Abstract. The Capelli identity is extended to the case of minors. The operators appearing in the generalized identities give the annihilator of the degenerate principal series for $GL(n)$ and characterizes the image of the Poisson transform of the hyperfunctions on several boundaries of $GL(n)$. Hypergeometric functions are defined through realizations of some special sections of the degenerate principal series and the realizations on boundaries of $GL(n)$ generalize Gelfand’s hypergeometric functions. Related Radon transforms for Grassmannians are discussed.

0. Introduction

The Capelli identity [C1] is an important fundamental tool in the classical invariant theory (cf. [We]). It should be remarked that the differential operator given in the identity is an invariant differential operator on $GL(n)$.

In this note, we first generalize the Capelli identity to the case of minors. In §3 the corresponding non-invariant differential operators will be shown to characterize the representations of $GL(n, \mathbb{R})$ belonging to the degenerate principal series.

Any simultaneous eigenfunction of invariant differential operators on a Riemannian symmetric space $G/K$ of a non-compact type has a Poisson integral representation of a hyperfunction section of a line bundle over the maximal boundary $G/P$ of a semisimple Lie group $G$. This was conjectured by [H2] and then [K-] solved it in general by formulating it as boundary value problems with regular singularities (cf. [KO]) under a smooth realization (cf. [O2]) of $G/K$.

A similar boundary value problem is naturally formulated for the general boundary $G/P_\Theta$ with any parabolic subgroup $P_\Theta$ of $G$. Combining the result [K-] with [KR], it is easy to see that the image of the Poisson transform of hyperfunction sections of a line bundle over a general boundary $G/P_\Theta$ is characterized by a suitable system of differential equations. Hence the main problem is to give an explicit description of the nice generators of the system.

In [O1] we gave nice generators of the system in the case of a certain boundary of $GL(n, \mathbb{R})$ and a conjecture for general semisimple Lie groups. On the other hand, [J1] and [J2] gave the generators in a less explicit way for general boundaries $G/P_\Theta$ in the case of trivial line bundle over $G/P_\Theta$. There are many related works for Shilov boundaries of bounded symmetric domains (cf. [BV], [KM], [L], [Sn]).

In §5 we will generalize [O1] and show that the generalized Capelli operators give the generators of the system in the case of the general boundary $G/P_\Theta$ of $GL(n, \mathbb{R})$ by using the fact that the regular representation on the solution space is isomorphic to a representation belonging to degenerate spherical principal series of $GL(n, \mathbb{R})$. These operators are closely connected with the operators given in [Sh] to characterize a singular representation of $U(n, n)$ realized on sections of a certain line bundle over $U(n, n)/U(n) \times U(n)$. 

Typeset by \texttt{AMS-\TeX}
Lastly we will consider a special vector in a realization of the representation characterized
by a finite-dimensional representation of a certain subgroup of \( G \). For example, a fixed vector
under the action of the maximal compact subgroup \( K \) of \( G \) is a zonal spherical function if
we consider the spherical representation, that is, the representation on the function space on
\( G/K \). If the subgroup is the diagonal matrices of \( GL(n, \mathbb{R}) \) and if we consider representations
of a certain degenerate principal series realized on boundaries, the functions coincide with
Gelfand’s hypergeometric functions.

In \( \S 6 \) we will define hypergeometric functions on a general reductive Lie group \( G \) and
give some examples when \( G = GL(n, \mathbb{R}) \), which are a generalization of Gelfand’s hypergeometric
functions introduced by \([G]\) and equations defined in \([GG]\). Then Corollary 6.8 is
fundamental for their analysis, which characterize the image of Radon transforms on real
Grassmann manifolds (cf. Remark 7.4 ii)).

In \( \S 3 \) we will restrict ourself to the case when \( G = GL(n, \mathbb{R}) \). The similar arguments
can be applied to \( GL(n, \mathbb{C}) \) or its other real forms (cf. \([OSn]\]). The study in this note is
also restricted to the case of \( GL(n) \) but we try to explain our results explicitly by using the
coordinates of \( GL(n, \mathbb{R}) \). Generalizations of our results including the study in the case of
other classical groups and further studies of hypergeometric functions will be discussed in
other papers.

1. Capelli identities

The classical Capelli identity can be considered as a quantization of the formula \( \det \ A \cdot \det \ B \) in the linear algebra. We quantize more general identities

\[
(1.1) \quad \det \left( \sum_{\nu=1}^{n} x_{\nu i} y_{\nu j} \right)_{1 \leq \nu, i \leq m} = \sum_{1 \leq \nu_1, \ldots, \nu_m \leq n} \det \left( x_{\nu j} \right)_{1 \leq \nu, j \leq m} \cdot \det \left( y_{\nu j} \right)_{1 \leq \nu, j \leq m} 
\]

for \( 2mn \) commutative variables \( x_{\nu i} \) and \( y_{\nu i} \) with \( 1 \leq i \leq m \) and \( 1 \leq \nu \leq n \) (cf. \([Si, II \S 5 \] Theorem 9)), where the left hand side of (1.1) is zero if \( m > n \), and we get

**Theorem 1.1.** (Generalized Capelli identities) Let \( I = \{i_{k}\}_{1 \leq k \leq m} \) and \( J = \{j_{\ell}\}_{1 \leq \ell \leq m} \) be sequences of positive integers. Then

\[
(1.2) \quad \det \left( \sum_{\nu=1}^{n} x_{\nu i_{k}} \frac{\partial}{\partial x_{\nu j_{\ell}}} + (m-\ell)\delta_{i_{k}j_{\ell}} \right)_{1 \leq k \leq m, 1 \leq \ell \leq m} = \begin{cases} \sum_{1 \leq \nu_1, \ldots, \nu_m \leq n} \det \left( x_{\nu p, i_{k}} \right)_{1 \leq p \leq m} \cdot \det \left( \frac{\partial}{\partial x_{\nu q, j_{\ell}}} \right)_{1 \leq q \leq m} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases}
\]

Here \( \delta_{ij} \) is the Kronecker symbol and for a matrix \( A = (A_{ij}) = (A_{ij})_{1 \leq i \leq m, 1 \leq j \leq m} \) with \((i, j)\) components \( A_{ij} \) in an associative algebra, we define

\[
(1.3) \quad \det A = \sum_{\sigma \in S_{m}} \text{sgn}(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(m)m},
\]

where \( S_{m} \) is the \( m \)-th symmetric group.

**Proof of Theorem 1.1.** First note that Theorem 1.1 is equivalent to (1.1) if \( I \cap J = \emptyset \). We will reduce the theorem to (1.1) by the induction on \( m \).
The theorem is trivial when $m = 1$. Suppose $m > 1$. Denoting $\det \begin{pmatrix} x_{\nu i \ell} \end{pmatrix}_{1 \leq \nu \leq n}^{1 \leq i \leq m}$ and $\partial / \partial x_{\nu j}$ by $\det (x_{\nu i_1, \ldots, \nu i_m})_{\{i_1, \ldots, i_m\}}$ and $\partial_{\nu j}$, respectively, from the hypothesis of the induction we deduce

$$\det \left( \sum_{\nu=1}^{n} x_{\nu i k} \partial_{\nu j k} + (m - \ell) \delta_{i k j} \right)_{1 \leq k \leq m, 1 \leq \ell \leq m}$$

$$= \sum_{\sigma \in \Sigma_m} \text{sgn}(\sigma) \left( \sum_{\nu=1}^{n} x_{\nu i_{\sigma(1)}} \partial_{\nu j 1} + (m - 1) \delta_{i_{\sigma(1)} j 1} \right)$$

$$\cdots \left( \sum_{\nu=1}^{n} x_{\nu i_{\sigma(m)}} \partial_{\nu j m} + (m - m) \delta_{i_{\sigma(m)} j m} \right)$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \left( \sum_{\nu=1}^{n} x_{\nu i k} \partial_{\nu j 1} + (m - 1) \delta_{i k j 1} \right)$$

$$\cdot \det (x_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{i_1, \ldots, i_{k-1}, i_{k-1} + 1, \ldots, i_m\}}) \cdot \det (\partial_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{j_2, \ldots, j_m\}}).$$

If $j_1 \notin I$, (1.4) equals

$$\sum_{k=1}^{m} (-1)^{k-1} \sum_{\nu=1}^{n} x_{\nu i k} \det (x_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{i_1, \ldots, i_{k-1}, i_{k-1} + 1, \ldots, i_m\}})$$

and the theorem follows from the corresponding equation in the commutative case (1.1).

Suppose there exists $i_\ell$ with $i_\ell = j_1$. Then the difference of (1.5) from (1.4) equals

$$\sum_{1 \leq \nu_2 < \ldots < \nu_m \leq n} (m - 1)(-1)^{\ell-1} \det (x_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{i_1, \ldots, i_{\ell-1}, i_\ell + 1, \ldots, i_m\}})$$

$$\cdot \det (\partial_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{j_2, \ldots, j_m\}})$$

$$+ \sum_{1 \leq \nu_2 < \ldots < \nu_m \leq n} \sum_{1 \leq k \leq m} (-1)^{k-1} x_{\nu i k}$$

$$\cdot [\partial_{\nu i k} \det (x_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{i_1, \ldots, i_{k-1}, i_{k-1} + 1, \ldots, i_m\}})] \cdot \det (\partial_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{j_2, \ldots, j_m\}}).$$

Note that if $\nu \notin \{\nu_2, \ldots, \nu_m\}$, then

$$[\partial_{\nu i k}, \det (x_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{i_1, \ldots, i_{k-1}, i_{k-1} + 1, \ldots, i_m\}})] = 0.$$ 

On the other hand, putting $J_N = \{\nu_2, \ldots, \nu_{N-1}, \nu_{N+1}, \ldots, \nu_m\}$ for $N = 2, \ldots, m$, we have

$$\sum_{k=1}^{m} (-1)^{k-1} x_{\nu N i k} \left[ \partial_{\nu N i \ell}, \det (x_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{i_1, \ldots, i_{\ell-1}, i_{\ell-1} + 1, \ldots, i_m\}}) \right]$$

$$= \sum_{k=1}^{\ell-1} (-1)^{k+N+\ell-1} x_{\nu N i k} \det (x_{\nu_{i_1}, \ldots, i_{\ell-1}, i_{\ell-1} + 1, \ldots, i_{k-1}, i_{k-1} + 1, \ldots, i_m})$$

$$+ \sum_{k=\ell+1}^{m} (-1)^{k+N+\ell} x_{\nu N i k} \det (x_{\nu_{i_1}, \ldots, i_\ell-1, i_\ell+1, \ldots, i_{k-1}, i_{k-1} + 1, \ldots, i_m})$$

$$= (-1)^\ell \det (x_{\nu_{i_2}, \ldots, \nu_{i_m}}_{\{i_1, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_m\}}).$$

Hence we can conclude that (1.6) is identically zero and we have the theorem as in the case when $j_1 \notin I$. □
Corollary 1.2. Under the notation in Theorem 1.1, we have

\[ (1.7) \]
\[
\det \left( \sum_{\nu=1}^{n} x_{\nu i k} \frac{\partial}{\partial x_{\nu j l}} + (n + 1 - \ell) \delta_{i j k l} \right)_{1 \leq k \leq m, 1 \leq \ell \leq m} = \begin{cases} 
\sum_{1 \leq \nu_{1} < \cdots < \nu_{m} \leq n} \det \left( \frac{\partial}{\partial x_{\nu p q}} \right)_{1 \leq p \leq m, 1 \leq q \leq m} \cdot \det \left( x_{\nu_{p} i q} \right)_{1 \leq p \leq m, 1 \leq q \leq m} & \text{if } m \leq n, \\
0 & \text{if } m > n
\end{cases}
\]

and

\[ (1.8) \]
\[
\det \left( \sum_{\nu=1}^{n} x_{\nu i k} \frac{\partial}{\partial x_{\nu j l}} + (m - \ell) \delta_{i j k l} \right)_{1 \leq k \leq m, 1 \leq \ell \leq m} = \det \left( \sum_{\nu=1}^{n} x_{\nu i k} \frac{\partial}{\partial x_{\nu j l}} + (\ell - 1) \delta_{i j k l} \right)_{1 \leq k \leq m, 1 \leq \ell \leq m}.
\]

Proof. Let \( \mathcal{W} \) denote the algebra generated by \( x_{i j} \) and \( \frac{\partial}{\partial x_{i j}} \), which is called a Weyl algebra. Applying the anti-automorphism of \( \mathcal{W} \) to (1.2) defined by \( x_{i j} \mapsto x_{i j} \) and \( \frac{\partial}{\partial x_{i j}} \mapsto -\frac{\partial}{\partial x_{i j}} \), we have

\[
\sum_{1 \leq \nu_{1} < \cdots < \nu_{m} \leq n} \det \left( -\frac{\partial}{\partial x_{\nu_{p} q}} \right)_{1 \leq p \leq m, 1 \leq q \leq m} \cdot \det \left( x_{\nu_{p} i q} \right)_{1 \leq p \leq m, 1 \leq q \leq m}
\]
\[
= \sum_{\sigma \in S_{m}} \text{sgn}(\sigma) \left( \sum_{\nu=1}^{n} -\frac{\partial}{\partial x_{\nu j m}} x_{\nu i (m)} + (m - m) \delta_{i (m) j m} \right)
\]
\[
\cdots \left( \sum_{\nu=1}^{n} -\frac{\partial}{\partial x_{1 j}} x_{\nu i (1)} + (m - 1) \delta_{i (1) j 1} \right)
\]
\[
= (-1)^{m} \left( \sum_{\sigma \in S_{m}} \text{sgn}(\sigma) \left( \sum_{\nu=1}^{n} x_{\nu i (m)} \frac{\partial}{\partial x_{\nu j m}} + n \delta_{i (m) j m} \right) \right)
\]
\[
\cdots \left( \sum_{\nu=1}^{n} x_{\nu i (1)} \frac{\partial}{\partial x_{1 j}} + (n - m + 1) \delta_{i (1) j 1} \right).
\]

Reversing the order of the indices in the above, we have (1.7).

Consider the automorphism of \( \mathcal{W} \) defined by \( x_{i j} \mapsto \frac{\partial}{\partial x_{i j}} \) and \( \frac{\partial}{\partial x_{i j}} \mapsto -x_{i j} \). Then it follows from (1.2) and (1.7) that

\[
\det \left( \sum_{\nu=1}^{n} x_{\nu j k} \frac{\partial}{\partial x_{\nu i l}} + (m - \ell) \delta_{j k i l} \right)_{1 \leq k \leq m, 1 \leq \ell \leq m} = \sum_{1 \leq \nu_{1} < \cdots < \nu_{m} \leq n} \det \left( x_{\nu_{p} q} \right)_{1 \leq p \leq m, 1 \leq q \leq m} \cdot \det \left( \frac{\partial}{\partial x_{\nu_{p} j l}} \right)_{1 \leq p \leq m, 1 \leq q \leq m}
\]
\[
= (-1)^{m} \det \left( \sum_{\nu=1}^{n} -\frac{\partial}{\partial x_{\nu i k}} x_{\nu j l} + (n + 1 - \ell) \delta_{i j k l} \right)_{1 \leq k \leq m, 1 \leq \ell \leq m}
\]
\[
= \det \left( \sum_{\nu=1}^{n} x_{\nu j l} \frac{\partial}{\partial x_{\nu i k}} + (\ell - 1) \delta_{i j k l} \right)_{1 \leq k \leq m, 1 \leq \ell \leq m}.
\]
Combining this with Theorem 1.1 and exchanging $I$ and $J$, we have (1.8). □

**Remark 1.3.** If $m = n$, Theorem 1.1 is reduced to the Capelli identity (cf. [C1])

$$\det \left( \sum_{\nu=1}^{n} x_{\nu i} \frac{\partial}{\partial x_{\nu j}} + (m - j) \delta_{ij} \right)_{1 \leq i, j \leq n} = \det \left( x_{ij} \right)_{1 \leq i, j \leq n} \cdot \det \left( \frac{\partial}{\partial x_{ij}} \right)_{1 \leq i, j \leq n}.$$

In general, we have the $m$-th order Capelli identity (cf. [C2])

$$\sum_{1 \leq i_{1} < \cdots < i_{m} \leq n} \det \left( \sum_{\nu=1}^{n} x_{\nu i} \frac{\partial}{\partial x_{\nu i_{\ell}}} + (m - \ell) \delta_{i_{\ell} i_{\ell}} \right)_{1 \leq i_{\ell} \leq m}$$

$$= \sum_{1 \leq i_{1} < \cdots < i_{m} \leq n} \det \left( x_{i_{k} j_{\ell}} \right)_{1 \leq k \leq m} \cdot \det \left( \frac{\partial}{\partial x_{i_{k} j_{\ell}}} \right)_{1 \leq k, \ell \leq m}.$$

### 2. Capelli operators

**Definition 2.1.** Let $E_{ij}$ be the $n \times n$ matrix whose $(\mu, \nu)$ element equals $\delta_{i\mu} \delta_{j\nu}$ and consider $E_{ij}$ as an element of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Let $I = \{i_{\mu}\}_{1 \leq \mu \leq m}$ and $J = \{j_{\nu}\}_{1 \leq \nu \leq m}$ be sequences of $m$ positive integers with $1 \leq i_{\mu} \leq n$ and $1 \leq j_{\nu} \leq n$. Then define

$$D_{IJ}^{\mu \nu} = E_{i_{\mu} j_{\nu}},$$

$$\bar{D}_{IJ}^{\mu \nu} = \bar{D}_{IJ}^{\mu \nu}(\lambda) = E_{i_{\mu} j_{\nu}} + (\lambda + m - \nu) \delta_{i_{\mu} j_{\nu}},$$

$$D_{IJ} = D_{IJ}(\lambda) = \det \left( D_{IJ}^{\mu \nu} \right)_{1 \leq \mu, \nu \leq m}.$$

Here $\lambda$ is an indeterminate which commutes with elements of $\mathfrak{g}$ and $D_{IJ}(\lambda)$ is an element of $\mathfrak{U} = U(\mathfrak{g}) \otimes \mathbb{C}[\lambda]$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

We naturally identify $E_{ij}$ with the left invariant vector filed on $G = GL(n, \mathbb{C})$. Then $E_{ij}$ is of the form

$$(2.1) \quad E_{ij} = \sum_{\nu=1}^{n} x_{\nu i} \frac{\partial}{\partial x_{\nu j}}$$

under the coordinates $\left( x_{ij} \right)_{1 \leq i, j \leq n} \in G$ and the left hand side of (1.2) is identified with $D_{IJ}(0)$. Hence we call $D_{IJ}$ and $D_{IJ}(0)$ generalized Capelli operators.

**Lemma 2.2.** Put $\tau(I) = \{i_{\tau(\mu)}\}_{1 \leq \mu \leq m}$ and $\tau'(J) = \{j_{\tau'(\nu)}\}_{1 \leq \nu \leq m}$ for $\tau, \tau' \in \mathfrak{S}_{m}$. Then

$$D_{IJ}(\lambda) = \text{sgn} (\tau) \text{sgn} (\tau') D_{\tau(I) \tau'(J)}(\lambda).$$

**Proof.** We can prove the lemma by a direct calculation but we remark that it is a corollary of Theorem 1.1. In fact, by the identification (2.1), it follows from Theorem 1.1 that $D_{IJ}(0) = \text{sgn} (\tau) \text{sgn} (\tau') D_{\tau(I) \tau'(J)}(0)$. Then applying an automorphism of $U(\mathfrak{g})$ defined by

$$(2.2) \quad S_{\lambda} : U(\mathfrak{g}) \ni E_{ij} \mapsto E_{ij} + \lambda \delta_{ij}$$

to this equality, we have Lemma 2.2 for any $\lambda \in \mathbb{C}$. □

This lemma immediately implies
Corollary 2.3. If there exist integers \( k \) and \( \ell \) with \( 1 \leq k < \ell \leq m \) which satisfy \( i_k = i_\ell \) or \( j_k = j_\ell \), then \( D_{I,J}(\lambda) = 0 \).

Proposition 2.4. Suppose \( I = \{i_\mu\}_{1 \leq \mu \leq m} \) and \( J = \{j_\nu\}_{1 \leq \nu \leq m} \) satisfy \( i_\mu \neq i_\nu' \) and \( j_\nu \neq j_\nu' \) for \( 1 \leq \mu < \mu' \leq m \) and \( 1 \leq \nu < \nu' \leq m \). Let \( k \) and \( \ell \) be positive integers with \( 1 \leq k \leq n \) and \( 1 \leq \ell \leq n \). Then

\[
[E_{k\ell}, D_{I,J}(\lambda)] = D_1 - D_2
\]

with

\[
D_1 = \begin{cases} 
0 & \text{if } \ell \notin I, \\
D_{\{i_1, \ldots, i_{\mu-1}, k, i_{\mu+1}, \ldots, i_m\}}(\lambda) & \text{if } \ell = i_\mu, \\
0 & \text{if } k \notin J,
\end{cases}
\]

\[
D_2 = \begin{cases} 
0 & \text{if } \ell \notin I, \\
D_{\{j_1, \ldots, j_{\nu-1}, \ell, j_{\nu+1}, \ldots, j_m\}}(\lambda) & \text{if } k = j_\nu.
\end{cases}
\]

Proof. We may assume \( \ell \in I \) and \( k \in J \) because otherwise the lemma is clear from the relation \( [E_{k\ell}, E_{ij}] = \delta_{ij} E_{kj} - \delta_{kj} E_{ij} \) and the definition of \( D_{I,J} \). Suppose \( i_\mu(\ell) = \ell \) and \( j_\nu(k) = k \). Putting \( I' = \{i_1, \ldots, i_{\mu(\ell)-1}, k, i_{\mu(\ell)+1}, \ldots, i_m\} \), \( J' = \{j_1, \ldots, j_{\nu(k)-1}, \ell, j_{\nu(k)+1}, \ldots, j_m\} \), \( I'' = \{i_1, \ldots, i_{\mu(\ell)-1}, i_{\mu(\ell)+1}, \ldots, i_m\} \) and \( J'' = \{j_1, \ldots, j_{\nu(k)-1}, j_{\nu(k)+1}, \ldots, j_m\} \), we have

\[
[E_{k\ell}, D_{I,J}] = \left( D_{ij}^{I,J} - (\lambda + m - j)\delta_{i\mu(\ell)}\delta_{j\nu(k)} \right)_{1 \leq i,j \leq m}^{1 \leq \mu, \nu \leq m}
\]

\[
- \left( D_{ij}^{I',J'} - (\lambda + m - j)\delta_{i\mu(\ell)}\delta_{j\nu(k)} \right)_{1 \leq i,j \leq m}^{1 \leq \mu, \nu \leq m}
\]

\[
= (D_{I,J} - (-1)^{\mu(\ell)+\nu(k)}(\lambda + m - \nu(k))D_{I'',J''}) - (D_{I,J} - (-1)^{\mu(\ell)+\nu(k)}(\lambda + m - \nu(k))D_{I'',J''})
\]

\[
= D_{I,J} - D_{I,J},
\]

and the proposition. \( \square \)

Lemma 2.2, Corollary 2.3 and Proposition 2.4 imply

Corollary 2.5. i) Put

\[
J(m, \lambda) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \mathbb{C}_{D_{\{i_1, \ldots, i_m\}}(\lambda)} \mathbb{C}_{D_{\{j_1, \ldots, j_m\}}(\lambda)}.
\]

Then \( \text{gl}(n, \mathbb{C}), J(m, \lambda) \subset J(m, \lambda) \).

ii) If \( \ell \notin I \) and \( k \notin J \), then \( [E_{k\ell}, D_{I,J}(\lambda)] = 0 \).

iii) If \( k \in I \cap J \) and \( \ell \in I \cap J \), then \( [E_{k\ell}, D_{I,J}(\lambda)] = 0 \).

Proposition 2.6. i) Under the above notation

\[
J(m + 1, \lambda) \subset \text{D}J(m, \lambda) \cap \text{D}J(m, \lambda + 1) \quad \text{for } m = 1, \ldots, n - 1.
\]

ii) Let \( I = \{i_1, \ldots, i_m\} \) and \( J = \{j_1, \ldots, j_m\} \) with \( 1 \leq i_1 < \cdots < i_m \leq n \) and \( 1 \leq j_1 < \cdots < j_m \leq n \). Put \( I \cap J = \{i_{\mu_1}, \ldots, i_{\mu_L}\} = \{j_{\nu_1}, \ldots, j_{\nu_L}\} \) with \( 1 \leq \mu_1 < \cdots < \mu_L \leq n \) and \( 1 \leq \nu_1 < \cdots < \nu_L \leq n \). Then

\[
D_{I,J}(\lambda) - D_{I,J}(\lambda - 1) = \sum_{k=1}^{L} D_{\{i_1, \ldots, i_{\mu_k-1}, i_{\mu_k+1}, \ldots, i_m\}, j_1, \ldots, j_{\nu_k-1}, j_{\nu_k+1}, \ldots, j_m}(\lambda).
\]
iii) Under the anti-automorphism $a$ of $U(g)$ satisfying $X \mapsto -X$ for $X \in g$, $D_{IJ}(\lambda)$ changes into $(-1)^m D_{IJ}(1 - m - \lambda)$ with $m = \#I$.

iv) Under the automorphism $t$ of $U(g)$ satisfying $E_{ij} \mapsto -E_{ji}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$, $D_{IJ}(\lambda)$ changes into $(-1)^m D_{IJ}(1 - m - \lambda)$ with $m = \#I$.

Proof. Note that Corollary 2.5 shows $\mathfrak{U}J(m, \lambda + 1) = J(m, \lambda + 1)\mathfrak{U}$. The Laplace expansions with respect to the first and the last columns of $J(m + 1, \lambda)$ imply i). ii) is clear from Lemma 2.2 with

$$a(E_{i_{\sigma(1)}j_1} + (\lambda + m - 1)\delta_{i_{\sigma(1)}j_1}) \cdots (E_{i_{\sigma(m)}j_m} + (\lambda + m - 1)\delta_{i_{\sigma(m)}j_m})$$

$$= (-E_{i_{\sigma(1)}j_1} + (\lambda + m - 1)\delta_{i_{\sigma(1)}j_1}) \cdots (-E_{i_{\sigma(m)}j_m} + (\lambda + m - 1)\delta_{i_{\sigma(m)}j_m})$$

$$= (-1)^m (E_{i_{\sigma(1)}j_1} + (1 - m - \lambda) + m - 1)\delta_{i_{\sigma(m)}j_m})$$

$$= \cdots (-1)^m E_{i_{\sigma(1)}j_1} + ((1 - m - \lambda) + m - 1)\delta_{i_{\sigma(m)}j_m}).$$

Put $D_{IJ}^k(\lambda) = \det (E_{ij} + (\lambda + m - q - c^q_k)\delta_{ipjq})_{1 \leq p \leq m}$, where $c^q_k = 1$ if $q \leq k$ and 0 otherwise. Then it easily follows from (1.3) that $D_{IJ}^k(\lambda) - D_{IJ}^l(\lambda)$ are summands of the right hand side of (2.5) and hence we have iii).

By (1.8) we have

$$\sum_{\sigma \in \mathfrak{S}_m} (-E_{j_1i_{\sigma(1)}} + (1 - m)\delta_{j_1i_{\sigma(1)}}) \cdots (-E_{j_mi_{\sigma(m)}} + (1 - m)\delta_{j_mi_{\sigma(m)}})$$

$$= (-1)^m \sum_{\sigma \in \mathfrak{S}_m} (E_{i_{\sigma(1)}j_1} + (1 - m)\delta_{j_1i_{\sigma(1)}}) \cdots (E_{i_{\sigma(m)}j_m}) + (m - m)\delta_{j_mi_{\sigma(m)}}).$$

Applying $S_{1-m-\lambda}$ given in (2.2) to this, we get iv). □

Definition 2.7. (Harish-Chandra homomorphism) Put $a = C E_{11} + \cdots + C E_{nn}$, $\bar{n} = \sum_{1 \leq i < j \leq n} C E_{ij}$ and $n = \sum_{1 \leq i < j \leq n} C E_{ij}$. For $D \in \mathfrak{U}$ we define elements $\gamma' (D)$ and $\gamma(D)$ of $U(a) \otimes \mathbb{C}[\lambda]$ so that $D - \gamma'(D) \in \mathfrak{U}a + n\mathfrak{U}$ and $\gamma(D) = \iota_{\rho}(\gamma'(D))$. Here $\iota_{\rho}$ is an algebra endomorphism of $U(a) \otimes \mathbb{C}[\lambda]$ which satisfies $\iota_{\rho}(\lambda) = \lambda$ and $\iota_{\rho}(E_{ii}) = E_{ii} + i - \frac{n + 1}{2}$.

Lemma 2.8. For $I = \{i_\mu\}_{1 \leq \mu \leq m}$ and $J = \{j_\nu\}_{1 \leq \nu \leq m}$ we put

$$E_{IJ} = E_{i_1j_1}E_{i_2j_2}\cdots E_{i_mj_m}.$$ 

Then $\gamma(E_{IJ}) = 0$ if $\sigma(I) \neq J$ for any $\sigma \in \mathfrak{S}_m$.

Proof. We will prove the lemma by the induction on $m$.

Suppose there exists $k$ which satisfies $i_k < j_k$ and $i_\mu \geq j_\mu$ for $\mu > k$. If $k = m$, the lemma is clear. If $k < m$,

$$E_{i_1j_1} \cdots E_{i_kj_k} E_{i_{k+1}j_{k+1}} \cdots E_{i_mj_m} = E_{i_1j_1} \cdots E_{i_{k+1}j_{k+1}} E_{i_kj_k} \cdots E_{i_mj_m}$$

$$+ \delta_{i_{k+1}j_{k+1}} E_{i_1j_1} \cdots E_{i_{k+1}j_{k+1}} \cdots E_{i_mj_m} - \delta_{i_kj_k} E_{i_1j_1} \cdots E_{i_{k+1}j_{k+1}} \cdots E_{i_mj_m}$$

and the lemma is proved by the induction on the lexicographic order of $(m, m - k)$.

On the other hand, if there exists $k$ with $i_k > j_k$ and $i_\mu \leq j_\mu$ for $\mu < k$, we have similarly the lemma by the induction on $(m, k)$.

Since the assumption of the lemma assures the existence of $k$ with $i_k \neq j_k$, we have the lemma. □
Proposition 2.9. Let $I = \{i_\mu\}_{1 \leq \mu \leq m}$ and $J = \{j_\nu\}_{1 \leq \nu \leq m}$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq n$. Then $\gamma(D_{IJ}(\lambda)) = 0$ if $I \neq J$ and

$$\gamma(D_{II}(\lambda)) = m \prod_{\nu=1}^{m} (E_{i_\nu i_\nu} + \lambda + i_\nu + m - \nu - \frac{n+1}{2}).$$

Proof. The definition of $D_{IJ}$ and Lemma 2.8 implies $\gamma(D_{IJ}) = 0$ if $I \neq J$. Note that $D_{II} = \sum_{\sigma \in S_m} \text{sgn}(\sigma) \bar{D}_{\sigma(1)} D^{(1)}_{\sigma(2)} \cdots D^{(m)}_{\sigma(m)}$ and that $D_{\sigma(1)}^{(1)} = E_{i_{\sigma(1)} i_{\sigma(1)}} n$ if $\sigma(1) \neq 1$. Hence we have

$$D_{II} \equiv \sum_{\sigma \in S_m, \sigma(1)=1} \text{sgn}(\sigma)(E_{i_{\sigma(1)} i_{\sigma(1)}} + \lambda + m - 1)\bar{D}_{\sigma(2)} D^{(2)}_{\sigma(3)} \cdots D^{(m)}_{\sigma(m)} \mod nU$$

by the induction on $m$ and we have the proposition. □

The following result is well-known.

Corollary 2.10. i) Regard $D_{\{1, \ldots, n\}\{1, \ldots, n\}}(\lambda)$ as a polynomial function of $\lambda$ and denote it by $D(\lambda)$. Then the coefficients of the terms $\lambda^k$ in $D(\lambda)$ for $k = 1, \ldots, n$ generate the center of $U(g)$ and

$$\gamma(D(\lambda)) = \prod_{i=1}^{n} (E_{ii} + \lambda + \frac{n+1}{2}).$$

ii) The $m$-th order Capelli operator (1.9), which will be denoted by $D_m$, is in the center of $U(g)$ and satisfies

$$\gamma(D_m) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \prod_{\nu=1}^{m} (E_{i_\nu i_\nu} + i_\nu - \nu + m - \frac{n+1}{2}).$$

The operators $D_1, \ldots, D_n$ generate the center of $U(g)$.

Changing the indices in Proposition 2.9 by $\begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix}$, we have

Corollary 2.11. Under the notation in Proposition 2.9

$$D_{IJ} \in \bar{n}U + \bar{u}n \ \text{if} \ I \neq J,$$

$$D_{II} \equiv \prod_{\nu=1}^{m} (E_{i_\nu i_\nu} + \lambda + \nu - 1) \mod \bar{n}U.$$

Here mod $\bar{n}U$ may be replaced by mod $\bar{u}n$.

Lastly we remark
Lemma 2.12. If $D \in U(\mathfrak{g})$ satisfies $\gamma(D) = \gamma([X, D]) = 0$ for any $X \in \mathfrak{g}$, then $D = 0$.

Proof. Let $\mathfrak{g}^*$ and $\mathfrak{a}^*$ be the dual spaces of $\mathfrak{g}$ and $\mathfrak{a}$, respectively, and identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form of $\mathfrak{g}$. Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$, which is identified with the space of polynomial functions on $\mathfrak{g}^*$. Denote by $S^{(m)}(\mathfrak{g})$ the set of polynomials in $S(\mathfrak{g})$ with degree at most $m$. Let $\Lambda$ be the map of symmetrization of $S(\mathfrak{g})$, which maps the element $X_1 \cdots X_m \in S(\mathfrak{g})$ with $X_i \in \mathfrak{g}$ to $\sum_{\sigma \in \mathfrak{S}_m} \frac{1}{m!} X_{\sigma(1)} \cdots X_{\sigma(m)} \in U(\mathfrak{g})$.

Suppose $D \neq 0$. Let $m$ be the smallest number with $\Lambda^{-1}(D) \in S^{(m)}(\mathfrak{g})$ and let $\bar{D}$ be the non-zero homogeneous element in $S^{(m)}(\mathfrak{g})$ with $\Lambda^{-1}(\bar{D}) - D \in S^{(m-1)}(\mathfrak{g})$. Since $\Lambda$ is Ad($g$)-equivariant, the assumption implies Ad($g$)$\bar{D}|_{\mathfrak{a}^*} = 0$ for $g \in G$. Hence by denoting $U = \bigcup_{g \in G} \text{Ad}(g)\mathfrak{a}^*$, we have $D|_U = 0$ and $\bar{D} = 0$ because $U$ is open dense in $\mathfrak{g}^*$, which leads a contradiction. □

3. Degenerate principal series

In this section we will see that the generalized Capelli operators define the annihilators of degenerate principal series representations of $G$, where $G$ is $GL(n, \mathbb{C})$ or its real form. For simplicity we assume $G = GL(n, \mathbb{R})$ hereafter in this note if otherwise stated. Then $K = O(n)$ is a maximal compact subgroup of $G$.

Given a positive integer $L$ and a sequence of positive integers $\Theta = \{n_1, \ldots , n_L\}$ with $0 = n_0 < n_1 < \cdots < n_l = n$, we define a parabolic subgroup

$$P_\Theta = \left\{ p = \begin{pmatrix} g_1 & \ast & \ast & \cdots & \ast \\ \ast & g_2 & \ast & \cdots & \ast \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \ast & \ast & \cdots & \ast & g_L \end{pmatrix} \in GL(n, \mathbb{R}); g_k \in GL(n_k - n_{k-1}, \mathbb{R}) \right\}.$$  \hspace{1cm} (3.1)

Note that the subgroup $P_{\{1,\ldots , n\}} = \{ (x_{ij}) \in G; x_{ij} = 0 \text{ if } i < j \}$ is a minimal parabolic subgroup of $G$, which will be simply denoted by $P$.

The space $\mathcal{B}(G)$ of hyperfunctions on $G$ is a left $G$-module by $G \times \mathcal{B}(G) \ni (g, f(x)) \mapsto (\pi_g(f))(x) = f(g^{-1}x)$. The space $\mathcal{A}(G)$ of real analytic functions on $G$ and the space $\mathcal{C}^\infty(G)$ of $C^\infty$-functions on $G$ are $G$-submodules of $\mathcal{B}(G)$.

For $\mu = (\mu_1, \ldots , \mu_L) \in \mathbb{C}^L$ and $\varepsilon = (\varepsilon_1, \ldots , \varepsilon_L) \in \{0, 1\}^L$, we define a $G$-submodule

$$\mathcal{B}(G/P_\Theta; \mu, \varepsilon) = \{ f \in \mathcal{B}(G); f(xp) = \tau_{\mu, \varepsilon}(p^{-1})f(x) \text{ for } p \in P_\Theta \}$$  \hspace{1cm} (3.2)

belonging to degenerate principal series of $G$, where

$$\tau_{\mu, \varepsilon}(p) = \text{sgn} (\det g_1)^{\varepsilon_1} | \det g_1|^{-\mu_1} \cdots \text{sgn} (\det g_L)^{\varepsilon_L} | \det g_L|^{-\mu_L} \text{ for } p \in P_\Theta$$  \hspace{1cm} (3.3)

is a character of $P_\Theta$ under the expression of $p$ in (3.1).

Note that $U(\mathfrak{g})$ is a subalgebra of the ring of differential operators on $G$ which satisfies $\mathfrak{g} \times \mathcal{B}(G) \ni (D, f) \mapsto (Df)(x) = \frac{d}{dt} f(x \exp t D)|_{t=0}$. Then we have

Theorem 3.1. For $\mu \in \mathbb{C}^L$ we define an algebra homomorphism $\chi_{\Theta, \mu}$ of $U(\mathfrak{a})$ to $\mathbb{C}$ so that

$$\chi_{\Theta, \mu}(E_{jj}) = \mu_k + j - \frac{n + 1}{2} \text{ if } n_{k-1} < j \leq n_k.$$  \hspace{1cm} (3.4)

Then for $u \in \mathcal{B}(G/P_\Theta; \mu, \varepsilon)$

$$D_m u = \chi_{\Theta, \mu}(\gamma(D_m)) u \text{ for } m = 1, \ldots , n$$
and

\[(3.5) \quad D_{I,J}(-\mu_k - n_{k-1})u = 0 \quad \text{for} \quad \#I = \#J = n - n_k + n_{k-1} + 1 \quad \text{and} \quad k = 1, \ldots, L\]

under the notation in Corollary 2.10.

**Proof.** For simplicity, we put \(V_\mu = B(G/P_\emptyset; \mu, \varepsilon)\) in this proof. Let consider the function

\[
h_\xi(x^{-1}) = \sgn \left( \det (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \right)^{\xi_1} \det (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}^{\xi_2} \cdots \sgn \left( \det (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \right)^{\xi_L} \det (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}^{\xi_L}
\]

with \(-\mu = (\xi_1 + \cdots + \xi_L, \xi_2 + \cdots + \xi_L, \ldots, \xi_L)\). If \(\Re \xi_k\) is sufficiently large for \(k = 1, \ldots, L\), \(h_\xi(x)\) is a sufficiently differentiable \(\tilde{N}\)-invariant function in \(V_\mu\), where \(\tilde{N} = \exp(\sum_{i<j} \mathbb{R} E_{ij})\).

For any \(D \in U(\mathfrak{a})\) and \(\tilde{D} \in D + \tilde{n}U(\mathfrak{g}) + U(\mathfrak{g})n\), we denote by \(\tilde{D}_\xi\) the image of \(D\) under the algebra homomorphism of \(U(\mathfrak{a})\) to \(\mathbb{C}\) with \((E_{ii})_\xi = \xi_i\) for \(i = 1, \ldots, m\) by putting \(\xi_j = \mu_k\) if \(n_{k-1} < j \leq n_k\). Then we have

\[
Dh_\xi(\tilde{n}) = \tilde{D}_\xi h_\xi(\tilde{n}) \quad \text{for} \quad \tilde{n} \in \tilde{N}
\]

and it follows from Corollary 2.11 that \(\tilde{D}_{I,J}(\lambda)_\xi = \prod_{\nu=1}^{m} (\zeta_{i_\nu} + \lambda + \nu - 1)\) if \(I = \{i_1, \ldots, i_m\}\) with \(1 \leq i_1 < \ldots < i_m \leq n\). Put \(m = n - n_k + n_{k-1} + 1\). Since \(n_{k-1} < n_{k-1} + 1 \leq n - (n - (n_{k-1} + 1)) = n_k\), we have \(\zeta_{i_\nu} + (-\mu_k - n_{k-1}) + \nu - 1 = 0\) when \(\nu = n_{k-1} + 1\), and therefore \(\tilde{D}_{I,J}(-\mu_k - n_{k-1})_\xi = 0\) if \(\#I = \#J = m\).

Suppose \(D_1^\mu, \ldots, D_N^\mu\) are elements of \(U(\mathfrak{g})\) which holomorphically depend on \(\mu\) and satisfy

\[
D_k^\mu h_\xi(\tilde{n}) = 0 \quad \text{and} \quad \text{Ad}(g)D_k^\mu \in \sum_{j=1}^{N} U(\mathfrak{g})D_j^\mu \quad \text{for} \quad k = 1, \ldots, N, \tilde{n} \in \tilde{N} \quad \text{and} \quad g \in G.
\]

Then if we prove \(D_k^\mu V_\mu = 0\), we have (3.4) and similarly (3.5) from Corollaries 2.5 and 2.10.

Note that \(D_k^\mu h_\xi(\tilde{n}p) = 0\) for \(p \in P_\emptyset\) because \(Dh_\xi(\tilde{n}p) = \tau_{\mu,\varepsilon}(p^{-1})(\text{Ad}(p)D\xi)(\tilde{n})\) for \(D \in U(\mathfrak{g})\). Since \(NP_\emptyset\) is open dense in \(G\), \(D_k^\mu h_\xi = 0\) if \(\Re \xi_k\) is sufficiently large for \(k = 1, \ldots, L\). If moreover \(\xi_k\) are generic, \(V_\mu\) is an irreducible \(G\)-module and hence in this case we have \(D_k^\mu u = 0\) for any \(u \in V_\mu\). For any fixed \(\mu' \in \mathbb{C}^L\) and a real analytic function \(u \in V_{\mu'}\), we can define \(u_\mu \in V_\mu\) with \(u_\mu |_{K} = u |_{K}\). Since \(u_\mu\) holomorphically depends on \(\mu \in \mathbb{C}^L\), we have \(D_k^\mu u_\mu = 0\) for any \(\mu \in \mathbb{C}^L\). Since \(A(G) \cap V_\mu\) is dense in \(V_\mu\), we have \(D_k^\mu u = 0\) for any \(u \in V_{\mu'}\). \(\square\)

**Proposition 3.2.** i) If

\[(\mu_i + n_i) - (\mu_j + n_{j-1} + 1) \notin \{0, 1, 2, \ldots, n_i - n_{i-1} + n_j - n_{j-1} - 2\} \quad \text{for} \quad 1 \leq i < j \leq L,
\]

then

\[
(3.6) \quad D_m - \chi_{\Theta,\mu}(\gamma(D_m)) \in \sum_{k=1}^{L} \Delta J(n - n_k + n_{k-1} + 1, -\mu_k - n_{k-1}).
\]

ii) For an integer \(m\) with \(0 < m \leq n\), the system of differential equations

\[
D_{I,J}(0)u = 0 \quad \text{for} \quad \#I = \#J = m
\]
on $G$ is equivalent to
\[
\det \left( \frac{\partial}{\partial x_{ij}} \right)_{i \in I, j \in J} u = 0 \quad \text{for} \quad \#I = \#J = m.
\]

Proof. Theorem 1.1 proves ii) because $\det \left( \det(x_{ij})_{i \in I, j} \right)_{I J} = \det(x_{ij})^N \neq 0$ with $\#I = \#J = m$ and $N = \frac{n!}{(n-1)!n(m-n)!}$.

Proposition 2.6 i) and Corollary 2.10 show that the right hand side of (3.6) contains $D(-\mu_k - n_{k-1} - \nu)u = 0$ for $k = 1, \ldots, L$ and $\nu = 0, \ldots, n_k - n_{k-1} - 1$ under the notation there, which is equivalent to $D(\lambda)u = 0$ for any $\lambda \in \mathbb{C}$ and we have (3.6) because the relation
\[
\begin{align*}
\{(\mu_i + \nu_i) - (\mu_j + \nu_j); & n_i - 1 + 1 \leq \nu_i \leq n_i, n_j - 1 + 1 \leq \nu_j \leq n_j \} \\
= & \{(\mu_i + \nu_i) - (\mu_j + \nu_j - 1 + 1) - \nu; 0 \leq \nu \leq n_i - n_{i-1} + n_j - n_{j-1} - 2\}
\end{align*}
\]
shows that the numbers $-\mu_k - n_{k-1} - \nu$ are different to each other. \qed

4. Intertwining operators

Other realizations of degenerate principal series we will investigate are given by some $G$-homomorphisms, namely, by intertwining operators, which are integral operators with kernel functions because $G/P_\Theta$ is compact. We will review them in this section.

Retain the notation in §3 and define Lie subalgebras of $g$:
\[
\begin{align*}
n_\Theta &= \sum_{k=1}^{L} \sum_{n_{k-1} < i \leq n_k} \mathbb{R} E_{ij}, \\
\bar{n}_\Theta &= \sum_{k=1}^{L} \sum_{n_{k-1} < i \leq n_k} \mathbb{R} E_{ji}, \\
\mu_\Theta &= \sum_{k=1}^{L} \sum_{n_{k-1} < i \leq n_k} \mathbb{R} E_{ij}.
\end{align*}
\]

Then $\text{Lie}(P_\Theta) = \mu_\Theta \oplus n_\Theta$ is a Levi decomposition. We put $N_\Theta = \exp(n_\Theta)$ and $\bar{N}_\Theta = \exp(\bar{n}_\Theta)$.

If $\Theta = \{1, 2, \ldots, n\}$, then $P_\Theta, n_\Theta, \bar{n}_\Theta, N_\Theta$ and $\bar{N}_\Theta$ are simply denoted by $P, n, \bar{n}, N, \bar{N}$, respectively.

Let $a^*_c$ be the complex dual of $a$, which equals $\sum_{j=1}^{n} \mathbb{C} e_j$ by denoting $e_i(E_{jj}) = \delta_{ij}$. Let $\rho$ and $\rho_\Theta$ be elements of $a^*_c$ corresponding to the restrictions of $\frac{1}{2} \text{trace}(\text{ad})$ on $n$ and $n_\Theta$, respectively, and put $\rho(\Theta) = \rho - \rho_\Theta$. Then we have
\[
\begin{align*}
\rho &= \frac{1}{2} \sum_{1 \leq i < j \leq n} (e_j - e_i) = \sum_{j=1}^{n} (j - \frac{n+1}{2}) e_j, \\
\rho(\Theta) &= \frac{1}{2} \sum_{k=1}^{L} \sum_{n_{k-1} < i < j \leq n_k} (e_j - e_i) = \sum_{k=1}^{L} \sum_{n_{k-1} < \ell \leq n_k} (\ell - \frac{n_{k-1} + n_k + 1}{2}) e_\ell, \\
\rho_\Theta &= \sum_{k=1}^{L} \sum_{n_{k-1} < \ell \leq n_k} \frac{n_{k-1} + n_k - n}{2} e_\ell = \sum_{k=1}^{L} \frac{n_{k-1} + n_k - n}{2} f_k
\end{align*}
\]
by denoting
\[
f_k = \sum_{n_{k-1} < \ell \leq n_k} e_\ell.
\]

We identify $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and $\mu = (\mu_1, \ldots, \mu_L) \in \mathbb{C}^L$ with the elements $\sum \lambda_j e_j \in a^*_c$ and $\sum \mu_k f_k \in a^*_c$, respectively.

Putting $A(G/P_\Theta; \mu, \varepsilon) = A(G) \cap B(G/P_\Theta; \mu, \varepsilon)$ and
\[
\mu^*_\Theta = -\mu - 2\rho_\Theta = (n - n_0 - n_1 - \mu_1, \ldots, n - n_{L-1} - n_L - \mu_L),
\]

we have a $G$-invariant bilinear form
\[(4.3)\quad \mathcal{B}(G/P_\Theta; \mu, \epsilon) \times \mathcal{A}(G/P_\Theta; \mu_\Theta^*, \epsilon) \ni (f, \phi) \mapsto \langle f, \phi \rangle_\Theta = \int_K f(k)\phi(k)dk\]
with the normalized Haar measure $dk$ on $K$, which follows from the $G$-invariant integral
\[(4.4)\quad \mathcal{B}(G/P_\Theta; -2\rho_\Theta, 0) \ni f \mapsto \int_K f(k)dk.
\]
For a close subgroup $Q$ of $G$ and an irreducible finite dimensional representation $(\tau, V_\tau)$ of $Q$, we have an associated representation space
\[(4.5)\quad \mathcal{B}(G/Q; \tau) = \{ f \in \mathcal{B}(G) \otimes V_\tau; f(xq) = \tau(q^{-1})f(x) \text{ for } q \in Q \}.
\]
Then for an element $T \in \mathcal{B}(G/P_\Theta; \mu_\Theta, \epsilon) \otimes V$ satisfying
\[(4.6)\quad T(qx) = \tau(q)T(x) \quad \text{for } q \in Q,
\]
we have a $G$-homomorphism
\[(4.7)\quad \mathcal{B}(G/P_\Theta; \mu, \epsilon) \ni f \mapsto (Tf)(x) = \int_K f(k)T(x^{-1}k)dk \in \mathcal{B}(G/Q; \tau).
\]
Here we note that $(Tf)(x) = \langle f, \pi_x(T) \rangle_\Theta = \langle \pi_{x^{-1}}(f), T \rangle_\Theta = \int_K f(k)T(k)dk$.

It is natural to choose $Q$ so that $Q \setminus G/P$ has an open orbit because the dimension of the intertwining operators of any irreducible Harish-Chandra module of $G$ to $\mathcal{B}(G/Q; \tau)$ is of finite dimension (cf. [O3]).

Suppose $Q$ is also a parabolic subgroup. Since
\[(4.8)\quad \mathcal{B}(G/P_\Theta; \mu, \epsilon) \subset \mathcal{B}(G/P; \zeta, \epsilon')
\]
if
\[(4.9)\quad \sum_{k=1}^L \mu_k f_k = \sum_{\ell=1}^L \zeta_\ell \epsilon_\ell \quad \text{and} \quad \epsilon'_k = \epsilon_\ell \quad \text{for} \quad n_{\ell-1} < k \leq n_\ell,
\]
the intertwining operators in the case when $P_\Theta = Q = P$ are fundamental. There exist standard intertwining operators (cf. [Kn, Chap. 7]) parametrized by the Weyl group $W$ of $G$ or equivalently by the double cosets $P \setminus G/P$, which is isomorphic to the $n$-th symmetric group $S_n$.

Fix $w \in W \simeq S_n$ and identify $w$ with the representative in $G$ whose $(i, j)$-component equals $\delta_{w(i)j}$. Hence $w(x_{ij})w^{-1} = (x_{w(i)w(j)})$ for $(x_{ij}) \in G$. By denoting $\tilde{N}_w = w^{-1}Nw \cap \tilde{N}$, we have
\[(4.10)\quad \tilde{N}_w = \{ (x_{ij})_{1 \leq i \leq j \leq n} \in G; x_{ij} = \delta_{ij} \text{ if } i \leq j \text{ or } w^{-1}(i) < w^{-1}(j) \}.
\]
Consider the integral
\[(4.11)\quad (T_wf)(x) = \int_{\tilde{N}_w} f(xwn_w)dn_w
\]
with the normalized Haar measure $dn_w$ on $\tilde{N}_w$, which is a constant multiple of the usual measure $\prod dx_{ij}$ under the coordinates in (4.10). If $\text{Re}(\mu_k - \mu_{k+1})$ is sufficiently large, this integral converges for any continuous function $f \in \mathcal{B}(G/P; \mu, \epsilon)$ and its kernel function defines the intertwining operator
\[(4.12)\quad T_w : \mathcal{B}(G/P; \mu, \epsilon) \to \mathcal{B}(G/P; \mu', \epsilon')
\]
with
\[(4.13)\quad \mu' = w(\mu + \rho) - \rho = (\mu_{w^{-1}(1)} + w^{-1}(1) - 1, \ldots, \mu_{w^{-1}(n)} + w^{-1}(n) - n),
\]
and
\[(4.14)\quad \epsilon' = (\epsilon_{w^{-1}(1)}, \ldots, \epsilon_{w^{-1}(n)}).
\]
This intertwining operator $T_w$ is defined for any $\mu \in \mathbb{C}^n$ by the analytic continuation of the kernel function.
5. Poisson transforms

In this section we study the realization of degenerate principal series on the Riemannian symmetric space \( G/K \). First introduce the Poisson kernel

\[
\Phi^\mu_{\Theta}(x) = \Phi_{n_1}(x)^{\xi_1} \cdots \Phi_{n_L}(x)^{\xi_L},
\]

\[
\Phi_m(x) = \sum_{1 \leq \nu_1 < \cdots < \nu_m \leq n} \left| \det \left( x_{\nu_{ij}} \right) \right|_{1 \leq i \leq m}^{2},
\]

\[(5.1)\]

\[2\xi_k = (\mu_k + n_{k-1}) - (\mu_{k+1} + n_{k+1}) \quad \text{for} \quad k = 1, \ldots, L - 1,
\]

\[2\xi_L = \mu_L + n_{L-1}.\]

It is easy to see that

\[
\Phi^\mu_{\Theta}(pxk) = \tau_{\Theta,n^*_m}(p)\Phi^\mu_{\Theta}(x) \quad \text{for} \quad p \in P_\Theta \text{ and } k \in K
\]

and therefore we have a \( G \)-homomorphism

\[
\mathcal{P}_{\Theta}^\mu : \mathcal{B}(G/P_\Theta; \mu) \ni f \mapsto (\mathcal{P}_{\Theta}^\mu f)(x) = \int_K f(xk)dk = \int_K f(k)\Phi^\mu_{\Theta}(k^{-1}x)dk
\]

\[(5.2)\]

as was stated in \( \S 5 \). This is called the Poisson transform. Here for simplicity we put \( \mathcal{B}(G/P_\Theta; \mu) = \mathcal{B}(G/P_\Theta; \mu; \varepsilon) \) if \( \varepsilon = \{0, \ldots, 0\} \).

The map \( \mathcal{P}_{\Theta}^\mu \) was studied in [H2], [H3] and [K-] when \( P_\Theta \) is a minimal parabolic subgroup and it was proved that \( \mathcal{P}_{\Theta}^\mu \) is injective if and only if \( e(\zeta) \neq 0 \) and in this case the image is the totality of the real analytic functions \( u \) on \( G/K \) which satisfy

\[
D_m u = \chi_{\zeta}(D_m)u \quad \text{for} \quad m = 1, \ldots, n.
\]

(5.3)

Here \( e(\zeta) \) corresponds to the denominator of Harish-Chandra's \( c \)-function given by

\[
e(\zeta) = \prod_{1 \leq i < j \leq n} \Gamma\left( \frac{\zeta_j - \zeta_i + j - i + 3}{4} \right)^{-1} \Gamma\left( \frac{\zeta_j - \zeta_i + j - i + 1}{4} \right)^{-1}
\]

(cf. [H2]) and \( e(\zeta) \neq 0 \) if and only if

\[
(\zeta_i + i) - (\zeta_j + j) \notin \{1, 3, 5, 7, \ldots\} \quad \text{for} \quad 1 \leq i < j \leq n.
\]

Hence it follows from (4.8), (4.9) and (3.7) that \( \mathcal{P}_{\Theta}^\mu \) is injective if

\[
(\mu_i + n_i) - (\mu_j + n_{j-1} + 1) \notin \{1, 2, 3, 4, \ldots\} \quad \text{for} \quad 1 \leq i < j \leq L.
\]

(5.4)

Theorem 5.1. i) Any \( u \in \text{Im} \mathcal{P}_{\Theta}^\mu \) is real analytic and satisfies (3.4) and (3.5).

ii) If the condition

\[
(\mu_i + n_i) - (\mu_j + n_{j-1} + 1) \notin \{0, 1, 2, 3, 4, \ldots\} \quad \text{for} \quad 1 \leq i < j \leq L
\]

(5.5)

holds, \( \mathcal{P}_{\Theta}^\mu \) is a topological \( G \)-isomorphism onto the subspace of \( C^\infty(G/K) \) which is the totality of solutions of the system of equations (3.5).
Example 5.2. Suppose $G = SL(2, \mathbb{R})$ and $P = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \in G \right\}$. Then the natural action on $\mathbb{P}_C^1 = (\mathbb{C}^2 - \{0\})/\mathbb{C} \ni \left[ \begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix} \right]$ is given by the left multiplication. The fixed point group with respect to $i = z_1/z_2$ equals $SO(2)$ and the symmetric space $G/SO(2)$ is identified with the upper half plane $H_+ = \left\{ \left[ \begin{smallmatrix} x + iy \\ 1 \end{smallmatrix} \right] \in \mathbb{P}_C^1; y > 0 \right\}$. Since
\[
\left( \sqrt{y} \frac{x}{\sqrt{y}} \begin{array}{l}
\frac{1}{\sqrt{y}}
\end{array} \right) \left[ \begin{array}{l}
i \\
1
\end{array} \right] = \left[ \sqrt{y} \frac{x + iy}{\sqrt{y}} \right] = \left[ \begin{smallmatrix} x + yi \\ 1 \end{smallmatrix} \right] \in H_+ \subset \mathbb{P}_C^1,
\]
we have $\Psi = \left( \sqrt{y} \frac{x}{\sqrt{y}} \right)^2 + (\frac{x}{\sqrt{y}})^2 = \frac{x^2 + y^2}{y^2}$. Hence if $\mu_1 = \mu_2 = 0$, we get the usual Poisson kernel $\Phi_{(1,2)}(u,v) = \frac{y}{x^2 + y^2}$ for $H_+$ and Theorem 5.1 says the isomorphism of the space of harmonic functions on $H_+$ onto that of hyperfunctions on the circle $G/P \simeq \mathbb{R} \cup \{\infty\} \subset \mathbb{P}_C^1$.

Example 5.3. Suppose $\Theta = \{n\}$. Then $P_\Theta = G$ and $\mathcal{B}(G/P_\Theta; \mu) = \mathbb{C}|\det x|^\mu$ for generic $\mu \in \mathbb{C}$ and we have the equality corresponding to (3.4):
\[
D_{\{1,\ldots,n\}}(\lambda)|\det x|^\mu = (\mu + \lambda)(\mu + \lambda + 1) \cdots (\mu + \lambda + n - 1)|\det x|^\mu.
\]
If we put $\lambda = 0$ and $\mu = s + 1$, this corresponds to Cayley’s formula
\[
\det \left( \frac{\partial}{\partial x_{ij}} \right)_{1 \leq i \leq n \leq n} (\det x)^{s+1} = (s+1)(s+2) \cdots (s+n)(\det x)^s
\]
in view of the Capelli identity. Note that this equality defines the $b$-function of $\det x$ and hence $\mu$ is the meromorphic parameter of $|\det x|^\mu$ and its poles are contained in $\{-1, -2, -3, \ldots\}$ (cf. [B], [Sm]).

Proof of Theorem 5.1. Since the Poisson kernel $\Phi_{\Theta}(u,v)$ is real analytic, any $u \in \text{Im} \mathcal{P}_{\Theta}^\mu$ is also real analytic. For $c \in \mathbb{C}$, $\text{Ad}(g)J(m,c) = J(m,c)$ (cf. Corollary 2.5) and the condition $Du = 0$ for any $D \in J(m,c)$ is equivalent to $\pi_m(D)u = 0$ for any $D \in J(m,c)$ (cf. Proposition 2.6 iii) and note that $\pi$ is the left regular representation). Hence i) follows from Theorem 3.1 and the $G$-equivariance of $\mathcal{P}_{\Theta}^\mu$.

Put $\mathcal{A}(G/K; \zeta) = \{u \in \mathcal{A}(G); u(gk) = u(g) \text{ and } D_mu = \zeta_{\{1,\ldots,n\},\zeta}(\gamma(D_m))u \text{ for } k \in K \text{ and } m = 1, \ldots, n\}$. Then $\mathcal{P}_{\{1,\ldots,n\}}^\zeta$ is a topological $G$-isomorphism of $\mathcal{B}(G/P; \zeta)$ onto $\mathcal{A}(G/K; \zeta)$. This is proved in [K] by constructing a map $\beta$ of taking the boundary values, which gives the inverse of $\mathcal{P}_{\{1,\ldots,n\}}^\zeta$. Here we note that $\chi_{\Theta,\mu} = \zeta_{\{1,\ldots,n\},\zeta}$.

Suppose $u \in \mathcal{A}(G/K; \zeta)$ satisfies (3.5). Since $\beta$ is $G$-equivariant, Corollary 2.5 i) assures $D_{IJ}(-\mu_k - n_{k-1})\beta(u) = 0$ for $\#I = \#J = n - n_k + n_{k-1} + 1$ and $k = 1, \ldots, L$. Fix $k$ with $n_k - n_{k-1} > 1$, choose $i_o$ and $j_o$ with $n_{k-1} < i_o < j_o \leq n_k$ and put $I = \{n, n - 1, \ldots, n_k + 1, i_o, n_{k-1}, n_{k-1} - 1, \ldots, 1\}$ and $J = \{n, n - 1, \ldots, n_k + 1, j_o, n_{k-1}, n_{k-1} - 1, \ldots, 1\}$. Then it is clear from the definition of $D_{IJ}$ that
\[
D_{IJ}(-\mu_k - n_{k-1}) \equiv \prod_{\nu=n_k+1}^{n_{k-1}} (E_{\nu\nu} - \mu_k - n_k + \nu) \cdot E_{i_o,j_o} \cdot \prod_{\nu=1}^{n_{k-1}} (E_{\nu\nu} - \mu_k - n_{k-1} + \nu - 1) \mod U(\mathfrak{g})n
\]
and for $\phi \in \mathcal{B}(G/P; \zeta)$ we have
\[
D_{IJ}(-\mu_k - n_{k-1})\phi = \prod_{\nu=n_k+1}^{n_{k-1}} (\zeta_{\nu} - \mu_k - n_k + \nu) \cdot E_{i_o,j_o} \cdot \prod_{\nu=1}^{n_{k-1}} (\zeta_{\nu} - \mu_k - n_{k-1} + \nu - 1)\phi.
\]
If $\ell < k$ and $n_{\ell-1} < \nu \leq n_\ell$, then $\zeta_\nu - \mu_k - n_{k-1} + \nu - 1 = (\mu_\ell + \nu) - (\mu_k + n_{k-1} + 1) \neq 0$, which follows from (5.5) because of (3.7). Similarly if $\ell \geq k$ and $n_\ell < \nu \leq n_{\ell+1}$, then $\zeta_\nu - \mu_k - n_k + \nu = -((\mu_k + n_k) - (\mu_{k+1} + \nu)) \neq 0$. Hence $E_{1,1,\nu}(u) = 0$. Thus we have $X^\nu(u) = 0$ for any $X \in \text{Lie}(P_\Theta) \cap \mathfrak{n}$, which means $\beta^\nu(u) \in \mathcal{B}(G/P_\Theta; \mu)$ and we have the theorem from Proposition 3.2 i). □

Remark 5.4. i) Theorem 5.1 in the case $\Theta = \{1, n\}$ is given in [O1], where it is conjectured that in general there exists a system of operators $D$ satisfying a suitable condition for $\gamma(D)$ and characterizing the image of the Poisson transform. When $G = GL(n, \mathbb{R})$, the conjecture corresponds to Theorem 5.1 and Corollary 2.11.

ii) If $\text{Re}\, \mu_j < \text{Re}\, \mu_{j+1} + 1$ for $j = 1, \ldots, L - 1$, then (5.5) is valid.

iii) In [J1] and [J2] some differential equations characterizing $\text{Im}P_\Theta^0$ are given, which are less explicit than ours.

iv) Let $w_\Theta \in \mathfrak{S}_n$ with $\text{Ad}(w_\Theta)\mathfrak{n} \cap \mathfrak{n} = \text{Lie}(P_\Theta) \cap \mathfrak{n}$. As is remarked in [O1], $\text{Im}P_\Theta^\mu = \text{Im}P_\Theta^{\mu'}$ with $\zeta_j' = \zeta_{w_\Theta(j)} + w_\Theta(j) - j$ and hence for $\eta \in \mathbb{C}^n$, it is interesting to study the system of differential equations characterizing $\text{Im}P_\Theta^{\eta}$.

6. Hypergeometric functions

In general, suppose $G$ is a linear reductive Lie group and moreover suppose that we are given three closed subgroups $P_\Theta$, $Q_1$ and $Q_2$ of $G$ and their finite dimensional representations $(\tau_\Theta, V_\Theta)$, $(\tau_1, V_1)$ and $(\tau_2, V_2)$, respectively, which satisfy the conditions

\begin{align}
\text{(6.1)} & \\
& P_\Theta \text{ is a parabolic subgroup of } G, \\
\text{(6.2)} & \text{ the double cosets } Q_j \backslash G/P_\Theta \text{ have open cosets for } j = 1 \text{ and } 2.
\end{align}

Define the degenerate principal series as in §3:

$$B(G/P_\Theta; \tau_\Theta) = \{ \phi(g) \in B(G) \otimes V_\Theta; \phi(gp) = \tau_\Theta(p^{-1})\phi(g) \text{ for } p \in P_\Theta \}.$$ 

Let $P_\Theta = L_\Theta N_\Theta$ be a Levi decomposition of $P_\Theta$. Fix a Cartan involution $\theta$ of $G$ with $\theta(L_\Theta) = L_\Theta$. Then $K = \{ g \in G; \theta(g) = g \}$ is a maximal compact subgroup of $G$. Let $V_\Theta^*$ be the dual space of $V_\Theta$ and let $\tau^*_\Theta$ be the representation of $P_\Theta$ on $V_\Theta^*$ such that

$$B(G/P_\Theta; \tau_\Theta) \times A(G/P_\Theta; \tau^*_\Theta) \ni (\phi, \psi) \mapsto \langle \phi, \psi \rangle_{\Theta} = \int_k \langle \phi(k), \psi(k) \rangle dk \in \mathbb{C}$$

defines a $G$-invariant bilinear form. Note that $(\tau^*_\Theta, V_\Theta^*)$ is the tensor product of the contragredient representation of $\tau_\Theta$ and the character det(Ad) of $P_\Theta$ on $\mathfrak{n}_\Theta = \text{Lie}(N_\Theta)$, and that if $\langle \phi(k), \psi(k) \rangle$ is an integrable function on $K$,

$$\langle \phi, \psi \rangle_{\Theta} = \int_{N_\Theta} \langle \phi(n), \psi(n) \rangle dn$$

with a suitably normalized Haar measure $dn$ on $N_\Theta = \theta(N_\Theta)$.

Definition 6.1. For given functions $\phi \in B(G/P_\Theta; \tau_\Theta) \otimes V_1$ and $\psi \in B(G/P_\Theta; \tau^*_\Theta) \otimes V_2$ satisfying

\begin{align}
\phi(q_1 x) = \tau_1(q_1) \phi(x) & \quad \text{for } q_1 \in Q_1, \\
\psi(q_2 x) = \tau_2(q_2) \psi(x) & \quad \text{for } q_2 \in Q_2,
\end{align}

we call a $V_1 \otimes V_2$-valued function

$$\Phi_{\phi, \psi}(x) = \int_k \langle \phi(xk), \psi(k) \rangle dk$$
on $G$ a hypergeometric function.

By the $G$-invariance of the bilinear form we have
\begin{equation}
\Phi_{\phi,\psi}(g_1 x g_2) = \tau_1(q_1) \tau_2(q_2^{-1}) f(x) \quad \text{for } (g_1, q_2) \in Q_1 \times Q_2
\end{equation}
and for elements $D$ of the universal enveloping algebra $U(g)$ of $\mathbb{C} \otimes \text{Lie}(G)$
\begin{equation}
\pi_D(\Phi_{\phi,\psi}) = 0 \quad \text{if } \pi_D(\phi) = 0.
\end{equation}
Note that the following map defines a $G$-homomorphism (cf. (4.7)):
\begin{equation}
\mathcal{B}(G/P_\Theta; \tau_\Theta) \ni f \mapsto (\mathcal{T}_\psi f)(x) = \int_K (f(xk), \psi(k)) dk \in \mathcal{B}(G/Q_2; \tau_2).
\end{equation}

**Example 6.2.** Suppose $P_\Theta$ is a minimal parabolic subgroup. If $Q_1$ and $Q_2$ are maximal compact subgroups, the integral representations of the corresponding hypergeometric functions are Eisenstein integrals. In particular, if $\tau_1$ and $\tau_2$ are trivial representations, we have the integral representations of zonal spherical functions.

If the parameter of the representation becomes degenerate, the zonal spherical function satisfies more differential equations (cf. [Kr] for an example). When $G = GL(n, \mathbb{R})$ and the parameter corresponds to the degenerate principal series for $P_{[1,n]}$, it satisfies equations
\begin{equation}
\text{with } \#I = \#J = 2 \quad \text{(cf. Theorem } 5.1)\text{ and the radial part of this zonal spherical function is given by Lauricella’s } F_D \text{ (cf. [E]).}
\end{equation}
Now we will consider the case where $G = GL(n, \mathbb{R})$ and furthermore $P_\Theta$ and $Q_2$ are parabolic subgroups. In particular, we examine the case where they are maximal, namely, $\Theta = \{\ell, n\}$ and $Q_2 = P_{[k,n]}$. Suppose $\ell < k < n$ and $n = m \ell$ with a positive integer $m$ satisfying $m > 1$. To assure the existence of the nontrivial intertwining operator (6.9), we assume $\tau_\Theta = \tau_{\mu,0}$ with $\mu = (k,0) \in \mathbb{C}^2$. In this case, the integral transformation (4.11) with $w = \left( \begin{array}{cccc} 1 & 2 & \cdots & \ell \\ k-\ell+1 & k-\ell+2 & \cdots & k+1 \\ k & k+1 & \cdots & n \end{array} \right)$ converges for any continuous function $f \in \mathcal{B}(G/P_\Theta; \tau_\Theta)$. Note that the corresponding kernel function is a measure whose support is the compact double coset in $Q_2 \backslash G/P_\Theta$ and $N_w = \{(x_{ij}) \in G; x_{ij} = \delta_{ij} \text{ if } i \geq \ell \text{ or } j \leq \ell \text{ or } j > k\}$.

**Lemma 6.3.** Under the notation above we have a $G$-homomorphism
\begin{equation}
\mathcal{T}_w : \mathcal{B}(G/P_{[\ell,n]}; (k,0)) \to \mathcal{B}(G/P_{[k,n]}; (\ell,0)).
\end{equation}

**Proof.** Let $f \in \mathcal{B}(G/P_{[\ell,n]}; (k,0))$. From (4.9), (4.12) and (4.13) we have $\mathcal{T}_w f \in \mathcal{B}(G/P_{[k,n]}; \mu)$ with $\mu = (\ell(e_1 + \cdots + e_k))$. If $k < i < j$, then $w^{-1} E_{ij} w$ commutes with any element of $N_w$ and we have $E_{ij} f = 0$. By the natural identification $GL(k, \mathbb{R}) \simeq GL(k, \mathbb{R}) \otimes I_{n-k} \subset GL(n, \mathbb{R})$, we have an imbedding $N_w \subset GL(k, \mathbb{R})$. Here $I_{n-k}$ is the identity matrix of size $n-k$.

For a continuous function $\phi$ on $GL(k, \mathbb{R})$ satisfying
\[ \phi(np) = | \det g_1 |^{\beta_1} | \det g_2 |^{\beta_2} \phi(n) \quad \text{for } p = \left( \begin{array}{cc} g_1 & 0 \\ 0 & g_2 \end{array} \right) \in SL(k, \mathbb{R}) \]
with $g_1 \in GL(\ell, \mathbb{R})$ and $g_2 \in GL(k-\ell, \mathbb{R})$, the integration $\int_{N_w} \phi(n) dn$ or equivalently, $\int_{SO(k)} \phi(k) dk$ is left $SL(k, \mathbb{R})$-invariant if $\beta_2 - \beta_1 = -k$, which corresponds to $-2 \rho_\Theta$ (cf. (4.1) and (4.4)). Hence by putting $\phi(x) = f(gx)$ with $g \in G$, we clearly have the right $(SL(k, \mathbb{R}) \otimes I_{n-k})$-invariance of $\mathcal{T}_w f$.

The invariances we have proved imply $\mathcal{T}_w f \in \mathcal{B}(G/P_{[k,n]}; (\ell,0))$. □
Through the anti-automorphism $G \ni g \mapsto g^{-1} \in G$, $\mathcal{B}(G/P_{\ell,n}; (k,0))$ is canonically identified with the space of hyperfunctions $\phi$ on the $\ell n$-dimensional manifold

$$M(\ell, n) = \left\{ (t_{ij}) = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \cdots & \cdots & \cdots \\ t_{\ell 1} & \cdots & t_{\ell n} \end{pmatrix}; t_{ij} \in \mathbb{R} \text{ and rank}(t_{ij}) = \ell \right\}$$

which satisfy

$$\phi(g(t_{ij})) = |\det g|^{-k} \phi((t_{ij})) \quad \text{for} \quad g \in GL(\ell, \mathbb{R}).$$

Similarly $\mathcal{B}(G/P_{k,n}; (\ell,0))$ is canonically identified with the space of hyperfunctions $\Phi$ on the $kn$-dimensional manifold

$$M(k, n) = \left\{ (y_{ij}) = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \cdots & \cdots & \cdots \\ y_{k1} & \cdots & y_{kn} \end{pmatrix}; y_{ij} \in \mathbb{R} \text{ and rank}(y_{ij}) = k \right\}$$

which satisfy

$$\Phi(g(y_{ij})) = |\det g|^{-k} \Phi((y_{ij})) \quad \text{for} \quad g \in GL(k, \mathbb{R}).$$

Denoting $GL(\ell, \mathbb{R})_+ = \{ g \in GL(\ell, \mathbb{R}); \det g > 0 \}$, we choose $Q_1 = GL(\ell, \mathbb{R})_+ \times \cdots \times GL(\ell, \mathbb{R})_+ \subset GL(n, \mathbb{R})$ and define $\tau_1$ by

$$\tau_1(g) = (\det g_1)^{-\alpha_1} \cdots (\det g_m)^{-\alpha_m} \quad \text{for} \quad g = (g_1, \ldots, g_m) \in Q_1,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m$ with the condition

$$\alpha_1 + \cdots + \alpha_m + k = 0.$$

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \{1, -1\}^m$. Using a function

$$|s|^\alpha_\epsilon = \left\{ \begin{array}{ll} |s|^\alpha & \text{if } \epsilon_\nu s > 0, \\ 0 & \text{if } \epsilon_\nu s \leq 0 \end{array} \right.$$ 

on $\mathbb{R}$, we give a function

$$\phi(t) = \frac{1}{2} \left( \prod_{p=1}^{m} \left| \det (t_{ij}) \right|_{p-1 < \ell \leq \ell}^{\alpha_\epsilon_p} + \prod_{p=1}^{m} \left| \det (t_{ij}) \right|_{(p-1)\ell < j \leq \ell \leq \ell}^{\alpha_\epsilon_p} \right)$$

in Definition 6.1, which belongs to $\mathcal{B}(G/P_{\ell,n}; (k,0))$.

Put $x_{w(i)w(j)} = y_{ij}$ for $i = 1, \ldots, \ell$ and $j = 1, \ldots, n$. Then in $GL(n, \mathbb{R})$

$$\left( \begin{array}{cccc} t_{11} & t_{1k} & \cdots & t_{1k} \\ \cdots & \cdots & \cdots & \cdots \\ t_{\ell 1} & t_{\ell k} & \cdots & t_{\ell k} \\ 1 & 1 & \cdots & 1 \end{array} \right) w^{-1} \left( \begin{array}{cccc} y_{11} & y_{1n} & \cdots & y_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{k1} & y_{kn} & \cdots & y_{kn} \\ \cdots & \cdots & \cdots & \cdots \end{array} \right) = \left( \begin{array}{cccc} \sum_{\nu=1}^{k} t_{1\nu} x_{\nu 1} & \cdots & \sum_{\nu=1}^{k} t_{1\nu} x_{\nu n} \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots \\ \sum_{\nu=1}^{k} t_{\ell \nu} x_{\nu 1} & \cdots & \sum_{\nu=1}^{k} t_{\ell \nu} x_{\nu n} \end{array} \right) w^{-1}$$

and we have our hypergeometric function

$$\Phi(\alpha, \epsilon; x) = \int_{\mathbb{R}^{(k-1)}} \prod_{p=1}^{m} \left| \det \left( \sum_{\nu=1}^{k} t_{\nu x_{\nu j}} \right) \right|_{1 \leq \nu \leq \ell}^{\alpha_\epsilon_p} \prod_{1 \leq \nu \leq \ell, 1 \leq \nu \leq \ell} \prod_{1 \leq \nu \leq \ell} dt_{ij}$$

with the convention

$$t_{ij} = \delta_{ij} \quad \text{if } 1 \leq j \leq \ell.$$
Theorem 6.4. The hypergeometric function $\Phi(\alpha, \epsilon; x)$ on $M(k, n) = \{(x_{ij}); x_{ij} \in \mathbb{R}, 1 \leq i \leq k, 1 \leq j \leq n, \text{rank}(x_{ij}) = k\}$ satisfies the following equations.

i) A right $GL(\ell, \mathbb{R}^+) \times \cdots \times GL(\ell, \mathbb{R}^+)$-invariance:

$$\Phi(xg) = \prod_{p=1}^{m} (\det g_p)^{\alpha_p} \Phi(x) \quad \text{for g = } g_1 \otimes \cdots \otimes g_m \text{ with } g_p \in GL(\ell, \mathbb{R}^+).$$

ii) A left $GL(k, \mathbb{R})$-invariance:

$$\Phi(gx) = |\det g|^{-\epsilon} \Phi(x) \quad \text{for } g \in GL(k, \mathbb{R}).$$

iii) Generalized Capelli operators:

$$\det \left( \frac{\partial}{\partial x_{i\mu,j\nu}} \right)_{1 \leq \mu \leq \ell + 1, 1 \leq \nu \leq \ell + 1} \Phi(x) = 0$$

for $1 \leq i_1 < \cdots < i_{\ell+1} \leq k$ and $1 \leq j_1 < \cdots < j_{\ell+1} \leq n$.

Proof. The $GL(k, \mathbb{R})$-dependence is clear from our argument. Note that $\phi$ satisfies the left $GL(\ell, \mathbb{R}^+) \times \cdots \times GL(\ell, \mathbb{R}^+)$-invariance and the equations $D_{I,J}(-\ell)\phi = 0$ for $#I = #J = \ell + 1$ (cf. Theorem 3.1). The other equations follow from the the $G$-equivariance of the intertwining operator $T_w$ combining with the coordinate transformation $G \ni g \mapsto g^{-1}$, Proposition 2.6 iii) and Proposition 3.2 ii). \(\square\)

The two invariants in Theorem 6.4 are infinitesimally as follows:

$$\sum_{i=1}^{k} x_{i\mu,\ell+p} \frac{\partial \Phi}{\partial x_{i\mu,\ell+p}} = \alpha_p \delta_{\mu\ell} \Phi \quad \text{for } 1 \leq \mu \leq \ell, 1 \leq \nu \leq \ell, 0 \leq p < m,$$

$$\sum_{j=1}^{n} x_{\nu j,\ell} \frac{\partial \Phi}{\partial x_{\nu j,\ell}} = -\ell \delta_{ij} \Phi \quad \text{for } 1 \leq i \leq k, 1 \leq j \leq k.$$

Note that $GL(n, \mathbb{R})/P_{\ell,n} \simeq O(n)/O(\ell) \times O(n-\ell)$. Hence the definition of $\Phi(\alpha, \epsilon; x)$ is an integration of a function on $O(n)/O(\ell) \times O(n-\ell)$ over its submanifold $O(k)/O(\ell) \times O(k-\ell)$.

Remark 6.5. Suppose $\ell = 1$. The integration can be rewritten as

$$\Phi(\alpha, \epsilon; x) = C \int \cdots \int_{t_1^2 + \cdots + t_k^2 = 1} \prod_{p=1}^{m} \left| \sum_{p=1}^{k} t_p x_{pj} \right|^{\alpha_p} \tilde{\omega}$$

with a suitable constant $C$ and

$$\tilde{\omega} = \sum_{p=1}^{k} (-1)^{p+1} t_p dt_1 \wedge dt_2 \wedge \cdots \wedge dt_{p-1} \wedge dt_{p+1} \wedge \cdots \wedge dt_k.$$

This integral representation coincides with the one given in [G] and the corresponding equations in Theorem 6.4 with $\ell = 1$ are same as in [GG]. This hypergeometric function is also studied by [A].

Remark 6.6. Let $(\tau_p, V_p)$ be representation of $GL(\ell, \mathbb{R})$ for $p = 1, \ldots, m$ and put $\tau = (\tau_1, \ldots, \tau_m)$ and $V = V_1 \otimes \cdots \otimes V_m$. Choose $v \in V$ satisfying

$$\tau_1(g) \otimes \cdots \otimes \tau_m(g)v = |\det g|^{k} v \quad \text{for } g \in GL(\ell, \mathbb{R}).$$
and define endomorphisms of $V_p$ for $\ell \times \ell$-matrices $(x_{ij})$:

$$\pi_p(x)|_{\epsilon_p} = \begin{cases} \pi_p(x)^{-1} & \text{if } \epsilon_p \det x > 0, \\
0 & \text{if } \epsilon_p \det x \leq 0. \end{cases}$$

We have $V$-valued hypergeometric functions

$$Φ(τ, ϵ, v; x) = \int_{R(ℓ−ℓ)} |π_1\left(\sum_{ν=1}^k t_{iν}x_{νj}\right)_{1≤i≤ℓ, 0<j≤ℓ}| \otimes \cdots \otimes |π_m\left(\sum_{ν=1}^k t_{iν}x_{νj}\right)_{(m−1)ℓ<j≤mℓ}| v \prod_{1≤i≤ℓ, 1≤j≤k} \int dt_{ij}. $$

Then $Φ(τ, ϵ, v; x)$ satisfy

$$Φ(σg) = τ(g)^{-1}Φ(σ) \text{ for } g ∈ GL(ℓ, R) \times \cdots \times GL(ℓ, R)-$$

and the equations given in Theorem 6.4 ii) and iii).

For the analysis of hypergeometric functions in the case where $G = P(ℓ,n)$ and $Q_2 = P(k,n)$, the following theorem is essential. Its proof will be given in §7.

**Theorem 6.7.** The intertwining operator (6.10) in Lemma 6.3 is a topological $G$-isomorphism onto the solution space $V_{k,n}^\ell$ of the system (6.21) under our realization using (6.13) and (6.14) if

$$0 < ℓ < k < n \text{ and } ℓ + k < n.$$  

**Corollary 6.8.** Let $Q$ be a closed subgroup of $GL(n, R)$ and let $(τ, V)$ be a finite dimensional representation of $Q$. If (6.30) holds, the integral transformation $R : φ ↦ ∫_{SO(k)} φ(kx)dk$ is a bijection between $S(ℓ, n; τ)$ and $S(k, n; ℓ, τ)$. Here $S(ℓ, n; τ)$ is the space of $V$-valued hyperfunctions $φ$ on $M(ℓ, n)$ satisfying (6.12) and

$$φ(tg) = τ(g)^{-1}φ(t) \text{ for } g ∈ Q.$$ 

Moreover $S(k, n; ℓ, τ)$ is the space of $V$-valued hyperfunctions $Φ(x)$ on $M(k, n)$ satisfying the equations given in Theorem 6.4 ii) and iii) and

$$Φ(σg) = τ(g)^{-1}Φ(σ) \text{ for } g ∈ Q.$$ 

In this corollary ‘hyperfunctions’ can be replaced by ‘Schwartz’s distributions’ or ‘$C^∞$-functions’ or ‘real analytic functions’, which is clear by our way of the proof.

Lastly we give other examples of $Q_1$ for the same $P_\emptyset$ and $Q_2$.

**Example 6.9.** For $A = (A_{ij}) ∈ gl(n, R)$ with $n = m\ell$ and for $μ = 1, \ldots, m$ and $ν = 1, \ldots, m$, put

$$u_1 = \{ A \in gl(n, R) ; M_{μν}(A) = 0 \text{ if } μ > ν, \quad
M_{μ, ν+1}(A) = M_{μ+1, ν+1}(A) \text{ for } i = 1, \ldots, m−2 \text{ and } ν = 1, \ldots, m−i−1, \quad
M_{μν}(A) = (A_{(μ−1)ℓ+i, (ν−1)ℓ+j})_{1≤i≤ℓ, 1≤j≤ℓ} \}$$

If $Q_1$ equals the closed subgroup of $GL(n, R)$ with the Lie algebra $u_1$, condition (6.2) is valid and we can define hypergeometric functions with respect to the character of $Q_1$ which has $m − 1$ continuous parameters. When $ℓ = 1$, the hypergeometric functions correspond to those discussed in [GRS] and [KHT].
7. Radon transforms

Retain the notation in the previous section and put $X_j = O(n)/O(j) \times O(n-j)$ with $0 < j < n$. Since $G = K P_\Theta = K Q_2$ with $K = O(n)$, we may restrict the intertwining operator (6.10) on $K$ and the restriction

\[ R_k^\ell : \mathcal{B}(X_\ell) \ni \phi \mapsto \langle R_k^\ell \phi \rangle (g) = \int_{O(k)} \phi(gk) dk \in \mathcal{B}(X_k) \]

is a Radon transform for a real Grassmannian (cf. [Kn, Chap. IV §]). Here $O(k) \simeq O(k) \otimes I_{n-k} \subset O(n)$ and we assume (6.30).

The Radon transforms, in particular, their inverses and the range characterization, were originated by [F], [R] and [Jf] in special cases and later studied by [H1], [GGR], [Gr1], [Gr2], [Go1], [Go2], [Ku], [I] etc. in more general cases. The characterization of $\text{Im } R_k^\ell$ stated in Theorem 6.7 is not clear from these references (cf. [Gr1]) and hence in this section we will give the proof of Theorem 6.7 for the sake of completeness.

Fix an irreducible representation $\pi_\Lambda$ of $O(n)$. Note that the dimension of $O(n)$-homomorphism of $\pi_\Lambda$ to $\mathcal{B}(X_\ell)$ is at most one because $X_\ell$ are connected symmetric spaces. Put $X_j = SO(n)/SO(j) \times SO(n-j)$ and $g_0 = \text{diag}(-1,1,\ldots,1,-1) \in SO(n)$. Then $X_j$ is a universal covering of $X_j$ and the fundamental group of $X_j$ equals $\mathbb{Z}/2\mathbb{Z}$. The function on $X_j$ is identified with the function on $X_j$ which is invariant under the involution $SO(n) \ni x \mapsto g_0 x g_0$.

Suppose $\pi_\Lambda$ is isomorphic to $\mathcal{B}(X_\ell)$. Thanks to the assumption (6.30), Cartan-Helgason’s theorem (cf. [Wa, Theorem 3.3.11]) says that $\pi_\Lambda$ has an $O(k) \times O(n-k)$-fixed vector $\phi_\Lambda$, which can be normalized by $\phi_\Lambda(e) = 1$ because of Lemma 7.1. Then $R_k^\ell \phi_\Lambda(e) = 1$ and therefore $R_k^\ell \pi_\Lambda \neq \{0\}$. Since $\text{Ker } R_k^\ell$ is $O(n)$-invariant, the map $R_k^\ell$ is injective.

Let $n'$ be a maximal positive integer with $2n' \leq n$. Put $F_{\mu\nu} = E_{\mu\nu} - E_{\nu\mu}$ and $\bar{v} = \nu + (n-n')$. Let $t$ be a maximal torus of $\mathfrak{o}(n,\mathbb{C})$ spanned by $H_\nu = F_{\nu\bar{v}}$ for $\nu = 1,\ldots,n'$ and define $f_\mu \in \mathfrak{t}_c$ by $f_\mu(H_\nu) = -i\delta_{\mu\nu}$. Then $\{f_1,f_2,\ldots,f_{n-1}-f_n,g_0\}$ is a fundamental system of the roots for the pair $(\mathfrak{o}(n),t)$, where $g_n = f_{n-1} + f_n$ if $2n' = n$ and $g_n = f_n$ otherwise. Moreover for $1 \leq \mu < \nu \leq n'$, $X_{\mu\nu}^+ = Y_{\mu\nu}^+ - (\pm Y_{\nu\mu}^-)$ is root vectors for the positive roots $f_\mu \pm f_\nu$, respectively, by putting $Y_{\mu\nu}^+ = F_{\mu\nu} - iF_{\nu\mu}$ and $Y_{\mu\nu}^- = F_{\nu\mu} - iF_{\mu\nu}$ (cf. [Kn, Chap. IV §1 Example 2]).

Note that $V_{k,n}^\ell$ is $O(n)$-invariant. Suppose $k \leq \frac{n}{2}$ and $\pi_\Lambda$ is contained in $V_{k,n}^\ell$. Let $\lambda_1 f_1 + \cdots + \lambda_{n-1} f_{n-1} + f_n$ be the corresponding highest weight and let $v_{\lambda_\ell}$ be the highest weight vector in $V_{k,n}^\ell$. Cartan-Helgason’s theorem and the covering map $\tilde{X}_j \to X_j$ say that $\lambda_1 \geq \cdots \geq \lambda_k \geq \lambda_{k+1} = \cdots = \lambda_{k' + 1} = 0$ and that $\lambda_1$ are even integers (cf. [Gr1], [St]). Suppose there exists $\nu$ satisfying $\lambda_\nu \neq 0$ and $\ell < \nu \leq k$. Then (7.2) proves $D_{1,\ldots,\ell,\nu} d_{1,\ldots,\ell,\nu} v_{\lambda_\ell}(e) = (-\lambda_1 e - \ell) \cdots (-\lambda_\ell e - \ell)(-\lambda_{\nu} e)v_{\lambda_\nu}(e)$ and therefore $v_{\lambda_\nu}(e) = 0$. This means $v_{\lambda_\nu} = 0$ because $v_{\lambda_\nu}$ is real analytic and $\mathbb{C} \otimes g = \mathbb{C} \otimes \text{Lie}(P_{k,n}) + \mathbb{C} \otimes t + \sum_{1 \leq \mu < \nu < n'} C_{X_{\mu\nu}^+,X_{\mu\nu}^-}$. Thus we have $\lambda_\nu = 0$ for $\nu > \ell$. Using again Cartan-Helgason’s theorem, we can conclude that $\pi_\Lambda$ has an $O(\ell) \times O(n-\ell)$-fixed vector and that $\text{Im } R_k^\ell$ is dense in $V_{k,n}^\ell$. By the isomorphism $X_{k} \simeq X_{n-k}$, the same conclusion holds in the case where $k \geq \frac{n}{2}$. Then through the imbedding (4.8) as a closed subspace of the Fréchet-Schwartz space $\mathcal{B}(K)$ (cf. [Kn]), Theorem 6.7 follows from Lemma 7.3 and the open mapping theorem.

**Lemma 7.1.** The intertwining function $\phi_\Lambda$ (cf. [Ho]) satisfies $\phi_\Lambda(e) \neq 0$.

**Proof.** We will prove the lemma in the same way as in the proof of [OS, Proposition 4.2]. So suppose $\phi_\Lambda(e) = 0$, put $\mathfrak{t}_\ell = \mathfrak{o}(j) \oplus \mathfrak{o}(n-j) \subset \mathfrak{o}(n)$ and let $\mathfrak{q}_\ell$ be the orthogonal compliment of $\mathfrak{t}_\ell$ in $\mathfrak{o}(n)$ with respect to the Killing form. Put $g' = \mathfrak{t}_\ell \cap \mathfrak{q}_\ell \cap \mathfrak{k} \simeq \mathfrak{o}(n-k+\ell) \oplus \mathfrak{o}(k-\ell)$. Fix a maximal abelian subspace $\mathfrak{u}_\ell$ of $\mathfrak{q}_\ell \cap \mathfrak{k}$. Note that $\dim \mathfrak{u}_\ell = \ell$.

Let $D \in U(\mathfrak{o}(n))$ and define $D' \in U(g')$ with $D - D' \in (\mathfrak{t}_k \cap \mathfrak{q}_\ell) U(g) + U(g)(\mathfrak{t}_\ell \cap \mathfrak{q}_k)$ and put $\tilde{D} = \int_{O(\ell) \times O(k-\ell) \times O(n-k)} A_D(k) D' dk$. Then $(D \phi_\Lambda)(e) = (D' \phi_\Lambda)(e) = (\tilde{D} \phi_\Lambda)(e)$. 20
Note that \( \tilde{D} \) defines an invariant differential operator on the symmetric space \( X_\alpha = O(n - k + \ell) \times O(k - \ell)/O(\ell) \times O(n - k) \times O(k - \ell) \). Moreover if \( D_\alpha \in U(\mathfrak{o}(n)) \) is \( \mathfrak{t}_\ell \)-invariant, then \( D'_\alpha \) is \( \mathfrak{t}_\ell \cap \mathfrak{t}_k \)-invariant. Note that the restricted root system for \( X_\ell \) and that of \( X_\alpha \) are of type \( B_\ell \). Hence we can choose a \( \mathfrak{t}_\ell \)-invariant element \( \tilde{D} \in U(\mathfrak{o}(n)) \) such that \( \tilde{D}' \) and \( \tilde{D} \) define the same element of \( U(\mathfrak{t}_\ell) \) under the Harish-Chandra homomorphism associated to the symmetric space \( X_\alpha \). Since there exists \( \lambda \in \mathbb{C} \) with \( \tilde{D}\phi_\lambda = \lambda \phi_\lambda \), we have \( (\tilde{D}\phi_\lambda)(\epsilon) = (\tilde{D}'\phi_\lambda)(\epsilon) = (\tilde{D}\phi_\lambda)(\epsilon) = \lambda \phi_\lambda(\epsilon) = 0 \). Since \( \phi_\lambda \) is a real analytic function on a connected manifold, this implies \( \phi_\lambda = 0 \) and leads a contradiction. \( \square \)

**Lemma 7.2.** For the set \( \{\nu_1, \ldots, \nu_{\ell+1}\} \) of positive integers satisfying \( 1 \leq \nu_1 < \ldots < \nu_{\ell+1} \leq \frac{n}{2} \),

\[
(7.2) \quad D_{\{\nu_1, \nu_2, \ldots, \nu_{\ell+1}\}} \equiv (H_{\nu_1} - \ell t_i)(H_{\nu_2} - (\ell - 1)i) \cdots (H_{\nu_{\ell+1}}) \mod \sum_{1 \leq p < q \leq \ell + 1} (F_{\nu_p \nu_q} \mathbf{1} + F_{\nu_p \nu_q} \mathbf{1} + \mathbf{1} X_{\nu_p \nu_q}^+ + \mathbf{1} X_{\nu_p \nu_q}^-) + \sum_{1 \leq p \leq \ell + 1} E_{\nu_p \nu_q} \mathbf{1}.
\]

**Proof.** We show (7.2) by the induction on \( \ell \). We may assume \( \nu_j = j \). Then

\[
D_{\{1, \ldots, \ell+1\}} = \sum_{p=1}^{\ell+1} (-1)^{\ell+j+1} D_{\{1, \ldots, p-1, p+1, \ldots, \ell+1\}} F_{p, \ell+1} + \sum_{1 \leq p \leq \ell+1} E_{p, \ell+1} \mathbf{1}.
\]

\[
= \sum_{p=1}^{\ell} (-1)^{\ell+j+1} D_{\{1, \ldots, p-1, p+1, \ldots, \ell+1\}} (E_{p, \ell+1}^1 + iE_{p, \ell+1} - iE_{p, \ell+1}^1)
+ (H_1 - (\ell - 1)i) \cdots (H_{\ell-1} - i)(H_{\ell} H_{\ell+1})

= \sum_{p=1}^{\ell} (-1)^{\ell+j+1} D_{\{1, \ldots, p-1, p+1, \ldots, \ell+1\}} (E_{p, \ell+1}^1 + iE_{p, \ell+1} - iE_{p, \ell+1}^1)
+ (H_1 - (\ell - 1)i) \cdots (H_{\ell-1} - i)(H_{\ell} H_{\ell+1})

= \sum_{p=1}^{\ell} (-1)^{\ell+j+1} D_{\{1, \ldots, p-1, p+1, \ldots, \ell+1\}} (E_{p, \ell+1}^1 + iE_{p, \ell+1} - iE_{p, \ell+1}^1)
+ (H_1 - (\ell - 1)i) \cdots (H_{\ell-1} - i)(H_{\ell} H_{\ell+1})

= \sum_{p=1}^{\ell} (-1)^{\ell+j+1} D_{\{1, \ldots, p-1, p+1, \ldots, \ell+1\}} (E_{p, \ell+1}^1 + iE_{p, \ell+1} - iE_{p, \ell+1}^1)
+ (H_1 - (\ell - 1)i) \cdots (H_{\ell-1} - i)(H_{\ell} H_{\ell+1})

= (H_1 - \ell i) \cdots (H_{\ell} - i) H_{\ell+1}. \quad \square
\]

**Lemma 7.3.**

i) Consider the intertwining operator

\[
T_w^t : \mathcal{B}(G/P; \mu + pt, \epsilon) \to \mathcal{B}(G/P; w(\mu + pt) - \rho, \epsilon')
\]

in (4.12) with a parameter \( t \in \mathbb{C} \). Fix an integer \( N \) such that \( T_w^t = t^N T_w^t \) holomorphically depends on \( t \) for \( |t| \ll 1 \). Then the image of \( T_w^0 \) is closed in \( \mathcal{B}(G/P; w(\mu + \rho) - \rho, \epsilon') \).

ii) The same result as above holds even if we replace the space of hyperfunctions by other functions spaces, such as Schwartz’s distributions, \( C^\infty \)-functions or real analytic functions.

**Proof.** Suppose \( |t| \ll 1 \). Consider the inverse intertwining operator

\[
T_w^{t-1} : \mathcal{B}(G/P; w(\mu + pt) - \rho, \epsilon') \to \mathcal{B}(G/P; \mu + pt, \epsilon)
\]

and fix a positive integer \( N' \) so that \( S_w^{t} = t^{N'} T_w^{t-1} \) is holomorphic for \( t \). We identify these spaces of hyperfunctions with subspaces of \( \mathcal{B}(K) \) which do not depend on \( t \). Since
\( \hat{T}_w^t \) and \( S_w^t \) are topological \( G \)-isomorphisms for \( t \neq 0 \), there exists a nonzero holomorphic function \( c(t) \) such that \( S_w^t \circ \hat{T}_w^t = c(t) \text{id} \). Let \( m \) be the order of zero of \( c(t) \) at \( t = 0 \). Then
\[
\text{Im} \hat{T}_w^0 = \bigcap_{n=1}^{m-1} \ker \left( \frac{\partial^n}{\partial t^n} S_w^0 \big|_{t=0} \right) \text{ is closed in } \mathcal{B}(K).
\]

ii) is clear because our proof similarly works on other function spaces. □

**Remark 7.4.** i) A proof of the injectivity of Radon transforms is given in [Gr1, §6]. But the problem seems to be insufficient since the conclusion \( \phi_A(e) \neq 0 \) in Lemma 7.1 is stated just as a consequence of the Frobenius reciprocity theorem.

ii) Theorem 6.7 (or Corollary 6.8 with \( Q = \{ e \} \)) characterizes the image of the Radon transform \( R_k^f \) on the real Grassmann manifold \( X_f \) (cf. (7.1)). Note that our proof naturally gives an inversion formula (cf. [GGR]). In fact, \( c(t) \) in the proof of Lemma 7.3 is known (cf. [GK], [Kn, Chap. VII §5]).

**References**


T. Oshima, *Boundary value problems for various boundaries of symmetric spaces*, RIMS Kōkyūroku, Kyoto Univ. 281 (1976), 211–226. (Japanese)


T. Oshima and N. Shimeno, in preparation.


TOSHI OSHIMA, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, KOMABA, TOKYO 153, JAPAN.