\alpha(\mathbf{m})$ is  and the lists in §2.4 show that the shape of $\alpha(\mathbf{m})$ is the second one in <u>*Case 2*</u>.

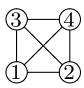
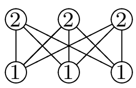
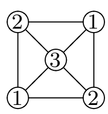
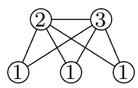
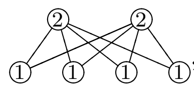
Suppose $k_0 \geq 3$, Then $d_0(1, 3) = 3$ or $d_0(2, 3) = 3$ by the relation (26). Hence the lists in §2.4 show that $k_0 \leq 3$ and moreover that if $k_0 = 3$, the shape of $\alpha(\mathbf{m})$ is the third one in <u>*Case 2*</u>. $\square$

Lemma 3.32. *If* $\max\{d_0(j, j') \mid j,\ j' \in \mathcal{J}_0\} \leq 2$, *the shape of* $\alpha(\mathbf{m})$ *is one of the shapes in* <u>*Case 3*</u> *in* Theorem 3.29.

Proof. Define the coset decomposition of $\mathcal{J}_0$ by the following relation: for distinct $j,\ j' \in \mathcal{J}_0$, j and j' are in the same coset if and only if $d_0(j, j') = 1$.

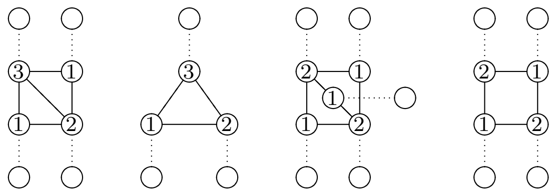
Put $\tilde{\mathcal{J}}_0 = \mathcal{J}_0 \cup \{(j, 1, 1) \mid j = 1, \ldots, p\}$ and define the coset decomposition $\tilde{\mathcal{J}}_0 = \coprod_{j=1}^{q} J(q)$ so that the coset is one of the cosets of $\mathcal{J}_0$ or $\{(j, 1, 1) \mid j = 1, \ldots, p\}$. We may assume $\#J(1) \geq \#J(2) \geq \cdots \geq \#J(q) \geq 1$.

Then we have $q \leq 3$, $\#J(2) \leq 2$ and if $q = 3$, then $\#J(2) = 1$ and $\#J(1) \leq 2$. Moreover if $\#J(2) = 2$, then $\#J(1) \leq 3$. In fact, if this is not valid, $\operatorname{supp} \alpha(\mathbf{m})$ contains a set of simple roots with the Dynkin diagram

 or  or  or  or ,

which contradicts to the classification in §2.4. Here ⓘ corresponds to a simple root in $J(i)$.

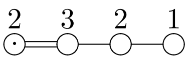
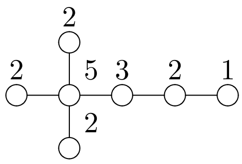
If $q = 1$ or $q = 2$ and $\#J(2) = 1$, the Dynkin diagram of the support of $\alpha(\mathbf{m})$ is star-shaped. Otherwise it is one of the following:



Hence we have the lemma from the classification in §2.4. □

Remark 3.33. *We mention about a related work by H. Kawakami, A. Nakamura and H. Sakai in* [5]*. They consider systems of first order differential equations with index of rigidity* -2 *whose singular points are regular singular or unramified irregular singular points. These equations are obtained by the confluence of singular points from Fuchsian systems of first order differential equations with index of rigidity* -2 *whose spectral types are basic in the sense of* Definition 3.16*. We notice that spectral types can be defined for systems of first order differential equations* (*see* [9] *for instance*)*.*

We regard these spectral types as elements in lattices of spectral types and write their shapes as in §3*. Then the list of shapes of these spectral types in* [5] *and our list of shapes of basic pairs with index* -2 *coincide with each other.*

This coincidence is no more valid in the case when the index of rigidity is -4*. Let* P *be a differential operator with the shape of the spectral type*  *, which represents a basic root with index* -4*. Then* P *is of order* 5 *and has an unramified irregular singular point. The operator* P *is obtained by a confluence of four regular singular points of a Fuchsian differential equation with the shape of the spectral type*  *, which does not corresponds to a basic root. Note that any Fuchsian differential equation of order* 5 *with a basic spectral type and index* -4 *has only three singular points* (see [9, 10])*.*

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