Resolution of singularities of KZ-type equations

Toshio OSHIMA（大島利雄）

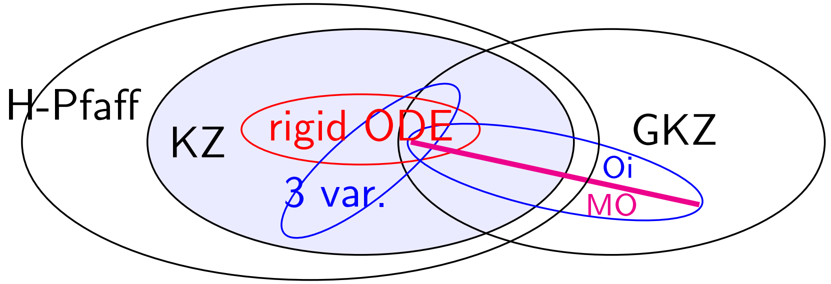
University of Tokyo

Differential equations seminar

International Workshop on SIT

January 15, 2026

KZ-type equation



$$\mathcal{M}: \frac{\partial u}{\partial x_i} = \sum_{\substack{1\le\nu\le n\\ \nu\ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u \qquad (i \in L_n := \{1,\dots,n\})$$

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \ A_{ij} = A_{ji} \in \mathrm{Mat}(N,\mathbb{C}),\ A_{ii} = 0$$

with integrability condition $[A_{ij}, A_{k\ell}] = [A_{ij}, A_{ik} + A_{jk}] = 0$

KZ-type eq. $\hookleftarrow$ rigid Fuchsian ODE (Haraoka) $\frac{du}{dx} = \frac{A_{12}}{x-x_1}u + \cdots + \frac{A_{1n}}{x-x_n}u$

middle convolutions and additions $\to$ accessory parameters and local structures

KZ-type eq. : $\mathrm{Sp}\,\mathcal{M}$ $\leftarrow$ ODE: spectral type / (generalized) Riemann scheme

$$A_{i\infty} := -(A_{i1} + A_{i2} + \cdots + A_{in}) \qquad (i \in L_n)$$

$$\widetilde{L}_n := L_n \cup \{\infty\}, \quad A_i = A_\emptyset = 0$$

$$A_{i_1\dots i_k} := \sum_{1\le p<q\le k} A_{i_p i_q} \quad (\{i_1,\dots,i_k\} \subset \widetilde{L}_n)$$ (generalized) residue matrices

$$\boxed{[A_I, A_J] = 0 \quad \big(I \subset J \text{ or } I \supset J \text{ or } I \cap J = \emptyset,\ I, J \subset \widetilde{L}_n\big)}$$

$[A_I, A_{L_n}] = 0 \ \Rightarrow\ A_{L_n} = \kappa$: scalar if $\mathcal{M}$ is irreducible $\quad (\kappa = 0:$ homogeneous$)$

$$[A_{12}, A_{1234\ldots k}] = [A_{12}, A_{12} + \sum_{j=3}^{k}(A_{1j} + A_{2j}) + \sum_{2\le i<j\le k} A_{ij}] = 0,$$

$$\sum_{i=1}^{n} A_{i\infty} = \sum_{i=1}^{n}\Big(-\sum_{j=1}^{n} A_{ij}\Big) = -2A_{1\cdots n} = -2A_{L_n},$$

$$A_{k\ldots n\infty} = A_{k\ldots n} + \sum_{i=k}^{n} A_{i\infty} = \frac{1}{2}\sum_{i=k}^{n}\sum_{j=k}^{n} A_{ij} - \sum_{i=k}^{n}\sum_{j=1}^{n} A_{ij}$$

$$= -\sum_{i=k}^{n}\sum_{j=1}^{k-1} A_{ij} - \frac{1}{2}\sum_{i=k}^{n}\sum_{j=k}^{n} A_{ij},$$

$$A_{1\ldots k-1} - A_{k\ldots n\infty} = \frac{1}{2}\Big(\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} A_{ij} + 2\sum_{i=k}^{n}\sum_{j=1}^{k-1} A_{ij} + \sum_{i=k}^{n}\sum_{j=k}^{n} A_{ij}\Big) = A_{L_n}$$

- $I \subset L_n \quad \Rightarrow \quad A_I = A_{\tilde{L}_n \setminus I} + \kappa$

- $[A, B] = 0 \;\Rightarrow\; \exists$ Simultaneous (generalized) eigenvalue class :

$$A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 3 \end{pmatrix} \quad B = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 4 \end{pmatrix} \quad \Rightarrow \; [A] = \{[1]_1, [2]_2, [3]_1\} = \{1, [2]_2, 3\}$$

$$[A : B] = \{[1:0]_1, [2:0]_2, [3:4]_1\} = \{[1:0], [2:0]_2, [3,4]\}$$

- $[B_i, B_j] = 0 \;\; (i, j = 1, .., r) \;\Rightarrow\; [B_1 : \cdots : B_r] = \{[\lambda_{1,1} : \cdots : \lambda_{r,1}]_{m_1}, \cdots\}$

Def. $\mathcal{I} = \{I_\nu \mid \nu = 1, \ldots, r\}$ is a commuting family of subsets of a finite set L

$\overset{\text{def.}}{\iff}$ $I_\nu \subset L$, $|I_\nu| > 1$ and ($I_\nu \subset I_{\nu'}$ or $I_\nu \supset I_{\nu'}$ or $I_\nu \cap I_{\nu'} = \emptyset$)

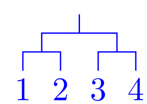
$\mathcal{I}$ is maximal $\overset{\text{def.}}{\iff}$ ($\mathcal{I}$, $\mathcal{I}'$ are these families with $\mathcal{I} \subset \mathcal{I}' \Rightarrow \mathcal{I} = \mathcal{I}'$)

Def. $\mathcal{L}_n := \{$maximal commuting families of subsets of $L_n\}$

$$\boxed{\operatorname{Sp}\mathcal{M} := \left\{[A_{I_1} : \cdots : A_{I_{n-1}}] \,\middle|\, \mathcal{I} = \{I_1, \ldots, I_{n-2}, I_{n-1} = L_n\}\right\}_{\mathcal{I} \in \mathcal{L}_n}}$$

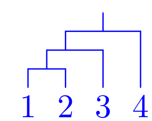
Fact. $\mathcal{L}_n \simeq \{$single-elimination tournaments of teams labeled by $L_n\}$

$$|\mathcal{L}_n| = (2n-3)!! \qquad |\mathcal{I}| = n - 1 \quad (\mathcal{I} \in \mathcal{L}_n)$$



: $\frac{4!}{2^3} = 3$ cases

$\{\{1,2\},\{3,4\},\{1,2,3,4\}\}$



: $\frac{4!}{2} = 12$ cases

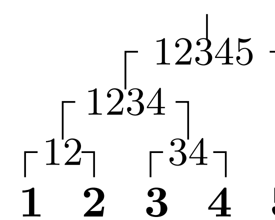
$\{\{1,2\},\{1,2,3\},\{1,2,3,4\}\}$

Ex. $L_4 = \{1,2,3,4\}$, $|\mathcal{L}_4| = 15$

$$\operatorname{Sp}\mathcal{M} = \left\{[A_{ij} : A_{k\ell} : A_{ijk\ell}],\ [A_{ij} : A_{ijk} : A_{ijk\ell}] \,\middle|\, \{i,j,k,\ell\} = \{1,2,3,4\}\right\}$$

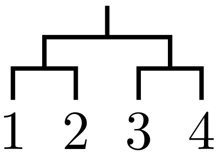
$$L_5 = \{1,2,3,4,5\}, \quad |\mathcal{L}_5| = 105$$

$$\mathcal{I} = \left\{\{1,2\},\{1,2,3,4\},\{3,4\},\{1,2,3,4,5\}\right\} \in \mathcal{L}_5$$



$$\left\{[A_{ij} : A_{k\ell} : A_{ijk\ell}],\ [A_{ij} : A_{ijk} : A_{\ell m}],\ [A_{ij} : A_{ijk} : A_{ijk\ell}] \,\middle|\, \{i,j,k,\ell,m\} = L_5\right\}$$

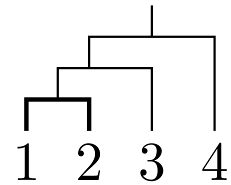
4 teams



: 1 pattern

: $\frac{4!}{2^3} = 3$ cases

$\{\{1,2\},\{3,4\}\}$



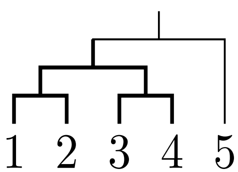
: 4 patterns

: $\frac{4!}{2} = 12$ cases

$\{\{1,2,3\},\{1,2\}\}$

2 types, 5 (=1+4) patterns, 15 (=3+12) tournaments, 4 (=1+3) win-types

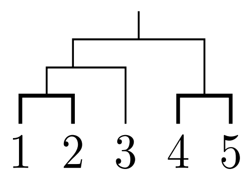
5 teams



: 2 patterns

: $\frac{5!}{2^3} = 15$ cases

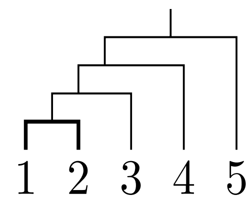
$\Big\{\{1,2\},\{1,2,3,4\},\{3,4\}\Big\}$



: 4 patterns

: $\frac{5!}{2^2} = 30$ cases

$\Big\{\{1,2,3\},\{1,2\},\{4,5\}\Big\}$



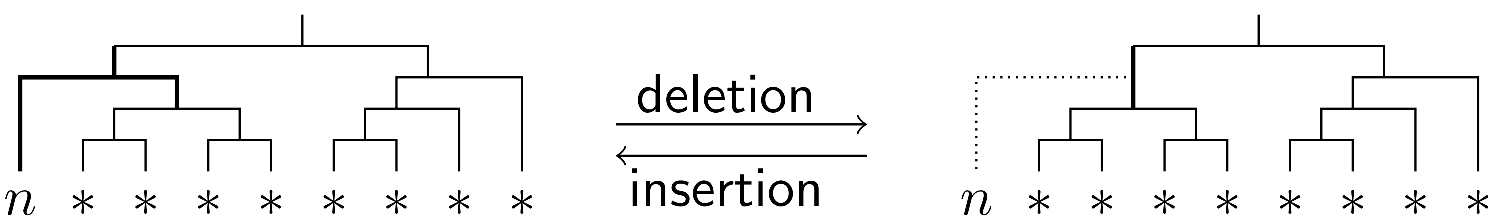
: 8 patterns

: $\frac{5!}{2} = 60$ cases

$\Big\{\{1,2,3\},\{1,2\},\{1,2,3,4\}\Big\}$

3 types, 14 patterns, 105 (=15+30+60) tournaments, 9 (=2+3+4) win-types

n teams $\qquad\qquad$ $n-1$ teams



Desingularization

KZ-type equation (Pfaffian form) : $du = \Omega u, \quad \Omega = \sum_{1 \le i < j \le n} A_{ij}\, d\log(x_i - x_j)$

$\mathcal{I} = \{I_1, \ldots, I_{n-2}, L_n\} \in \mathcal{L}_n$: a maximal commuting family of L_n

$J,\ J' \in \tilde{\mathcal{I}} := \mathcal{I} \cup \bigcup_{\nu \in L_n} \{\{\nu\}\}$ with $L_n = J \sqcup J'$: semi-final games

$$J = \{j_0, \ldots, j_k\},\ J' = \{j'_0, \ldots, j'_{k'}\} \quad (k + k' = n - 2)$$

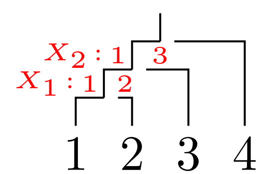
A singular point : $x_{j_0} = \cdots = x_{j_k}$ and $x_{j'_0} = \cdots = x_{j'_{k'}} \quad (k,\ k' \ge 0)$

A local coordinate : $(x_{j_1}, \ldots, x_{j_k}, x_{j'_1}, \ldots, x_{j'_{k'}})$ with $x_{j_0} = 0$ and $x_{j'_0} = 1$

$\{n_i, n'_i\}$: players of the game I_i (determined after the tournament)
(n_i is the loser of I_i)

Def. $X = (X_1, \ldots, X_{n-2})$: local coordinate defined by

$$\boxed{x_{n_i} - x_{n'_i} = \prod_{I_i \subset I_\nu \neq L_n} X_\nu}$$



Theorem. $\Omega - \sum_{i=1}^{n-2} A_{I_i}\, d\log X_i$ is non-singular around the origin of X

Lemma. $I^o_{i,j}$: minimal subset in $\mathcal{I}$ for $i,\, j \in L_n$

$x_I := x_{n_i} - x_{n'_i}$ $(I = I_i)$ with $I \in \mathcal{I}$ are linearly independent over $\mathbb{C}$ and

$$x_i - x_j = \sum_{I\in\mathcal{I}} \epsilon^I_{i,j} x_I \quad \text{with} \quad \begin{cases} \epsilon^I_{i,j} = 0 & (I \supsetneqq I^o_{i,j} \text{ or } I \cap I^o_{i,j} = \emptyset) \\ \epsilon^I_{i,j} \in \{1, -1\} & (I = I^o_{i,j}) \\ \epsilon^I_{i,j} \in \mathbb{Z} & (I \subsetneqq I^o_{i,j}) \end{cases}$$

Proof of Lemma: Induction on $|I^o_{i,j}|$. $i = i_0$ or $I^o_{i,i_0} \subsetneqq I^o_{i,j}$, $\;j = j_0$ or $I^o_{j,j_0} \subsetneqq I^o_{i,j}$.

$$x_i - x_j = (x_{i_0} - x_{j_0}) + (x_i - x_{i_0}) - (x_j - x_{j_0}).$$

Lemma shows $$x_i - x_j = f_{i,j}(X) \cdot \prod_{\{i,j\}\subset I\in\mathcal{I}\setminus\{L_n\}} X_I$$

$f_{i,j}$: a polynomial of X_I ($I \in \mathcal{I}$ and $I_{i,j} \subset I \neq L_n$) with $f_{i,j}(0) = 1$ or -1.

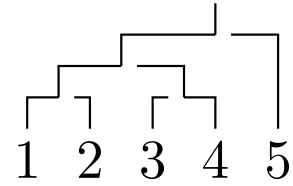
Note that $\{i,j\} \subset I \in \mathcal{I} \Leftrightarrow I^o_{i,j} \subset I \in \mathcal{I}$ and $d\log(fg) = d\log f + d\log g$.

Hence,

$$\sum_{1\le i<j\le n} A_{ij} \frac{d(x_i - x_j)}{x_i - x_j} - \sum_{1\le i<j\le n} \sum_{\{i,j\}\subset I\in\mathcal{I}\setminus\{L_n\}} A_{ij} \frac{dX_I}{X_I}$$

is regular in a neighborhood of the origin.

$$\mathcal{I} = \{\{1,2\},\{3,4\},\{1,2,3,4\},\{1,2,3,4,5\}\}$$



$$x_4 - x_1 = x_{1234} = X_{1234}, \quad \frac{d(x_1 - x_4)}{x_1 - x_4} = \frac{dX_{1234}}{X_{1234}}$$

$$x_2 - x_1 = x_{12} = X_{12}X_{1234}, \quad \frac{d(x_1 - x_2)}{x_1 - x_2} = \frac{dX_{12}}{X_{12}} + \frac{dX_{1234}}{X_{1234}}$$

$$x_3 - x_4 = x_{34} = X_{34}X_{1234}, \quad \frac{d(x_3 - x_4)}{x_3 - x_4} = \frac{dX_{34}}{X_{34}} + \frac{dX_{1234}}{X_{1234}}$$

$$I^o_{13} = I^o_{23} = I^o_{24} = \{1234\}$$

$$x_1 - x_3 = (x_1 - x_4) - (x_3 - x_4) = -x_{1234} - x_{34} = -(1 + X_{34})X_{1234}$$

$$x_2 - x_3 = (x_2 - x_1) - (x_3 - x_4) + (x_1 - x_4) = x_{12} - x_{34} - x_{1234}$$

$$= -(1 - X_{12} + X_{34})X_{1234}$$

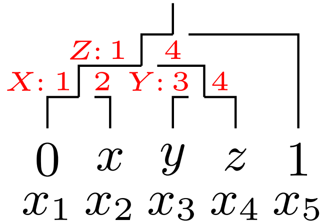
$$x_2 - x_4 = (x_2 - x_1) - (x_4 - x_1) = -x_{12} - x_{1234} = -(1 + X_{12})X_{1234}$$

$$\frac{d(x_1 - x_3)}{x_1 - x_3} \equiv \frac{d(x_2 - x_3)}{x_2 - x_3} \equiv \frac{d(x_2 - x_4)}{x_2 - x_4} \equiv \frac{dX_{1234}}{X_{1234}}, \quad \frac{d(1 + X_{34})}{1 + X_{34}} \equiv 0$$

$$\Omega = \sum_{1 \le i < j \le 5} A_{ij} \frac{d(x_i - x_j)}{x_i - x_j} \equiv A_{12}\frac{dX_{12}}{X_{12}} + A_{34}\frac{dX_{34}}{X_{34}} + A_{1234}\frac{dX_{1234}}{X_{1234}}$$

$$(X, Y, Z) = (X_{12}, X_{34}, X_{1234}), \quad (x_1, x_2, x_3, x_4) = (0, x, y, z, 1)$$

Ex. $(x,y,z)=(0,0,0)$:



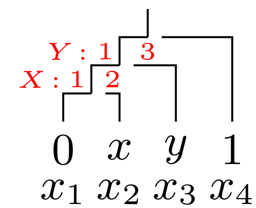
$$\begin{cases} x_4-x_1=z=Z \\ x_2-x_1=x=XZ \\ x_3-x_4=y-z=YZ \end{cases}$$

$$\begin{cases} X=\frac{x}{z} \\ Y=\frac{y-z}{z} \\ Z=z \end{cases} \qquad \begin{cases} y=(1+Y)Z \\ x-y=(X-Y-1)Z \\ x-z=(X-1)Z \end{cases}$$

$$\Omega \equiv \Omega' = A_{x0}\frac{dX}{X}+A_{yz}\frac{dY}{Y}+A_{xyz0}\frac{dZ}{Z} \qquad (|x|,\ |y-z| \ll |z| \ll 1)$$

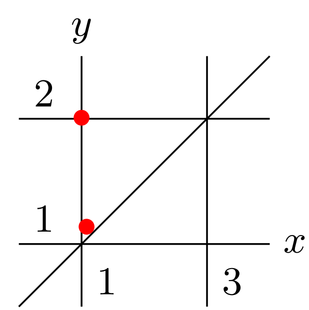
Ex. $\Omega = A_{x0}\frac{dx}{x}+A_{y0}\frac{dy}{y}+A_{xy}\frac{d(x-y)}{x-y}+A_{x1}\frac{d(x-1)}{x-1}+A_{y1}\frac{d(y-1)}{y-1} \qquad (n=4)$

- $(x,y)=(0,0)$: $\begin{cases} x_3-x_1=y=Y \\ x_2-x_1=x=XY \end{cases} \quad \begin{cases} x_2-x_3=(X-1)Y \\ (X,Y)=(\frac{x}{y},y) \end{cases}$

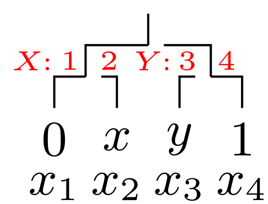


$$\frac{dx}{x}=\frac{dX}{X}+\frac{dY}{Y},\quad \frac{dy}{y}=\frac{dY}{Y},\quad \frac{d(x-y)}{x-y}=\frac{dY}{Y}+\frac{d(X-1)}{X-1},\quad \{I_1,I_2\}=\{\underbrace{\{1,2\}}_{X},\underbrace{\{1,2,3\}}_{Y}\}$$

$$\Omega' := \sum_{i=1}^{n-2} A_{I_i}\,d\log X_i = A_{x0}\frac{dX}{X}+A_{xy0}\frac{dY}{Y} \qquad (|x| \ll |y| \ll 1)$$



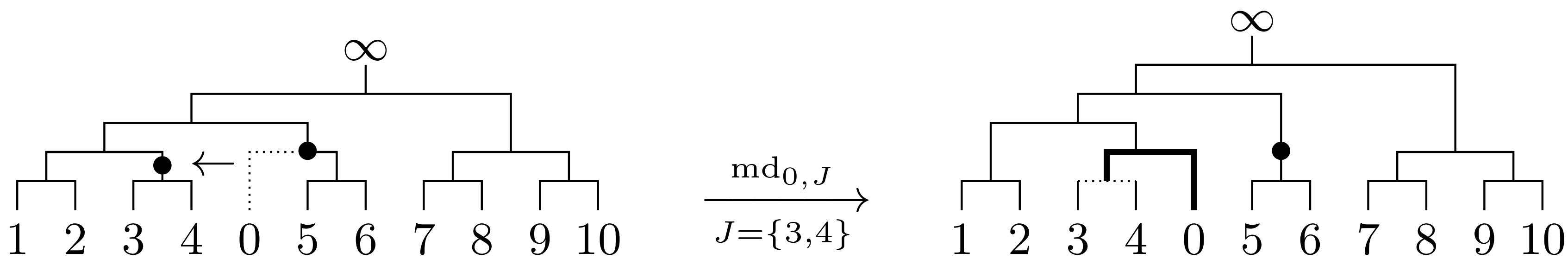
- $(x,y)=(0,1)$: $\begin{cases} x_2-x_1=x=X \\ x_3-x_4=y-1=Y \end{cases} \quad \begin{cases} X=x \\ Y=y-1 \end{cases}$



$$\Omega' = A_{x0}\frac{dX}{X}+A_{y1}\frac{dY}{Y} \qquad (|x|,\ |y-1| \ll 1)$$

Transformation of $\mathrm{Sp}\,\mathcal{M}$ induced by middle convolutins is stated in terms of single-elimination tournaments.

$\mathrm{Sp}\,\mathcal{M}$ is necessary to get the eigenvalues of each residue matrices of $\mathrm{mc}_{x_i,\mu}\mathcal{M}$



$J = \{3,4\} : \{3,4\} \to \{0,3,4\},\ \{1,2,3,4\} \to \{0,1,2,3,4\},\ \{0,5,6\} \to \{5,6\}$

KZ-type equations vs single-elimination tournaments

KZ-type equation with n variables	Tournament of n teams
Family of maximal commuting residue matrices	Tournament
Local coordinate for desingularization	Result of a tournament
Spectra of KZ-type equation	Set of all tournaments
Variable of middle convolution	Winner of tournament
Base of upper triangulation of the family	Result of a tournament with the winner
Middle convolution	Deletion and Insertion of the winner
Kernels to define middle convolution	Basic/Top insertion of the winner

Remarks

- $\mathrm{Sp}\,\mathcal{M}$ gives local structure at every strata of the desingularization of $\mathcal{M}$.
  It is a generalization of the Riemann scheme of linear Fuchsian ODE's.
- Rigidity and accessory parameters are defined on KZ-type systems through $\mathrm{Sp}\,\mathcal{M}$.
- $\mathrm{Sp}\,\mathcal{M}$ is generalized to Pfaffian systems with logarithmic singularites along hyperplane arrangements.

- Characterize basic KZ-type equations !
  (basic: cannot reduce the rank by middle convolutions+additions)
  Is there a non-trivial rigid basic KZ-type equation?
  Is there a basic KZ-type equation with more than three (essential) variables?

Thank you for your attention!

[Om] T. Oshima, Middle convolutions of KZ-type equations and single-elimination tournaments, arXiv:2504.09003.