GENERALIZED CAPELLI IDENTITIES AND BOUNDARY VALUE PROBLEMS FOR $GL(n)$

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Dedicated to Professor Hikosaburo Komatsu on the occasion of his 60th birthday

Abstract. The Capelli identity is extended to the case of minors. The operators appearing in the generalized identities give the annihilator of the degenerate principal series for $GL(n)$ and characterizes the image of the Poisson transform of the hyperfunctions on several boundaries of $GL(n)$. Hypergeometric functions are defined through realizations of some special sections of the degenerate principal series and the realizations on boundaries of $GL(n)$ generalize Gelfand's hypergeometric functions. Related Radon transforms for Grassmannians are discussed.

0. Introduction

The Capelli identity [C1] is an important fundamental tool in the classical invariant theory (cf. [We]). It should be remarked that the differential operator given in the identity is an invariant differential operator on $GL(n)$.

In this note, we first generalize the Capelli identity to the case of minors. In §3 the corresponding non-invariant differential operators will be shown to characterize the representations of $GL(n,\mathbb{R})$ belonging to the degenerate principal series.

Any simultaneous eigenfunction of invariant differential operators on a Riemannian symmetric space G/K of a non-compact type has a Poisson integral representation of a hyperfunction section of a line bundle over the maximal boundary G/P of a semisimple Lie group G. This was conjectured by [H2] and then [K-] solved it in general by formulating it as boundary value problems with regular singularities (cf. [KO]) under a smooth realization (cf. [O2]) of G/K.

A similar boundary value problem is naturally formulated for the general boundary G/P_Θ with any parabolic subgroup P_Θ of G. Combining the result [K-] with [KR], it is easy to see that the image of the Poisson transform of hyperfunction sections of a line bundle over a general boundary G/P_Θ is characterized by a suitable system of differential equations. Hence the main problem is to give an explicit description of the nice generators of the system.

In [O1] we gave nice generators of the system in the case of a certain boundary of $GL(n,\mathbb{R})$ and a conjecture for general semisimple Lie groups. On the other hand, [J1] and [J2] gave the generators in a less explicit way for general boundaries G/P_Θ in the case of trivial line bundle over G/P_Θ. There are many related works for Shilov boundaries of bounded symmetric domains (cf. [BV], [KM], [L], [Sn]).

In §5 we will generalize [O1] and show that the generalized Capelli operators give the generators of the system in the case of the general boundary G/P_Θ of $GL(n,\mathbb{R})$ by using the fact that the regular representation on the solution space is isomorphic to a representation belonging to degenerate spherical principal series of $GL(n,\mathbb{R})$. These operators are closely connected with the operators given in [Sh] to characterize a singular representation of $U(n,n)$ realized on sections of a certain line bundle over $U(n,n)/U(n)\times U(n)$.

Lastly we will consider a special vector in a realization of the representation characterized by a finite-dimensional representation of a certain subgroup of G. For example, a fixed vector under the action of the maximal compact subgroup K of G is a zonal spherical function if we consider the spherical representation, that is, the representation on the function space on G/K. If the subgroup is the diagonal matrices of $GL(n,\mathbb{R})$ and if we consider representations of a certain degenerate principal series realized on boundaries, the functions coincide with Gelfand's hypergeometric functions.

In §6 we will define hypergeometric functions on a general reductive Lie group G and give some examples when $G = GL(n,\mathbb{R})$, which are a generalization of Gelfand's hypergeometric functions introduced by [G] and equations defined in [GG]. Then Corollary 6.8 is fundamental for their analysis, which characterize the image of Radon transforms on real Grassmann manifolds (cf. Remark 7.4 ii)).

In §3 we will restrict ourself to the case when $G = GL(n,\mathbb{R})$. The similar arguments can be applied to $GL(n,\mathbb{C})$ or its other real forms (cf. [OSn]). The study in this note is also restricted to the case of $GL(n)$ but we try to explain our results explicitly by using the coordinates of $GL(n,\mathbb{R})$. Generalizations of our results including the study in the case of other classical groups and further studies of hypergeometric functions will be discussed in other papers.

1. Capelli identities

The classical Capelli identity can be considered as a *quantization* of the formula $\det {}^tAB = \det A \cdot \det B$ in the linear algebra. We *quantize* more general identities

$$\det\Big(\sum_{\nu=1}^{n} x_{\nu i}y_{\nu j}\Big)_{\substack{1\le i\le m\\ 1\le j\le m}} = \sum_{1\le\nu_1<\cdots<\nu_m\le n} \det\Big(x_{\nu_i j}\Big)_{\substack{1\le i\le m\\ 1\le j\le m}} \cdot \det\Big(y_{\nu_i j}\Big)_{\substack{1\le i\le m\\ 1\le j\le m}} \tag{1.1}$$

for $2mn$ commutative variables $x_{\nu i}$ and $y_{\nu i}$ with $1 \le i \le m$ and $1 \le \nu \le n$ (cf. [Si, II §5 Theorem 9]), where the left hand side of (1.1) is zero if $m > n$, and we get

Theorem 1.1. (Generalized Capelli identities) *Let $I = \{i_k\}_{1\le k\le m}$ and $J = \{j_\ell\}_{1\le \ell\le m}$ be sequences of positive integers. Then*

$$\begin{aligned}
&\det\Big(\sum_{\nu=1}^{n} x_{\nu i_k}\frac{\partial}{\partial x_{\nu j_\ell}} + (m-\ell)\delta_{i_k j_\ell}\Big)_{\substack{1\le k\le m\\ 1\le \ell\le m}} \\
&\qquad = \begin{cases} \displaystyle\sum_{1\le\nu_1<\cdots<\nu_m\le n} \det\Big(x_{\nu_p i_q}\Big)_{\substack{1\le p\le m\\ 1\le q\le m}} \cdot \det\Big(\frac{\partial}{\partial x_{\nu_p j_q}}\Big)_{\substack{1\le p\le m\\ 1\le q\le m}} & \text{if } m\le n,\\ 0 & \text{if } m>n.\end{cases}
\end{aligned} \tag{1.2}$$

Here δ_{ij} is the Kronecker symbol and for a matrix $A = (A_{ij}) = \Big(A_{ij}\Big)_{\substack{1\le i\le m\\ 1\le j\le m}}$ with (i,j) components A_{ij} in an associative algebra, we define

$$\det A = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}(\sigma) A_{\sigma(1)1}A_{\sigma(2)2}\cdots A_{\sigma(m)m}, \tag{1.3}$$

where $\mathfrak{S}_m$ is the m-th symmetric group.

Proof of Theorem 1.1. First note that Theorem 1.1 is equivalent to (1.1) if $I \cap J = \emptyset$. We will reduce the theorem to (1.1) by the induction on m.

The theorem is trivial when $m = 1$. Suppose $m > 1$. Denoting $\det\Big(x_{\nu_p i_q}\Big)_{\substack{1\le p\le m\\ 1\le q\le m}}$ and $\partial/\partial x_{\nu j}$ by $\det(x_{\{\nu_1,\dots,\nu_m\}\{i_1,\dots,i_m\}})$ and $\partial_{\nu j}$, respectively, from the hypothesis of the induction we deduce

$$
\begin{aligned}
&\det\Big(\sum_{\nu=1}^{n} x_{\nu i_k}\partial_{\nu j_\ell} + (m-\ell)\delta_{i_k j_\ell}\Big)_{\substack{1\le k\le m\\ 1\le \ell\le m}}\\
&= \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}(\sigma)(\sum_{\nu=1}^{n} x_{\nu i_{\sigma(1)}}\partial_{\nu j_1} + (m-1)\delta_{i_{\sigma(1)} j_1})\\
&\qquad \cdots(\sum_{\nu=1}^{n} x_{\nu i_{\sigma(m)}}\partial_{\nu j_m} + (m-m)\delta_{i_{\sigma(m)} j_m})\\
&= \sum_{k=1}^{m}(-1)^{k-1}(\sum_{\nu=1}^{n} x_{\nu i_k}\partial_{\nu j_1} + (m-1)\delta_{i_k j_1})\\
&\qquad \sum_{1\le\nu_2<\dots<\nu_m\le n} \det(x_{\{\nu_2,\dots,\nu_m\}\{i_1,\dots,i_{k-1},i_{k-1},\dots,i_m\}})\cdot\det(\partial_{\{\nu_2,\dots,\nu_m\}\{j_2,\dots,j_m\}}).
\end{aligned}
\tag{1.4}
$$

If $j_1 \notin I$, (1.4) equals

$$
\begin{aligned}
&\sum_{k=1}^{m}(-1)^{k-1}\sum_{\nu=1}^{n}\sum_{1\le\nu_2<\dots<\nu_m\le n} x_{\nu i_k}\det(x_{\{\nu_2,\dots,\nu_m\}\{i_1,\dots,i_{k-1},i_{k+1},\dots,i_m\}})\\
&\qquad\qquad\cdot\partial_{\nu j_1}\det(\partial_{\{\nu_2,\dots,\nu_m\}\{j_2,\dots,j_m\}})
\end{aligned}
\tag{1.5}
$$

and the theorem follows from the corresponding equation in the commutative case (1.1).

Suppose there exists i_ℓ with $i_\ell = j_1$. Then the difference of (1.5) from (1.4) equals

$$
\begin{aligned}
&\sum_{1\le\nu_2<\dots<\nu_m\le n}(m-1)(-1)^{\ell-1}\det(x_{\{\nu_2,\dots,\nu_m\}\{i_1,\dots,i_{\ell-1},i_{\ell+1},\dots,i_m\}})\\
&\quad\cdot\det(\partial_{\{\nu_2,\dots,\nu_m\}\{j_2,\dots,j_m\}})\\
&\quad+\sum_{1\le\nu_2<\dots<\nu_m\le n}\sum_{1\le k\le m}\sum_{\nu=1}^{n}(-1)^{k-1}x_{\nu i_k}\\
&\quad\cdot[\partial_{\nu i_\ell},\det(x_{\{\nu_2,\dots,\nu_m\}\{i_1,\dots,i_{k-1},i_{k+1},\dots,i_m\}})]\det(\partial_{\{\nu_2,\dots,\nu_m\}\{j_2,\dots,j_m\}}).
\end{aligned}
\tag{1.6}
$$

Note that if $\nu \notin \{\nu_2,\dots,\nu_m\}$, then

$$[\partial_{\nu i_\ell},\det(x_{\{\nu_2,\dots,\nu_m\}\{i_1,\dots,i_{k-1},i_{k+1},\dots,i_m\}})] = 0.$$

On the other hand, putting $J_N = \{\nu_2,\dots,\nu_{N-1},\nu_{N+1},\dots,\nu_m\}$ for $N = 2,\dots,m$, we have

$$
\begin{aligned}
&\sum_{k=1}^{m}(-1)^{k-1}x_{\nu_N i_k}[\partial_{\nu_N i_\ell},\det(x_{\{\nu_2\dots,\nu_m\}\{i_1,\dots,i_{k-1},i_{k+1},\dots,i_m\}})]\\
&=\sum_{k=1}^{\ell-1}(-1)^{k+N+\ell-1}x_{\nu_N i_k}\det(x_{J_N\{i_1,\dots,i_{k-1},i_{k+1},\dots,i_{\ell-1},i_{\ell+1},\dots,i_m\}})\\
&\quad+\sum_{k=\ell+1}^{m}(-1)^{k+N+\ell}x_{\nu_N i_k}\det(x_{J_N\{i_1,\dots,i_{\ell-1},i_{\ell+1},\dots,i_{k-1},i_{k+1},\dots,i_m\}})\\
&=(-1)^{\ell}\det(x_{\{\nu_2,\dots,\nu_m\}\{i_1,\dots,i_{\ell-1},i_{\ell+1},\dots,i_m\}}).
\end{aligned}
$$

Hence we can conclude that (1.6) is identically zero and we have the theorem as in the case when $j_1 \notin I$. $\square$

Corollary 1.2. *Under the notation in* Theorem 1.1, *we have*

(1.7)
$$\det\Big(\sum_{\nu=1}^{n} x_{\nu i_k}\frac{\partial}{\partial x_{\nu j_\ell}} + (n+1-\ell)\delta_{i_k j_\ell}\Big)_{\substack{1\le k\le m\\1\le \ell\le m}}$$
$$= \begin{cases} \displaystyle\sum_{1\le \nu_1<\cdots<\nu_m\le n} \det\Big(\frac{\partial}{\partial x_{\nu_p j_q}}\Big)_{\substack{1\le p\le m\\1\le q\le m}}\cdot\det\Big(x_{\nu_p i_q}\Big)_{\substack{1\le p\le m\\1\le q\le m}} & \text{if } m\le n,\\ 0 & \text{if } m>n \end{cases}$$

and

(1.8)
$$\det\Big(\sum_{\nu=1}^{n} x_{\nu i_k}\frac{\partial}{\partial x_{\nu j_\ell}} + (m-\ell)\delta_{i_k j_\ell}\Big)_{\substack{1\le k\le m\\1\le \ell\le m}}$$
$$= \det\Big(\sum_{\nu=1}^{n} x_{\nu i_\ell}\frac{\partial}{\partial x_{\nu j_k}} + (\ell-1)\delta_{i_\ell j_k}\Big)_{\substack{1\le k\le m\\1\le \ell\le m}}.$$

Proof. Let $\mathcal{W}$ denote the algebra generated by x_{ij} and $\frac{\partial}{\partial x_{ij}}$, which is called a Weyl algebra. Applying the anti-automorphism of $\mathcal{W}$ to (1.2) defined by $x_{ij}\mapsto x_{ij}$ and $\frac{\partial}{\partial x_{ij}}\mapsto -\frac{\partial}{\partial x_{ij}}$, we have

$$\sum_{1\le \nu_1<\cdots<\nu_m\le n} \det\Big(-\frac{\partial}{\partial x_{\nu_p j_q}}\Big)_{\substack{1\le p\le m\\1\le q\le m}}\cdot\det\Big(x_{\nu_p i_q}\Big)_{\substack{1\le p\le m\\1\le q\le m}}$$
$$= \sum_{\sigma\in\mathfrak{S}_m}\operatorname{sgn}(\sigma)\big(\sum_{\nu=1}^{n} -\frac{\partial}{\partial x_{\nu j_m}}x_{\nu i_{\sigma(m)}} + (m-m)\delta_{i_{\sigma(m)} j_m}\big)$$
$$\cdots\big(\sum_{\nu=1}^{n} -\frac{\partial}{\partial x_{\nu j_1}}x_{\nu i_{\sigma(1)}} + (m-1)\delta_{i_{\sigma(1)} j_1}\big)$$
$$= (-1)^m\sum_{\sigma\in\mathfrak{S}_m}\operatorname{sgn}(\sigma)\big(\sum_{\nu=1}^{n} x_{\nu i_{\sigma(m)}}\frac{\partial}{\partial x_{\nu j_m}} + n\delta_{i_{\sigma(m)} j_m}\big)$$
$$\cdots\big(\sum_{\nu=1}^{n} x_{\nu i_{\sigma(1)}}\frac{\partial}{\partial x_{\nu j_1}} + (n-m+1)\delta_{i_{\sigma(1)} j_1}\big).$$

Reversing the order of the indices in the above, we have (1.7).

Consider the automorphism of $\mathcal{W}$ defined by $x_{ij}\mapsto \frac{\partial}{\partial x_{ij}}$ and $\frac{\partial}{\partial x_{ij}}\mapsto -x_{ij}$. Then it follows from (1.2) and (1.7) that

$$\det\Big(\sum_{\nu=1}^{n} x_{\nu j_k}\frac{\partial}{\partial x_{\nu i_\ell}} + (m-\ell)\delta_{j_k i_\ell}\Big)_{\substack{1\le k\le m\\1\le \ell\le m}}$$
$$= \sum_{1\le \nu_1<\cdots<\nu_m\le n} \det\Big(x_{\nu_p j_q}\Big)_{\substack{1\le p\le m\\1\le q\le m}}\cdot\det\Big(\frac{\partial}{\partial x_{\nu_p i_q}}\Big)_{\substack{1\le p\le m\\1\le q\le m}}$$
$$= (-1)^m\det\Big(\sum_{\nu=1}^{n} -\frac{\partial}{\partial x_{\nu i_k}}x_{\nu j_\ell} + (n+1-\ell)\delta_{i_k j_\ell}\Big)_{\substack{1\le k\le m\\1\le \ell\le m}}$$
$$= \det\Big(\sum_{\nu=1}^{n} x_{\nu j_\ell}\frac{\partial}{\partial x_{\nu i_k}} + (\ell-1)\delta_{i_k j_\ell}\Big)_{\substack{1\le k\le m\\1\le \ell\le m}}.$$

Combining this with Theorem 1.1 and exchanging I and J, we have (1.8). $\square$

Remark 1.3. If $m = n$, Theorem 1.1 is reduced to the Capelli identity (cf. [C1])

$$\det\Big(\sum_{\nu=1}^{n} x_{\nu i}\frac{\partial}{\partial x_{\nu j}} + (m-j)\delta_{ij}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}} = \det\Big(x_{ij}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}} \cdot \det\Big(\frac{\partial}{\partial x_{ij}}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}}.$$

In general, we have the m-th order Capelli identity (cf. [C2])

$$\text{(1.9)} \qquad \sum_{1\le i_1<\cdots<i_m\le n} \det\Big(\sum_{\nu=1}^{n} x_{\nu i_k}\frac{\partial}{\partial x_{\nu i_\ell}} + (m-\ell)\delta_{k\ell}\Big)_{\substack{1\le k\le m\\ 1\le \ell\le m}}$$
$$= \sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le j_1<\cdots<j_m\le n}} \det\Big(x_{i_k j_\ell}\Big)_{\substack{1\le k\le m\\ 1\le \ell\le m}} \cdot \det\Big(\frac{\partial}{\partial x_{i_k j_\ell}}\Big)_{\substack{1\le k\le m\\ 1\le \ell\le m}}.$$

2. Capelli operators

Definition 2.1. Let E_{ij} be the $n\times n$ matrix whose (μ,ν) element equals $\delta_{i\mu}\delta_{j\nu}$ and consider E_{ij} as an element of $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$. Let $I = \{i_\mu\}_{1\le\mu\le m}$ and $J = \{j_\nu\}_{1\le\nu\le m}$ be sequences of m positive integers with $1 \le i_\mu \le n$ and $1 \le j_\nu \le n$. Then define

$$\begin{aligned} D^{IJ}_{\mu\nu} &= E_{i_\mu j_\nu},\\ \bar{D}^{IJ}_{\mu\nu} &= \bar{D}^{IJ}_{\mu\nu}(\lambda) = E_{i_\mu j_\nu} + (\lambda + m - \nu)\delta_{i_\mu j_\nu},\\ D_{IJ} &= D_{IJ}(\lambda) = \det\Big(\bar{D}^{IJ}_{\mu\nu}\Big)_{\substack{1\le\mu\le m\\ 1\le\nu\le m}}. \end{aligned}$$

Here λ is an indeterminate which commutes with elements of $\mathfrak{g}$ and $D_{IJ}(\lambda)$ is an element of $\mathfrak{U} = U(\mathfrak{g}) \otimes \mathbb{C}[\lambda]$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

We naturally identify E_{ij} with the left invariant vector filed on $G = GL(n,\mathbb{C})$. Then E_{ij} is of the form

$$\text{(2.1)} \qquad E_{ij} = \sum_{\nu=1}^{n} x_{\nu i}\frac{\partial}{\partial x_{\nu j}}$$

under the coordinates $\Big(x_{ij}\Big)_{\substack{1\le i\le n\\ 1\le j\le n}} \in G$ and the left hand side of (1.2) is identified with $D_{IJ}(0)$. Hence we call D_{IJ} and $D_{IJ}(0)$ *generalized Capelli operators.*

Lemma 2.2. *Put $\tau(I) = \{i_{\tau(\mu)}\}_{1\le\mu\le m}$ and $\tau'(J) = \{j_{\tau'(\nu)}\}_{1\le\nu\le m}$ for τ, $\tau' \in \mathfrak{S}_m$. Then*

$$D_{IJ}(\lambda) = \operatorname{sgn}(\tau)\operatorname{sgn}(\tau')D_{\tau(I)\tau'(J)}(\lambda).$$

Proof. We can prove the lemma by a direct calculation but we remark that it is a corollary of Theorem 1.1. In fact, by the identification (2.1), it follows from Theorem 1.1 that $D_{IJ}(0) = \operatorname{sgn}(\tau)\operatorname{sgn}(\tau')D_{\tau(I)\tau'(J)}(0)$. Then applying an automorphism of $U(\mathfrak{g})$ defined by

$$\text{(2.2)} \qquad S_\lambda : U(\mathfrak{g}) \ni E_{ij} \mapsto E_{ij} + \lambda\delta_{ij}$$

to this equality, we have Lemma 2.2 for any $\lambda \in \mathbb{C}$. $\square$

This lemma immediately implies

Corollary 2.3. *If there exist integers k and ℓ with $1 \le k < \ell \le m$ which satisfy $i_k = i_\ell$ or $j_k = j_\ell$, then $D_{IJ}(\lambda) = 0$.*

Proposition 2.4. *Suppose $I = \{i_\mu\}_{1\le\mu\le m}$ and $J = \{j_\nu\}_{1\le\nu\le m}$ satisfy $i_\mu \ne i_{\mu'}$ and $j_\nu \ne j_{\nu'}$ for $1 \le \mu < \mu' \le m$ and $1 \le \nu < \nu' \le m$. Let k and ℓ be positive integers with $1 \le k \le n$ and $1 \le \ell \le n$. Then*

$$[E_{k\ell}, D_{IJ}(\lambda)] = D_1 - D_2$$

with

$$D_1 = \begin{cases} 0 & \text{if } \ell \notin I, \\ D_{\{i_1,\dots,i_{\mu-1},k,i_{\mu+1},\dots,i_m\}J}(\lambda) & \text{if } \ell = i_\mu, \end{cases}$$
$$D_2 = \begin{cases} 0 & \text{if } k \notin J, \\ D_{I\{j_1,\dots,j_{\nu-1},\ell,j_{\nu+1},\dots,j_m\}}(\lambda) & \text{if } k = j_\nu. \end{cases}$$

Proof. We may assume $\ell \in I$ and $k \in J$ because otherwise the lemma is clear from the relation $[E_{k\ell}, E_{ij}] = \delta_{i\ell}E_{kj} - \delta_{jk}E_{i\ell}$ and the definition of D_{IJ}. Suppose $i_{\mu(\ell)} = \ell$ and $j_{\nu(k)} = k$. Putting $I' = \{i_1, \dots, i_{\mu(\ell)-1}, k, i_{\mu(\ell)+1}, \dots, i_m\}$, $J' = \{j_1, \dots, j_{\nu(k)-1}, \ell, j_{\nu(k)+1}, \dots, j_m\}$, $I'' = \{i_1, \dots, i_{\mu(\ell)-1}, i_{\mu(\ell)+1}, \dots, i_m\}$ and $J'' = \{j_1, \dots, j_{\nu(k)-1}, j_{\nu(k)+1}, \dots, j_m\}$, we have

$$\begin{aligned}
[E_{k\ell}, D_{IJ}] &= \det\Big(D^{I'J}_{ij} - (\lambda + m - j)\delta_{i\mu(\ell)}\delta_{j\nu(k)}\Big)_{\substack{1\le i\le m\\1\le j\le m}} \\
&\quad - \det\Big(D^{IJ'}_{ij} - (\lambda + m - j)\delta_{i\mu(\ell)}\delta_{j\nu(k)}\Big)_{\substack{1\le i\le m\\1\le j\le m}} \\
&= \big(D_{I'J} - (-1)^{\mu(\ell)+\nu(k)}(\lambda + m - \nu(k))D_{I''J''}\big) \\
&\quad - \big(D_{IJ'} - (-1)^{\mu(\ell)+\nu(k)}(\lambda + m - \nu(k))D_{I''J''}\big) \\
&= D_{I'J} - D_{IJ'}
\end{aligned}$$

and the proposition. $\square$

Lemma 2.2, Corollary 2.3 and Proposition 2.4 imply

Corollary 2.5. i) *Put*

$$J(m,\lambda) = \sum_{\substack{1\le i_1<\dots<i_m\le n\\1\le j_1<\dots<j_m\le n}} \mathbb{C}D_{\{i_1,\dots,i_m\}\{j_1,\dots,j_m\}}(\lambda). \tag{2.3}$$

Then $[\mathfrak{gl}(n,\mathbb{C}), J(m,\lambda)] \subset J(m,\lambda)$.

ii) *If $\ell \notin I$ and $k \notin J$, then $[E_{k\ell}, D_{IJ}(\lambda)] = 0$.*

iii) *If $k \in I \cap J$ and $\ell \in I \cap J$, then $[E_{k\ell}, D_{IJ}(\lambda)] = 0$.*

Proposition 2.6. i) *Under the above notation*

$$J(m+1,\lambda) \subset \mathfrak{U}J(m,\lambda) \cap \mathfrak{U}J(m,\lambda+1) \quad \textit{for} \;\; m = 1, \dots, n-1. \tag{2.4}$$

ii) *Let $I = \{i_1, \dots, i_m\}$ and $J = \{j_1, \dots, j_m\}$ with $1 \le i_1 < \dots < i_m \le n$ and $1 \le j_1 < \dots < j_m \le n$. Put $I \cap J = \{i_{\mu_1}, \cdots, i_{\mu_L}\} = \{j_{\nu_1}, \cdots, j_{\nu_L}\}$ with $1 \le \mu_1 < \dots < \mu_L \le n$ and $1 \le \nu_1 < \dots < \nu_L \le n$. Then*

$$D_{IJ}(\lambda) - D_{IJ}(\lambda - 1) = \sum_{k=1}^{L} D_{\{i_1,\dots,i_{\mu_k-1},i_{\mu_k+1},\dots,i_m\}\{j_1,\dots,j_{\nu_k-1},j_{\nu_k+1},\dots,j_m\}}(\lambda). \tag{2.5}$$

iii) *Under the anti-automorphism a of $U(\mathfrak{g})$ satisfying $X \mapsto -X$ for $X \in \mathfrak{g}$, $D_{IJ}(\lambda)$ changes into $(-1)^m D_{IJ}(1-m-\lambda)$ with $m = \#I$.*

iv) *Under the automorphism t of $U(\mathfrak{g})$ satisfying $E_{ij} \mapsto -E_{ji}$ for $1 \le i \le n$ and $1 \le j \le n$, $D_{IJ}(\lambda)$ changes into $(-1)^m D_{JI}(1-m-\lambda)$ with $m = \#I$.*

Proof. Note that Corollary 2.5 shows $\mathfrak{U}J(m, \lambda+1) = J(m, \lambda+1)\mathfrak{U}$. The Laplace expansions with respect to the first and the last columns of $J(m+1, \lambda)$ imply i). ii) is clear from Lemma 2.2 with

$$\begin{aligned}
&a\big((E_{i_{\sigma(1)}j_1} + (\lambda+m-1)\delta_{i_{\sigma(1)}j_1}) \cdots (\bar{E}_{i_{\sigma(m)}j_m} + (\lambda+m-m)\delta_{i_{\sigma(m)}j_m})\big) \\
&\quad = (-E_{i_{\sigma(m)}j_m} + (\lambda+m-m)\delta_{i_{\sigma(m)}j_m}) \cdots (-E_{i_{\sigma(1)}j_1} + (\lambda+m-1)\delta_{i_{\sigma(1)}j_1}) \\
&\quad = (-1)^m (E_{i_{\sigma(m)}j_m} + ((1-m-\lambda)+m-1)\delta_{i_{\sigma(m)}j_m}) \\
&\qquad \cdots (E_{i_{\sigma(1)}j_1} + ((1-m-\lambda)+m-m)\delta_{i_{\sigma(1)}j_1}).
\end{aligned}$$

Put $D^k_{IJ}(\lambda) = \det\big(E_{i_pj_q} + (\lambda+m-q-c^q_k)\delta_{i_pj_q}\big)_{\substack{1\le p\le m\\1\le q\le m}}$, where $c^q_k = 1$ if $q \le k$ and 0 otherwise. Then it easily follows from (1.3) that $D^{k-1}_{IJ}(\lambda) - D^k_{IJ}(\lambda)$ are summands of the right hand side of (2.5) and hence we have iii).

By (1.8) we have

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_m} (-E_{j_1i_{\sigma(1)}} + (1-1)\delta_{j_1i_{\sigma(1)}}) \cdots (-E_{j_mi_{\sigma(m)}} + (1-m)\delta_{j_mi_{\sigma(m)}}) \\
&\quad = (-1)^m \sum_{\sigma\in\mathfrak{S}_m} (E_{i_{\sigma(1)j_1}} + (m-1)\delta_{j_1i_{\sigma(1)}}) \cdots (E_{i_{\sigma m j_m)}} + (m-m)\delta_{j_mi_{\sigma(m)}}).
\end{aligned}$$

Applying $S_{1-m-\lambda}$ given in (2.2) to this, we get iv). □

Definition 2.7. (Harish-Chandra homomorphism) Put $\mathfrak{a} = \mathbb{C}E_{11} + \cdots + \mathbb{C}E_{nn}$, $\bar{\mathfrak{n}} = \sum_{1\le i<j\le n} \mathbb{C}E_{ij}$ and $\mathfrak{n} = \sum_{1\le j<i\le n} \mathbb{C}E_{ij}$. For $D \in \mathfrak{U}$ we define elements $\gamma'(D)$ and $\gamma(D)$ of $U(\mathfrak{a}) \otimes C[\lambda]$ so that $D - \gamma'(D) \in \mathfrak{U}\bar{\mathfrak{n}} + \mathfrak{n}\mathfrak{U}$ and $\gamma(D) = \iota_\rho(\gamma'(D))$. Here ι_ρ is an algebra endomorphism of $U(\mathfrak{a}) \otimes \mathbb{C}[\lambda]$ which satisfies $\iota_\rho(\lambda) = \lambda$ and $\iota_\rho(E_{ii}) = E_{ii} + i - \frac{n+1}{2}$.

Lemma 2.8. *For $I = \{i_\mu\}_{1\le\mu\le m}$ and $J = \{j_\nu\}_{1\le\nu\le m}$ we put*

$$E_{IJ} = E_{i_1j_1}E_{i_2j_2}\cdots E_{i_mj_m}.$$

Then $\gamma(E_{IJ}) = 0$ if $\sigma(I) \ne J$ for any $\sigma \in \mathfrak{S}_m$.

Proof. We will prove the lemma by the induction on m.

Suppose there exists k which satisfies $i_k < j_k$ and $i_\mu \ge j_\mu$ for $\mu > k$. If $k = m$, the lemma is clear. If $k < m$,

$$\begin{aligned}
E_{i_1j_1}\cdots E_{i_kj_k}E_{i_{k+1}j_{k+1}}\cdots E_{i_mj_m} &= E_{i_1j_1}\cdots E_{i_{k+1}j_{k+1}}E_{i_kj_k}\cdots E_{i_mj_m} \\
&\quad + \delta_{i_{k+1}j_k}E_{i_1j_1}\cdots E_{i_kj_{k+1}}\cdots E_{i_mj_m} - \delta_{i_kj_{k+1}}E_{i_1j_1}\cdots E_{i_{k+1}j_k}\cdots E_{i_mj_m}
\end{aligned}$$

and the lemma is proved by the induction on the lexicographic order of $(m, m-k)$.

On the other hand, if there exists k with $i_k > j_k$ and $i_\mu \le j_\mu$ for $\mu < k$, we have similarly the lemma by the induction on (m, k).

Since the assumption of the lemma assures the existence of k with $i_k \ne j_k$, we have the lemma. □

Proposition 2.9. *Let* $I = \{i_\mu\}_{1\le\mu\le m}$ *and* $J = \{j_\nu\}_{1\le\nu\le m}$ *with* $1 \le i_1 < i_2 < \cdots < i_m \le n$ *and* $1 \le j_1 < j_2 < \cdots < j_m \le n$. *Then* $\gamma(D_{IJ}(\lambda)) = 0$ *if* $I \ne J$ *and*

$$\gamma(D_{II}(\lambda)) = \prod_{\nu=1}^{m}(E_{i_\nu i_\nu} + \lambda + i_\nu + m - \nu - \frac{n+1}{2}).$$

Proof. The definition of D_{IJ} and Lemma 2.8 implies $\gamma(D_{IJ}) = 0$ if $I \ne J$. Note that

$$D_{II} = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}(\sigma)\bar{D}^{II}_{\sigma(1)1}\cdots\bar{D}^{II}_{\sigma(m)m}$$

and that $D^{II}_{\sigma(1)1} = E_{i_{\sigma(1)}i_1} \in \mathfrak{n}$ if $\sigma(1) \ne 1$. Hence we have

$$\begin{aligned}
D_{II} &\equiv \sum_{\sigma\in\mathfrak{S}_m,\ \sigma(1)=1} \operatorname{sgn}(\sigma)(E_{i_1 i_1} + \lambda + m - 1)\bar{D}^{II}_{\sigma(2)2}\cdots\bar{D}^{II}_{\sigma(m)m} \mod \mathfrak{n}\mathfrak{U}\\
&= (E_{i_1 i_1} + \lambda + m - 1)D_{\{i_2,i_3,\cdots,i_m\}\{i_2,i_3,\cdots,i_m\}}\\
&\equiv \prod_{\nu=1}^{m}(E_{i_\nu i_\nu} + \lambda + m - \nu) \mod \mathfrak{n}\mathfrak{U}
\end{aligned}$$

by the induction on m and we have the proposition. $\square$

The following result is well-known.

Corollary 2.10. i) *Regard* $D_{\{1,\ldots,n\}\{1,\ldots,n\}}(\lambda)$ *as a polynomial function of* λ *and denote it by* $D(\lambda)$. *Then the coefficients of the terms* λ^k *in* $D(\lambda)$ *for* $k = 1, \ldots, n$ *generate the center of* $U(\mathfrak{g})$ *and*

$$\gamma(D(\lambda)) = \prod_{i=1}^{n}(E_{ii} + \lambda + \frac{n+1}{2}).$$

ii) *The* m*-th order Capelli operator* (1.9), *which will be denoted by* D_m, *is in the center of* $U(\mathfrak{g})$ *and satisfies*

$$\gamma(D_m) = \sum_{1\le i_1<\cdots<i_m\le n}\prod_{\nu=1}^{m}(E_{i_\nu i_\nu} + i_\nu - \nu + m - \frac{n+1}{2}).$$

The operators $D_1, \ldots, D_n$ *generate the center of* $U(\mathfrak{g})$.

Changing the indices in Proposition 2.9 by $\left(\begin{smallmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1\end{smallmatrix}\right)$, we have

Corollary 2.11. *Under the notation in* Proposition 2.9

$$\begin{aligned}
&D_{IJ} \in \bar{\mathfrak{n}}\mathfrak{U} + \mathfrak{U}\mathfrak{n} \quad \text{if } I \ne J,\\
&D_{II} \equiv \prod_{\nu=1}^{m}(E_{i_\nu i_\nu} + \lambda + \nu - 1) \mod \bar{n}\mathfrak{U}.
\end{aligned}$$

Here mod $\bar{n}\mathfrak{U}$ *may be replaced by* mod $\mathfrak{U}\mathfrak{n}$.

Lastly we remark

Lemma 2.12. *If $D \in U(\mathfrak{g})$ satisfies $\gamma(D) = \gamma([X, D]) = 0$ for any $X \in \mathfrak{g}$, then $D = 0$.*

Proof. Let $\mathfrak{g}^*$ and $\mathfrak{a}^*$ be the dual spaces of $\mathfrak{g}$ and $\mathfrak{a}$, respectively, and identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form of $\mathfrak{g}$. Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$, which is identified with the space of polynomial functions on $\mathfrak{g}^*$. Denote by $S^{(m)}(\mathfrak{g})$ the set of polynomials in $S(\mathfrak{g})$ with degree at most m. Let Λ be the map of symmetrization of $S(\mathfrak{g})$, which maps the element $X_1 \cdots X_m \in S(\mathfrak{g})$ with $X_i \in \mathfrak{g}$ to $\sum_{\sigma \in \mathfrak{S}_m} \frac{1}{m!} X_{\sigma(1)} \cdots X_{\sigma(m)} \in U(\mathfrak{g})$.

Suppose $D \neq 0$. Let m be the smallest number with $\Lambda^{-1}(D) \in S^{(m)}(\mathfrak{g})$ and let $\bar{D}$ be the non-zero homogeneous element in $S^{(m)}(\mathfrak{g})$ with $\Lambda^{-1}(D) - \bar{D} \in S^{(m-1)}(\mathfrak{g})$. Since Λ is $\mathrm{Ad}(g)$-equivariant, the assumption implies $\mathrm{Ad}(g)\bar{D}|_{\mathfrak{a}^*} = 0$ for $g \in G$. Hence by denoting $U = \bigcup_{g \in G} \mathrm{Ad}(g)\mathfrak{a}^*$, we have $\bar{D}|_U = 0$ and $\bar{D} = 0$ because U is open dense in $\mathfrak{g}^*$, which leads a contradiction. $\square$

3. Degenerate principal series

In this section we will see that the generalized Capelli operators define the annihilators of degenerate principal series representations of G, where G is $GL(n, \mathbb{C})$ or its real form. For simplicity we assume $G = GL(n, \mathbb{R})$ hereafter in this note if otherwise stated. Then $K = O(n)$ is a maximal compact subgroup of G.

Given a positive integer L and a sequence of positive integers $\Theta = \{n_1, \ldots, n_L\}$ with $0 = n_0 < n_1 < \cdots < n_l = n$, we define a parabolic subgroup

$$P_\Theta = \left\{ p = \begin{pmatrix} g_1 & & & \text{\huge 0} \\ * & g_2 & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & g_L \end{pmatrix} \in GL(n, \mathbb{R}); g_k \in GL(n_k - n_{k-1}, \mathbb{R}) \right\}. \tag{3.1}$$

Note that the subgroup $P_{\{1,\ldots,n\}} = \{(x_{ij}) \in G; x_{ij} = 0 \text{ if } i < j\}$ is a minimal parabolic subgroup of G, which will be simply denoted by P.

The space $\mathcal{B}(G)$ of hyperfunctions on G is a left G-module by $G \times \mathcal{B}(G) \ni (g, f(x)) \mapsto (\pi_g(f))(x) = f(g^{-1}x)$. The space $\mathcal{A}(G)$ of real analytic functions on G and the space $\mathcal{C}^\infty(G)$ of C^∞-functions on G are G-submodules of $\mathcal{B}(G)$.

For $\mu = (\mu_1, \ldots, \mu_L) \in \mathbb{C}^L$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_L) \in \{0, 1\}^L$, we define a G-submodule

$$\mathcal{B}(G/P_\Theta; \mu, \varepsilon) = \{f \in \mathcal{B}(G); f(xp) = \tau_{\mu,\varepsilon}(p^{-1})f(x) \text{ for } p \in P_\Theta\} \tag{3.2}$$

belonging to degenerate principal series of G, where

$$\tau_{\mu,\varepsilon}(p) = \mathrm{sgn}(\det g_1)^{\varepsilon_1} |\det g_1|^{-\mu_1} \cdots \mathrm{sgn}(\det g_L)^{\varepsilon_L} |\det g_L|^{-\mu_L} \text{ for } p \in P_\Theta \tag{3.3}$$

is a character of P_Θ under the expression of p in (3.1).

Note that $U(\mathfrak{g})$ is a subalgebra of the ring of differential operators on G which satisfies $\mathfrak{g} \times \mathcal{B}(G) \ni (D, f) \mapsto (Df)(x) = \frac{d}{dt} f(x \exp tD)|_{t=0}$. Then we have

Theorem 3.1. *For $\mu \in \mathbb{C}^L$ we define an algebra homomorphism $\chi_{\Theta,\mu}$ of $U(\mathfrak{a})$ to $\mathbb{C}$ so that*

$$\chi_{\Theta,\mu}(E_{jj}) = \mu_k + j - \frac{n+1}{2} \quad \text{if} \quad n_{k-1} < j \leq n_k.$$

Then for $u \in \mathcal{B}(G/P_\Theta; \mu, \varepsilon)$

$$D_m u = \chi_{\Theta,\mu}(\gamma(D_m))u \quad \text{for} \quad m = 1, \ldots, n \tag{3.4}$$

and

(3.5) $D_{IJ}(-\mu_k - n_{k-1})u = 0$

$$\text{for } \ \#I = \#J = n - n_k + n_{k-1} + 1 \ \text{and} \ \ k = 1, \ldots, L$$

under the notation in Corollary 2.10.

Proof. For simplicity, we put $V_\mu = \mathcal{B}(G/P_\Theta; \mu, \varepsilon)$ in this proof. Let consider the function

$$h_\xi(x^{-1}) = \operatorname{sgn}\big(\det\big(x_{ij}\big)_{\substack{1\le i\le n_1\\1\le j\le n_1}}\big)^{\varepsilon_1}\big|\det\big(x_{ij}\big)_{\substack{1\le i\le n_1\\1\le j\le n_1}}\big|^{\xi_1} \cdots \operatorname{sgn}\big(\det\big(x_{ij}\big)_{\substack{1\le i\le n_L\\1\le j\le n_L}}\big)^{\varepsilon_L}\big|\det\big(x_{ij}\big)_{\substack{1\le i\le n_L\\1\le j\le n_L}}\big|^{\xi_L}$$

with $-\mu = (\xi_1 + \cdots + \xi_L, \xi_2 + \cdots + \xi_L, \ldots, \xi_L)$. If $\operatorname{Re}\xi_k$ is sufficiently large for $k = 1, \ldots, L$, $h_\xi(x)$ is a sufficiently differentiable $\bar{N}$-invariant function in V_μ, where $\bar{N} = \exp(\sum_{i<j} \mathbb{R}E_{ij})$.

For any $D \in U(\mathfrak{a})$ and $\tilde{D} \in D + \bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$, we denote by $\tilde{D}_\zeta$ the image of D under the algebra homomorphism of $U(\mathfrak{a})$ to $\mathbb{C}$ with $(E_{ii})_\zeta = \zeta_i$ for $i = 1, \ldots, m$ by putting $\zeta_j = \mu_k$ if $n_{k-1} < j \le n_k$. Then we have

$$\tilde{D}h_\xi(\bar{n}) = \tilde{D}_\zeta h_\xi(\bar{n}) \quad \text{for } \bar{n} \in \bar{N}$$

and it follows from Corollary 2.11 that $\tilde{D}_{II}(\lambda)_\zeta = \prod_{\nu=1}^m (\zeta_{i_\nu} + \lambda + \nu - 1)$ if $I = \{i_1, \ldots, i_m\}$ with $1 \le i_1 < \ldots < i_m \le n$. Put $m = n - n_k + n_{k-1} + 1$. Since $n_{k-1} < i_{n_{k-1}+1} \le n - (m - (n_{k-1} + 1)) = n_k$, we have $\zeta_{i_\nu} + (-\mu_k - n_{k-1}) + \nu - 1 = 0$ when $\nu = n_{k-1} + 1$, and therefore $\tilde{D}_{IJ}(-\mu_k - n_{k-1})_\zeta = 0$ if $\#I = \#J = m$.

Suppose $D_1^\mu, \ldots, D_N^\mu$ are elements of $U(\mathfrak{g})$ which holomorphically depend on μ and satisfy

$$D_k^\mu h_\xi(\bar{n}) = 0 \ \text{ and } \ \operatorname{Ad}(g)D_k^\mu \in \sum_{j=1}^N U(\mathfrak{g})D_j^\mu \quad \text{for } k = 1, \ldots, N, \ \bar{n} \in \bar{N} \text{ and } g \in G.$$

Then if we prove $D_k^\mu V_\mu = 0$, we have (3.4) and similarly (3.5) from Corollaries 2.5 and 2.10.

Note that $D_k^\mu h_\xi(\bar{n}p) = 0$ for $p \in P_\Theta$ because $Dh_\xi(\bar{n}p) = \tau_{\mu,\varepsilon}(p^{-1})(\operatorname{Ad}(p)Dh_\xi)(\bar{n})$ for $D \in U(\mathfrak{g})$. Since $\bar{N}P_\Theta$ is open dense in G, $D_k^\mu h_\xi = 0$ if $\operatorname{Re}\xi_k$ is sufficiently large for $k = 1, \ldots, L$. If moreover ξ_k are generic, V_μ is an irreducible G-module and hence in this case we have $D_k^\mu u = 0$ for any $u \in V_\mu$. For any fixed $\mu' \in \mathbb{C}^L$ and a real analytic function $u \in V_{\mu'}$, we can define $u_\mu \in V_\mu$ with $u_\mu|_K = u|_K$. Since u_μ holomorphically depends on $\mu \in \mathbb{C}^L$, we have $D_k^\mu u_\mu = 0$ for any $\mu \in \mathbb{C}^L$. Since $\mathcal{A}(G) \cap V_\mu$ is dense in V_μ, we have $D_k^{\mu'} u = 0$ for any $u \in V_{\mu'}$. $\square$

Proposition 3.2. i) *If*

$$(\mu_i + n_i) - (\mu_j + n_{j-1} + 1) \notin \{0, 1, 2, \ldots, n_i - n_{i-1} + n_j - n_{j-1} - 2\} \quad \text{for} \ \ 1 \le i < j \le L,$$

then

$$D_m - \chi_{\Theta,\mu}(\gamma(D_m)) \in \sum_{k=1}^L \mathfrak{U}J(n - n_k + n_{k-1} + 1, -\mu_k - n_{k-1}). \tag{3.6}$$

ii) *For an integer m with $0 < m \le n$, the system of differential equations*

$$D_{IJ}(0)u = 0 \quad \text{for } \ \#I = \#J = m$$

on G is equivalent to

$$\det\Big(\frac{\partial}{\partial x_{ij}}\Big)_{i\in I,j\in J} u = 0 \quad \textit{for} \ \ \#I = \#J = m.$$

Proof. Theorem 1.1 proves ii) because $\det\big(\det(x_{ij})_{i\in I,j\in J}\big)_{IJ} = \det(x_{ij})^N \neq 0$ with $\#I = \#J = m$ and $N = \frac{n!}{(m-1)!(n-m)!}$.

Proposition 2.6 i) and Corollary 2.10 show that the right hand side of (3.6) contains $D(-\mu_k - n_{k-1} - \nu)u = 0$ for $k = 1, \dots, L$ and $\nu = 0, \dots, n_k - n_{k-1} - 1$ under the notation there, which is equivalent to $D(\lambda)u = 0$ for any $\lambda \in \mathbb{C}$ and we have (3.6) because the relation

$$\begin{aligned}(3.7)\qquad &\{(\mu_i + \nu_i) - (\mu_j + \nu_j); n_{i-1} + 1 \le \nu_i \le n_i,\ n_{j-1} + 1 \le \nu_j \le n_j\}\\ &\quad = \{(\mu_i + n_i) - (\mu_j + n_{j-1} + 1) - \nu; 0 \le \nu \le n_i - n_{i-1} + n_j - n_{j-1} - 2\}\end{aligned}$$

shows that the numbers $-\mu_k - n_{k-1} - \nu$ are different to each other. □

4. Intertwining operators

Other realizations of degenerate principal series we will investigate are given by some G-homomorphisms, namely, by intertwining operators, which are integral operators with kernel functions because G/P_Θ is compact. We will review them in this section.

Retain the notation in §3 and define Lie subalgebras of $\mathfrak{g}$:

$$\mathfrak{n}_\Theta = \sum_{k=1}^{L} \sum_{\substack{n_{k-1}<i\le n_k\\ j\le n_{k-1}}} \mathbb{R}E_{ij},\ \ \bar{\mathfrak{n}}_\Theta = \sum_{k=1}^{L} \sum_{\substack{n_{k-1}<i\le n_k\\ j\le n_{k-1}}} \mathbb{R}E_{ji},\ \ \mathfrak{l}_\Theta = \sum_{k=1}^{L} \sum_{\substack{n_{k-1}<i\le n_k\\ n_{k-1}<j\le n_k}} \mathbb{R}E_{ij}.$$

Then $\mathrm{Lie}(P_\Theta) = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta$ is a Levi decomposition. We put $N_\Theta = \exp(\mathfrak{n}_\Theta)$ and $\bar{N}_\Theta = \exp(\bar{\mathfrak{n}}_\Theta)$. If $\Theta = \{1, 2, \dots, n\}$, then P_Θ, $\mathfrak{n}_\Theta$, $\bar{\mathfrak{n}}_\Theta$, N_Θ and $\bar{N}_\Theta$ are simply denoted by P, $\mathfrak{n}$, $\bar{\mathfrak{n}}$, N, $\bar{N}$, respectively.

Let $\mathfrak{a}_{\mathbb{C}}^*$ be the complex dual of $\mathfrak{a}$, which equals $\sum_{j=1}^n \mathbb{C}e_j$ by denoting $e_i(E_{jj}) = \delta_{ij}$. Let ρ and ρ_Θ be elements of $\mathfrak{a}_{\mathbb{C}}^*$ corresponding to the restrictions of $\frac{1}{2}\,\mathrm{trace(ad)}$ on $\mathfrak{n}$ and $\mathfrak{n}_\Theta$, respectively, and put $\rho(\Theta) = \rho - \rho_\Theta$. Then we have

$$\begin{aligned}\rho &= \frac{1}{2}\sum_{1\le i<j\le n}(e_j - e_i) = \sum_{j=1}^{n}\Big(j - \frac{n+1}{2}\Big)e_j,\\ (4.1)\qquad \rho(\Theta) &= \frac{1}{2}\sum_{k=1}^{L}\sum_{n_{k-1}<i<j\le n_k}(e_j - e_i) = \sum_{k=1}^{L}\sum_{n_{k-1}<\ell\le n_k}\Big(\ell - \frac{n_{k-1}+n_k+1}{2}\Big)e_\ell,\\ \rho_\Theta &= \sum_{k=1}^{L}\sum_{n_{k-1}<\ell\le n_k}\frac{n_{k-1}+n_k-n}{2}e_\ell = \sum_{k=1}^{L}\frac{n_{k-1}+n_k-n}{2}f_k\end{aligned}$$

by denoting

$$f_k = \sum_{n_{k-1}<\ell\le n_k} e_\ell.$$

We identify $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and $\mu = (\mu_1, \dots, \mu_L) \in \mathbb{C}^L$ with the elements $\sum \lambda_j e_j \in \mathfrak{a}_{\mathbb{C}}^*$ and $\sum \mu_k f_k \in \mathfrak{a}_{\mathbb{C}}^*$, respectively.

Putting $\mathcal{A}(G/P_\Theta; \mu, \varepsilon) = \mathcal{A}(G) \cap \mathcal{B}(G/P_\Theta; \mu, \varepsilon)$ and

$$(4.2)\qquad \mu_\Theta^* = -\mu - 2\rho_\Theta = (n - n_0 - n_1 - \mu_1, \dots, n - n_{L-1} - n_L - \mu_L),$$

we have a G-invariant bilinear form

$$\mathcal{B}(G/P_\Theta;\mu,\varepsilon)\times\mathcal{A}(G/P_\Theta;\mu^*_\Theta,\varepsilon)\ni(f,\phi)\mapsto\langle f,\phi\rangle_\Theta=\int_K f(k)\phi(k)dk \tag{4.3}$$

with the normalized Haar measure dk on K, which follows from the G-invariant integral

$$\mathcal{B}(G/P_\Theta;-2\rho_\Theta,0)\ni f\mapsto\int_K f(k)dk. \tag{4.4}$$

For a close subgroup Q of G and an irreducible finite dimensional representation (τ,V_τ) of Q, we have an associated representation space

$$\mathcal{B}(G/Q;\tau)=\{f\in\mathcal{B}(G)\otimes V_\tau;\ f(xq)=\tau(q^{-1})f(x)\text{ for }q\in Q\}. \tag{4.5}$$

Then for an element $T\in\mathcal{B}(G/P_\Theta;\mu^*_\Theta,\varepsilon)\otimes V$ satisfying

$$T(qx)=\tau(q)T(x)\quad\text{for }q\in Q, \tag{4.6}$$

we have a G-homomorphism

$$\mathcal{B}(G/P_\Theta;\mu,\varepsilon)\ni f\mapsto(\mathcal{T}f)(x)=\int_K f(k)T(x^{-1}k)dk\in\mathcal{B}(G/Q;\tau). \tag{4.7}$$

Here we note that $(\mathcal{T}f)(x)=\langle f,\pi_x(T)\rangle_\Theta=\langle\pi_{x^{-1}}(f),T\rangle_\Theta=\int_K f(xk)T(k)dk$.

It is natural to choose Q so that $Q\backslash G/P$ has an open orbit because the dimension of the intertwining operators of any irreducible Harish-Chandra module of G to $\mathcal{B}(G/Q;\tau)$ is of finite dimension (cf. [O3]).

Suppose Q is also a parabolic subgroup. Since

$$\mathcal{B}(G/P_\Theta;\mu,\varepsilon)\subset\mathcal{B}(G/P;\zeta,\varepsilon') \tag{4.8}$$

if

$$\sum_{k=1}^L\mu_k f_k=\sum_{\ell=1}^L\zeta_n e_\ell\quad\text{and}\quad\epsilon'_k=\epsilon_\ell\quad\text{for}\quad n_{\ell-1}<k\le n_\ell, \tag{4.9}$$

the intertwining operators in the case when $P_\Theta=Q=P$ are fundamental. There exist standard intertwining operators (cf. [Kn, Chap. 7]) parametrized by the Weyl group W of G or equivalently by the double cosets $P\backslash G/P$, which is isomorphic to the n-th symmetric group $\mathfrak{S}_n$.

Fix $w\in W\simeq\mathfrak{S}_n$ and identify w with the representative in G whose (i,j)-component equals $\delta_{w(i)j}$. Hence $w\bigl(x_{ij}\bigr)w^{-1}=\bigl(x_{w(i)w(j)}\bigr)$ for $\bigl(x_{ij}\bigr)\in G$. By denoting $\bar N_w=w^{-1}Nw\cap\bar N$, we have

$$\bar N_w=\Bigl\{\Bigl(x_{ij}\Bigr)_{\substack{1\le i\le n\\1\le j\le n}}\in G;x_{ij}=\delta_{ij}\quad\text{if}\quad i\le j\text{ or }w^{-1}(i)<w^{-1}(j)\Bigr\}. \tag{4.10}$$

Consider the integral

$$(\mathcal{T}_w f)(x)=\int_{\bar N_w}f(xwn_w)dn_w \tag{4.11}$$

with the normalized Haar measure dn_w on $\bar N_w$, which is a constant multiple of the usual measure $\prod dx_{ij}$ under the coordinates in (4.10). If $\mathrm{Re}(\mu_{k-1}-\mu_k)$ is sufficiently large, this integral converges for any continuous function $f\in\mathcal{B}(G/P;\mu,\varepsilon)$ and its kernel function defines the intertwining operator

$$\mathcal{T}_w:\mathcal{B}(G/P;\mu,\varepsilon)\to\mathcal{B}(G/P;\mu',\varepsilon') \tag{4.12}$$

with

$$\begin{aligned}&\mu'=w(\mu+\rho)-\rho=(\mu_{w^{-1}(1)}+w^{-1}(1)-1,\dots,\mu_{w^{-1}(n)}+w^{-1}(n)-n),\\&\varepsilon'=(\varepsilon_{w^{-1}(1)},\dots,\varepsilon_{w^{-1}(n)}).\end{aligned} \tag{4.13}$$

This intertwining operator $\mathcal{T}_w$ is defined for any $\mu\in\mathbb{C}^n$ by the analytic continuation of the kernel function.

5. Poisson transforms

In this section we study the realization of degenerate principal series on the Riemannian symmetric space G/K. First introduce the Poisson kernel

$$
\begin{aligned}
&\Phi^{\mu}_{\Theta}(x) = \Phi_{n_1}(x)^{\xi_1}\cdots\Phi_{n_L}(x)^{\xi_L},\\
&\Phi_m(x) = \sum_{1\le\nu_1<\cdots<\nu_m\le n}\left|\det\Bigl(x_{i\nu_j}\Bigr)_{\substack{1\le i\le m\\1\le j\le m}}\right|^2,\\
&\quad 2\xi_k = (\mu_k+n_{k-1})-(\mu_{k+1}+n_{k+1})\quad\text{for } k=1,\dots,L-1,\\
&\quad 2\xi_L = \mu_L+n_{L-1}.
\end{aligned}
\tag{5.1}
$$

It is easy to see that

$$\Phi^{\mu}_{\Theta}(pxk) = \tau_{\Theta,\mu^*_\Theta}(p)\Phi^{\mu}_{\Theta}(x)\quad\text{for } p\in P_\Theta \text{ and } k\in K$$

and therefore we have a G-homomorphism

$$
\begin{aligned}
\mathcal{P}^{\mu}_{\Theta} : \mathcal{B}(G/P_\Theta;\mu)\ni f\mapsto (\mathcal{P}^{\mu}_{\Theta}f)(x) = \int_K f(xk)dk = \int_K f(k)\Phi^{\mu}_{\Theta}(k^{-1}x)dk&\\
\in\mathcal{B}(G/K)&
\end{aligned}
\tag{5.2}
$$

as was stated in §5. This is called the Poisson transform. Here for simplicity we put $\mathcal{B}(G/P_\Theta;\mu)=\mathcal{B}(G/P_\Theta;\mu,\varepsilon)$ if $\varepsilon=\{0,\dots,0\}$.

The map $\mathcal{P}^{\mu}_{\Theta}$ was studied in [H2], [H3] and [K-] when P_Θ is a minimal parabolic subgroup and it was proved that $\mathcal{P}^{\zeta}_{\{1,\dots,n\}}$ is injective if and only if $e(\zeta)\neq 0$ and in this case the image is the totality of the real analytic functions u on G/K which satisfy

$$D_m u = \chi_\zeta(D_m)u\quad\text{for}\ \ m=1,\dots,n. \tag{5.3}$$

Here $e(\zeta)^{-1}$ corresponds to the denominator of Harish-Chandra's c-function given by

$$e(\zeta) = \prod_{1\le i<j\le n}\Gamma\Bigl(\frac{\zeta_j-\zeta_i+j-i+3}{4}\Bigr)^{-1}\Gamma\Bigl(\frac{\zeta_j-\zeta_i+j-i+1}{4}\Bigr)^{-1}$$

(cf. [H2]) and $e(\zeta)\neq 0$ if and only if

$$(\zeta_i+i)-(\zeta_j+j)\notin\{1,3,5,7,\dots\}\quad\text{for } 1\le i<j\le n.$$

Hence it follows from (4.8), (4.9) and (3.7) that $\mathcal{P}^{\mu}_{\Theta}$ is injective if

$$(\mu_i+n_i)-(\mu_j+n_{j-1}+1)\notin\{1,2,3,4,\dots\}\quad\text{for } 1\le i<j\le L. \tag{5.4}$$

Theorem 5.1. i) *Any* $u\in\operatorname{Im}\mathcal{P}^{\mu}_{\Theta}$ *is real analytic and satisfies* (3.4) *and* (3.5).

ii) *If the condition*

$$(\mu_i+n_i)-(\mu_j+n_{j-1}+1)\notin\{0,1,2,3,4,\dots\}\quad\textit{for } 1\le i<j\le L \tag{5.5}$$

holds, $\mathcal{P}^{\mu}_{\Theta}$ *is a topological* G*-isomorphism onto the subspace of* $C^\infty(G/K)$ *which is the totality of solutions of the system of equations* (3.5).

Example 5.2. Suppose $G = SL(2,\mathbb{R})$ and $P = \left\{ \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \in G \right\}$. Then the natural action on $\mathbb{P}^1_{\mathbb{C}} = (\mathbb{C}^2 - \{0\})/\mathbb{C}^\times \ni \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ is given by the left multiplication. The fixed point group with respect to $i = z_1/z_2$ equals $SO(2)$ and the symmetric space $G/SO(2)$ is identified with the upper half plane $H_+ = \left\{ \left[\begin{smallmatrix} x+yi \\ 1 \end{smallmatrix}\right] \in \mathbb{P}^1_{\mathbb{C}}; y > 0 \right\}$. Since

$$\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{y}i + \frac{x}{\sqrt{y}} \\ \frac{1}{\sqrt{y}} \end{bmatrix} = \begin{bmatrix} x + yi \\ 1 \end{bmatrix} \in H_+ \subset \mathbb{P}^1_{\mathbb{C}},$$

we have $\Psi_1 = \left(\sqrt{y}\right)^2 + \left(\frac{x}{\sqrt{y}}\right)^2 = \frac{x^2+y^2}{y}$. Hence if $\mu_1 = \mu_2 = 0$, we get the usual Poisson kernel $\Phi^\mu_{\{1,2\}} = \frac{y}{x^2+y^2}$ for H_+ and Theorem 5.1 says the isomorphism of the space of harmonic functions on H_+ onto that of hyperfunctions on the circle $G/P \simeq \mathbb{R} \cup \{\infty\} \subset \mathbb{P}^1_{\mathbb{C}}$.

Example 5.3. Suppose $\Theta = \{n\}$. Then $P_\Theta = G$ and $\mathcal{B}(G/P_\Theta; \mu) = \mathbb{C}|\det x|^\mu$ for generic $\mu \in \mathbb{C}$ and we have the equality corresponding to (3.4):

$$D_{\{1,\dots,n\}\{1,\dots,n\}}(\lambda)|\det x|^\mu = (\mu + \lambda)(\mu + \lambda + 1)\cdots(\mu + \lambda + n - 1)|\det x|^\mu.$$

If we put $\lambda = 0$ and $\mu = s + 1$, this corresponds to Cayley's formula

$$\det\Big(\frac{\partial}{\partial x_{ij}}\Big)_{\substack{1\le i\le n \\ 1\le j\le n}} (\det x)^{s+1} = (s+1)(s+2)\cdots(s+n)(\det x)^s$$

in view of the Capelli identity. Note that this equality defines the b-function of $\det x$ and hence μ is the meromorphic parameter of $|\det x|^\mu$ and its poles are contained in $\{-1, -2, -3, \dots\}$ (cf. [B], [Sm]).

Proof of Theorem 5.1. Since the Poisson kernel Φ^μ_Θ is real analytic, any $u \in \operatorname{Im}\mathcal{P}^\mu_\Theta$ is also real analytic. For $c \in \mathbb{C}$, $\operatorname{Ad}(g)J(m,c) = J(m,c)$ (cf. Corollary 2.5) and the condition $Du = 0$ for any $D \in J(m,c)$ is equivalent to $\pi_{a(D)}u = 0$ for any $D \in J(m,c)$ (cf. Proposition 2.6 iii) and note that π is the left regular representation). Hence i) follows from Theorem 3.1 and the G-equivariance of $\mathcal{P}^\mu_\Theta$.

Put $\mathcal{A}(G/K;\zeta) = \{u \in \mathcal{A}(G); u(gk) = u(g)$ and $D_m u = \chi_{\{1,\dots,n\},\zeta}(\gamma(D_m))u$ for $k \in K$ and $m = 1,\dots,n\}$. Then $\mathcal{P}^\zeta_{\{1,\dots,n\}}$ is a topological G-isomorphism of $\mathcal{B}(G/P;\zeta)$ onto $\mathcal{A}(G/K;\zeta)$. This is proved in [K-] by constructing a map β^ζ of taking the boundary values, which gives the inverse of $\mathcal{P}^\zeta_{\{1,\dots,n\}}$. Here we note that $\chi_{\Theta,\mu} = \chi_{\{1,\dots,n\},\zeta}$.

Suppose $u \in \mathcal{A}(G/K;\zeta)$ satisfies (3.5). Since β^ζ is G-equivariant, Corollary 2.5 i) assures $D_{IJ}(-\mu_k - n_{k-1})\beta^\zeta(u) = 0$ for $\#I = \#J = n - n_k + n_{k-1} + 1$ and $k = 1,\dots,L$. Fix k with $n_k - n_{k-1} > 1$, choose i_o and j_o with $n_{k-1} < i_o < j_o \le n_k$ and put $I = \{n, n-1, \dots, n_k + 1, i_o, n_{k-1}, n_{k-1} - 1, \dots, 1\}$ and $J = \{n, n-1, \dots, n_k + 1, j_o, n_{k-1}, n_{k-1} - 1, \dots, 1\}$. Then it is clear from the definition of D_{IJ} that

$$D_{IJ}(-\mu_k - n_{k-1}) \equiv \prod_{\nu=n_k+1}^{n} (E_{\nu\nu} - \mu_k - n_k + \nu) \cdot E_{i_o j_o} \cdot \prod_{\nu=1}^{n_{k-1}} (E_{\nu\nu} - \mu_k - n_{k-1} + \nu - 1) \mod U(\mathfrak{g})\mathfrak{n}$$

and for $\phi \in \mathcal{B}(G/P;\zeta)$ we have

$$D_{IJ}(-\mu_k - n_{k-1})\phi = \prod_{\nu=n_k+1}^{n} (\zeta_\nu - \mu_k - n_k + \nu) \cdot E_{i_o j_o} \cdot \prod_{\nu=1}^{n_{k-1}} (\zeta_\nu - \mu_k - n_{k-1} + \nu - 1)\phi.$$

If $\ell < k$ and $n_{\ell-1} < \nu \le n_\ell$, then $\zeta_\nu - \mu_k - n_{k-1} + \nu - 1 = (\mu_\ell + \nu) - (\mu_k + n_{k-1} + 1) \neq 0$, which follows from (5.5) because of (3.7). Similarly if $\ell \ge k$ and $n_\ell < \nu \le n_{\ell+1}$, then $\zeta_\nu - \mu_k - n_k + \nu = -\big((\mu_k + n_k) - (\mu_{\ell+1} + \nu)\big) \neq 0$. Hence $E_{i_o j_o}\beta^\zeta(u) = 0$. Thus we have $X\beta^\zeta(u) = 0$ for any $X \in \mathrm{Lie}(P_\Theta) \cap \bar{\mathfrak{n}}$, which means $\beta^\zeta(u) \in \mathcal{B}(G/P_\Theta; \mu)$ and we have the theorem from Proposition 3.2 i). $\square$

Remark 5.4. i) Theorem 5.1 in the case $\Theta = \{1, n\}$ is given in [O1], where it is conjectured that in general there exists a system of operators D satisfying a suitable condition for $\gamma(D)$ and characterizing the image of the Poisson transform. When $G = GL(n, \mathbb{R})$, the conjecture corresponds to Theorem 5.1 and Corollary 2.11.

ii) If $\mathrm{Re}\,\mu_j < \mathrm{Re}\,\mu_{j+1} + 1$ for $j = 1, \ldots, L-1$, then (5.5) is valid.

iii) In [J1] and [J2] some differential equations characterizing $\mathrm{Im}\,\mathcal{P}^0_\Theta$ are given, which are less explicit than ours.

iv) Let $w_\Theta \in \mathfrak{S}_n$ with $\mathrm{Ad}(w_\Theta)\bar{\mathfrak{n}} \cap \mathfrak{n} = \mathrm{Lie}(P_\Theta) \cap \mathfrak{n}$. As is remarked in [O1], $\mathrm{Im}\,\mathcal{P}^\mu_\Theta = \mathrm{Im}\,\mathcal{P}^{\zeta'}_{\{1,\ldots,n\}}$ with $\zeta'_j = \zeta_{w_\Theta(j)} + w_\Theta(j) - j$ and hence for $\eta \in \mathbb{C}^n$, it is interesting to study the system of differential equations characterizing $\mathrm{Im}\,\mathcal{P}^\eta_{\{1,\ldots,n\}}$.

6. Hypergeometric functions

In general, suppose G is a linear reductive Lie group and moreover suppose that we are given three closed subgroups P_Θ, Q_1 and Q_2 of G and their finite dimensional representations (τ_Θ, V_Θ), (τ_1, V_1) and (τ_2, V_2), respectively, which satisfy the conditions

(6.1) $\quad P_\Theta$ is a parabolic subgroup of G,

(6.2) $\quad$ the double cosets $Q_j\backslash G/P_\Theta$ have open cosets for $j = 1$ and 2.

Define the degenerate principal series as in §3:

$$\mathcal{B}(G/P_\Theta; \tau_\Theta) = \{\phi(g) \in \mathcal{B}(G) \otimes V_\Theta; \phi(gp) = \tau_\Theta(p^{-1})\phi(g) \text{ for } p \in P_\Theta\}.$$

Let $P_\Theta = L_\Theta N_\Theta$ be a Levi decomposition of P_Θ. Fix a Cartan involution θ of G with $\theta(L_\Theta) = L_\Theta$. Then $K = \{g \in G; \theta(g) = g\}$ is a maximal compact subgroup of G. Let V^*_Θ be the dual space of V_Θ and let τ^*_Θ be the representation of P_Θ on V^*_Θ such that

$$\mathcal{B}(G/P_\Theta; \tau_\Theta) \times \mathcal{A}(G/P_\Theta; \tau^*_\Theta) \ni (\phi, \psi) \mapsto \langle \phi, \psi \rangle_\Theta = \int_K \langle \phi(k), \psi(k) \rangle dk \in \mathbb{C} \tag{6.3}$$

defines a G-invariant bilinear form. Note that $(\tau^*_\Theta, V^*_\Theta)$ is the tensor product of the contragredient representation of τ_Θ and the character $\det(\mathrm{Ad})$ of P_Θ on $\mathfrak{n}_\Theta = \mathrm{Lie}(N_\Theta)$, and that if $\langle \phi(k), \psi(k) \rangle$ is an integrable function on K,

$$\langle \phi, \psi \rangle_\Theta = \int_{\bar{N}_\Theta} \langle \phi(n), \psi(n) \rangle dn \tag{6.4}$$

with a suitably normalized Haar measure dn on $\bar{N}_\Theta = \theta(N_\Theta)$.

Definition 6.1. For given functions $\phi \in \mathcal{B}(G/P_\Theta; \tau_\Theta) \otimes V_1$ and $\psi \in \mathcal{B}(G/P_\Theta; \tau^*_\Theta) \otimes V_2$ satisfying

$$\begin{aligned} \phi(q_1 x) &= \tau_1(q_1)\phi(x) \quad \text{for } q_1 \in Q_1, \\ \psi(q_2 x) &= \tau_2(q_2)\psi(x) \quad \text{for } q_2 \in Q_2, \end{aligned} \tag{6.5}$$

we call a $V_1 \otimes V_2$-valued function

$$\Phi_{\phi,\psi}(x) = \int_K \langle \phi(xk), \psi(k) \rangle dk \tag{6.6}$$

on G a *hypergeometric function.*

By the G-invariance of the bilinear form we have

$$\Phi_{\phi,\psi}(q_1xq_2) = \tau_1(q_1)\tau_2(q_2^{-1})f(x) \quad \text{for } (q_1,q_2) \in Q_1 \times Q_2 \tag{6.7}$$

and for elements D of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathbb{C} \otimes \mathrm{Lie}(G)$

$$\pi_D(\Phi_{\phi,\psi}) = 0 \quad \text{if } \pi_D(\phi) = 0. \tag{6.8}$$

Note that the following map defines a G-homomorphism (cf. (4.7)):

$$\mathcal{B}(G/P_\Theta;\tau_\Theta) \ni f \mapsto (\mathcal{T}_\psi f)(x) = \int_K \langle f(xk), \psi(k)\rangle dk \in \mathcal{B}(G/Q_2;\tau_2). \tag{6.9}$$

Example 6.2. Suppose P_Θ is a minimal parabolic subgroup. If Q_1 and Q_2 are maximal compact subgroups, the integral representations of the corresponding hypergeometric functions are Eisenstein integrals. In particular, if τ_1 and τ_2 are trivial representations, we have the integral representations of zonal spherical functions.

If the parameter of the representation becomes degenerate, the zonal spherical function satisfies more differential equations (cf. [Kr] for an example). When $G = GL(n,\mathbb{R})$ and the parameter corresponds to the degenerate principal series for $P_{\{1,n\}}$, it satisfies equations (3.5) with $\#I = \#J = 2$ (cf. Theorem 5.1) and the radial part of this zonal spherical function is given by Lauricella's F_D (cf. [E]).

Now we will consider the case where $G = GL(n,\mathbb{R})$ and furthermore P_Θ and Q_2 are parabolic subgroups. In particular, we examine the case where they are maximal, namely, $\Theta = \{\ell, n\}$ and $Q_2 = P_{\{k,n\}}$. Suppose $\ell < k < n$ and $n = m\ell$ with a positive integer m satisfying $m > 1$. To assure the existence of the nontrivial intertwining operator (6.9), we assume $\tau_\Theta = \tau_{\mu,0}$ with $\mu = (k,0) \in \mathbb{C}^2$. In this case, the integral transformation (4.11) with $w = \left(\begin{smallmatrix} 1 & 2 & \cdots & \ell & \ell+1 & \ell+2 & \cdots & k & k+1 & \cdots & n \\ k-\ell+1 & k-\ell+2 & \cdots & k & 1 & 2 & \cdots & k-\ell & k+1 & \cdots & n \end{smallmatrix}\right)$ converges for any continuous function $f \in \mathcal{B}(G/P_\Theta;\tau_\Theta)$. Note that the corresponding kernel function is a measure whose support is the compact double coset in $Q_2\backslash G/P_\Theta$ and $\bar{N}_w = \big\{\big(x_{ij}\big) \in G; x_{ij} = \delta_{ij} \ \text{ if } \ i \ge \ell \text{ or } j \le \ell \text{ or } j > k\big\}$.

Lemma 6.3. *Under the notation above we have a G-homomorphism*

$$\mathcal{T}_w : \mathcal{B}(G/P_{\{\ell,n\}};(k,0)) \to \mathcal{B}(G/P_{\{k,n\}};(\ell,0)). \tag{6.10}$$

Proof. Let $f \in \mathcal{B}(G/P_{\{\ell,n\}};(k,0))$. From (4.9), (4.12) and (4.13) we have $\mathcal{T}_w f \in \mathcal{B}(G/P;\mu)$ with $\mu = \ell(e_1+\cdots+e_k)$. If $k < i < j$, then $w^{-1}E_{ij}w$ commutes with any element of $\bar{N}_w$ and we have $E_{ij}f = 0$. By the natural identification $GL(k,\mathbb{R}) \simeq GL(k,\mathbb{R}) \otimes I_{n-k} \subset GL(n,\mathbb{R})$, we have an imbedding $\bar{N}_w \subset GL(k,\mathbb{R})$. Here I_{n-k} is the identity matrix of size $n-k$.

For a continuous function ϕ on $GL(k,\mathbb{R})$ satisfying

$$\phi(np) = |\det g_1|^{\beta_1}|\det g_2|^{\beta_2}\phi(n) \quad \text{for} \quad p = \left(\begin{smallmatrix} g_1 & 0 \\ * & g_2 \end{smallmatrix}\right) \in SL(k,\mathbb{R})$$

with $g_1 \in GL(\ell,\mathbb{R})$ and $g_2 \in GL(k-\ell,\mathbb{R})$, the integration $\int_{\bar{N}_w}\phi(n)dn$ or equivalently, $\int_{SO(k)}\phi(k)dk$ is left $SL(k,\mathbb{R})$-invariant if $\beta_2-\beta_1 = -k$, which corresponds to $-2\rho_\Theta$ (cf. (4.1) and (4.4)). Hence by putting $\phi(x) = f(gx)$ with $g \in G$, we clearly have the right $(SL(k,\mathbb{R})\otimes I_{n-k})$-invariance of $\mathcal{T}_w f$.

The invariances we have proved imply $\mathcal{T}_w f \in \mathcal{B}(G/P_{\{k,n\}};(\ell,0))$. $\square$

Through the anti-automorphism $G \ni g \mapsto g^{-1} \in G$, $\mathcal{B}(G/P_{\{\ell,n\}};(k,0))$ is canonically identified with the space of hyperfunctions ϕ on the ℓn-dimensional manifold

$$M(\ell,n) = \Big\{\Big(t_{ij}\Big) = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \multicolumn{3}{c}{\dotfill} \\ t_{\ell 1} & \cdots & t_{\ell n} \end{pmatrix};\ t_{ij} \in \mathbb{R} \ \text{ and } \ \operatorname{rank}\Big(t_{ij}\Big) = \ell\Big\} \tag{6.11}$$

which satisfy

$$\phi(g(t_{ij})) = |\det g|^{-k}\phi((t_{ij})) \quad \text{for} \quad g \in GL(\ell,\mathbb{R}). \tag{6.12}$$

Similarly $\mathcal{B}(G/P_{\{k,n\}};(\ell,0))$ is canonically identified with the space of hyperfunctions Φ on the kn-dimensional manifold

$$M(k,n) = \Big\{\Big(y_{ij}\Big) = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \multicolumn{3}{c}{\dotfill} \\ y_{k1} & \cdots & y_{kn} \end{pmatrix};\ y_{ij} \in \mathbb{R} \ \text{ and } \ \operatorname{rank}\Big(y_{ij}\Big) = k\Big\} \tag{6.13}$$

which satisfy

$$\Phi(g(y_{ij})) = |\det g|^{-\ell}\Phi((y_{ij})) \quad \text{for} \quad g \in GL(k,\mathbb{R}). \tag{6.14}$$

Denoting $GL(\ell,\mathbb{R})_+ = \{g \in GL(\ell,\mathbb{R}); \det g > 0\}$, we choose $Q_1 = GL(\ell,\mathbb{R})_+ \times \cdots \times GL(\ell,\mathbb{R})_+ \subset GL(n,\mathbb{R})$ and define τ_1 by

$$\tau_1(g) = (\det g_1)^{-\alpha_1}\cdots(\det g_m)^{-\alpha_m} \quad \text{for} \quad g = (g_1,\ldots,g_m) \in Q_1,$$

where $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{C}^m$ with the condition

$$\alpha_1 + \cdots + \alpha_m + k = 0. \tag{6.15}$$

Let $\epsilon = (\epsilon_1,\ldots,\epsilon_m) \in \{1,-1\}^m$. Using a function

$$|s|^{\alpha_p}_{\epsilon_p} = \begin{cases} |s|^{\alpha_p} & \text{if } \ \epsilon_p s > 0, \\ 0 & \text{if } \ \epsilon_p s \le 0 \end{cases} \tag{6.16}$$

on $\mathbb{R}$, we give a function

$$\phi(t) = \frac{1}{2}\Big(\prod_{p=1}^{m}\Big|\det\Big(t_{ij}\Big)_{\substack{1\le i\le\ell \\ (p-1)\ell<j\le p\ell}}\Big|^{\alpha_p}_{\epsilon_p} + \prod_{p=1}^{m}\Big|\det\Big(t_{ij}\Big)_{\substack{1\le i\le\ell \\ (p-1)\ell<j\le p\ell}}\Big|^{\alpha_p}_{-\epsilon_p}\Big) \tag{6.17}$$

in Definition 6.1, which belongs to $\mathcal{B}(G/P_{\{\ell,n\}};(k,0))$.

Put $x_{w(i)w(j)} = y_{ij}$ for $i = 1,\ldots,\ell$ and $j = 1,\ldots,n$. Then in $GL(n,\mathbb{R})$

$$\begin{pmatrix} t_{11} & t_{1k} & & \\ \cdots & \cdots & & \\ t_{\ell 1} & t_{\ell k} & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} w^{-1} \begin{pmatrix} y_{11} \cdots y_{1n} \\ \cdots\cdots \\ y_{k1} \cdots y_{kn} \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \end{pmatrix} = \begin{pmatrix} \sum_{\nu=1}^k t_{1\nu}x_{\nu 1} \cdots \sum_{\nu=1}^k t_{1\nu}x_{\nu n} \\ \cdots\cdots\cdots \\ \sum_{\nu=1}^k t_{\ell\nu}x_{\nu 1} \cdots \sum_{\nu=1}^k t_{\ell\nu}x_{\nu n} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{pmatrix} w^{-1} \tag{6.18}$$

and we have our hypergeometric function

$$\Phi(\alpha,\epsilon;x) = \int_{\mathbb{R}^{\ell(k-\ell)}} \prod_{p=1}^{m}\Big|\det\Big(\sum_{\nu=1}^{k} t_{i\nu}x_{\nu j}\Big)_{\substack{1\le i\le\ell \\ (p-1)\ell<j\le p\ell}}\Big|^{\alpha_p}_{\epsilon_p} \prod_{1\le i\le\ell<j\le k} dt_{ij} \tag{6.19}$$

with the convention

$$t_{ij} = \delta_{ij} \quad \text{if} \quad 1 \le j \le \ell. \tag{6.20}$$

Theorem 6.4. *The hypergeometric function $\Phi(\alpha,\epsilon;x)$ on $M(k,n)=\{(x_{ij});x_{ij}\in\mathbb{R},\ 1\le i\le k,\ 1\le j\le n,\ \operatorname{rank}(x_{ij})=k\}$ satisfies the following equations.*

i) *A right $GL(\ell,\mathbb{R})_+\times\cdots\times GL(\ell,\mathbb{R})_+$-invariance:*

$$\Phi(xg)=\Big(\prod_{p=1}^{m}(\det g_p)^{\alpha_p}\Big)\Phi(x)\quad \textit{for}\quad g=g_1\otimes\cdots\otimes g_m\quad \textit{with}\quad g_p\in GL(\ell,\mathbb{R})_+.$$

ii) *A left $GL(k,\mathbb{R})$-invariance:*

$$\Phi(gx)=|\det g|^{-\ell}\Phi(x)\quad \textit{for}\quad g\in GL(k,\mathbb{R}).$$

iii) *Generalized Capelli operators:*

$$\det\Big(\frac{\partial}{\partial x_{i_\mu j_\nu}}\Big)_{\substack{1\le\mu\le\ell+1\\1\le\nu\le\ell+1}}\Phi(x)=0 \tag{6.21}$$
$$\textit{for}\ \ 1\le i_1<\cdots<i_{\ell+1}\le k\ \textit{and}\ 1\le j_1<\cdots<j_{\ell+1}\le n.$$

Proof. The $GL(k,\mathbb{R})$-dependence is clear from our argument. Note that ϕ satisfies the left $GL(\ell,\mathbb{R})_+\times\cdots\times GL(\ell,\mathbb{R})_+$-invariance and the equations $D_{IJ}(-\ell)\phi=0$ for $\#I=\#J=\ell+1$ (cf. Theorem 3.1). The other equations follow from the the G-equivariance of the intertwining operator $\mathcal{T}_w$ combining with the coordinate transformation $G\ni g\mapsto g^{-1}$, Proposition 2.6 iii) and Proposition 3.2 ii). $\square$

The two invariances in Theorem 6.4 are infinitesimally as follows:

(6.22)
$$\sum_{i=1}^{k}x_{i,\mu+\ell p}\frac{\partial\Phi}{\partial x_{i,\nu+\ell p}}=\alpha_p\delta_{\mu\nu}\Phi\quad\text{for}\ \ 1\le\mu\le\ell,\ 1\le\nu\le\ell,\ 0\le p<m,$$
$$\sum_{\nu=1}^{n}x_{i\nu}\frac{\partial\Phi}{\partial x_{j\nu}}=-\ell\delta_{ij}\Phi\quad\text{for}\ \ 1\le i\le k,\ 1\le j\le k. \tag{6.23}$$

Note that $GL(n,\mathbb{R})/P_{\{\ell,n\}}\simeq O(n)/O(\ell)\times O(n-\ell)$. Hence the definition of $\Phi(\alpha,\epsilon;x)$ is an integration of a function on $O(n)/O(\ell)\times O(n-\ell)$ over its submanifold $O(k)/O(\ell)\times O(k-\ell)$.

Remark 6.5. Suppose $\ell=1$. The integration can be rewritten as

$$\Phi(\alpha,\epsilon;x)=C\int_{t_1^2+\cdots+t_k^2=1}\prod_{j=1}^{n}\Big|\sum_{p=1}^{k}t_px_{pj}\Big|_{\epsilon_p}^{\alpha_p}\hat{\omega} \tag{6.24}$$

with a suitable constant C and

$$\hat{\omega}=\sum_{p=1}^{k}(-1)^{p+1}t_pdt_1\wedge dt_2\wedge\cdots\wedge dt_{p-1}\wedge dt_{p+1}\wedge\cdots\wedge dt_k. \tag{6.25}$$

This integral representation coincides with the one given in [G] and the corresponding equations in Theorem 6.4 with $\ell=1$ are same as in [GG]. This hypergeometric function is also studied by [A].

Remark 6.6. Let (τ_p,V_p) be representation of $GL(\ell,\mathbb{R})$ for $p=1,\dots,m$ and put $\tau=(\tau_1,\dots,\tau_m)$ and $V=V_1\otimes\cdots\otimes V_m$. Choose $v\in V$ satisfying

$$\tau_1(g)\otimes\cdots\otimes\tau_m(g)v=|\det g|^kv\quad\text{for}\quad g\in GL(\ell,\mathbb{R}) \tag{6.26}$$

and define endomorphisms of V_p for $\ell \times \ell$-matrices (x_{ij}):

$$|\pi_p(x)|_{\epsilon_p} = \begin{cases} \tau_p^{-1}(x) & \text{if} \quad \epsilon_p \det x > 0, \\ 0 & \text{if} \quad \epsilon_p \det x \le 0. \end{cases} \tag{6.27}$$

We have V-valued hypergeometric functions

$$\begin{aligned}\Phi(\tau,\epsilon,v;x) = \int_{\mathbb{R}^{\ell(k-\ell)}} \Big|\pi_1\Big(\sum_{\nu=1}^{k} t_{i\nu}x_{\nu j}\Big)_{\substack{1\le i\le\ell\\ 0<j\le\ell}}\Big|_{\epsilon_1} \otimes \\ \cdots \otimes \Big|\pi_m\Big(\sum_{\nu=1}^{k} t_{i\nu}x_{\nu j}\Big)_{\substack{1\le i\le\ell\\ (m-1)\ell<j\le m\ell}}\Big|_{\epsilon_m} v \prod_{1\le i\le\ell<j\le k} dt_{ij}.\end{aligned} \tag{6.28}$$

Then $\Phi(\tau,\epsilon,v;x)$ satisfy

$$\Phi(xg) = \tau^{-1}(g)\Phi(x) \quad \text{for} \quad g \in GL(\ell,\mathbb{R})_+ \times \cdots \times GL(\ell,\mathbb{R})_+ \tag{6.29}$$

and the equations given in Theorem 6.4 ii) and iii).

For the analysis of hypergeometric functions in the case where $G = P_{\{\ell,n\}}$ and $Q_2 = P_{\{k,n\}}$, the following theorem is essential. Its proof will be given in §7.

Theorem 6.7. *The intertwining operator* (6.10) *in* Lemma 6.3 *is a topological G-isomorphism onto the solution space $V^{\ell}_{k,n}$ of the system* (6.21) *under our realization using* (6.13) *and* (6.14) *if*

$$0 < \ell < k < n \quad \textit{and} \quad \ell + k < n. \tag{6.30}$$

Corollary 6.8. *Let Q be a closed subgroup of $GL(n,\mathbb{R})$ and let (τ, V) be a finite dimensional representation of Q. If* (6.30) *holds, the integral transformation $\mathcal{R} : \phi \mapsto \int_{SO(k)} \phi(kx)dk$ is a bijection between $S(\ell,n;\tau)$ and $S(k,n;\ell,\tau)$. Here $S(\ell,n;\tau)$ is the space of V-valued hyperfunctions ϕ on $M(\ell,n)$ satisfying* (6.12) *and*

$$\phi(tg) = \tau(g^{-1})\phi(t) \quad \textit{for} \;\; g \in Q. \tag{6.31}$$

Moreover $S(k,n;\ell,\tau)$ is the space of V-valued hyperfunctions $\Phi(x)$ on $M(k,n)$ satisfying the equations given in Theorem 6.4 ii) *and* iii) *and*

$$\Phi(xg) = \tau(g^{-1})\Phi(x) \quad \textit{for} \;\; g \in Q. \tag{6.32}$$

In this corollary 'hyperfunctions' can be replaced by 'Schwartz's distributions' or 'C^∞-functions' or 'real analytic functions', which is clear by our way of the proof.

Lastly we give other examples of Q_1 for the same P_Θ and Q_2.

Example 6.9. For $A = (A_{ij}) \in \mathfrak{gl}(n,\mathbb{R})$ with $n = m\ell$ and for $\mu = 1,\dots,m$ and $\nu = 1,\dots,m$, put

$$\begin{aligned}\mathfrak{u}_1 = \{A \in \mathfrak{gl}(n,\mathbb{R});\, M_{\mu\nu}(A) = 0 \;\text{ if }\; \mu > \nu,\\ M_{\nu,\nu+i}(A) = M_{\nu+1,\nu+i+1}(A) \;\text{ for }\; i = 1,\dots,m-2 \;\text{ and }\; \nu = 1,\dots,m-i-1\},\\ M_{\mu\nu}(A) = (A_{(\mu-1)\ell+i,(\nu-1)\ell+j})_{\substack{1\le i\le\ell\\1\le j\le\ell}}.\end{aligned}$$

If Q_1 equals the closed subgroup of $GL(n,\mathbb{R})$ with the Lie algebra $\mathfrak{u}_1$, condition (6.2) is valid and we can define hypergeometric functions with respect to the character of Q_1 which has $m-1$ continuous parameters. When $\ell = 1$, the hypergeometric functions correspond to those discussed in [GRS] and [KHT].

7. Radon transforms

Retain the notation in the previous section and put $X_j = O(n)/O(j) \times O(n-j)$ with $0 < j < n$. Since $G = KP_\Theta = KQ_2$ with $K = O(n)$, we may restrict the intertwining operator (6.10) on K and the restriction

$$\mathcal{R}_k^\ell : \mathcal{B}(X_\ell) \ni \phi \mapsto (\mathcal{R}_k^\ell \phi)(g) = \int_{O(k)} \phi(gk)dk \in \mathcal{B}(X_k) \tag{7.1}$$

is a Radon transform for a real Grassmannian (cf. [H4, Chap. 1]). Here $O(k) \simeq O(k) \otimes I_{n-k} \subset O(n)$ and we assume (6.30).

The Radon transforms, in particular, their inversions and the range characterization, were originated by [F], [R] and [Jf] in special cases and later studied by [H1], [GGR], [Gr1],[Gr2], [Go1], [Go2], [Ka], [I] etc. in more general cases. The characterization of $\operatorname{Im}\mathcal{R}_k^\ell$ stated in Theorem 6.7 is not clear from these references (cf. [Gr1]) and hence in this section we will give the proof of Theorem 6.7 for the sake of completeness.

Fix an irreducible representation π_Λ of $O(n)$. Note that the dimension of $O(n)$-homomorphism of π_Λ to $\mathcal{B}(X_j)$ is at most one because X_j are connected symmetric spaces. Put $\tilde{X}_j = SO(n)/SO(j) \times SO(n-j)$ and $g_o = \operatorname{diag}(-1, 1 \ldots, 1, -1) \in SO(n)$. Then $\tilde{X}_j$ is a universal covering of X_j and the fundamental group of X_j equals $\mathbb{Z}/2\mathbb{Z}$. The function on X_j is identified with the function on $\tilde{X}_j$ which is invariant under the involution $SO(n) \ni x \mapsto g_o x g_o$.

Suppose π_Λ is isomorphic to $V_\Lambda \subset \mathcal{B}(X_\ell)$. Thanks to the assumption (6.30), Cartan-Helgason's theorem (cf. [Wa, Theorem 3.3.11]) says that V_Λ has an $O(k) \times O(n-k)$-fixed vector ϕ_Λ, which can be normalized by $\phi_\Lambda(e) = 1$ because of Lemma 7.1. Then $\mathcal{R}_k^\ell \phi_\Lambda(e) = 1$ and therefore $\mathcal{R}_k^\ell V_\Lambda \neq \{0\}$. Since $\operatorname{Ker}\mathcal{R}_k^\ell$ is $O(n)$-invariant, the map $\mathcal{R}_k^\ell$ is injective.

Let n' be a maximal positive integer with $2n' \leq n$. Put $F_{\mu\nu} = E_{\mu\nu} - E_{\nu\mu}$ and $\bar{\nu} = \nu + (n - n')$. Let $\mathfrak{t}$ be a maximal torus of $\mathfrak{o}(n, \mathbb{C})$ spanned by $H_\nu = F_{\nu\bar{\nu}}$ for $\nu = 1, \ldots, n'$ and define $f_\mu \in \mathfrak{t}_\mathbb{C}^*$ by $f_\mu(H_\nu) = -i\delta_{\mu\nu}$. Then $\{f_1 - f_2, \ldots, f_{n-1} - f_n, g_n\}$ is a fundamental system of the roots for the pair $(\mathfrak{o}(n), \mathfrak{t})$, where $g_n = f_{n-1} + f_n$ if $2n' = n$ and $g_n = f_n$ otherwise. Moreover for $1 \leq \mu < \nu \leq n'$, $X_{\mu\nu}^\pm = Y_{\mu\nu}^+ - (\pm Y_{\mu\nu}^-)$ are root vectors for the positive roots $f_\mu \pm f_\nu$, respectively, by putting $Y_{\mu\nu}^+ = F_{\mu\bar{\nu}} - iF_{\bar{\mu}\bar{\nu}}$ and $Y_{\mu\nu}^- = F_{\nu\bar{\mu}} - iF_{\mu\nu}$ (cf. [Kn, Chap. IV §1 Example 2]).

Note that $V_{k,n}^\ell$ is $O(n)$-invariant. Suppose $k \leq \frac{n}{2}$ and π_Λ is contained in $V_{k,n}^\ell$. Let $\lambda_1 f_1 + \cdots + \lambda_{n'} f_{n'}$ be the corresponding highest weight and let v_Λ be the highest weight vector in $V_{k,n}^\ell$. Cartan-Helgason's theorem and the covering map $\tilde{X}_j \to X_j$ say that $\lambda_1 \geq \cdots \geq \lambda_k \geq \lambda_{k+1} = \cdots = \lambda_{n'} = 0$ and that λ_j are even integers (cf. [Gr1], [Sr]). Suppose there exists ν satisfying $\lambda_\nu \neq 0$ and $\ell < \nu \leq k$. Then (7.2) proves $D_{\{1,\ldots,\ell,\nu\}\{\bar{1},\ldots,\bar{\ell},\bar{\nu}\}} v_\lambda(e) = (-\lambda_1 i - \ell i) \cdots (-\lambda_\ell i - i)(-\lambda_\nu i) v_\lambda(e)$ and therefore $v_\lambda(e) = 0$. This means $v_\lambda = 0$ because v_λ is real analytic and $\mathbb{C} \otimes \mathfrak{g} = \mathbb{C} \otimes \operatorname{Lie}(P_{\{k,n\}}) + \mathbb{C} \otimes \mathfrak{t} + \sum_{1 \leq \mu < \nu \leq n'} \mathbb{C}X_{\mu\nu}^+ + \mathbb{C}X_{\mu\nu}^-$. Thus we have $\lambda_\nu = 0$ for $\nu > \ell$. Using again Cartan-Helgason's theorem, we can conclude that π_Λ has an $O(\ell) \times O(n-\ell)$-fixed vector and that $\operatorname{Im}\mathcal{R}_k^\ell$ is dense in $V_{k,n}^\ell$. By the isomorphism $X_k \simeq X_{n-k}$, the same conclusion holds in the case where $k \geq \frac{n}{2}$. Then through the imbedding (4.8) as a closed subspace of the Fréchet-Schwartz space $\mathcal{B}(K)$ (cf. [Km]), Theorem 6.7 follows from Lemma 7.3 and the open mapping theorem.

Lemma 7.1. *The intertwining function* ϕ_Λ (cf. [Ho]) *satisfies* $\phi_\Lambda(e) \neq 0$.

Proof. We will prove the lemma in the same way as in the proof of [OS, Proposition 4.2]. So suppose $\phi_\Lambda(e) = 0$, put $\mathfrak{k}_j = \mathfrak{o}(j) \oplus \mathfrak{o}(n-j) \subset \mathfrak{o}(n)$ and let $\mathfrak{q}_j$ be the orthogonal compliment of $\mathfrak{k}_j$ in $\mathfrak{o}(n)$ with respect to the Killing form. Put $\mathfrak{g}' = \mathfrak{k}_\ell \cap \mathfrak{k}_k + \mathfrak{q}_\ell \cap \mathfrak{q}_k \simeq \mathfrak{o}(n-k+\ell) \oplus \mathfrak{o}(k-\ell)$. Fix a maximal abelian subspace $\mathfrak{t}_\ell$ of $\mathfrak{q}_\ell \cap \mathfrak{q}_k$. Note that $\dim \mathfrak{t}_\ell = \ell$.

Let $D \in U(\mathfrak{o}(n))$ and define $D' \in U(\mathfrak{g}')$ with $D - D' \in (\mathfrak{k}_k \cap \mathfrak{q}_\ell)U(\mathfrak{g}) + U(\mathfrak{g})(\mathfrak{k}_\ell \cap \mathfrak{q}_k)$ and put $\bar{D} = \int_{O(\ell) \times O(k-\ell) \times O(n-k)} \operatorname{Ad}(k) D' dk$. Then $(D\phi_\Lambda)(e) = (D'\phi_\Lambda)(e) = (\bar{D}\phi_\Lambda)(e)$.

Note that $\bar{D}$ defines an invariant differential operator on the symmetric space $X_o = O(n-k+\ell) \times O(k-\ell)/O(\ell) \times O(n-k) \times O(k-\ell)$. Moreover if $D_o \in U(\mathfrak{o}(n))$ is $\mathfrak{k}_\ell$-invariant, then D'_o is $\mathfrak{k}_\ell \cap \mathfrak{k}_k$-invariant. Note that the restricted root system for X_ℓ and that of X_o are of type B_ℓ. Hence we can choose a $\mathfrak{k}_\ell$-invariant element $\tilde{D} \in U(\mathfrak{o}(n))$ such that $\tilde{D}'$ and $\bar{D}$ define the same element of $U(\mathfrak{t}_\ell)$ under the Harish-Chandra homomorphism associated to the symmetric space X_o. Since there exists $\lambda \in \mathbb{C}$ with $\tilde{D}\phi_\Lambda = \lambda\phi_\Lambda$, we have $(D\phi_\Lambda)(e) = (\tilde{D}'\phi_\Lambda)(e) = (\tilde{D}\phi_\Lambda)(e) = \lambda\phi_\Lambda(e) = 0$. Since ϕ_Λ is a real analytic function on a connected manifold, this implies $\phi_\Lambda = 0$ and leads a contradiction. $\square$

Lemma 7.2. *For the set* $\{\nu_1, \dots, \nu_{\ell+1}\}$ *of positive integers satisfying* $1 \le \nu_1 < \dots < \nu_{\ell+1} \le \frac{n}{2}$,

$$
(7.2) \quad D_{\{\nu_1,\nu_2,\dots,\nu_{\ell+1}\}\{\overline{\nu_1},\overline{\nu_2},\dots,\overline{\nu_{\ell+1}}\}} \equiv (H_{\nu_1} - \ell i)(H_{\nu_2} - (\ell-1)i)\cdots(H_{\nu_{\ell+1}})
$$
$$
\mod \sum_{1\le p<q\le \ell+1} (F_{\nu_p\nu_q}\mathfrak{U} + F_{\overline{\nu_p}\,\overline{\nu_q}}\mathfrak{U} + \mathfrak{U}X^-_{\nu_p\nu_q} + \mathfrak{U}X^+_{\nu_p\nu_q}) + \sum_{\substack{1\le p\le \ell+1\\ 1\le q\le \ell+1}} E_{\overline{\nu_p}\nu_q}\mathfrak{U}.
$$

Proof. We show (7.2) by the induction on ℓ. We may assume $\nu_j = j$. Then

$$
\begin{aligned}
D_{\{1,\dots,\ell+1\}\{\bar{1},\dots,\overline{\ell+1}\}} &= \sum_{p=1}^{\ell+1}(-1)^{\ell+j+1} D_{\{1,\dots,p-1,p+1,\dots,\ell+1\}\{\bar{1},\dots,\bar{\ell}\}} E_{p,\overline{\ell+1}} \\
&\equiv \sum_{p=1}^{\ell}(-1)^{\ell+p+1} D_{\{1,\dots,p-1,p+1,\dots,\ell+1\}\{\bar{1},\dots,\bar{\ell}\}} (E_{\overline{\ell+1},p} + iE_{\overline{p},\overline{\ell+1}} - iE_{\overline{\ell+1},\overline{p}}) \\
&\quad + (H_1 - (\ell-1)i)\cdots(H_{\ell-1} - i)(H_\ell)H_{\ell+1} \\
&\equiv i\sum_{p=1}^{\ell}(-1)^{\ell+p+1} D_{\{1,\dots,p-1,p+1,\dots,\ell+1\}\{\bar{1},\dots,\overline{p-1},\overline{\ell+1},\overline{p+1},\cdots,\bar{\ell}\}} \\
&\quad + (H_1 - (\ell-1)i)\cdots(H_{\ell-1} - i)(H_\ell)H_{\ell+1} \\
&\equiv -i\sum_{p=1}^{\ell}(H_1 - (\ell-1)i)\cdots(H_{p-1} - (\ell-p+1)i)(H_{p+1} - (\ell-p)i)\cdots H_{\ell+1} \\
&\quad + (H_1 - (\ell-1)i)\cdots(H_{\ell-1} - i)(H_\ell)H_{\ell+1} \\
&= (H_1 - \ell i)\cdots(H_\ell - i)H_{\ell+1}. \quad \square
\end{aligned}
$$

Lemma 7.3. i) *Consider the intertwining operator*

$$
\mathcal{T}^t_w : \mathcal{B}(G/P; \mu + \rho t, \epsilon) \to \mathcal{B}(G/P; w(\mu + \rho + \rho t) - \rho, \epsilon')
$$

in (4.12) *with a parameter* $t \in \mathbb{C}$. *Fix an integer* N *such that* $\tilde{\mathcal{T}}^t_w = t^N \mathcal{T}^t_w$ *holomorphically depends on* t *for* $|t| \ll 1$. *Then the image of* $\tilde{\mathcal{T}}^0_w$ *is closed in* $\mathcal{B}(G/P; w(\mu + \rho) - \rho, \epsilon')$.

ii) *The same result as above holds even if we replace the space of hyperfunctions by other functions spaces, such as Schwartz's distributions,* C^∞*-functions or real analytic functions.*

Proof. Suppose $|t| \ll 1$. Consider the inverse intertwining operator

$$
(7.3) \qquad \mathcal{T}^t_{w^{-1}} : \mathcal{B}(G/P; w(\mu + \rho + \rho t) - \rho, \epsilon') \to \mathcal{B}(G/P; \mu + \rho t, \epsilon)
$$

and fix a positive integer N' so that $\mathcal{S}^t_w = t^{N'}\mathcal{T}^t_{w^{-1}}$ is holomorphic for t. We identify these spaces of hyperfunctions with subspaces of $\mathcal{B}(K)$ which do not depend on t. Since

$\tilde{\mathcal{T}}_w^t$ and $\mathcal{S}_w^t$ are topological G-isomorphisms for $t \neq 0$, there exists a nonzero holomorphic function $c(t)$ such that $\mathcal{S}_w^t \circ \tilde{\mathcal{T}}_w^t = c(t)\,\mathrm{id}$. Let m be the order of zero of $c(t)$ at $t = 0$. Then $\operatorname{Im} \tilde{\mathcal{T}}_w^0 = \bigcap_{\nu=0}^{m-1} \operatorname{Ker}(\frac{d^\nu}{dt^\nu}\mathcal{S}_w^t|_{t=0})$ is closed in $\mathcal{B}(K)$.

ii) is clear because our proof similarly works on other function spaces. □

Remark 7.4. i) A proof of the injectivity of Radon transforms is given in [Gr1, §6]. But the proof seems to be insufficient since the conclusion $\phi_\Lambda(e) \neq 0$ in Lemma 7.1 is stated just as a consequence of the Frobenius reciprocity theorem.

ii) Theorem 6.7 (or Corollary 6.8 with $Q = \{e\}$) characterizes the image of the Radon transform $\mathcal{R}_k^\ell$ on the real Grassmann manifold X_ℓ (cf. (7.1)). Note that our proof naturally gives an inversion formula (cf. [GGR]). In fact, $c(t)$ in the proof of Lemma 7.3 is known (cf. [GK], [Kn, Chap. VII §5]).

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