

Generalized hypergeometric functions with several variables

Saiei-Jaeyeong Matsubara-Heo^{a,1}, Toshio Oshima^{b,2}

^a*Faculty of Advanced Science and Technology, Kumamoto University, Japan*

^b*Center for Mathematics and Data Science, Josai University, Japan*

Abstract

We introduce a hypergeometric series with several variables, which generalizes Appell's, Lauricella's and Kampé de Fériet's hypergeometric series, and study the system of differential equations that it satisfies. We determine the singularities, the rank and the condition for the reducibility of the system. We give complete local solutions of the system at many singular points of the system and solve the connection problem among these local solutions. Under some assumptions, the system is written as a KZ equation. We determine its spectral type in the direction of coordinates as well as simultaneous eigenspace decompositions of residue matrices. The system may or may not be rigid in the sense of N. Katz viewed as an ordinary differential equation in some direction. We also show that the system is a special case of Gel'fand-Kapranov-Zelevinsky system. From this point of view, we discuss multivariate generalizations.

Contents

1	Introduction	2
2	Generalized hypergeometric functions of two variables	6
2.1	Generalized Hypergeometric Functions of one variable	6
2.2	The system $\mathcal{M}_{\alpha',\beta',\gamma'}$ satisfied by Hypergeometric Series	8
2.3	Some Conditions	10
2.4	Examples: Known Series	10
2.5	Integral Representation	11
3	Global structure of hypergeometric functions	12
3.1	Coordinate transformations	12
3.2	Local Solutions	15
3.3	Connection Problem	17

¹e-mail: saiei@educ.kumamoto-u.ac.jp

²e-mail: oshima@ms.u-tokyo.ac.jp

4 Rank and singular set	20
4.1 Rank of the system $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$	20
4.2 Singular set of the system $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$	23
5 KZ equation	26
5.1 KZ equation and middle convolution	26
5.2 Example: Generalization of Appell's F_1	32
5.3 Appell's F_1	40
5.4 Example: Generalization of Appell's F_2 and F_3	43
5.5 Appell's F_2 and F_3	46
6 Many variables and GKZ system	49
6.1 The system $\mathcal{M}_{\alpha'_1,\dots,\alpha'_n,\gamma'}^{\alpha_1,\dots,\alpha_n,\gamma}$	49
6.2 The singular set	51
6.3 GKZ system	54
6.4 The system $\mathcal{M}_{\alpha'_1,\dots,\alpha'_n,\gamma'}^{\alpha_1,\dots,\alpha_n,\gamma}$ from the view point of GKZ system	55
6.5 The structure of the secondary fan and irreducibility	59
6.6 Local solutions and the connection problem	61
6.7 Braid monodromy	65

1. Introduction

In this paper, we study a bivariate hypergeometric series

$$F_{p',q',r'}^{p,q,r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; x, y \right) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_{m+n}}{(1-\alpha')_m (1-\beta')_n (1-\gamma')_{m+n}} x^m y^n,$$

$$\alpha \in \mathbb{C}^p, \beta \in \mathbb{C}^q, \gamma \in \mathbb{C}^r, \alpha' \in \mathbb{C}^{p'}, \beta' \in \mathbb{C}^{q'}, \gamma' \in \mathbb{C}^{r'}$$

in the case when this series satisfies a system of differential equations without an irregular singularity, which means $p' - p = q' - q = r - r'$. The notation such as $(\alpha)_m$ is defined in (2.1). J. Kampé de Fériet's hypergeometric (HG in short) function given in [2] is a special case of our hypergeometric function, namely, it is

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_{m+n}}{(\alpha')_m (\beta')_n (\gamma')_{m+n}} \frac{x^m y^n}{m!n!} = F_{p'+1,p'+1,r'}^{p,p,r} \left(\begin{matrix} \alpha & \beta & \gamma \\ 1-\alpha',0 & 1-\beta',0 & 1-\gamma' \end{matrix}; x, y \right).$$

Classical Appell's hypergeometric functions (cf. [2]) are all special cases of $F_{p',q',r'}^{p,q,r}(x, y)$. Another example of $F_{p',q',r'}^{p,q,r}(x, y)$ was investigated in [43], which corresponds to the case $p = q = p' = q' = 1$ and $r = r'$. In [36] the second author studies certain integral transformations of convergent power series and introduces $F_{p,q,r}^{p,q,r}(x, y)$ together with some related results as an application of the transformations without proof.

We investigate $F_{p',q',r'}^{p,q,r}(x, y)$ from two different points of views. One is the viewpoint of a system of partial differential equations. We introduce a system

$\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ satisfied by $F_{p', q', r'}^{p, q, r}(x, y)$. The system resembles Appell's system, in the sense that we can determine its irreducibility, singular locus, and some of its connection coefficients. It is the symmetry of the system that is stressed in this paper. There are distinguished coordinate transformations which exchange the role of parameters. These transformations naturally act on the blowing-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two points $(0, 0)$ and (∞, ∞) . The divisors $\{x = 0\}, \{x = \infty\}, \{y = 0\}, \{y = \infty\}$ and the exceptional divisors form a hexagon and the transformations act transitively on its vertices. In the course of the study, it is also convenient to introduce another series

$$G_{p', q', r'}^{p, q, r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; x, y \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_{m-n}}{(1-\alpha')_m (1-\beta')_n (1-\gamma')_{m-n}} x^m y^n.$$

Then, at the vertices of the hexagon, we can construct a basis of series solutions using $F_{p', q', r'}^{p, q, r}$ and $G_{p', q', r'}^{p, q, r}$.

The connection problem among the vertices of the hexagon can be solved only by using Gamma functions and trigonometric functions. The idea of the computation is *boundary value problem* which goes back to a paper by the second author [19]. In fact, the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ has regular singularities along an edge of the hexagon and the boundary value of the solution to the system is defined according to the characteristic exponent. We reduce the connection problem along an edge to that of the boundary value. The same idea also used in the calculation of c -function of Heckman-Opdam's hypergeometric function ([39]).

At the end of this paper, we show that our system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ can be seen as a particular case of Gel'fand-Kapranov-Zelevinsky system ([11]). From this point of view, a multivariate generalization $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ is introduced and analyzed. Through Gale duality in combinatorics ([44, §6.3]), the secondary fan of the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ is described. Although our system is strictly more general than Appell-Lauricella's system, the secondary structure is the same. From this point of view, some of the results on the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ can be generalized to $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$. Results of this paper can be related to the literature of Gel'fand-Kapranov-Zelevinsky system. For example, the result on the connection problem can be seen as a generalization of [40] or a special case of [24]. When the singular locus of $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ is a braid arrangement, its monodromy representation gives rise to a representation of the pure braid group. Since the monodromy representation of Lauricella's F_D system corresponds to the so-called Gassner representation ([4, Example 4 of §3.2], see also [22, §5.2]), our result constructs a generalization of Gassner representation from the viewpoint of hypergeometric system.

The other point of view is that of Katz's theory. For an ordinary differential equation on the Riemann sphere, Katz's theory provides a way to understand the global analysis ([20], [32]). An invariant called *rigidity index* and an integral transform called *middle convolution* play a fundamental role in Katz's theory. The local structure of an ordinary differential equation is given by its local behavior called *generalized Riemann scheme*. An ordinary differential equation

is said to be *rigid* when the Riemann scheme uniquely determines the equation. Katz [20] proves that an ordinary differential equation is rigid if and only if it is constructed from the trivial equation $\frac{du}{dx}(x) = 0$ by a successive application of middle convolutions and gauge transformations by a power function (*addition*).

As the next step to Katz's theory, Y. Haraoka proposes to extend middle convolutions to Knizhnik-Zamolodchikov equations (KZ equations in short) in [14] and [15]. KZ equation is a particular class of integrable connections. Any rigid ordinary differential equation can be extended to a KZ equation. When $r = r'$, the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ is isomorphic to a KZ equation. We will show in §5.2 that no restriction of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ to a line $\{x = \text{constant}\}$ or $\{y = \text{constant}\}$ is rigid if $(p-1)(q-1)(r-1) \neq 0$. On the other hand, the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ is constructed from the trivial KZ system $\frac{\partial u}{\partial x}(x, y) = \frac{\partial u}{\partial y}(x, y) = 0$ by a successive application of middle convolutions and additions. The trick is to use several directions when we perform middle convolutions. Thus, the restriction of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ to $x = \{\text{constant}\}$ or $y = \{\text{constant}\}$ provides a non-trivial example of non-rigid ordinary differential equation whose global monodromy can be determined explicitly.

From the viewpoint of KZ equation, it is more natural to study the blowing-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at three points $(0, 0)$, $(1, 1)$ and (∞, ∞) . It has 10 distinguished divisors and at each of 15 normal crossing points, the corresponding pair of residue matrices admits a simultaneous eigenspace decomposition. We use [34] to see the change of this data via middle convolutions. The vertices of the hexagon corresponds to the six points where each simultaneous eigenspace is one-dimensional. Moreover, in some cases, there are more than six singular points where this happens.

The generalized Riemann scheme of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ enjoys symmetry. Let us write $F_{p, q, r}$ for $F_{p, q, r}^{p, q, r}$, $I_{p, q}$ for $F_{p, q, 0}^{p-1, q-1, 1}$ and $J_{p, q}$ for $F_{p-1, q-1, 1}^{p, q, 0}$. Besides the case $r = r'$, the case when $p' = p + 1, q' = q + 1, r = 1, r' = 0$ and the case $p = p' + 1, q = q' + 1, r = 0, r' = 1$ both satisfy KZ equations. The following picture shows the symmetry of equations:

$$\begin{array}{ccc}
\boxed{\text{Gauss' HG : } F(a, b, c; x)} & \longrightarrow & \boxed{{}_p F_{p-1}(\boldsymbol{\alpha}; \boldsymbol{\beta}; x) \text{ (cf. §2.1)}} \\
S_{\{0, 1, \infty\}} & & S_{\{0, \infty\}} \\
\\
\boxed{\text{Appell's HG : } F_1(a; b, c, d; x, y)} & \longrightarrow & \boxed{F_{p, q, r}(x, y) \left(\begin{array}{ccc} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{array}; x, y \right) \text{ (cf. §5.2)}} \\
S_{\{x, y, 1, 0, \infty\}} & & S_{\{x, y, 1\}} \times S_{\{0, \infty\}} \\
\\
\boxed{\text{Appell's HG : } F_2, F_3} & \longrightarrow & \boxed{I_{p, q}(x, y), J_{p, q}(x, y) \text{ (cf. §5.4)}} \\
W_{B_2} & & S_{\{x, y\}}
\end{array}$$

Here, $S_{\{x, y, 1, 0, \infty\}}$ means the group of permutations of 5 points $\{x, y, 1, 0, \infty\}$ on the Riemann sphere, which is the permutation group S_5 . In the above, $S_{\{x, y, 1, 0, \infty\}}$ acts on Appell's F_1 as coordinate transformations (cf. (3.1)) and gives a symmetry of F_1 . On the other hand, F_2 and F_3 have coordinate symme-

tries which are identified with elements of the group W_{B_2} , the reflection group of type B_2 (cf. Remark 5.17). The horizontal arrow means generalization. Thus, the principal example in this paper $F_{p,q,r}$ is a direct generalization of the well-known function Appell's F_1 . We emphasize that any restriction of Appell's F_1 system to $x = \{\text{constant}\}$ or $y = \{\text{constant}\}$ gives rise to a rigid ordinary differential equation.

Katz's theory for KZ equation is still being developed. For example, no complete description of the change of simultaneous eigenspace decomposition under middle convolution is available except two variable case [34]. When the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ is a KZ equation, it should be a good example of a *rigid* system, of which no sophisticated definition exists so far.

The content of each section is as follows.

In §2.1 we review some basic properties of the generalized hypergeometric function ${}_pF_{p-1}(\alpha; \beta; x)$. Most results stated in the following sections up to §4 are reduced to the results given in §2.1. In §2.2 we introduce a system of differential equations $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ satisfied by $F_{p', q', r'}^{p, q, r} \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix}; x, y \right)$. In §2.3 we list some conditions on p, q, r, p', q', r' used in this paper. In §2.4 we define another series $G_{p', q', r'}^{p, q, r}$ which will appear in an analytic continuation of $F_{p', q', r'}^{p, q, r}$ and we show some known series including Appell's hypergeometric series and some Horn's series are examples of our two series. In §2.5 we give an integral representation of $F_{p', q', r'}^{p, q, r}$ using integral transformations in [36]. Some of the transformations are essentially middle convolutions.

In §3.1 we study some coordinate transformations and its action on the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$. In §3.3 we solve the connection problem among the vertices of the hexagon.

In §4.1 we determine the rank of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$, the number of independent local solutions to the system at a point, which equals the number of local solution we have constructed.

In §4.2 we determine the singular points of the system. We call the number $L := r - r'$ the level of the system. If $L = 0$ or $L = 1$, the singular points are a union of lines. If $L > 1$, the singular points are a union of lines and a set defined by zeros of an irreducible polynomial $f_L(x, y)$ with degree L .

In §5.1 we review basic results on KZ equations based on [34]. The system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ becomes a KZ equation under some conditions. Of particular interest are the system satisfied by $F_{p,q,r}$ in §5.2 and the one satisfied by $I_{p,q}$ or $J_{p,q}$ in §5.4. In particular, $F_{p,q,1}$, $F_{p,1,r}$ and $I_{p,q}$ satisfy rigid ordinary differential equations for the variable x . We show that the condition of the irreducibility of the system follows from a result given by [33].

Appell's F_1, F_2, F_3 systems are special examples of these with more symmetry. More symmetry produces formulas on the hypergeometric series like Kummer's formula of Gauss' hypergeometric series. They are discussed in §5.3 and in §5.5.

In §6 we introduce hypergeometric series $F_{p'_1, \dots, p'_n, r'}^{p_1, \dots, p_n, r}(x_1, \dots, x_n)$ with more than two variables. Then Lauricella's hypergeometric series (cf. [23]) is a spe-

cial case of the series. In §6.1, we give the system of differential equations $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ satisfied by the series and the integral representation of the series.

In §6.2, we determine the singular set of the system. The formula generalizes known results for Appell-Lauricella's system [16] and [25].

In §6.3, we review some basic properties and terminologies on GKZ system.

In §6.4, we relate the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ to a GKZ system.

In §6.5, we describe the secondary fan of the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ and determine the condition for the irreducibility by applying the criterion of [41].

In §6.6, we derive a connection formula for $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ which generalizes the same result for the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$.

In §6.7, we show that the connection problem discussed in §6.6 determines the monodromy representation of the pure braid group when $p_i = p'_i$ for $i = 1, \dots, n$ and $r = r'$.

2. Generalized hypergeometric functions of two variables

2.1. Generalized Hypergeometric Functions of one variable

We review the generalized hypergeometric function ${}_pF_{p-1}(x)$ of one variable (for example, see [32, §13.4]). We introduce the following notation for $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{C}^p$, $c \in \mathbb{C}$ and $a \in \mathbb{C}$:

$$\begin{aligned} (\alpha)_m &:= \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = (\alpha_1)_m (\alpha_2)_m \cdots (\alpha_p)_m, \quad \Gamma(\alpha) := \Gamma(\alpha_1) \cdots \Gamma(\alpha_p), \\ c - \alpha &= (c - \alpha_1, \dots, c - \alpha_p), \quad |\alpha| = \alpha_1 + \cdots + \alpha_p, \quad (1-a)_{-m} = \frac{(-1)^m}{(a)_m}. \end{aligned} \quad (2.1)$$

In general the generalized hypergeometric series

$${}_pF_q(\alpha, \beta; x) := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \frac{x^n}{n!} \quad (\alpha \in \mathbb{C}^p, \beta \in \mathbb{C}^q) \quad (2.2)$$

is a convergent power series if $p \leq q + 1$ and $\beta_i \notin \{0, -1, -2, \dots\}$. The series satisfies a differential equation without an irregular singularity if $p = q + 1$. A convergent power series

$$F_p(\alpha'; x) := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(1 - \alpha')_n} x^n \quad (\alpha, \alpha' \in \mathbb{C}^p) \quad (2.3)$$

satisfies an inhomogeneous ordinary differential equation

$$\prod_{i=1}^p (\vartheta - \alpha'_i) u \equiv x \prod_{i=1}^p (\vartheta + \alpha_i) u \pmod{\mathbb{C}} \quad (\vartheta := x \frac{d}{dx}). \quad (2.4)$$

In fact, by comparing the terms of x^n in the above equation, we have

$$\begin{aligned} u = \sum_{n=0}^{\infty} c_n x^n &\Rightarrow \prod_{i=1}^p (n - \alpha'_i) c_n x^n = x \prod_{i=1}^p (n - 1 + \alpha_i) c_{n-1} x^{n-1} \quad (n > 0) \\ &\Rightarrow c_n = \frac{\prod_{i=1}^p (n - 1 + \alpha_i)}{\prod_{i=1}^{p'} (n - \alpha'_i)} c_{n-1} = \cdots = \frac{(\boldsymbol{\alpha})_n}{(1 - \boldsymbol{\alpha}')_n} c_0. \end{aligned}$$

Note that $(1)_n = n!$ and if $(\boldsymbol{\alpha}')_1 (= \alpha'_1 \cdots \alpha'_{p'}) = 0$, the modulo notation " \equiv " in (2.4) can be replaced by the true equality " $=$ ". Since

$$P(\vartheta) x^\lambda u(x) = x^\lambda P(\vartheta + \lambda) u(x) \quad (2.5)$$

for a polynomial $P(\vartheta)$ of ϑ , the functions

$$\begin{aligned} F_p \left(\begin{matrix} \boldsymbol{\alpha}' \\ \boldsymbol{\alpha}' \end{matrix}; \pm; x \right) &:= (\pm x)^{\alpha'_i} F_p \left(\begin{matrix} \boldsymbol{\alpha} + \boldsymbol{\alpha}'_i \\ \boldsymbol{\alpha}' - \boldsymbol{\alpha}'_i \end{matrix}; x \right) \quad (i = 1, \dots, p) \\ &= (\pm x)^{\alpha'_i} {}_pF_{p-1}(\boldsymbol{\alpha} + \boldsymbol{\alpha}'_i; \{1 - \alpha'_\nu + \alpha'_i : \nu \neq i\}; x) \end{aligned} \quad (2.6)$$

give p solutions to the generalized hypergeometric equation

$$\mathcal{M}_{\boldsymbol{\alpha}'}^\alpha : \prod_{i=1}^p (\vartheta - \alpha'_i) = x \prod_{i=1}^p (\vartheta + \alpha_i) u \quad (2.7)$$

if $\alpha'_i - \alpha'_j \notin \mathbb{Z}$ for any $i \neq j$. In the sequel, we often use a convention such as

$$(\vartheta + \boldsymbol{\alpha}) := \prod_{i=1}^p (\vartheta + \alpha_i). \quad (2.8)$$

The generalized Riemann scheme of the equation (2.7) is

$$\left\{ \begin{array}{ccc} x = 0 & x = 1 & x = \infty \\ \alpha'_1 & [0]_{(p-1)} & \alpha_1 \\ \vdots & & \vdots \\ \alpha'_p & \delta & \alpha_p \end{array} \right\}, \quad [0]_{(p-1)} := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ p-1 \end{pmatrix},$$

$$\sum_{\nu=1}^p \alpha_\nu + \sum_{\nu=1}^p \alpha'_\nu + \delta = p - 1 \quad (\text{Fuchs condition}).$$

Here $[0]_{(p-1)}$ means the existence of $p-1$ linearly independent local holomorphic solutions at $x = 1$. The condition for the irreducibility of the equation equals

$$\alpha'_i + \alpha_j \notin \mathbb{Z} \quad (1 \leq i \leq p, 1 \leq j \leq p).$$

Solutions at $x = \infty$ are given by

$$F_p \left(\begin{matrix} \boldsymbol{\alpha}' \\ \boldsymbol{\alpha} \end{matrix}; \alpha_j; \pm; \frac{1}{x} \right) := (\pm \frac{1}{x})^{\alpha_j} {}_pF_{p-1}(\boldsymbol{\alpha}' + \alpha_j, \{1 - \alpha_\nu + \alpha_j : \nu \neq j\}; \frac{1}{x})$$

if $\alpha_i - \alpha_j \notin \mathbb{Z}$ for any $i \neq j$.

The connection formula between $x = 0$ and ∞ is given by

$$F_p \left(\begin{matrix} \alpha \\ \alpha' \end{matrix}; \alpha'_i; -; x \right) = \sum_{j=1}^p c(0 : \alpha'_i \rightsquigarrow \infty : \alpha_j) F_p \left(\begin{matrix} \alpha' \\ \alpha \end{matrix}; \alpha_j; -; \frac{1}{x} \right) \quad (x \in \mathbb{C} \setminus [0, \infty)),$$

$$c(0 : \alpha'_i \rightsquigarrow \infty : \alpha_j) = \prod_{\substack{\nu \neq i \\ 1 \leq \nu \leq p}} \frac{\Gamma(1 + \alpha'_i - \alpha'_\nu)}{\Gamma(1 - \alpha'_\nu - \alpha_j)} \prod_{\substack{\nu \neq j \\ 1 \leq \nu \leq p}} \frac{\Gamma(\alpha'_\nu - \alpha_j)}{\Gamma(\alpha'_i + \alpha'_\nu)}.$$

2.2. The system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ satisfied by Hypergeometric Series

For a given set of parameters $\alpha \in \mathbb{C}^p$, $\beta \in \mathbb{C}^q$, $\gamma \in \mathbb{C}^r$, $\alpha' \in \mathbb{C}^{p'}$, $\beta' \in \mathbb{C}^{q'}$, $\gamma' \in \mathbb{C}^{r'}$, we set

$$F_{p', q', r'}^{p, q, r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; x, y \right) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_{m+n}}{(1 - \alpha')_m (1 - \beta')_n (1 - \gamma')_{m+n}} x^m y^n. \quad (2.9)$$

Here, we note that

$$F_{p', q', r'}^{p, q, r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; x, y \right) = F_{p'+1, q'+1, r'}^{p+1, q+1, r} \left(\begin{matrix} \alpha, 1 & \beta, 1 & \gamma \\ \alpha, 0 & \beta', 0 & \gamma' \end{matrix}; x, y \right).$$

As in the last section, it is easy to see that the $p'q'$ functions

$$F_{p', q', r'}^{p, q, r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; \alpha'_i, \beta'_j; x, y \right) := x^{\alpha'_i} y^{\beta'_j} F_{p', q', r'}^{p, q, r} \left(\begin{matrix} \alpha + \alpha'_i & \beta + \beta'_j & \gamma + \alpha'_i + \beta'_j \\ \alpha' - \alpha'_i & \beta' - \beta'_j & \gamma' - \alpha'_i - \beta'_j \end{matrix}; x, y \right) \quad (2.10)$$

for $i = 1, \dots, p'$ and $j = 1, \dots, q'$ are solutions to the system of differential equations

$$\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} : P_1 u = P_2 u = P_{12} u = 0 \quad (2.11)$$

with

$$P_1 = (\vartheta_x - \alpha')(\vartheta_x + \vartheta_y - \gamma') - x(\vartheta_x + \alpha)(\vartheta_x + \vartheta_y + \gamma),$$

$$P_2 = (\vartheta_y - \beta')(\vartheta_x + \vartheta_y - \gamma') - y(\vartheta_y + \beta)(\vartheta_x + \vartheta_y + \gamma),$$

$$P_{12} = x(\vartheta_x + \alpha)(\vartheta_y - \beta') - y(\vartheta_y + \beta)(\vartheta_x - \alpha').$$

Here, we use the notation $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$, $\vartheta_x = x\partial_x$ and $\vartheta_y = y\partial_y$ and the convention (2.8). For $\epsilon_1, \epsilon_2 = \pm 1$, $i = 1, \dots, p'$ and $j = 1, \dots, q'$ we set

$$F_{p', q', r'}^{p, q, r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; \alpha'_i, \beta'_j; \epsilon_1, \epsilon_2; x, y \right) := (\epsilon_1 x)^{\alpha'_i} (\epsilon_2 y)^{\beta'_j} F_{p', q', r'}^{p, q, r} \left(\begin{matrix} \alpha + \alpha'_i & \beta + \beta'_j & \gamma + \alpha'_i + \beta'_j \\ \alpha' - \alpha'_i & \beta' - \beta'_j & \gamma' - \alpha'_i - \beta'_j \end{matrix}; x, y \right). \quad (2.12)$$

Remark 2.1. If $\alpha'_{p'} = \beta'_{q'} = 0$, the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ is reduced to $\bar{\mathcal{M}}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} : \bar{P}_1 u = \bar{P}_2 u = \bar{P}_{12} u = 0$ with

$$\begin{aligned}\bar{P}_1 &= \partial_x \prod_{i=1}^{p'-1} (\vartheta_x - \alpha'_i) \prod_{k=1}^{r'} (\vartheta_x + \vartheta_y - \gamma'_k) - \prod_{i=1}^p (\vartheta_x + \alpha_i) \prod_{k=1}^r (\vartheta_x + \vartheta_y + \gamma_k), \\ \bar{P}_2 &= \partial_y \prod_{j=1}^{q'-1} (\vartheta_y - \beta'_j) \prod_{k=1}^{r'} (\vartheta_x + \vartheta_y - \gamma'_k) - \prod_{j=1}^q (\vartheta_y + \beta_j) \prod_{k=1}^r (\vartheta_x + \vartheta_y + \gamma_k), \\ \bar{P}_{12} &= \partial_y \prod_{i=1}^p (\vartheta_x + \alpha_i) \prod_{j=1}^{q'-1} (\vartheta_y - \beta'_j) - \partial_x \prod_{j=1}^q (\vartheta_y + \beta_j) \prod_{i=1}^{p'-1} (\vartheta_x - \alpha'_i).\end{aligned}$$

We mention that the contiguity relation can be readily seen from the definition of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$. Let $\text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \right)$ be the space of holomorphic solutions to the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$. We denote by $\text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} |_{\alpha_i \mapsto \alpha_i + 1} \right)$ the solution space of the same system with its parameter α_i replaced by $\alpha_i + 1$ and the other parameters unchanged. We use similar notation for other parameters.

Proposition 2.2. *There are induced morphisms*

$$\begin{aligned}\vartheta_x + \alpha_i &: \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \right) \rightarrow \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} |_{\alpha_i \mapsto \alpha_i + 1} \right) & (i = 1, \dots, p), \\ \vartheta_x + \beta_i &: \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \right) \rightarrow \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} |_{\beta_i \mapsto \beta_i + 1} \right) & (i = 1, \dots, q), \\ \vartheta_x + \vartheta_y + \gamma_k &: \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \right) \rightarrow \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} |_{\gamma_k \mapsto \gamma_k + 1} \right) & (k = 1, \dots, r), \\ \vartheta_x - \alpha'_i &: \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \right) \rightarrow \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} |_{\alpha'_i \mapsto \alpha'_i + 1} \right) & (i = 1, \dots, p'), \\ \vartheta_x - \beta'_i &: \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \right) \rightarrow \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} |_{\beta'_i \mapsto \beta'_i + 1} \right) & (i = 1, \dots, q'), \\ \vartheta_x + \vartheta_y - \gamma'_k &: \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \right) \rightarrow \text{Sol} \left(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} |_{\gamma'_k \mapsto \gamma'_k + 1} \right) & (k = 1, \dots, r').\end{aligned}$$

Proof. Using the identity $\vartheta_x \circ x = x \circ (\vartheta_x + 1)$, we obtain relations

$$\begin{aligned}P_1|_{\alpha_i \mapsto \alpha_i + 1} \circ (\vartheta_x + \alpha_i) &= (\vartheta_x + \alpha_i) \circ P_1 \\ P_2|_{\alpha_i \mapsto \alpha_i + 1} &= P_2 \\ P_{12}|_{\alpha_i \mapsto \alpha_i + 1} \circ (\vartheta_x + \alpha_i) &= (\vartheta_x + \alpha_i) \circ P_{12}.\end{aligned}$$

Thus, the first line of the proposition is proved. The other lines can be proved in a similar way. \square

Remark 2.3. *Under the irreducibility condition on Theorem 6.9 and the condition (2.14), the maps above are isomorphisms.*

2.3. Some Conditions

We introduce the following conditions for the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$.

$$p + r = p' + r', \quad q + r = q' + r' \quad (\text{F})$$

$$p = p', \quad q = q', \quad r = r' \quad (\text{F1})$$

$$p = p' - 1, \quad q = q' - 1, \quad r = 1, \quad r' = 0 \quad (\text{F2})$$

$$p' = p - 1, \quad q' = q - 1, \quad r = 0, \quad r' = 1 \quad (\text{F3})$$

$$p = p' - 2, \quad q = q' - 2, \quad r = 2, \quad r' = 0 \quad (\text{F4})$$

$$(\alpha')_1 = (\beta')_1 = 0 \quad (\text{T})$$

Note that the condition $(\alpha')_1 = 0$ in (T) is that at least one of $\alpha'_1, \dots, \alpha'_{p'}$ is zero. We always assume the condition (F) in this paper and call the number

$$L := r - r' \quad (2.13)$$

the level of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$. Moreover, we assume

$$p + p' > 1, \quad q + q' > 1, \quad r + r' > 0. \quad (2.14)$$

Under the conditions (T) and (2.14) the conditions (F1), (F2), (F3) and (F4) with smallest possible (p, q) correspond to Appell's F_1 , F_2 , F_3 and F_4 , respectively. Namely, in these cases, $\left\{ \begin{smallmatrix} p & q & r \\ p' & q' & r' \end{smallmatrix} \right\}$ are as follows:

	(F1)	(F2)	(F3)	(F4)
$\left\{ \begin{smallmatrix} p & q & r \\ p' & q' & r' \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} p & q & r \\ p & q & r \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} p-1 & q-1 & 1 \\ p & q & 0 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} p & q & 0 \\ p-1 & q-1 & 1 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} p-2 & q-2 & 2 \\ p & q & 0 \end{smallmatrix} \right\}$
Appell's HG	$\left\{ \begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} 2 & 2 & 0 \\ 1 & 1 & 1 \end{smallmatrix} \right\}$	$\left\{ \begin{smallmatrix} 0 & 0 & 2 \\ 2 & 2 & 0 \end{smallmatrix} \right\}$

2.4. Examples: Known Series

We introduce series

$$G_{p', q', r'}^{p, q, r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; x, y \right) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_{m-n}}{(1 - \alpha')_m (1 - \beta')_n (1 - \gamma')_{m-n}} x^m y^n$$

and

$$\begin{aligned} G_{p', q'}^{p, q, r, r'} \left(\begin{matrix} \alpha & \beta & \gamma & \gamma' \\ \alpha' & \beta' & \gamma' \end{matrix}; x, y \right) &:= G_{p', q', r'}^{p, q, r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; x, (-1)^{r'} y \right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_{m-n} (\gamma')_{n-m}}{(1 - \alpha')_m (1 - \beta')_n} x^m y^n. \end{aligned}$$

Appell's hypergeometric functions and some Horn's hypergeometric functions

(cf. [7]) are examples of the functions we have introduced:

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = F_{1,1,1}^{1,1,1} \left(\begin{matrix} \beta & \beta' & \alpha \\ 0 & 0 & 1-\gamma \end{matrix}; x, y \right), \quad (2.15)$$

$$F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y) = F_{2,2,0}^{1,1,1} \left(\begin{matrix} \beta & \beta' & \alpha \\ 0, 1-\gamma & 0, 1-\gamma' & \end{matrix}; x, y \right), \quad (2.16)$$

$$F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) = F_{1,1,1}^{2,2,0} \left(\begin{matrix} \alpha, \alpha' & \beta, \beta' \\ 0 & 0 & 1-\gamma \end{matrix}; x, y \right), \quad (2.17)$$

$$F_4(\alpha; \beta; \gamma, \gamma'; x, y) = F_{2,2,0}^{0,0,2} \left(\begin{matrix} \emptyset & \emptyset & \alpha, \beta \\ 0, 1-\gamma & 0, 1-\gamma' & \emptyset \end{matrix}; x, y \right), \quad (2.18)$$

$$G_2(\alpha, \alpha'; \beta, \beta'; x, y) = G_{1,1}^{1,1,1,1} \left(\begin{matrix} \alpha & \alpha' & \beta' & \beta \\ 0 & 0 & & \end{matrix}; x, y \right), \quad (2.19)$$

$$H_2(\alpha, \beta, \gamma, \delta, \epsilon; x, y) = G_{2,1}^{1,2,1,0} \left(\begin{matrix} \beta & \gamma, \delta & \alpha \\ 0, 1-\epsilon & 0 & \end{matrix}; x, y \right). \quad (2.20)$$

2.5. Integral Representation

We recall the following integral transformations in [36]. We set $\mathbf{x} := (x_1, \dots, x_n)$.

Definition 2.4.

$$K_{\mathbf{x}}^{\mu, \lambda} u := \frac{\Gamma(|\boldsymbol{\lambda}| + \mu)}{\Gamma(\mu)\Gamma(\boldsymbol{\lambda})} \int_{\substack{t_1 > 0, \dots, t_n > 0 \\ t_1 + \dots + t_n < 1}} t^{\boldsymbol{\lambda}-1} (1 - |\mathbf{t}|)^{\mu-1} u(t_1 x_1, \dots, t_n x_n) dt_1 \dots dt_n,$$

$$L_{\mathbf{x}}^{\mu, \lambda} u := \frac{\Gamma(\mu + n)\Gamma(\boldsymbol{\lambda})}{(2\pi i)^n \Gamma(|\boldsymbol{\lambda}| + \mu)} \int_{\frac{1}{n+1} - i\infty}^{\frac{1}{n+1} + i\infty} \dots \int_{\frac{1}{n+1} - i\infty}^{\frac{1}{n+1} + i\infty} t^{1-\boldsymbol{\lambda}} (1 - |\mathbf{t}|)^{-\mu-n} u\left(\frac{x_1}{t_1}, \dots, \frac{x_n}{t_n}\right) \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n},$$

$$(\tilde{K}_{\mathbf{x}}^{\mu, \lambda} u)(x) := \frac{\Gamma(\mu + \lambda)}{\Gamma(\mu)\Gamma(\lambda)} \int_0^1 t^{\lambda-1} (1-t)^{\mu-1} u(tx_1, \dots, tx_n) dt.$$

Remark 2.5. By meromorphic continuations of convergent integrals with respect to $\boldsymbol{\lambda}$ and μ , the integral transformations $K_{\mathbf{x}}^{\mu, \lambda}$ and $L_{\mathbf{x}}^{\mu, \lambda}$ are defined in [36]. They are same transformations as the above ones up to constant multiples.

Putting

$$(T_{\mathbf{x} \rightarrow R(\mathbf{x})} u)(\mathbf{x}) := u(R(\mathbf{x}))$$

for a coordinate transformation $\mathbf{x} \mapsto R(\mathbf{x})$, we have an identity

$$\tilde{K}_{\mathbf{x}}^{\mu, \lambda} = T_{\mathbf{x} \rightarrow (x_1, \frac{x_1}{x_2}, \dots, \frac{x_1}{x_n})}^{-1} \circ K_{x_1}^{\mu, \lambda} \circ T_{\mathbf{x} \rightarrow (x_1, \frac{x_1}{x_2}, \dots, \frac{x_1}{x_n})}. \quad (2.21)$$

These transformations have the following properties:

	$K_{x,y}^{\mu-\alpha-\beta, (\alpha, \beta)}$	$L_{x,y}^{\mu-\alpha-\beta, (\alpha, \beta)}$	$\tilde{K}_{x,y}^{\mu-\gamma, \gamma}$	$K_x^{\mu-\alpha, \alpha}$
$x^m y^n \mapsto$	$\frac{(\alpha)_m (\beta)_n}{(\mu)_{m+n}} x^m y^n$	$\frac{(\mu)_{m+n}}{(\alpha)_m (\beta)_n} x^m y^n$	$\frac{(\gamma)_{m+n}}{(\mu)_{m+n}} x^m y^n$	$\frac{(\alpha)_m}{(\mu)_m} x^m y^n$

Hence, we have the following theorem.

Theorem 2.6. Assume $\alpha'_{p'} = \beta'_{q'} = 0$. Then, the following identity holds.

$$\begin{aligned}
& F_{p',q',r'}^{p,q,r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; x, y \right) \\
&= \prod_{i=1}^{p'-1} K_x^{1-\alpha'_i-\alpha_i, \alpha_i} \prod_{j=1}^{q'-1} K_y^{1-\beta'_j-\beta_j, \beta_j} \prod_{k=1}^r \tilde{K}_{x,y}^{1-\gamma'_k-\gamma_k, \gamma_k} \\
&\quad \prod_{k=1}^{r'-r} K_{x,y}^{1-\gamma'_{r+k}-\alpha'_{p'+k-1}-\beta'_{q'+k-1}, (\alpha'_{p'+k-1}, \beta'_{q'+k-1})} (1-x)^{-\alpha_p} (1-y)^{-\beta_q} \\
&\quad \text{if } r \leq r', \\
&= \prod_{i=1}^p K_x^{1-\alpha'_i-\alpha_i, \alpha_i} \prod_{j=1}^q K_y^{1-\beta'_j-\beta_j, \beta_j} \prod_{k=1}^{r'} \tilde{K}_{x,y}^{1-\gamma'_k-\gamma_k, \gamma_k} \\
&\quad \prod_{k=1}^{r-r'-1} L_{x,y}^{1-\gamma_{r+k}-\alpha'_{p+k}-\beta'_{q+k}, (\alpha'_{p+k}, \beta'_{q+k})} (1-x-y)^{-\gamma_r} \quad \text{if } r > r'.
\end{aligned}$$

3. Global structure of hypergeometric functions

3.1. Coordinate transformations

By a fractional linear transformation of the Riemann sphere \mathbb{P}^1 , 5 points x_0, x_1, x_2, x_3 and x_4 in \mathbb{P}^1 can be uniquely transformed to 5 points $x, y, 1, 0$ and ∞ . Some hypergeometric functions with the variables (x, y) has a symmetry corresponding to the permutation of 5 points x_0, x_1, x_2, x_3 and x_4 under this correspondence. This permutation group isomorphic to S_5 is realized by a group of coordinate transformations of (x, y) generated by 4 involutive elements.

$$\begin{aligned}
& \mathbb{P}^1 \ni (x_0, x_1, x_2, x_3, x_4) \mapsto (x, y, 1, 0, \infty) \\
& S_5 \ni x_0 \leftrightarrow x_1 \quad x_1 \leftrightarrow x_2 \quad x_2 \leftrightarrow x_3 \quad x_3 \leftrightarrow x_4 \\
& (x, y) \mapsto (y, x) \quad \left(\frac{x}{y}, \frac{1}{y} \right) \quad (1-x, 1-y) \quad \left(\frac{1}{x}, \frac{1}{y} \right)
\end{aligned} \tag{3.1}$$

The subgroup generated by the transformations $(x, y) \mapsto (y, x)$, $(\frac{x}{y}, \frac{1}{y})$ and $(\frac{1}{x}, \frac{1}{y})$ is isomorphic to $S_3 \times S_2$ and defines transformations of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ as follows.

In general, we denote by $T_{(x,y) \rightarrow (X,Y)} \mathcal{M}$ the direct image of a system \mathcal{M} by the transformation $(x, y) \mapsto (X, Y) = (y, x)$, $(\frac{1}{x}, \frac{1}{y})$ or $(\frac{x}{y}, \frac{1}{y})$. We calculate these transformations below.

Proposition 3.1. In this proposition (2.14) is not assumed. One has the following transformations of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$.

$$(1) T_{(x,y) \rightarrow (y,x)} \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} = \mathcal{M}_{\beta', \alpha', \gamma'}^{\beta, \alpha, \gamma}.$$

$$(2) T_{(x,y) \rightarrow (\frac{1}{x}, \frac{1}{y})} \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} = \mathcal{M}_{\alpha, \beta, \gamma}^{\alpha', \beta', \gamma'}.$$

$$(3) T_{(x,y) \rightarrow (\epsilon \frac{x}{y}, \frac{1}{y})} \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} = \mathcal{M}_{\alpha', \gamma', \beta'}^{\alpha, \gamma', \beta'}, \text{ where } \epsilon := (-1)^L.$$

Proof. (1) is obvious. Let us prove (2). Putting $(x, y) = (\frac{1}{X}, \frac{1}{Y})$, we have $\vartheta_x = -\vartheta_X$, $\vartheta_y = -\vartheta_Y$. The equation $P_1 u = 0$ is transformed into

$$\prod_{i=1}^p (\vartheta_X - \alpha_i) \prod_{k=1}^r (\vartheta_X + \vartheta_Y - \gamma_k) u = X \prod_{i=1}^{p'} (\vartheta_X + \alpha'_i) \prod_{k=1}^{r'} (\vartheta_X + \vartheta_Y + \gamma'_k) u.$$

By similar calculations for P_2 and P_{12} , we obtain a relation

$$T_{(x,y) \rightarrow (\frac{1}{x}, \frac{1}{y})} \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} = \mathcal{M}_{\alpha, \beta, \gamma}^{\alpha', \beta', \gamma'}. \quad (3.2)$$

As for (3), we use a relation $\vartheta_x = \vartheta_X$, $\vartheta_y = -\vartheta_X - \vartheta_Y$ and the proof is a direct computation. \square

By successive applications of the above transformations, we obtain the following theorem.

Theorem 3.2. *One has the following relations.*

$$\begin{aligned} T_{(x,y) \rightarrow (y,x)}^2 &= T_{(x,y) \rightarrow (\frac{1}{x}, \frac{1}{y})}^2 = T_{(x,y) \rightarrow (\frac{x}{y}, \frac{1}{y})}^2 = id, \\ (T_{(x,y) \rightarrow (y,x)} \circ T_{(x,y) \rightarrow (\frac{1}{x}, \frac{1}{y})})^2 &= (T_{(x,y) \rightarrow (\epsilon \frac{x}{y}, \frac{1}{y})} \circ T_{(x,y) \rightarrow (\frac{1}{x}, \frac{1}{y})})^2 = id, \\ (T_{(x,y) \rightarrow (y,x)} \circ T_{(x,y) \rightarrow (\epsilon \frac{x}{y}, \frac{1}{y})})^3 &= id. \end{aligned}$$

Then the group generated by the involutions $T_{(x,y) \rightarrow (y,x)}$, $T_{(x,y) \rightarrow (\frac{x}{y}, \frac{1}{y})}$ and $T_{(x,y) \rightarrow (\frac{1}{x}, \frac{1}{y})}$ is isomorphic to $S_3 \times S_2$ and the dihedral group of degree 6 with the order 12 which is the group of symmetries of a regular hexagon. This regular hexagon can be realized in a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ as in **Fig.1** (cf. §3.2). The transformation of $\mathcal{M}_{\alpha', \gamma', \beta'}^{\alpha, \beta, \gamma}$ by the element of this group and the corresponding coordinate transformation is given in the following table:

Systems satisfied by (X, Y) with $X = X(x, y)$, $Y = Y(x, y)$.

(X, Y)	(x, y)	$(\epsilon \frac{x}{y}, \frac{1}{y})$	$(\frac{1}{y}, \epsilon \frac{x}{y})$	$(\frac{1}{x}, \epsilon \frac{y}{x})$	$(\epsilon \frac{y}{x}, \frac{1}{x})$	(y, x)
(x, y)	(X, Y)	$(\epsilon \frac{X}{Y}, \frac{1}{Y})$	$(\epsilon \frac{Y}{X}, \frac{1}{X})$	$(\frac{1}{X}, \epsilon \frac{Y}{X})$	$(\frac{1}{Y}, \epsilon \frac{X}{Y})$	(Y, X)
$\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$	α, β, γ	α, γ', β'	γ', α, β'	γ', β, α'	β, γ', α'	β, α, γ
	α', β', γ'	α', γ, β	γ, α', β	γ, β', α	β', γ, α	β', α', γ'
(X, Y)	$(\frac{1}{x}, \frac{1}{y})$	$(\epsilon \frac{y}{x}, y)$	$(y, \epsilon \frac{y}{x})$	$(x, \epsilon \frac{x}{y})$	$(\epsilon \frac{x}{y}, x)$	$(\frac{1}{y}, \frac{1}{x})$
(x, y)	$(\frac{1}{X}, \frac{1}{Y})$	$(\epsilon \frac{Y}{X}, Y)$	$(\epsilon \frac{X}{Y}, X)$	$(X, \epsilon \frac{X}{Y})$	$(Y, \epsilon \frac{Y}{X})$	$(\frac{1}{Y}, \frac{1}{X})$
$\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$	α', β', γ'	α', γ, β	γ, α', β	γ, β', α	β', γ, α	β', α', γ'
	α, β, γ	α, γ', β'	γ', α, β'	γ', β, α'	β, γ', α'	β, α, γ

The transformations in Proposition 3.1 can be used to construct a local solution at a normal crossing point from that at another point. For example, applying the transformation $(x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$ to local solutions (2.10) at the origin, we obtain pq solutions

$$F_{p,q,r}^{p',q',r'} \left(\begin{array}{c} \alpha' \ \beta' \ \gamma' \\ \alpha \ \beta \ \gamma \end{array}; \alpha_i, \beta_j; \frac{1}{x}, \frac{1}{y} \right) \quad (1 \leq i \leq p, 1 \leq j \leq q)$$

in a neighborhood of (∞, ∞) . In the same manner, applying the transformation $(x, y) \mapsto (\epsilon \frac{x}{y}, \frac{1}{y})$ to local solutions (2.10) at the origin, we obtain $p'r$ solutions

$$F_{p',r,q}^{p,r',q'} \left(\begin{array}{c} \alpha \ \gamma' \ \beta' \\ \alpha' \ \gamma \ \beta \end{array}; \alpha'_i, \gamma_k; \epsilon \frac{x}{y}, \frac{1}{y} \right) \quad (1 \leq i \leq p', 1 \leq k \leq r)$$

which are defined along a line $y = \infty$.

Let us consider another transformation $(x, y) \mapsto (X, Y) = (x, \frac{1}{y})$. Then $\vartheta_x = \vartheta_X$, $\vartheta_y = -\vartheta_Y$ and the system $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$ in (X, Y) coordinate is

$$\begin{cases} \prod_{i=1}^{p'} (\vartheta_X - \alpha'_i) \prod_{k=1}^{r'} (\vartheta_X - \vartheta_Y - \gamma'_k) u = X \prod_{i=1}^p (\vartheta_X + \alpha_i) \prod_{k=1}^r (\vartheta_X - \vartheta_Y + \gamma_k) u, \\ \prod_{j=1}^q (\vartheta_Y - \beta_j) \prod_{k=1}^r (\vartheta_X - \vartheta_Y + \gamma_k) u = \epsilon Y \prod_{j=1}^{q'} (\vartheta_Y + \beta'_j) \prod_{k=1}^{r'} (\vartheta_X - \vartheta_Y - \gamma'_k) u, \\ \prod_{i=1}^{p'} (\vartheta_X - \alpha'_i) \prod_{j=1}^q (\vartheta_Y - \beta_j) u = \epsilon XY \prod_{i=1}^p (\vartheta_X + \alpha_i) \prod_{j=1}^{q'} (\vartheta_Y + \beta'_j) u. \end{cases}$$

Hence, we obtain the following proposition.

Proposition 3.3. *One has a relation*

$$T_{(x,y) \rightarrow (x, \frac{\epsilon}{y})} \mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma} = \mathcal{N}_{\alpha',\beta',\gamma'}^{\alpha,\beta',\gamma}, \quad (3.3)$$

where the system $\mathcal{N}_{\alpha',\beta',\gamma'}^{\alpha,\beta',\gamma}$ is defined as follows:

$$\begin{cases} (\vartheta_x - \alpha')(\vartheta_x - \vartheta_y - \gamma')u = x(\vartheta_x + \alpha)(\vartheta_x - \vartheta_y + \gamma)u, \\ (\vartheta_y - \beta)(\vartheta_x - \vartheta_y - \gamma)u = y(\vartheta_y - \beta')(\vartheta_x - \vartheta_y - \gamma')u, \\ (\vartheta_x - \alpha')(\vartheta_y - \beta)u = xy(\vartheta_x + \alpha)(\vartheta_y - \beta')u. \end{cases}$$

The local solution to this system in a neighborhood of the origin in (X, Y) coordinate corresponds to a local solution in a neighborhood of $(0, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1$. In fact, we obtain the following $p'q$ solutions to $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$:

$$\begin{aligned} G_{p',q,r'}^{p,q',r} \left(\begin{array}{c} \alpha \ \beta' \ \gamma' \\ \alpha' \ \beta \ \gamma \end{array}; \alpha'_i, \beta_j; X, \epsilon Y \right) \\ := x^{\alpha'_i} \left(\frac{\epsilon}{y} \right)^{\beta_j} G_{p',q,r'}^{p,q',r} \left(\begin{array}{c} \alpha + \alpha'_i \ \beta' + \beta_j \ \gamma + \alpha'_i - \beta_j \\ \alpha' - \alpha'_i \ \beta - \beta_j \ \gamma' - \alpha'_i + \beta_j \end{array}; x, \frac{\epsilon}{y} \right). \end{aligned}$$

For later use, we set

$$G_{p',q,r'}^{p,q',r} \left(\begin{matrix} \alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma' \end{matrix}; \alpha'_i, \beta_j; \epsilon_1, \epsilon_2; X, \epsilon Y \right) \\ := (\epsilon_1 x)^{\alpha'_i} (\epsilon_2 \frac{\epsilon}{y})^{\beta_j} G_{p',q,r'}^{p,q',r} \left(\begin{matrix} \alpha + \alpha'_i & \beta' + \beta_j & \gamma + \alpha'_i - \beta_j \\ \alpha' - \alpha'_i & \beta - \beta_j & \gamma' - \alpha'_i + \beta_j \end{matrix}; x, \frac{\epsilon}{y} \right)$$

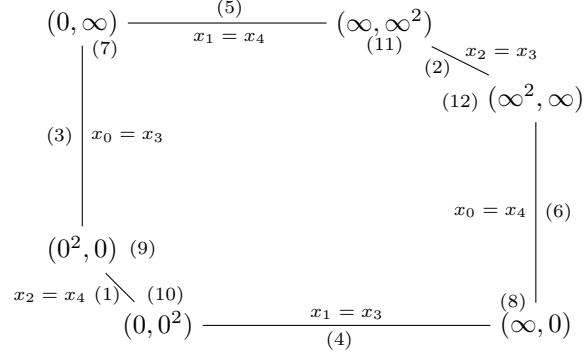
for $1 \leq i \leq p'$, $1 \leq j \leq q$ and $\epsilon_1, \epsilon_2 = \pm 1$.

3.2. Local Solutions

We consider $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$ on the space \tilde{X} which is constructed from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up two points $(0,0)$ and (∞, ∞) . The local coordinates are $(\frac{x}{y}, y)$ and $(x, \frac{y}{x})$ at $(0,0)$ and they are $(\frac{y}{x}, \frac{1}{y})$ and $(\frac{1}{x}, \frac{x}{y})$ at (∞, ∞) . Then for example, $(0^2, 0)$ represents the point of \tilde{X} corresponding to the origin under the coordinate $(\frac{x}{y}, y)$. Regarding **Fig.1** below, the transformations $(x, y) \mapsto (y, \epsilon \frac{y}{x})$ and $(x, y) \mapsto (y, x)$ correspond to the clockwise rotation of the hexagon by $\frac{\pi}{3}$ and the reflection of the hexagon keeping the edge $(0,0)$.

Fig.1

local solutions of 12 types



Owing to the coordinate transformations in §3.1, we have the following local solutions to $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$: for simplicity, we sometimes omit the suffices and superfixes of $F_{p',q',r'}^{p,q',r}$ and $G_{p',q',r'}^{p,q',r}$ and write α, β, \dots in place of α, β, \dots

$$p'q' : F \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; \alpha'_i, \beta'_j; x, y \right) \quad \text{around } (0, 0) \quad (1)$$

$$pq : F \left(\begin{matrix} \alpha' & \beta' & \gamma' \\ \alpha & \beta & \gamma \end{matrix}; \alpha_i, \beta_j; \frac{1}{x}, \frac{1}{y} \right) \quad \text{around } (\infty, \infty) \quad (2)$$

$$qr' : F \left(\begin{matrix} \gamma & \beta' & \alpha \\ \gamma' & \beta & \alpha' \end{matrix}; \gamma'_k, \beta_j; x, \epsilon \frac{x}{y} \right) \quad \text{along } x = 0 \quad (3)$$

$$pr' : F \left(\begin{matrix} \alpha' & \gamma & \beta \\ \alpha & \gamma' & \beta' \end{matrix}; \alpha_i, \gamma'_k; \epsilon \frac{y}{x}, y \right) \quad \text{along } y = 0 \quad (4)$$

$$p'r : F \left(\begin{matrix} \alpha & \gamma' & \beta' \\ \alpha' & \gamma & \beta \end{matrix}; \alpha'_i, \gamma'_k; \epsilon \frac{x}{y}, \frac{1}{y} \right) \quad \text{along } y = \infty \quad (5)$$

$$q'r : F \left(\begin{matrix} \gamma' & \beta & \alpha' \\ \gamma & \beta' & \alpha \end{matrix}; \gamma'_k, \beta'_j; \frac{1}{x}, \epsilon \frac{y}{x} \right) \quad \text{along } x = \infty \quad (6)$$

$$p'q : G \left(\begin{array}{ccc} \alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma' \end{array}; \alpha'_i, \beta_k; x, \epsilon \frac{1}{y} \right) \quad \text{at } (0, \infty) \quad (7)$$

$$pq' : G \left(\begin{array}{ccc} \alpha' & \beta & \gamma' \\ \alpha & \beta' & \gamma \end{array}; \alpha_i, \beta'_j; \frac{1}{x}, \epsilon y \right) \quad \text{at } (\infty, 0) \quad (8)$$

$$p'r' : G \left(\begin{array}{ccc} \alpha & \gamma & \beta' \\ \alpha' & \gamma' & \beta \end{array}; \alpha'_i, \gamma'_k; \epsilon \frac{x}{y}, \epsilon y \right) \quad \text{at } (0^2, 0) \quad (9)$$

$$q'r' : G \left(\begin{array}{ccc} \gamma & \beta & \alpha \\ \gamma' & \beta' & \alpha' \end{array}; \gamma'_k, \beta'_j; x, \frac{y}{x} \right) \quad \text{at } (0, 0^2) \quad (10)$$

$$qr : G \left(\begin{array}{ccc} \gamma' & \beta' & \alpha' \\ \gamma & \beta & \alpha \end{array}; \gamma_k, \beta_j; \frac{1}{x}, \frac{x}{y} \right) \quad \text{at } (\infty, \infty^2) \quad (11)$$

$$pr : G \left(\begin{array}{ccc} \alpha' & \gamma' & \beta \\ \alpha & \gamma & \beta' \end{array}; \alpha_i, \gamma_k; \epsilon \frac{1}{x}, \epsilon \frac{x}{y} \right) \quad \text{at } (\infty^2, \infty) \quad (12)$$

For example, Theorem 3.2 and (3.3) show a relation

$$\begin{aligned} T_{(x,y) \rightarrow (x, \frac{y}{x})} \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} &= T_{(x,y) \rightarrow (x, \frac{\epsilon}{y})} \circ T_{(x,y) \rightarrow (x, \epsilon \frac{y}{x})} \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \\ &= T_{(x,y) \rightarrow (x, \frac{\epsilon}{y})} \mathcal{M}_{\gamma', \beta', \alpha'}^{\gamma, \beta, \alpha} \\ &= \mathcal{N}_{\gamma', \beta', \alpha'}^{\gamma, \beta, \alpha} \end{aligned}$$

and then we have the above $q'r'$ solutions (10).

Each solution in (7)–(12) expressed by G is defined in a neighborhood of a normally crossing singular point in \tilde{X} , namely, one of the 6 vertices in **Fig.1** in §3.3. The solution in (1)–(6) expressed by F is defined in a neighborhood of one of 6 singular lines in \tilde{X} excluding normally crossing singular points, namely, one of six edges in **Fig.1**.

We examine the independent local solutions at a singular point in \tilde{X} and calculate the numbers of the solutions in the following table:

	(∞, ∞^2)	$(0, \infty)$	$(0^2, 0)$
sol.	(5)+(11)+(2)	(7)+(5)+(3)	(1)+(3)+(9)
#	$p'r + qr + pq$	$p'q + p'r + qr'$	$p'q' + qr' + p'r'$
(F1)	$pr + qr + pq$	$pq + pr + qr$	$pq + qr + pr$
(F2)	$p + (q-1) + (p-1)(q-1)$	$p(q-1) + p + 0$	$pq + 0 + 0$
(F3)	$0 + 0 + pq$	$(p-1)q + 0 + q$	$(p-1)(q-1) + q + (p-1)$
(F4)	$2p+2(q-2)+(p-2)(q-2)$	$p(q-2) + 2p + 0$	$pq + 0 + 0$

For example, this table indicates that there exist $p'q'$ solutions by (1), qr' solutions by (3) and $p'r'$ solutions by (9) at $(0^2, 0)$. Hence, for example, under the condition (F2), all the solutions we get at $(0^2, 0)$ belong to (1) and then such solutions are defined around $(0, 0)$, which are called solutions with a simple monodromy at $(0, 0)$ in [36, Definition 7.3].

For the points (∞^2, ∞) , $(\infty, 0)$ and $(0, 0^2)$, we obtain the corresponding table by the map $(x, y) \mapsto (y, x)$. Namely we change $(p, q, r; p', q', r')$, $(2m)$ and $(2m-1)$ to $(q, p, r; q', p', r')$, $(2m-1)$, $(2m)$ for $m = 2, \dots, 6$, respectively, in the above table.

Since $p' - p = q' - q = r - r'$, we have

$$\begin{aligned} p'r + qr + pq &= p'q + p'r + qr' = p'q' + qr' + p'r' \\ &= pq + qr + rp + (p + q + r)(r - r') + (r - r')^2, \end{aligned}$$

which equals $R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$ in (3.4). Thus, we obtain the following theorem.

Theorem 3.4. *In a neighborhood of any one of 6 points $(0^2, 0)$, $(0, \infty)$, (∞, ∞^2) , (∞^2, ∞) , $(\infty, 0)$ and $(0, 0^2)$ of \tilde{X} , one has*

$$R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix} := \begin{cases} \frac{1}{L}(p'q'r - pqr') & (L \neq 0), \\ pq + qr + rp & (L = 0) \end{cases} \quad (3.4)$$

local solutions expressed by the series $F_{p',q',r'}^{p,q,r}$ and $G_{p',q',r'}^{p,q,r}$ given in §1 and §2.4.

Remark 3.5. *It will be shown in §4.1 that the solutions in this theorem span the space of all local solutions to $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$ (cf. Corollary 4.2) for generic parameters.*

3.3. Connection Problem

We consider the connection problem between the local solutions in Theorem 3.4. Since the local solutions are constructed according to the symmetry (3.1) and **Fig.1**, it is sufficient only to calculate the connection problem from the point $(0^2, 0)$ to $(0, \infty)$ along a suitable path

$$(0, \infty) \ni t \mapsto (c(t), -t) \in (0, \infty) \times (-\infty, 0) \quad (3.5)$$

with small $c(t)$ so that the system $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$ has no singularities in

$$U = \{(x, y) \mid 0 < x < 2c(-y), y \in (-\infty, 0)\}. \quad (3.6)$$

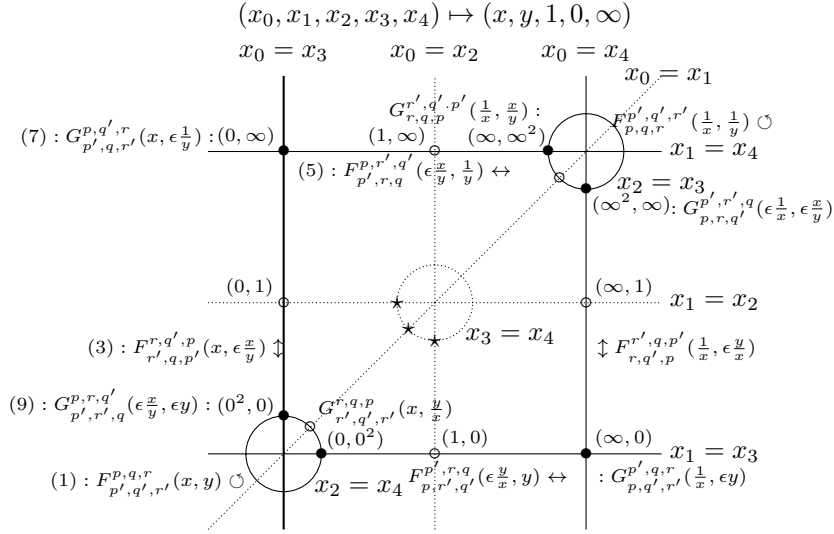
It follows from Theorem 4.7 that there exists a connected neighborhood U of $\{0\} \times (0, -\infty)$ such that $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$ has no singularities in $U \setminus \{0\} \times (0, -\infty)$.

The system $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$ has regular singularities along the wall $\{0\} \times (-\infty, 0)$ and its solution is ideally analytic (cf. [30, Definition 5.1, see also Theorem 5.3]). More precisely, the system has regular singularities along the set of walls $x = 0$ and $L_{(0,0)}$ in \tilde{X} with the edge $(0^2, 0)$. Here $L_{(0,0)}$ is the exceptional fiber of the blowing up of the origin. In general, if the edge is a point, the condition (5.3) in [30] is satisfied and then it follows from [30, Theorem 5.2 and 5.3] that any solution at the edge is in the spanning subspace of local solutions with series expansion. Then the connection problem is reduced to that of the boundary values of the solution in the sense of [38] and the study of induced equation on the boundary in the sense of [30]. A schematic diagram for the connection problem between $(0^2, 0)$ and $(0, \infty)$ is the following: we write $\mathcal{S}ol(\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma})|_{x=0}$ for the space of (single-valued) holomorphic solutions to $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$ near a generic point $(0, y) \in \mathbb{C}^2$ on the divisor $\{x = 0\}$. Let $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}|_{x=0}$ be the restriction of the

system $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}|_{x=0}$ in the sense of D -modules. Then, the unique solvability of the boundary value problem can be summarized as the commutativity of the following diagram:

$$\begin{array}{ccc}
\text{Sol}(\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}|_{x=0}) & \xrightarrow{y: 0 \rightsquigarrow \infty} & \text{Sol}(\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}|_{x=0}) \\
\downarrow x \rightsquigarrow 0 & & \downarrow x \rightsquigarrow 0 \\
\text{Sol}(\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}|_{x=0}) & \xrightarrow{y: 0 \rightsquigarrow \infty} & \text{Sol}(\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}|_{x=0}).
\end{array}$$

Here, the vertical arrow is the boundary value map onto the divisor $\{x = 0\}$ and the horizontal arrow is the analytic continuation along the path (3.5). Note that the boundary value map is an isomorphism. Connection formula for other exponents can be derived in the same manner by multiplying suitable power of x to the unknown function as far as no integral difference appears among exponents.



The space of local solutions at $(0^2, 0)$ (resp. $(0, \infty)$) is spanned by the solutions (1), (3) and (9) (resp. (7), (5) and (3)) given in §3.2. Since the solutions (3) are globally defined along $x = 0$, the connection problem is reduced to the problem expressing a solution in (1) or (9) by a linear combination of the solutions in (5) and (7). These solutions are classified by the characteristic exponent α'_i along the regular singularity $x = 0$.

An ideally analytic solution along $x = 0$ with the characteristic exponent α'_i has the form

$$x^{\alpha'_i} \phi(x, y).$$

Here $\phi(x, y)$ is holomorphic in a neighborhood of $\{0\} \times (-\infty, 0)$ and its boundary value with respect to the characteristic exponent α'_i is $\phi(0, y)$. Those solutions

are given as follows under the notation in §2.1. Here we note that $q+r = q'+r'$ and $-\epsilon$ means $-$ or $+$ according to $\epsilon = 1$ or -1 , respectively.

$$\begin{aligned}
(1) : & F \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; \alpha'_i, \beta'_j; +, -; x, y \right) \quad \text{around } (0, 0) \\
& = x^{\alpha'_i} (-y)^{\beta'_j} F \left(\begin{matrix} \alpha + \alpha'_i & \beta + \beta'_j & \gamma + \alpha'_i + \beta'_j \\ \alpha' - \alpha'_i & \beta' - \beta'_j & \gamma - \alpha'_i - \beta'_j \end{matrix}; x, y \right) \\
& \stackrel{x \rightarrow 0}{\sim} x^{\alpha'_i} F_{q+r} \left(\begin{matrix} \beta, \gamma + \alpha'_i \\ \beta', \gamma' - \alpha'_i \end{matrix}; \beta'_j; -; y \right) \quad (1 \leq i \leq p', 1 \leq j \leq q'), \\
(9) : & G \left(\begin{matrix} \alpha & \gamma & \beta' \\ \alpha' & \gamma' & \beta \end{matrix}; \alpha'_i, \gamma'_k; -\epsilon, -; \epsilon \frac{x}{y}, \epsilon y \right) \quad \text{at } (0^2, 0) \\
& = \left(-\frac{x}{y} \right)^{\alpha'_i} (-y)^{\gamma'_k} G \left(\begin{matrix} \alpha + \alpha'_i & \gamma + \gamma'_k & \beta' + \alpha'_i - \gamma'_k \\ \alpha' - \alpha'_i & \gamma' - \gamma'_k & \beta - \alpha'_i + \gamma'_k \end{matrix}; \epsilon \frac{x}{y}, \epsilon y \right) \\
& \stackrel{x \rightarrow 0}{\sim} x^{\alpha'_i} (-y)^{\gamma'_k - \alpha'_i} F_{q+r} \left(\begin{matrix} \gamma + \gamma'_k, \beta - \alpha'_i + \gamma'_k \\ \gamma' - \gamma'_k, \beta' + \alpha'_i - \gamma'_k \end{matrix}; y \right) \\
& = x^{\alpha'_i} F_{q+r} \left(\begin{matrix} \beta, \gamma + \alpha'_i \\ \beta', \gamma' - \alpha'_i \end{matrix}; \gamma'_k - \alpha'_i; -; y \right) \quad (1 \leq i \leq p', 1 \leq k \leq r'), \\
(5) : & F \left(\begin{matrix} \alpha & \gamma' & \beta' \\ \alpha' & \gamma & \beta \end{matrix}; \alpha'_i, \gamma_k; -\epsilon, -1; \epsilon \frac{x}{y}, \frac{1}{y} \right) \quad \text{along } y = \infty \\
& = \left(-\frac{x}{y} \right)^{\alpha'_i} \left(-\frac{1}{y} \right)^{\gamma_k} F \left(\begin{matrix} \alpha + \alpha'_i & \gamma' + \gamma'_k & \beta' + \alpha'_i + \gamma'_k \\ \alpha' - \alpha'_i & \gamma - \gamma'_k & \beta - \alpha'_i - \gamma'_k \end{matrix}; \epsilon \frac{x}{y}, \frac{1}{y} \right) \\
& \stackrel{x \rightarrow 0}{\sim} x^{\alpha'_i} F_{q+r} \left(\begin{matrix} \beta', \gamma' - \alpha'_i \\ \beta, \gamma + \alpha'_i \end{matrix}; \gamma_k + \alpha'_i; -; \frac{1}{y} \right) \quad (1 \leq i \leq p', 1 \leq k \leq r), \\
(7) : & G \left(\begin{matrix} \alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma' \end{matrix}; \alpha'_i, \beta_j; +, -\epsilon; x, \epsilon \frac{1}{y} \right) \quad \text{at } (0, \infty) \\
& = x^{\alpha'_i} \left(-\frac{1}{y} \right)^{\beta_j} G \left(\begin{matrix} \alpha + \alpha'_i & \beta' + \beta_j & \gamma + \alpha'_i - \beta_j \\ \alpha' - \alpha'_i & \beta - \beta_j & \gamma' - \alpha'_i + \beta_j \end{matrix}; x, \epsilon \frac{1}{y} \right) \\
& \stackrel{x \rightarrow 0}{\sim} x^{\alpha'_i} F_{q+r} \left(\begin{matrix} \beta', \gamma' - \alpha'_i \\ \beta, \gamma + \alpha'_i \end{matrix}; \beta_j; -; \frac{1}{y} \right) \quad (1 \leq i \leq p', 1 \leq j \leq q).
\end{aligned}$$

When we choose one of the the characteristic exponent α'_i , the corresponding induced equation is $\mathcal{M}_{\beta, \gamma + \alpha'_i}^{\beta', \gamma' - \alpha'_i}$ and the boundary value of the hypergeometric function $F \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; \alpha'_i, \beta'_j; +, -; x, y \right)$ is $F_{q+r} \left(\begin{matrix} \beta, \gamma + \alpha'_i \\ \beta', \gamma' - \alpha'_i \end{matrix}; \beta'_j; -; y \right)$. Note that the boundary value of the function with respect to α'_i , with $i \neq i'$ equals 0. Thus our connection problem is reduced to that of the system $\mathcal{M}_{\beta, \gamma + \alpha'_i}^{\beta', \gamma' - \alpha'_i}$ explained in §2.1 and we have the following theorem from the results in §2.1.

Theorem 3.6. *One has the following connection formula*

$$\begin{aligned}
F \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; \alpha'_i, \beta'_j; +, -; x, y \right) & = \sum_{k=1}^r a_{jk}^i F \left(\begin{matrix} \alpha & \gamma' & \beta' \\ \alpha' & \gamma & \beta \end{matrix}; \alpha'_i, \gamma_k; -\epsilon, -1; \epsilon \frac{x}{y}, \frac{1}{y} \right) \\
& + \sum_{\ell=1}^q b_{j\ell}^i G \left(\begin{matrix} \alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma' \end{matrix}; \alpha'_i, \beta_\ell; +, -\epsilon; x, \epsilon \frac{1}{y} \right),
\end{aligned}$$

$$a_{jk}^i = \frac{\prod_{\nu \neq j} \Gamma(1 + \beta'_j - \beta'_\nu) \prod_{\nu=1}^{r'} \Gamma(1 + \beta'_j - \gamma'_\nu + \alpha'_i) \prod_{\nu \neq k} \Gamma(\gamma_\nu - \gamma_k) \prod_{\nu=1}^q \Gamma(\beta_\nu - \alpha'_i - \gamma_k)}{\prod_{\nu \neq j} \Gamma(1 - \gamma_k - \alpha'_i - \beta'_\nu) \prod_{\nu=1}^{r'} \Gamma(1 - \gamma_k - \gamma'_\nu) \prod_{\nu \neq k} \Gamma(\beta'_j + \gamma_\nu + \alpha'_j) \prod_{\nu=1}^q \Gamma(\beta'_j + \beta_\nu)},$$

$$b_{j\ell}^i = \frac{\prod_{\nu \neq j} \Gamma(1 + \beta'_j - \beta'_\nu) \prod_{\nu=1}^{r'} \Gamma(1 + \beta'_j - \gamma'_\nu + \alpha'_i) \prod_{\nu \neq \ell} \Gamma(\beta_\nu - \beta_\ell) \prod_{\nu=1}^r \Gamma(\gamma_\nu + \alpha'_i - \beta_\ell)}{\prod_{\nu \neq j} \Gamma(1 - \beta'_\nu - \beta_\ell) \prod_{\nu=1}^{r'} \Gamma(1 - \gamma'_\nu + \alpha'_i - \beta_\ell) \prod_{\nu \neq \ell} \Gamma(\beta'_j + \beta_\nu) \prod_{\nu=1}^r \Gamma(\beta'_j + \gamma_\nu + \alpha'_i)}$$

and

$$G \left(\begin{matrix} \alpha & \gamma & \beta' \\ \alpha' & \gamma' & \beta \end{matrix}; \alpha'_i, \gamma'_k; -, -; \epsilon \frac{x}{y}, \epsilon y \right) = \sum_{\ell=1}^r a_{k\ell}^i F \left(\begin{matrix} \alpha & \gamma' & \beta' \\ \alpha' & \gamma & \beta \end{matrix}; \alpha'_i, \gamma_\ell; -\epsilon, -1; \epsilon \frac{x}{y}, \frac{1}{y} \right) \\ + \sum_{j=1}^q d_{kj}^i G \left(\begin{matrix} \alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma' \end{matrix}; \alpha'_i, \beta_j; +, -\epsilon; x, \epsilon \frac{1}{y} \right),$$

$$c_{k\ell}^i = \frac{\prod_{\nu \neq k} \Gamma(1 + \gamma'_k - \gamma'_\nu) \prod_{\nu=1}^{q'} \Gamma(1 + \gamma'_k - \alpha'_i - \beta'_\nu) \prod_{\nu \neq \ell} \Gamma(\gamma_\nu - \gamma_\ell) \prod_{\nu=1}^q \Gamma(\beta_\nu - \gamma_\ell - \alpha'_i)}{\prod_{\nu \neq k} \Gamma(1 - \gamma_\ell - \gamma'_\nu) \prod_{\nu=1}^{q'} \Gamma(1 - \gamma_\ell - \alpha'_i - \beta'_\nu) \prod_{\nu \neq \ell} \Gamma(\gamma'_k + \gamma_\nu) \prod_{\nu=1}^q \Gamma(\gamma'_k - \alpha'_i + \beta_\nu)},$$

$$d_{kj}^i = \frac{\prod_{\nu \neq k} \Gamma(1 + \gamma'_k - \gamma'_\nu) \prod_{\nu=1}^{q'} \Gamma(1 + \gamma'_k - \alpha'_i - \beta'_\nu) \prod_{\nu \neq j} \Gamma(\beta_\nu - \beta_j) \prod_{\nu=1}^r \Gamma(\gamma_\nu + \alpha'_i - \beta_j)}{\prod_{\nu \neq k} \Gamma(1 - \beta_j - \gamma'_\nu + \alpha'_i) \prod_{\nu=1}^{q'} \Gamma(1 - \beta_j - \beta'_\nu) \prod_{\nu \neq j} \Gamma(\gamma'_k - \alpha'_i + \beta_\nu) \prod_{\nu=1}^r \Gamma(\gamma'_k + \gamma_\nu)}$$

for $(x, y) \in U$ given by (3.6).

Owing to the coordinate transformations in Proposition 3.1, we have solutions of other connection problems.

4. Rank and singular set

4.1. Rank of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$

In this section, we prove that the rank of the hypergeometric equation $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ equals $R \left(\begin{matrix} p & q & r \\ p' & q' & r' \end{matrix} \right)$. We prepare some notation used in this section. Let K be the field of rational functions of the variables x and y over \mathbb{C} . We denote by $W(x, y)$ the ring of differential operators of these variables with coefficients in K . For a polynomial $D = \sum_{a, b \geq 0} c_{ab} \xi^a \eta^b \in K[\xi, \eta]$ and a differential operator $P = \sum_{a, b \geq 0} \tilde{c}_{ab} \partial_x^a \partial_y^b \in W(x, y)$, we set

$$\deg D := \max\{a + b \mid c_{ab} \neq 0\} \quad \text{and} \quad \text{ord } P := \max\{a + b \mid \tilde{c}_{ab} \neq 0\}.$$

For a non-negative integer m , we put

$$K[\xi, \eta]_m := \{D \in K[\xi, \eta] \mid \deg D \leq m\},$$

$$W(x, y)_m := \{P \in W(x, y) \mid \text{ord } P \leq m\}.$$

For a differential operator

$$P = \sum_{m+n \leq \text{ord } P} a_{m,n}(x,y) \partial_x^m \partial_y^n \in W(x,y),$$

the principal symbol of P is defined by

$$\sigma(P) := \sum_{m+n \leq \text{ord } P} a_{m,n}(x,y) \xi^m \eta^n.$$

We define the rank $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ of the system by

$$\text{rank } \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} := \dim_K W(x,y) / (W(x,y)P_1 + W(x,y)P_2 + W(x,y)P_{12}). \quad (4.1)$$

It is well-known that the rank (4.1) is equal to the dimension of the space of holomorphic solutions to the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ at a generic point ([17, §6.2, §6.3]).

Theorem 4.1. *The identity*

$$\text{rank } \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} = R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$$

holds for any $\alpha \in \mathbb{C}^p$, $\beta \in \mathbb{C}^q$, $\gamma \in \mathbb{C}^r$, $\alpha' \in \mathbb{C}^{p'}$, $\beta' \in \mathbb{C}^{q'}$ and $\gamma' \in \mathbb{C}^{r'}$.

Proof. Since the transformation $(x,y) \mapsto (\frac{1}{x}, \frac{1}{y})$ changes $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ into $\mathcal{M}_{\alpha, \beta, \gamma}^{\alpha', \beta', \gamma'}$, we may assume $L \geq 0$ for the proof. Note that

$$\begin{aligned} \text{rank } \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} &\leq \dim_K K[\xi, \eta] / (K[\xi, \eta]\sigma(P_1) + K[\xi, \eta]\sigma(P_2) + K[\xi, \eta]\sigma(P_{12})) \\ &= R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}. \end{aligned}$$

The above equality is given by Lemma 4.4.

On the other hand, in Theorem 3.4 we constructed $R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$ local solutions at $(0^2, 0)$. They meromorphically depend on the parameters and are linearly independent when the parameters are generic. By analytic continuation of suitable linear combinations of solutions with respect to a parameter [38, Proposition 2.21] assures that the dimension of local solutions at $(0^2, 0)$ which equals $\text{rank } \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ is not smaller than $R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$. Hence we have the theorem. \square

Corollary 4.2. *The $R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$ local solutions given in Theorem 3.4 span the space of local solutions to the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ if the parameters are generic. Moreover [31, Lemma 6.3] assures that for any fixed parameters we have precisely $R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$ linearly independent solutions which holomorphically depend on the parameters in a neighborhood of the fixed parameters. They are suitable linear combinations of the solutions given in Theorem 3.4.*

Suppose $L \geq 0$. For $\alpha \in \mathbb{C}^p$, $\beta \in \mathbb{C}^q$, $\gamma \in \mathbb{C}^{r'}$, $\alpha' \in \mathbb{C}^L$ and $\beta' \in \mathbb{C}^L$ we define the following elements in $\mathbb{C}[x, y, \xi, \eta]$:

$$\begin{aligned} D_1 &= \prod_{\nu=1}^p (x\xi - \alpha_\nu), \quad D_2 = \prod_{\nu=1}^q (y\eta - \beta_\nu), \quad D_3 = \prod_{\nu=1}^{r'} (x\xi + y\eta - \gamma_\nu), \\ D_4 &= \prod_{\nu=1}^L (x\xi - \alpha'_\nu) - x(x\xi + y\eta)^L, \quad D_5 = \prod_{\nu=1}^L (y\eta - \beta'_\nu) - y(x\xi + y\eta)^L, \\ D_6 &= xD_5 - yD_4, \quad Q_1 = D_1D_3D_4, \quad Q_2 = D_2D_3D_5, \quad Q_3 = D_1D_2D_6. \end{aligned}$$

In the case when parameters $\alpha_\nu, \beta_\nu, \dots$ are 0, we put

$$\bar{D}_i = D_i|_{\alpha=\beta=\gamma=\alpha'=\beta'=0}, \quad \bar{Q}_j = Q_j|_{\alpha=\beta=\gamma=\alpha'=\beta'=0} \quad (1 \leq i \leq 6, 1 \leq j \leq 3).$$

Then we note that $\sigma(P_1) = \bar{Q}_1$, $\sigma(P_2) = \bar{Q}_2$ and $\sigma(P_{12}) = \bar{Q}_3$.

Lemma 4.3. *Suppose there exist $C_i \in K[\xi, \eta]$ satisfying*

$$C_1D_1D_3D_4 + C_2D_2D_3D_5 + C_3D_1D_2D_6 = 0. \quad (4.2)$$

Then there exist $A, B \in K[\xi, \eta]$ satisfying

$$\begin{cases} C_1 = (AD_5 + yB)D_2, \\ C_2 = -(AD_4 + xB)D_1, \\ C_3 = BD_3. \end{cases} \quad (4.3)$$

Here we note that (4.2) is valid if C_j are given by (4.3).

Proof. Note that $K[\xi, \eta]$ is a unique factorization domain and D_i and D_j are mutually prime for $1 \leq i < j \leq 6$. Hence $C_1 \in K[\xi, \eta]D_2$, $C_2 \in K[\xi, \eta]D_1$ and $C_3 \in K[\xi, \eta]D_3$ and therefore $C_1 = B_1D_2$, $C_2 = B_2D_1$ and $C_3 = B_3D_3$ with $B_j \in K[\xi, \eta]$. Then $B_1D_4 + B_2D_5 + B_3D_6 = 0$ and therefore

$$(B_1 - yB_3)D_4 + (B_2 + xB_3)D_5 = 0.$$

Hence $B_1 + yB_3 = AD_5$ and $B_2 - xB_3 = -AD_4$ with $A \in K[\xi, \eta]$. Putting $B = B_3$, we have (4.3). \square

Lemma 4.4. *Put $I = K[\xi, \eta]Q_1 + K[\xi, \eta]Q_2 + K[\xi, \eta]Q_3$ and $\bar{I} = K[\xi, \eta]\bar{Q}_1 + K[\xi, \eta]\bar{Q}_2 + K[\xi, \eta]\bar{Q}_3$. Then $\dim_K K[\xi, \eta]/I$ does not depend on the parameters $\alpha, \beta, \gamma, \alpha'$ and β' and*

$$\dim_K K[\xi, \eta]/I = \dim_K K[\xi, \eta]/\bar{I} = R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}. \quad (4.4)$$

Proof. Let $D \in I$. Then

$$D = A_1Q_1 + A_2Q_2 + A_3Q_3 \quad (4.5)$$

with suitable $A_j \in K[\xi, \eta]$. Put $N = \max_{1 \leq j \leq 3} \deg A_j Q_j$.

Suppose $N > \deg D$. Let \bar{C}_j are homogeneous polynomials with $A_j - \bar{C}_j \in K[\xi, \eta]_{\deg A_j - 1}$ if $N = \deg A_j Q_j$. Put $\bar{C}_j = 0$ if $N > \deg A_j Q_j$. Then $\bar{C}_1 \bar{Q}_1 + \bar{C}_2 \bar{Q}_2 + \bar{C}_3 \bar{Q}_3 = 0$ and we have $\bar{A}, \bar{B} \in K[\xi, \eta]$ satisfying

$$\begin{cases} \bar{C}_1 = (A\bar{D}_5 + yB)\bar{D}_2, \\ \bar{C}_2 = -(A\bar{D}_4 + xB)\bar{D}_1, \\ \bar{C}_3 = B\bar{D}_3. \end{cases}$$

Replacing A_j by $A_j - C_j$ for $1 \leq j \leq 3$ with (4.3), we have (4.5) with smaller N . Thus, we may assume $N = \text{ord } D$. Then $D + K[\xi, \eta]_{\deg D - 1} \subset \bar{I} + K[\xi, \eta]_{\deg D - 1}$. Here $\bar{I} = K[\xi, \eta] \bar{Q}_1 + K[\xi, \eta] \bar{Q}_2 + K[\xi, \eta] \bar{Q}_3$. Since

$$(L \cap K[\xi, \eta]_m) / (I \cap K[\xi, \eta]_{m-1}) \simeq (\bar{I} \cap K[\xi, \eta]_m) / (\bar{I} \cap K[\xi, \eta]_{m-1})$$

and $\dim K[\xi, \eta] / I = \sum_{m=0}^{\infty} \dim(I \cap K[\xi, \eta]_m) / (I \cap K[\xi, \eta]_{m-1})$, $\dim K[\xi, \eta] / I$ does not depend on the parameters.

Note that $\dim K[\xi, \eta] / I$ is $\dim C[\xi, \eta] / \sum_{i=1}^3 \mathbb{C}[\xi, \eta] Q_i |_{(x,y)=(x_0,y_0)}$ for generic $(x_0, y_0) \in \mathbb{C}^2$. Hence we assume (x, y) is a generic point of \mathbb{C}^2 .

Put $V := \{(\xi, \eta) \in \mathbb{C}^2 / \mathbb{C}^\times \mid Q_1 = Q_2 = Q_3 = 0\}$ and $V_{ij} := \{(\xi, \eta) \in \mathbb{C}^2 / \mathbb{C}^\times \mid D_i = D_j = 0\}$. Then

$$V = V_{12} \cup V_{13} \cup V_{23} \cup V_{15} \cup V_{24} \cup V_{36} \cup V_{45}.$$

When $\alpha, \beta, \gamma, \alpha', \beta'$ and (x, y) are generic, the points in V defined by $Q_1 = Q_2 = Q_3 = 0$ is multiplicity free and the numbers of points of $V_{12}, V_{13}, V_{23}, V_{15}, V_{24}, V_{36}, V_{45}$ are $pq, pr', qr', pL, qL, r'L, L^2$, respectively. Since $pq + pr' + qr' + pL + qL + r'L + L^2 = R \begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}$, we have Lemma 4.4. \square

The proof of Theorem 4.1 shows the following.

Corollary 4.5. *The symbol ideal of the equation $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ in $W(x, y)$ is generated by $\sigma(P_1)$, $\sigma(P_2)$ and $\sigma(P_{12})$, namely,*

$$\begin{aligned} \{\sigma(P) \mid P \in W(x, y)P_1 + W(x, y)P_2 + W(x, y)P_{12}\} \\ = K[\xi, \eta]\sigma(P_1) + K[\xi, \eta]\sigma(P_2) + K[\xi, \eta]\sigma(P_{12}). \end{aligned}$$

4.2. Singular set of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$

In this section we examine the singularities of the solutions to $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$.

Definition 4.6. We define the singular set of $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ by

$$\begin{aligned} \text{Sing}(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}) := \{(x, y) \in \mathbb{C}^2 \mid \exists (\xi, \eta) \in \mathbb{C}^2 \setminus \{(0, 0)\} \text{ such that} \\ \sigma(P_1)(x, y, \xi, \eta) = \sigma(P_2)(x, y, \xi, \eta) = \sigma(P_{12})(x, y, \xi, \eta) = 0\}. \end{aligned}$$

We note that any local solution to $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$ has an analytic continuation along any path in $\mathbb{C}^2 \setminus \text{Sing}(\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma})$. It is more natural to take the closure of the singular set in a compactification $\mathbb{P}^1 \times \mathbb{P}^1$ of \mathbb{C}^2 . The divisors $\{x = \infty\}$ and $\{y = \infty\}$ are regarded as a subset of the singular set.

Theorem 4.7. *The singular set $\text{Sing}(\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma})$ is given by the union of the following irreducible subvarieties:*

(1) $L \neq 0$. Then, the subvarieties are as follows:

$$V_L := \left\{ \left(\frac{1}{(1+t)^L}, \frac{t^L}{(1+t)^L} \right) \in \mathbb{C} \times \mathbb{C} \mid t \in (\mathbb{C} \setminus \{-1\}) \cup \{\infty\} \right\}, \{x = 0\}, \{y = 0\}, \\ \{x = 1\} \ (qq' > 0), \quad \{y = 1\} \ (pp' > 0), \quad \{x = \epsilon y\} \ (r' > 0).$$

(2) $L = 0$. Then, the subvarieties are all linear:

$$\{x = 0\}, \{y = 0\}, \{x = 1\}, \{y = 1\}, \{x = y\}.$$

Proof. We may assume $L = r - r' \geq 0$ as in the proof of the last theorem. A point $(x, y) \in \mathbb{C}^2$ is in the singular set $\text{Sing}(\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma})$ if there exists $(\xi, \eta) \in \mathbb{C}^2$ such that

$$\begin{aligned} ((x\xi)^L - x(x\xi + y\eta)^L)(x\xi)^p(x\xi + y\eta)^{r'} &= 0 \\ ((y\eta)^L - y(x\xi + y\eta)^L)(y\eta)^q(x\xi + y\eta)^{r'} &= 0 \\ (y(x\xi)^L - x(y\eta)^L)(x\xi)^p(y\eta)^q &= 0. \end{aligned} \quad (4.6)$$

Suppose $L > 0$. Then, it is clear that $\{x = 0\} \cup \{y = 0\} \subset \text{Sing}(\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma})$. Let us take an element $(x, y, \xi, \eta) \in \mathbb{C}^2 \times \mathbb{C}^2 \setminus \{(0, 0)\}$ such that the equations

$$\begin{aligned} (\xi^L - x(\xi + \eta)^L)\xi^p(\xi + \eta)^{r'} &= (\eta^L - y(\xi + \eta)^L)\eta^q(\xi + \eta)^{r'} \\ &= (y\xi^L - x\eta^L)\xi^p\eta^q = 0 \end{aligned} \quad (4.7)$$

hold. Putting $(\xi, \eta) = (1, 0)$ in (4.7), we have

$$1 - x = y0^q = y0^q = 0$$

and therefore we have $x = 1$ if $y \neq 0$ and $q > 0$.

Suppose $\xi = -\eta \neq 0$ satisfies (4.7). We have $r' > 0$ and $y\xi^L - x\eta^L = 0$, namely, $x - (-1)^L y = 0$.

Suppose $\xi\eta(\xi + \eta) \neq 0$ and $xy \neq 0$ satisfies (4.7). Then (4.7) means

$$\xi^L - x(\xi + \eta)^L = \eta^L - y(\xi + \eta)^L = y\xi^L - x\eta^L = 0. \quad (4.8)$$

We may assume $(\xi, \tau) = (1, t)$ with $t \in \bar{\mathbb{C}}$ and then we have the theorem.

Suppose $L = 0$ and p, q and r' are positive, then (4.7) is reduced to

$$\begin{aligned} x(1-x)\xi(x\xi + y\eta) &= 0 \\ y(1-y)\eta(x\xi + y\eta) &= 0 \\ xy(x-y)\xi\eta &= 0. \end{aligned}$$

Hence, it is clear that

$$\text{Sing}(\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}) = \{x = 0\} \cup \{x = 1\} \cup \{y = 0\} \cup \{y = 1\} \cup \{x = y\}.$$

□

Remark 4.8. *If the parameters take special values, some singularities may vanish. For example the system satisfied by $(1 - x - y)^{-\alpha} = \sum_{m \geq 0, n \geq 0} \frac{(\alpha)_{m+n} x^m y^n}{m!n!} = F_{1,1,0}^{0,0,1} \left(\begin{matrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \end{matrix}; x, y \right)$ does not have singularities $\{x = 0\} \cup \{y = 0\}$. This example follows from the equality $(y(x\xi)^1 - x(y\eta)^1) = xy(\xi - \eta)$ in (4.7) and does not happen when $|r - r'| > 1$ (cf. Remark 2.1).*

Remark 4.9. *Below, we list some properties of the defining polynomial of V_L .*

i) *Let $f_L(x, y)$ be the resultant of the polynomials $x(1+t)^L - 1$ and $y(1+t)^L - t^L$ of t for $L > 0$ and put $f_0(x, y) = 1$ and $f_{-L}(x, y) = f_L(\frac{1}{x}, \frac{1}{y})$. Then*

$$\begin{aligned} V_L &= \{(x, y) \in \mathbb{C}^2 \mid f_L(x, y) = 0\}, \\ f_L(x, y) &= f_L(y, x) \quad (\Leftarrow t \mapsto \frac{1}{t}), \\ f_L(x, 0) &= 0 \Rightarrow x = 1, \\ f_L(\frac{1}{2^L}, \frac{1}{2^L}) &= 0 \quad (L \neq 0), \\ f_L(x, x) &= 0 \Leftrightarrow \frac{1}{x} = \sum_{\nu=1}^k \binom{k}{\nu} \cos \frac{2m\nu}{L} \pi \quad (m = 1, \dots, L) \text{ if } L > 0. \end{aligned}$$

Here $(x, y) = (\frac{1}{(1+t)^L}, \frac{t^L}{(1+t)^L})$ and the last line in the above follows from $t = \cos \frac{2m}{L} \pi + \sqrt{-1} \sin \frac{2m}{L} \pi$.

ii) *Direct calculation shows*

$$\begin{aligned} f_1(x, y) &= 1 - x + y, \\ f_2(x, y) &= (1 - x - y)^2 - 4xy, \\ f_3(x, y) &= (1 - x - y)^3 - 27xy, \\ f_4(x, y) &= (1 - x - y)^4 - 8xy(x^2 + y^2 + 14(x + y) + 17), \\ f_5(x, y) &= (1 - x - y)^5 - 625xy(x^2 + y^2 - 3(xy - x - y) + 1), \\ f_6(x, y) &= (1 - x - y)^6 - xy \cdot g_6(x, y), \\ g_6(x, y) &= 12(x^4 + y^4) + 40x^2y^2 + 2706(x^3 + y^3) + 20526xy(x + y) - 131637xy \\ &\quad + 20526(x + y) + 2766. \end{aligned}$$

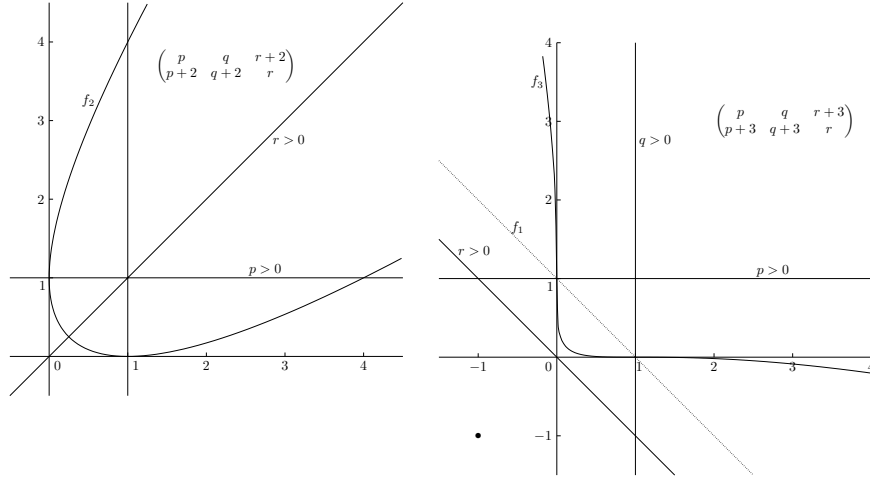
iii) *We give a different expression of $f_L(x, y)$ in Theorem 6.3.*

iv) *If $L > 0$, $f_{L+1}(x, y)$ is a factor of the discriminant of $f_L(\frac{x}{t}, \frac{y}{1-t}) = 0$ as a polynomial of t , which follows from the integral transformation $K_{x,y}^{\mu-\alpha-\beta, (\alpha, \beta)}$ in §2.5,*

For example, $f_1\left(\frac{x}{t}, \frac{y}{1-t}\right) = \frac{x}{1-t} + \frac{y}{1-t} = 1$ means $t^2 - (x - y + 1)t + x = 0$ and the discriminant equals $(x - y + 1)^2 - 4x = (x + y - 1)^2 - 4xy = f_2(x, y)$. The equation $f_2\left(\frac{x}{t}, \frac{y}{1-t}\right) = \left(\frac{x}{t} + \frac{y}{1-t} - 1\right)^2 - 4\frac{x}{t}\frac{y}{1-t} = 0$ means

$$2(x - y + 1)t^3 - (x^2 + 2xy + y^2 + 4x - 2y + 1)t^2 + 2x(x + y + 1)t - x^2$$

and the discriminant of this polynomial of t equals $256x^3y^3((x + y - 1)^3 + 27xy)$.



5. KZ equation

Katz ([20]) defines an operation called middle convolution on local systems. Dettweiler and Reiter ([6]) formulate it as an operation on the Fuchsian systems (5.5) and then Haraoka ([14]) extends it to an operation on KZ equations (5.2). The integral transformations $K_x^{\mu-\alpha, \alpha}$ and $\tilde{K}_{x,y}^{\mu-\lambda, \lambda}$ correspond to some transformations of differential equations generated by middle convolutions, additions and coordinate transformations and hence the middle convolutions are useful to study our hypergeometric functions. In particular, $F_{p',q',r'}^{p,q,r}(x,y)$ satisfying the condition (F1) or (F2) in §2.3 is realized as a component of a solution to a KZ equation. Studying this function from the viewpoint of KZ equation, we obtain several results, which are explained in this section. Some of the results are announced in [36] in the case satisfying (F1).

5.1. KZ equation and middle convolution

Any rigid Fuchsian differential equation on \mathbb{P}^1 with more than 3 singular points can be extended to a KZ equation (Knizhnik-Zamolodchikov) (cf. [21]) regarding singular points as new variables. This is proved by using middle convolutions and additions on KZ equations (cf. [6, 14]) and we obtain many *hypergeometric systems* and functions with several variables. When the number of singular points is 4, we get hypergeometric equations of two variables. Then, the

classical Appell's F_1 system corresponds to the spectral type 21, 21, 21, 21, and Appell's F_2 system corresponds to 211, 22, 31, 31 (cf. [33, §3]). This extension is also valid, when the original equation has unramified irregular singularities (cf. [35]).

The local structure of the ordinary differential equation (5.5) is given by the Riemann scheme and the equation is called rigid when the Riemann scheme uniquely determines the equation. For simplicity, we assume that the equation is non-resonant, namely, the residue matrices $A_{i,j}$ are semisimple and the difference of eigenvalues of any matrix $A_{i,j}$ is not a non-zero integer. The dimension of the moduli space describing the global structure with fixed local structure is given by the rigidity index defined by Katz [20] which can be computed by the spectral data of the Riemann Scheme. The spectral data are described by a list of multiplicities of eigenvalues of $A_{i,j}$. Katz [20] proves that if the equation is rigid, it is constructed from the trivial equation $\frac{du}{dx}(x) = 0$ by successive applications of middle convolutions and additions.

In the case of KZ equation, the Riemann scheme is not sufficient to describe the local structure of the equation. It determines the local structure at a generic point of the singularity defined by $x_i = x_j$. The local structure at a generic point defined by $\{x_i = x_j, x_k = x_\ell\}$ with $\#\{i, j, k, \ell\} = 4$ is also important. The point is the intersection of the hyperplanes defined by $x_i = x_j$ and $x_k = x_\ell$. In this case, the commutator $[A_{i,j}, A_{k,\ell}] = A_{i,j}A_{k,\ell} - A_{k,\ell}A_{i,j}$ is zero and the simultaneous eigenspace decomposition of these matrices is important.

Suppose $n = 3$. Then these two kinds of data are sufficient to describe the local structure of the equation. Moreover the second author [34] describes the transformation of these data by the middle convolution as in the case of ordinary differential equation due to [6]. Thus we get the Riemann scheme of the differential equation of the hypergeometric equation with (F1) or (F2) and the equation is rigid because these local data uniquely determine the equation. The spectral type of a KZ equation corresponding to that of the ordinary differential equation will be the data consisting of the multiplicities of eigenvalues of $A_{i,j}$ together with the multiplicities of eigenvalues of $A_{k,\ell}$ restricted to the eigenspaces of $A_{i,j}$ with $\#\{i, j, k, \ell\} = 4$. We note that there are 15 pairs of these matrices. We denote the eigenvalues of a square matrices A and the simultaneous eigenvalues of commuting matrices A and A' by

$$\begin{aligned} [A] &= \{[\lambda_1]_{m_1}, [\lambda_2]_{m_2}, \dots\}, \\ [A : A'] &= \{[\lambda_1 : \lambda'_1]_{m_1}, [\lambda_2 : \lambda'_2]_{m_2}, \dots\}. \end{aligned} \tag{5.1}$$

The second line means that the dimension of the simultaneous eigenspace with the eigenvalues λ_1 of A and λ'_1 of A' equals m_1 . We will simply denote $[\lambda_1 : \lambda'_1]_1$ by $[\lambda_1 : \lambda'_1]$.

From the data $[A_{i,j}, A_{k,\ell}]$ ($0 < i < j \leq 4$, $i < k < \ell \leq 4$, $j \neq k$, $j \neq \ell$), we obtain the Riemann scheme of the equation, which is the data $[A_{i,j}]$ ($0 \leq i < j \leq 4$) and the Riemann schemes and the spectral type of the ordinary differential equations on a singular line defined by $x_i = x_j$ which is satisfied by the boundary value of the solution with respect to an eigenvalue of $A_{i,j}$.

A KZ equation \mathcal{M} is

$$\mathcal{M} : \frac{\partial u}{\partial x_i} = \sum_{\substack{0 \leq \nu \leq n \\ \nu \neq i}} \frac{A_{i,\nu}}{x_i - x_\nu} u \quad (i = 0, \dots, n), \quad (5.2)$$

$$A_{i,j} = A_{j,i} \in M(N, \mathbb{C}) \quad (i, j \in \{0, 1, \dots, n+1\})$$

with

$$A_{i,i} = A_\emptyset = A_i = 0, \quad A_{i,n+1} := - \sum_{\nu=0}^n A_{i,\nu},$$

$$A_{i_1, i_2, \dots, i_k} := \sum_{1 \leq \nu < \nu' \leq k} A_{i_\nu, i_{\nu'}} \quad (\{i_1, \dots, i_k\} \subset \{0, \dots, n+1\}).$$

Here, $A_{i,j}$ is called the residue matrix along the hypersurface defined by $x_i = x_j$ and u is a column N -vector of unknown functions. We always assume the compatibility condition

$$[A_I, A_J] = 0 \quad \text{if } I \cap J = \emptyset \text{ or } I \subset J \text{ with } I, J \subset \{0, \dots, n+1\}. \quad (5.3)$$

Moreover, we assume \mathcal{M} is homogeneous (cf. [34]), namely,

$$A_I = 0 \quad (\#I = n+1). \quad (5.4)$$

The symmetric group S_{n+3} acts on the space of KZ equations (5.2) as the permutation group of the indices of the matrices $A_{i,j}$.

Any rigid Fuchsian ordinary differential equation

$$\frac{du}{dx} = \sum_{i=1}^n \frac{A_i}{x - x_i} u \quad (5.5)$$

can be extended to a KZ equation (5.2) by putting $x = x_0$ and $A_i = A_{0,i}$, which is proved by [14, 34]. Using the fractional linear transformation of \mathbb{P}^1 which maps to x_{n-1} , x_n and x_{n+1} to 0, 1 and ∞ , respectively, only $n-1$ variables x_0, \dots, x_{n-2} remain. These variables are treated as independent variables on the level of KZ equation (5.2).

In this section, we consider the case $n = 3$. Then, by the relation

$$A_{01} + A_{02} + A_{03} + A_{12} + A_{13} + A_{23} = 0, \quad (5.6)$$

a tuple $(A_{01}, A_{02}, A_{03}, A_{12}, A_{13})$ determines \mathcal{M} .

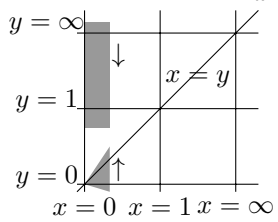
We use a normalization $(x_0, x_1, x_2, x_3, x_4) = (x, y, 1, 0, \infty)$ so that the equation \mathcal{M} is

$$\mathcal{M} : \begin{cases} \frac{\partial u}{\partial x} = \frac{A_{01}}{x-y} u + \frac{A_{02}}{x-1} u + \frac{A_{03}}{x} u, \\ \frac{\partial u}{\partial y} = \frac{A_{01}}{y-x} u + \frac{A_{12}}{y-1} u + \frac{A_{13}}{y} u. \end{cases} \quad (5.7)$$

Remark 5.1. *The involutive coordinate transformations*

$$\begin{aligned} (x_0, x_1, x_2, x_3, x_4) &\leftrightarrow (x_2, x_1, x_0, x_4, x_3) &\rightarrow (x, y) &\leftrightarrow (x, \frac{x}{y}) \\ (x_0, x_1, x_2, x_3, x_4) &\leftrightarrow (x_0, x_2, x_1, x_4, x_3) &\rightarrow (x, y) &\leftrightarrow (\frac{y}{x}, y) \end{aligned}$$

corresponding to elements of S_5 give the coordinates resolving the singularities of (5.7) at the origin (cf. Equation (3.1)). The transformation in the above first line gives the following correspondences in (x, y) -space.

$$\begin{aligned} \{|x| < \epsilon, |y| < C|x|\} &\leftrightarrow \{|x| < \epsilon, |y| > C^{-1}\} \\ x = y = 0 &\leftrightarrow x = 0 \end{aligned}$$


$$\mathcal{M} \leftrightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{A_{12}}{x-y} u + \frac{A_{02}}{x-1} u + \frac{A_{23}}{x} u, \\ \frac{\partial u}{\partial y} = \frac{A_{12}}{y-x} u + \frac{A_{01}}{y-1} u + \frac{A_{13}}{y} u. \end{cases}$$

The middle convolution mc_μ corresponds to an integral transformation

$$\begin{aligned} (I_0^\mu u)(x) &:= \frac{1}{\Gamma(\mu)} \int_0^x t^{\mu-1} u(x-t) dt \\ &= \frac{x^\mu}{\Gamma(\mu)} \int_0^1 (1-t)^{\mu-1} u(tx) dt. \end{aligned}$$

The middle convolution mc_μ of the KZ equation (5.2) is defined through the convolution $\tilde{\text{mc}}_\mu$ of (5.2), which is the equation satisfied by the vector of functions

$$(\tilde{\text{mc}}_\mu u)(x, y) := \begin{pmatrix} I_{x,0}^{\mu+1} \frac{u(x,y)}{x-y} \\ I_{x,0}^{\mu+1} \frac{u(x,y)}{y} \\ I_{x,0}^{\mu+1} \frac{u(x,y)}{x} \end{pmatrix} = \begin{pmatrix} \frac{1}{\Gamma(\mu+1)} \int_0^x (1-t)^\mu \frac{u(t,y)}{t-y} dt \\ \frac{1}{\Gamma(\mu+1)} \int_0^x (1-t)^\mu \frac{u(t,y)}{y} dt \\ \frac{1}{\Gamma(\mu+1)} \int_0^x (1-t)^\mu \frac{u(t,y)}{t} dt \end{pmatrix}$$

for the solution $u(x)$ to (5.2). In view of an equality $K_x^{\mu,\lambda} = x^{-\mu-\lambda} \circ I_{x,0}^\mu \circ x^\lambda$, we define

$$(\tilde{K}_x^{\mu,\lambda} u)(x, y) := \begin{pmatrix} K_x^{\mu+1,\lambda} \frac{xu(x,y)}{x-y} \\ K_x^{\mu+1,\lambda} \frac{xu(x,y)}{x-1} \\ K_x^{\mu+1,\lambda} u(x, y) \end{pmatrix}. \quad (5.8)$$

For a solution u to the equation (5.2), we put $\tilde{u} = \tilde{K}_x^{-\tau-\lambda, \lambda} u$. Then we have a system of equations

$$\frac{\partial \tilde{u}}{\partial x_i} = \sum_{\substack{0 \leq \nu \leq 3 \\ \nu \neq i}} \frac{\tilde{A}_{i,\nu}}{x_i - x_\nu} \tilde{u} \quad (0 \leq i \leq 3) \quad (5.9)$$

satisfied by \tilde{u} . These matrices $\tilde{A}_{i,j}$ are given by

$$\begin{aligned}
\tilde{A}_{01} &= \begin{pmatrix} A_{01} - \tau - \lambda & A_{02} & A_{03} + \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \tilde{A}_{02} &= \begin{pmatrix} 0 & 0 & 0 \\ A_{01} & A_{02} - \tau - \lambda & A_{03} + \lambda \\ 0 & 0 & 0 \end{pmatrix}, \\
\tilde{A}_{03} &= \begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ A_{01} & A_{02} & A_{03} \end{pmatrix}, & \tilde{A}_{04} &= \begin{pmatrix} -A_{01} + \lambda & -A_{02} & -A_{03} - \lambda \\ -A_{01} & -A_{02} + \lambda & -A_{03} - \lambda \\ -A_{01} & -A_{02} & -A_{03} \end{pmatrix}, \\
\tilde{A}_{12} &= \begin{pmatrix} A_{12} + A_{02} & -A_{02} & 0 \\ -A_{01} & A_{12} + A_{01} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}, & \tilde{A}_{13} &= \begin{pmatrix} A_{13} + A_{03} + \lambda & 0 & -A_{03} - \lambda \\ 0 & A_{13} & 0 \\ -A_{01} & 0 & A_{01} + A_{13} \end{pmatrix}, \\
\tilde{A}_{14} &= \begin{pmatrix} A_{23} + \tau & 0 & 0 \\ A_{01} & A_{02} + A_{03} + A_{23} & 0 \\ A_{01} & 0 & A_{02} + A_{03} + A_{23} \end{pmatrix}, \\
\tilde{A}_{23} &= \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{03} + A_{23} + \lambda & -A_{03} - \lambda \\ 0 & -A_{02} & A_{02} + A_{23} \end{pmatrix}, \\
\tilde{A}_{24} &= \begin{pmatrix} A_{01} + A_{13} + A_{03} & A_{02} & 0 \\ 0 & A_{13} + \tau & 0 \\ 0 & A_{02} & A_{01} + A_{13} + A_{03} \end{pmatrix}, \\
\tilde{A}_{34} &= \begin{pmatrix} A_{12} + A_{01} + A_{02} - \tau - \lambda & 0 & A_{03} + \lambda \\ 0 & A_{12} + A_{01} + A_{02} - \tau - \lambda & A_{03} + \lambda \\ 0 & 0 & A_{12} \end{pmatrix}.
\end{aligned}$$

Here we write A_{ij} in place of $\tilde{A}_{i,j}$ for simplicity.

We define a subspace $\mathcal{L} \subset \mathbb{C}^{3N}$ by

$$\begin{aligned}
\mathcal{L} &:= \begin{pmatrix} \ker A_{01} \\ \ker A_{02} \\ \ker(A_{03} + \lambda) \end{pmatrix} + \ker(\tilde{A}_{04} + \tau) \\
&= \begin{pmatrix} \ker A_{01} \\ \ker A_{02} \\ \ker(A_{03} + \lambda) \end{pmatrix} + \ker \begin{pmatrix} A_{01} - \tau - \lambda & A_{02} & A_{03} + \lambda \\ A_{01} & A_{02} - \tau - \lambda & A_{03} + \lambda \\ A_{01} & A_{02} & A_{03} - \tau \end{pmatrix}.
\end{aligned} \tag{5.10}$$

It satisfies $\tilde{A}_{i,j}\mathcal{L} \subset \mathcal{L}$. The matrices $\tilde{A}_{i,j}$ induce linear transformations on the quotient space $\mathbb{C}^{3N}/\mathcal{L}$. We fix a basis of the quotient space and write $\bar{A}_{i,j}$ for the matrices corresponding to the induced linear transformations. Then $\bar{A}_{i,j}$ are square matrices of size $3N - \dim \mathcal{L}$. Thus, we obtain another KZ equation

$$\bar{\mathcal{M}} : \frac{\partial \bar{u}}{\partial x_i} = \sum_{\substack{0 \leq \nu \leq 3 \\ \nu \neq i}} \frac{\bar{A}_{i,\nu}}{x_i - x_\nu} \bar{u} \quad (1 \leq i \leq 3). \tag{5.11}$$

If λ and τ are generic so that

$$\ker(A_{03} + \lambda) = 0 \text{ and } \ker(\tilde{A}_{04} + \tau) = 0, \tag{5.12}$$

we have

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \quad \mathcal{L}_1 = \begin{pmatrix} \ker A_{01} \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{L}_2 = \begin{pmatrix} 0 \\ \ker A_{02} \\ 0 \end{pmatrix}. \quad (5.13)$$

Remark 5.2. i) *Since the KZ equation \mathcal{M} have holomorphic parameters, we define middle convolution in a neighborhood of fixed parameters by defining the above kernels for the generic points in a neighborhood of the fixed parameters. Then the resulting system obtained by the middle convolution has also holomorphic parameters in the neighborhood. Then we may examine the reducibility of the resulting equation, which is related with the difference of the kernels for the fixed parameter.*

ii) *Since $\ker(A_{03} + \lambda) = 0$ in the above, we may assume that the residue class of the last component of \tilde{u} is the last component of \bar{u} and therefore if $\phi(x, y)$ is the last component of a solution to (5.2), then $\tilde{K}_x^{-\tau-\lambda, \lambda} \phi$ is the last component of a solution to $\tilde{\mathcal{M}}$.*

We write $\text{Ad}((x_0 - x_3)^\lambda)$ for the transformation defined by the replacement

$$A_{03} \mapsto A_{03} + \lambda, \quad A_{04} \mapsto A_{04} - \lambda, \quad A_{34} \mapsto A_{34} - \lambda$$

and $A_{ij} \mapsto A_{ij}$ for the other A_{ij} . Note that $\text{Ad}((x_0 - x_3)^\lambda)$ maps a KZ system to a KZ system, namely, the compatibility condition (5.3) is preserved. We have

$$\tilde{K}_x^{-\tau-\lambda, \lambda} = \text{Ad}((x_0 - x_3)^\tau) \circ \text{mc}_{-\tau-\lambda} \circ \text{Ad}((x_0 - x_3)^\lambda). \quad (5.14)$$

Here $\text{Ad}((x_0 - x_3)^\lambda)$ is the transformation of the system (5.2) corresponding to the transformation $u \mapsto (x_0 - x_3)^\lambda u$ and the transformation $\tilde{K}_x^{-\tau-\lambda, \lambda}$ keeps homogeneity (5.4).

The following theorem is a direct consequence of [34, Theorem 7.1].

Theorem 5.3. *Retain the above notation.*

Consider the transformation mc_μ in place of $\tilde{K}_x^{-\tau-\lambda, \lambda}$. Assume μ is generic. For $\{i, j, k\} = \{1, 2, 3\}$ we have (cf. (5.1))

$$\begin{aligned} [\tilde{A}_{0,k} : \tilde{A}_{i,j}] &= [A_{0,k} : A_{i,j}] \cup [-\mu : A_{i,j}] \cup [-\mu : A_{k,4}], \\ [\tilde{A}_{0,k} : \tilde{A}_{i,j}]|_{\mathcal{L}_\nu} &= \begin{cases} [-\mu : A_{k,4}]|_{\text{Ker } A_{0,\nu}} & (\nu = i, j), \\ [0 : A_{i,j}]|_{\text{Ker } A_{0,k}} & (\nu = k), \end{cases} \\ [\tilde{A}_{0,k} : \tilde{A}_{i,4}] &= [A_{0,k} : A_{i,4}] \cup [0 : A_{i,4}] \cup [0 : A_{j,k} - \mu], \\ [\tilde{A}_{0,k} : \tilde{A}_{i,4}]|_{\mathcal{L}_\nu} &= \begin{cases} [-\mu : A_{j,k} - \mu]|_{\text{Ker } A_{0,i}} & (\nu = i), \\ [0 : A_{i,4}]|_{\text{Ker } A_{0,j}} & (\nu = j), \\ [\mu : A_{i,4}]|_{\text{Ker } A_{0,k}} & (\nu = k), \end{cases} \\ [\tilde{A}_{0,4} : \tilde{A}_{i,j}] &= [A_{0,4} : A_{i,j}] \cup [0 : A_{i,j}] \cup [0 : A_{k,4}], \\ [\tilde{A}_{0,4} : \tilde{A}_{i,j}]|_{\mathcal{L}_\nu} &= \begin{cases} [0 : A_{k,4}]|_{\text{Ker } A_{0,\nu}} & (\nu = i, j), \\ [0 : A_{i,j}]|_{\text{Ker } A_{0,k}} & (\nu = k), \end{cases} \end{aligned}$$

$$\begin{aligned}
[\tilde{A}_{k,4} : \tilde{A}_{i,j}] &= [A_{i,j} : A_{i,j}] \cup [A_{k,4} + \mu : A_{i,j}] \cup [A_{k,4} : A_{k,4} + \mu], \\
[\tilde{A}_{k,4} : \tilde{A}_{i,j}]|_{\mathcal{L}_\nu} &= \begin{cases} [A_{k,4} + \mu : A_{k,4}]|_{\text{Ker } A_{0,\nu}} & (\nu = i, j), \\ [A_{i,j} : A_{i,j}]|_{\text{Ker } A_{0,k}} & (\nu = k). \end{cases}
\end{aligned}$$

If a function $u(x, y)$ is a component of a solution to a certain KZ equation, we simply say that $u(x, y)$ is realized as a solution to a KZ equation. Then Theorem 2.6 and the following remark assure that the hypergeometric function $F_{p',q',r'}^{p,q,r}(x, y)$ with the condition (F1) or (F2) is realized as a solution to a KZ equation.

Remark 5.4. i) Suppose $u(x, y)$ is realized as a solution to a KZ equation at the origin. Then $K_x^{\mu,\alpha}u$, $K_y^{\mu,\alpha}u$ and $\tilde{K}_{x,y}^{\mu,\alpha}u$ have the same property. Hence $F_{p,q,r}^{p,q,r}(x, y)$ is a solution to a KZ equation.

ii) Suppose $u(x, 1-y)$ is a component of a solution to a KZ equation at the origin. Then $K_x^{\mu,\alpha}u$ and $K_y^{\mu,\alpha}u$ have the same property.

5.2. Example: Generalization of Appell's F_1

In this section we examine the hypergeometric function

$$F_{p,q,r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; x, y \right) := F_{p,q,r}^{p,q,r} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; x, y \right) \quad \text{with } \alpha'_p = \beta'_q = 0. \quad (5.15)$$

Then this hypergeometric function is realized as a solution to the KZ equation (5.7) or (5.2) whose construction is given in the last section.

Theorem 5.5. The simultaneous eigenvalues with their multiplicities of the pair of commuting residue matrices at the 15 normally crossing points (cf. [34, §6]) are give by

$$\begin{aligned}
[A_{01} : A_{23}] &= \{[0 : \alpha_i + \beta_j], [0 : \gamma_k]_{p+q-1}, [-\alpha'' - \beta'' : \gamma_k]\}, \\
[A_{01} : A_{24}] &= \{[0 : \alpha'_i + \beta'_j], [0 : \gamma'_k]_{p+q-1}, [-\alpha'' - \beta'' : \gamma'_k]\}, \\
[A_{01} : A_{34}] &= \{[0 : 0]_{R-(p+q+r)+1}, [0 : -\alpha'' - \beta'' - \gamma''], \\
&\quad [-\alpha'' - \beta'' : -\alpha'' - \beta'' - \gamma''], [0 : -\beta'' - \gamma'']_{p-1}, \\
&\quad [0 : -\alpha'' - \gamma'']_{q-1}, [-\alpha'' - \beta'' : -\alpha'' - \beta'']_{r-1}\}, \\
[A_{02} : A_{13}] &= \{[-\alpha'' - \gamma'' : \beta'_j], [0 : \beta'_j]_{p+r-1}, [0 : \alpha_i + \gamma'_k]\}, \\
[A_{02} : A_{14}] &= \{[-\alpha'' - \gamma'' : \beta_j], [0 : \beta_j]_{p+r-1}, [0 : \alpha'_i + \gamma_k]\}, \\
[A_{02} : A_{34}] &= \{[0 : 0]_{R-(p+q+r)+1}, [-\alpha'' - \gamma'' : -\alpha'' - \beta'' - \gamma''], \\
&\quad [0 : -\alpha'' - \beta'' - \gamma''], [0 : -\beta'' - \gamma'']_{p-1}, \\
&\quad [-\alpha'' - \gamma'' : -\alpha'' - \gamma'']_{q-1}, [0 : -\alpha'' - \beta'']_{r-1}\}, \\
[A_{03} : A_{12}] &= \{[\alpha'_i : -\beta'' - \gamma''], [\alpha'_i : 0]_{q+r-1}, [\beta_j + \gamma'_k : 0]\}, \\
[A_{03} : A_{14}] &= \{[\alpha'_i : \beta_j], [\beta_j + \gamma'_k : \beta_j], [\alpha'_i : \alpha'_i + \gamma_k]\}, \\
[A_{03} : A_{24}] &= \{[\alpha'_i : \alpha'_i + \beta'_j], [\alpha'_i : \gamma'_k], [\beta_j + \gamma'_k : \gamma'_k]\}, \\
[A_{04} : A_{12}] &= \{[\alpha_i : -\beta'' - \gamma''], [\alpha_i : 0]_{q+r-1}, [\beta'_j + \gamma_k : 0]\},
\end{aligned}$$

$$\begin{aligned}
[A_{04} : A_{13}] &= \{[\alpha_i : \beta'_j], [\alpha_i : \alpha_i + \gamma'_k], [\beta'_j + \gamma_k : \beta'_j]\}, \\
[A_{04} : A_{23}] &= \{[\alpha_i : \alpha_i + \beta_j], [\alpha_i : \gamma_k], [\beta'_j + \gamma_k : \gamma_k]\}, \\
[A_{12} : A_{34}] &= \{[0 : 0]_{R-(p+q+r)+1}, [-\beta'' - \gamma'' : -\alpha'' - \beta'' - \gamma''], \\
&\quad [0 : -\alpha'' - \beta'' - \gamma''], [-\beta'' - \gamma'' : -\beta'' - \gamma'']_{p-1}, \\
&\quad [0 : -\alpha'' - \gamma'']_{q-1}, [0 : -\alpha'' - \beta'']_{r-1}\}, \\
[A_{13} : A_{24}] &= \{[\beta'_j : \alpha'_i + \beta'_j], [\beta'_j : \gamma'_k], [\alpha_i + \gamma'_k : \gamma'_k]\}, \\
[A_{14} : A_{23}] &= \{[\beta_j : \alpha_i + \beta_j], [\beta_j : \gamma_k], [\alpha'_i + \gamma_k : \gamma_k]\}. \tag{5.16}
\end{aligned}$$

Here

$$R := pq + qr + rp, \quad 1 \leq i \leq p, \quad 1 \leq j \leq q, \quad 1 \leq k \leq r, \tag{5.17}$$

$$\alpha'' := \sum_{i=1}^p (\alpha_i + \alpha'_i), \quad \beta'' := \sum_{j=1}^q (\beta_j + \beta'_j), \quad \gamma'' := \sum_{k=1}^r (\gamma_k + \gamma'_k). \tag{5.18}$$

Proof. When $p = q = 1$ and $r = 0$, the claim is obvious because the rank of the system equals 1 and characterized by the solution

$$F_{1,1,0} \begin{pmatrix} \alpha & \beta & 0 \\ \alpha' & \beta' & 0 \end{pmatrix}; \alpha', \beta'; x, y = x^{\alpha'} (1-x)^{-\alpha-\alpha'} y^{\beta} (1-y)^{-\beta-\beta'}.$$

Assuming the above list, we get the resulting list under the convolution and addition corresponding to $p \mapsto p+1$ which is realized by

$$\begin{aligned}
K_x^{-\alpha_0 - \alpha'_0, \alpha_0} &= C \text{Ad}(x^{\alpha_0}) \circ \text{mc}_{-\alpha'_0} \circ \text{Ad}(x^{\alpha'_0}), \\
\alpha_0, \alpha'_0 &\in \mathbb{C}, \quad \alpha''_0 = \alpha_0 + \alpha'_0 \text{ and a constant } C. \tag{5.19}
\end{aligned}$$

To avoid confusion of the notation, we assume the condition (T) and introduce parameters α_0 and α'_0 which will be α_{p+1} and α'_{p+1} , respectively. Hence $\alpha'' = \sum_{i=1}^p (\alpha'_i + \alpha'_i)$ etc. Then Theorem 5.3 and (5.19) show

$$\begin{aligned}
[\tilde{A}_{01} : \tilde{A}_{23}] &= [A_{01} - \alpha''_0 : A_{23}] \cup [0 : A_{23}] \cup [0, A_{14} + \alpha_0], \\
[A_{01} - \alpha''_0 : A_{23}] &= \{[-\alpha''_0, \alpha_i + \beta_j], [-\alpha''_0 : \gamma_k]_{p+q-1}, [-\alpha'' - \beta'' - \alpha''_0 : \gamma_k]\}, \\
[0 : A_{23}] &= \{[0 : \alpha_i + \beta_j], [0 : \gamma_k]_{p+q}\}, \\
[0, A_{14} + \alpha_0] &= \{[0 : \alpha_0 + \beta_j]_{p+r}, [0 : \alpha'_i + \gamma_k + \alpha_0]\}
\end{aligned}$$

and

$$\begin{aligned}
[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{L}_1} &= [A_{01} - \alpha''_0 : A_{23}]|_{\ker A_{01}} \\
&= \{[-\alpha''_0 : \alpha_i + \beta_j], [-\alpha''_0 : \gamma_k]_{p+q-1}\}, \\
[\tilde{A}_{01} : \tilde{A}_{23}]|_{\mathcal{L}_2} &= [0, A_{14} + \alpha_0]|_{\ker A_{02}} \\
&= \{[0 : \alpha_0 + \beta_j]_{p+r-1}, [0 : \alpha_0 + \alpha'_i + \gamma_k]\}.
\end{aligned}$$

Here we note that $\dim \mathcal{L}_1 = pq + qr + rp - r$ and $\dim \mathcal{L}_2 = pq + qr + rp - q$. Since \tilde{A}_{ij} are matrices induced from A_{ij} on the quotient space $\mathbb{C}^{3N}/(\mathcal{L}_1 \oplus \mathcal{L}_2)$, we have the required result

$$[\tilde{A}_{01} : \tilde{A}_{23}] = \{[0 : \alpha_i + \beta_j], [0 : \gamma_k]_{p+q}, [-\alpha'' - \beta'' : \gamma_k]\}$$

by putting $\alpha_{p+1} = \alpha_0$ and $\alpha'_{p+1} = \alpha'_0$.

This result can be read from the first part of the following table. The part without an underline means it is contained in $\mathcal{L}_1 \oplus \mathcal{L}_2$. The multiplicity without an underline means that $\mathcal{L}_1 \oplus \mathcal{L}_2$ contains it with a smaller multiplicity, which is obtained comparing it with the original list.

The other 14 cases are similarly obtained and the result is as follows and hence we get the required result under the transformation corresponding to $p \mapsto p + 1$.

$$\begin{aligned}
[\tilde{A}_{01} : \tilde{A}_{23}] &= \{[-\alpha''_0 : \alpha_i + \beta_j], [-\alpha''_0 : \gamma_k]_{p+q-1}, [-\alpha'' - \beta'' - \alpha''_0 : \gamma_k], [0 : \gamma_k]_{p+q}, \\
&\quad [0 : \alpha_i + \beta_j], [0 : \alpha_0 + \beta_j]_{p+r}, [0 : \alpha_0 + \gamma_k + \alpha'_i]\}, \\
[\tilde{A}_{01} : \tilde{A}_{24}] &= \{[-\alpha''_0 : \alpha'_i + \beta'_j], [-\alpha''_0 : \gamma'_k]_{p+q-1}, [-\alpha'' - \beta'' - \alpha''_0 : \gamma'_k], [0 : \gamma'_k]_{p+q}, \\
&\quad [0 : \alpha'_i + \beta'_j], [0 : \alpha'_0 + \beta'_j]_{p+r}, [0 : \alpha_i + \gamma'_k + \alpha'_0]\}, \\
[\tilde{A}_{01} : \tilde{A}_{34}] &= \{[-\alpha''_0 : -\beta'' - \gamma'' - \alpha''_0]_{p-1}, [-\alpha''_0 : -\alpha'' - \beta'' - \alpha''_0], \\
&\quad [-\alpha''_0 : -\alpha'' - \gamma'' - \alpha''_0]_{q-1}, [-\alpha''_0 : -\alpha''_0]_{R-(p+q+r)+1}, \\
&\quad [-\alpha'' - \beta'' - \alpha''_0 : -\alpha'' - \beta'' - \gamma'' - \alpha''_0], [-\alpha'' - \beta'' - \alpha''_0 : -\alpha'' - \beta'' - \alpha''_0]_{r-1}, \\
&\quad [0 : -\alpha'' - \beta'' - \gamma'' - \alpha''_0]_2, [0 : -\alpha'' - \gamma'' - \alpha''_0]_{q-1}, [0 : 0]_{R-p}, [0 : -\beta'' - \gamma'']_p, \\
&\quad [0 : -\alpha''_0]_{R-(p+q+r)+1}, [0 : -\alpha'' - \beta'' - \alpha''_0]_{r-1}, [0 : -\beta'' - \gamma'' - \alpha''_0]_{p-1}\}, \\
[\tilde{A}_{02} : \tilde{A}_{13}] &= \{[-\alpha'' - \gamma'' - \alpha''_0 : \beta'_j], [-\alpha''_0 : \beta'_j]_{p+r-1}, [-\alpha''_0 : \alpha_i + \gamma'_k], [0 : \beta'_j]_{p+r}, \\
&\quad [0 : \alpha_i + \gamma'_k], [0 : \alpha_0 + \gamma'_k]_{p+q}, [0 : \alpha_0 + \alpha'_i + \beta'_j]\}, \\
[\tilde{A}_{02} : \tilde{A}_{14}] &= \{[-\alpha'' - \gamma'' - \alpha''_0 : \beta_j], [-\alpha''_0 : \beta_j]_{p+r-1}, [-\alpha''_0 : \gamma_k + \alpha'_i], [0 : \beta_j]_{p+r}, \\
&\quad [0 : \alpha'_i + \gamma_k], [0 : \alpha'_0 + \gamma_k]_{p+q}, [0 : \alpha_i + \beta_j + \alpha'_0]\}, \\
[\tilde{A}_{02} : \tilde{A}_{34}] &= \{[-\alpha'' - \gamma'' - \alpha''_0, -\alpha'' - \beta'' - \gamma'' - \alpha''_0], [-\alpha'' - \gamma'' - \alpha''_0 : -\alpha'' - \gamma'' - \alpha''_0]_{q-1}, \\
&\quad [-\alpha''_0 : -\alpha'' - \beta'' - \gamma'' - \alpha''_0], [-\alpha''_0 : -\beta'' - \gamma'' - \alpha''_0]_{p-1}, \\
&\quad [-\alpha''_0 : -\alpha''_0]_{R-(p+q+1)+1}, [-\alpha''_0 : -\alpha'' - \beta'' - \alpha''_0], \\
&\quad [0 : -\alpha''_0]_{R-(p+q+1)+1}, [0 : -\alpha'' - \beta'' - \gamma'' - \alpha''_0]_2, [0 : -\alpha'' - \beta'' - \alpha''_0]_{r-1}, \\
&\quad [0 : -\beta'' - \gamma'' - \alpha''_0]_{p-1}, [0 : -\alpha'' - \gamma'' - \alpha''_0]_{q-1}, [0 : 0]_{R-p}, [0 : -\beta'' - \gamma'']_p\}, \\
[\tilde{A}_{03} : \tilde{A}_{12}] &= \{[\alpha'_i : -\beta'' - \gamma''], [\alpha'_i : 0]_{q+r-1}, [\beta_j + \gamma'_k : 0], [\alpha'_0 : 0]_{R-p}, [\alpha'_0 : -\beta'' - \gamma'']_p, \\
&\quad [\alpha'_0 : 0]_{R-(p+q+r)+1}, [\alpha'_0 : -\alpha'' - \beta'' - \gamma'']_2, [\alpha'_0 : -\alpha'' - \beta]_{r-1}, \\
&\quad [\alpha'_0 : -\beta'' - \gamma'']_{p-1}, [\alpha'_0 : -\alpha'' - \gamma'']_{q-1}\}, \\
[\tilde{A}_{03} : \tilde{A}_{14}] &= \{[\alpha'_i : \beta_j], [\beta_j + \gamma'_k : \beta_j], [\alpha'_i : \alpha'_i + \gamma_k], [\alpha'_0 : \beta_j]_{p+r}, [\alpha'_0 : \alpha'_i + \gamma_k], \\
&\quad [\alpha'_0 : \alpha'_0 + \gamma_k]_{p+q}, [\alpha'_0 : \beta_j + \alpha_i + \alpha'_0]\}, \\
[\tilde{A}_{03} : \tilde{A}_{24}] &= \{[\alpha'_i : \alpha'_i + \beta'_j], [\alpha'_i : \gamma'_k], [\beta_j + \gamma'_k : \gamma'_k], [\alpha'_0 : \gamma'_k]_{p+q}, [\alpha'_0 : \alpha'_i + \beta'_j], \\
&\quad [\alpha'_0 : \alpha'_0 + \beta'_j]_{p+r}, [\alpha'_0 : \alpha_i + \gamma'_k + \alpha'_0]\}, \\
[\tilde{A}_{04} : \tilde{A}_{12}] &= \{[\alpha_i : -\beta'' - \gamma''], [\alpha_i : 0]_{q+r-1}, [\beta'_j + \gamma_k : 0], [\alpha_0 : 0]_{R-p}, [\alpha_0 : -\beta'' - \gamma'']_p, \\
&\quad [\alpha_0 : 0]_{R-(p+q+r)+1}, [\alpha_0 : -\alpha'' - \beta'' - \gamma'']_2, [\alpha_0 : -\alpha'' - \beta]_{r-1}, \\
&\quad [\alpha_0 : -\beta'' - \gamma'']_{p-1}, [\alpha_0 : -\alpha'' - \gamma'']_{q-1}\}, \\
[\tilde{A}_{04} : \tilde{A}_{13}] &= \{[\alpha_i : \beta'_j], [\alpha_i : \alpha_i + \gamma'_k], [\beta'_j + \gamma_k : \beta'_j], [\alpha_0 : \beta'_j]_{p+r}, [\alpha_0 : \alpha_i + \gamma'_k], \\
&\quad [\alpha_0 : \alpha_0 + \gamma'_k]_{p+q}, [\alpha_0 : \alpha_0 + \alpha'_i + \beta'_j]\}, \\
[\tilde{A}_{04} : \tilde{A}_{23}] &= \{[\alpha_i : \alpha_i + \beta_j], [\alpha_i : \gamma_k], [\beta'_j + \gamma_k : \gamma_k], [\alpha_0 : \gamma_k]_{p+q}, [\alpha_0 : \alpha_i + \beta_j], \\
&\quad [\alpha_0 : \beta_j + \alpha_0]_{p+r}, [\alpha_0 : \alpha_0 + \gamma_k + \alpha'_i]\}
\end{aligned}$$

$$\begin{aligned}
[\tilde{A}_{12} : \tilde{A}_{34}] &= \{[-\beta'' - \gamma : -\alpha'' - \beta'' - \gamma'' - \alpha_0''], [-\beta'' - \gamma : -\beta'' - \gamma'' - \alpha_0 - \alpha_0']_{p-1}, \\
&\quad \underline{[0 : -\alpha'' - \beta'' - \gamma'' - \alpha_0'']}, \underline{[0 : -\alpha'' - \gamma'' - \alpha_0'']}_{q-1}, [0 : -\alpha_0 - \alpha_0']_{R-(p+q+r)+1}, \\
&\quad \underline{[0 : -\alpha'' - \beta'' - \alpha_0'']}_{r-1}, [0 : 0]_{R-p}, \underline{[-\beta'' - \gamma : -\beta'' - \gamma'']}_p, \\
&\quad [0 : -\alpha_0 - \alpha_0']_{R-(p+q+r)+1}, [-\alpha'' - \beta'' - \gamma : -\alpha'' - \beta'' - \gamma'' - \alpha_0'']_2, \\
&\quad [-\alpha'' - \beta : -\alpha'' - \beta'' - \alpha_0'']_{r-1}, \underline{[-\beta'' - \gamma : -\alpha'' - \beta'' - \gamma'' - \alpha_0'']}_{p-1}, \\
&\quad [-\alpha'' - \gamma : -\alpha'' - \gamma'' - \alpha_0'']_{q-1}\} \\
[\tilde{A}_{13} : \tilde{A}_{24}] &= \{[0 : \alpha_i' + \beta_j'], [\beta_j' + \gamma_k' : \gamma_k'], [\beta_j' : \alpha_i' + \beta_j'], [\alpha_i + \gamma_k' : \gamma_k'], [\beta_j' : \gamma_k'], \\
&\quad \underline{[\beta_j' : \alpha_0' + \beta_j']}_{p+r}, [\alpha_i + \gamma_k' : \alpha_i + \gamma_k' + \alpha_0'], \\
&\quad \underline{[\alpha_0 + \gamma_k' : \gamma_k']}_{p+q}, [\alpha_i' + \alpha_0 + \beta_j' : \alpha_i' + \beta_j']\}, \\
[\tilde{A}_{14} : \tilde{A}_{23}] &= \{[\beta_j : \alpha_i + \beta_j], [\beta_j : \gamma_k], [\alpha_i' + \gamma_k : \gamma_k], \underline{[\alpha_0' + \gamma_k : \gamma_k]}_{p+q}, [\alpha_i + \beta_j + \alpha_0' : \alpha_i + \beta_j], \\
&\quad \underline{[\beta_j : \alpha_0 + \beta_j]}_{p+r}, [\alpha_i' + \gamma_k : \alpha_i' + \alpha_0 + \gamma_k]\}.
\end{aligned}$$

By the symmetry of the numbers $\{p, q, r\}$ of the points $\{x_0, x_1, x_2\}$, we have the list given in Theorem 5.5 by the induction with respect to (p, q, r) starting from $(p, q, r) = (1, 1, 0)$. \square

Note that the Riemann scheme is easily obtained from the list in Theorem 5.5. Moreover the list above already contains the spectral type of the boundary equation. For example, the boundary equation for the diagonal line $x_0 = x_1$ with respect to the characteristic exponent 0 is

$$\frac{du}{dx} = \frac{A_{013}|_{\text{Ker}A_{01}}}{x}u + \frac{A_{012}|_{\text{Ker}A_{01}}}{x-1}u = -\frac{A_{24}|_{\text{Ker}A_{01}}}{x}u - \frac{A_{34}|_{\text{Ker}A_{01}}}{x-1}u \quad (5.20)$$

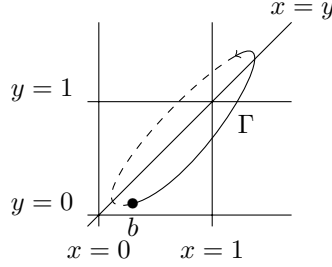
The residue matrix at infinity is

$$-A_{013}|_{\text{Ker}A_{01}} - A_{012}|_{\text{Ker}A_{01}} = -A_{23}|_{\text{Ker}A_{01}} = A_{014}|_{\text{Ker}A_{01}} \quad (5.21)$$

in view of the relation (5.6). The first three lines tell us the spectral type of the boundary equation. It is

$$(p+q-1)^r 1^{pq}, (R-p-q-r+1)(p-1)(q-1)1, (p+q-1)^r 1^{pq}$$

with the rigidity index $-2(p^2q - p^2 + pq^2 - 3pq + 2p - q^2 + 2q - 2)$, which is strictly smaller than 2 if and only if $p, q > 1$. Thus, the equation (5.20) may not be not rigid. However, the accessory parameters are suitably specialized so that it is globally analyzable in the following sense: a basis of local solutions near the origin of (5.20) can be identified with a subspace of the space of local solutions at $(0, 0^2)$ which is invariant by the monodromy along a loop γ_{01} going around the divisor $\{x = y\}$ once with the base point b near $(0, 0^2)$. The monodromy of (5.20) around $x = 1$ can be computed from the monodromy of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ along a loop Γ going around the divisors $\{x = 0\}, \{y = 1\}, \{x = y\}$ all at once with the base point b . Under the notation of §6.7, $\Gamma = \gamma_{01} * \gamma_{02} * \gamma_{12}$ where $*$ is the product in the fundamental group. The discussion of §6.7 shows that we can compute the monodromy along Γ .



On the other hand, the boundary equation with respect to the characteristic exponent $-\alpha'' - \beta''$ is given as above after the application of the homogenized addition $\text{Ad}\left(\left(\frac{x_0-x_1}{x_2-x_3}\right)^{\alpha''+\beta''}\right)$ which keeps the condition (5.4) and the equation has the rigid spectral type $1^r, 1^r, (r-1)1$.

Remark 5.6. *The calculation of the list for given integers p, q and r is supported by the Library [37] of the computer algebra system Risa/Asir ([28]). If $(p, q, r) = (4, 3, 2)$, it can be done as follows.*

```
[0] K432=os_md.mc2grs(0, ["K", [4,3,2]]);
[1] os_md.mc2grs(K432, "show" |dviout=-1);
[2] os_md.mc2grs(K432, "show0" |dviout=-1);
[3] os_md.mc2grs(K432, "get" |dviout=-1);
[4] os_md.mc2grs(K432, "spct" |dviout=-1, div=5);
[5] os_md.mc2grs(K432, "rest" |dviout=-1);
[6] os_md.mc2grs(K432, "rest0" |dviout=-1);
```

Here we get the list K432 in [0]. By [1], [2], [3], [4], [5] and [6], we get the list as in Theorem 5.5, the multiplicities of eigenvalues for 15 normally crossing points, the Riemann scheme (cf. [34]), the spectral type of 10 residue matrices (cf. [34]), the Riemann scheme of the boundary values with respect to 10 boundaries and the spectral types of the boundary values with respect to 10 boundaries, respectively, in a $\text{T}_{\text{E}}\text{X}$ format. Here the boundary values are defined with respect to the characteristic exponents of the regular singularities along the boundaries (cf. [19]). The option `div=5` indicate that the Riemann scheme is divided into two parts so that the first part contains 5 columns as is given in this section. If `dviout=-1` is replaced by `dviout=1` in the above, the result is transformed into a PDF file and displayed.

Remark 5.7. *The boundary equation of $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ on $x = 0$ with respect to the characteristic exponent α'_i equals $\mathcal{M}_{\beta', \gamma' - \alpha'_i}^{\beta, \gamma + \alpha'_i}$, which has the characteristic exponent $q + r - \beta'' - \gamma'' - 1$ with the free multiplicity at $y = 1$ and the connection coefficients between the local solutions at $y = 0$ and the local solution with the characteristic exponent $q + r - \beta'' - \gamma'' - 1$ at $y = 1$. They equal the connection coefficients of $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ between the local solutions at $(0^2, 0)$ and the local solutions at $(0, 1)$ with the characteristic exponents $(\alpha'_i, q + r - \beta'' - \gamma'' - 1)$ and the corresponding connection problem of $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ is solved. By the symmetry*

of the system we can solve the similar connection problems for 6 edges in **Fig. 1** in §3.3.

The KZ system \mathcal{M} defined in (5.7) corresponds to a $W(x, y)$ -module. The coefficient matrices $A_{i,j}$ are $R \times R$ matrices acting on \mathbb{C}^R by left multiplication. For any $i = 1, \dots, R$, let $v_i = (0, \dots, 1, \dots, 0) \in \mathbb{C}^R$ be the i -th unit column vector and we regard it as an element of $\mathbb{C}(x, y)^R$. Then, we define an action of partial differential operators on an element of the form $a(x, y)v_i$ by

$$\begin{aligned} \frac{\partial}{\partial x} (a(x, y)v_i) &:= \frac{\partial a}{\partial x} (x, y)v_i + a(x, y) \left(\frac{A_{01}}{x-y} + \frac{A_{02}}{x-1} + \frac{A_{03}}{x} \right) v_i, \\ \frac{\partial}{\partial y} (a(x, y)v_i) &:= \frac{\partial a}{\partial y} (x, y)v_i + a(x, y) \left(\frac{A_{01}}{y-x} + \frac{A_{12}}{y-1} + \frac{A_{13}}{y} \right) v_i \end{aligned} \quad (5.22)$$

for any $a(x, y) \in \mathbb{C}(x, y)$ and $i = 1, \dots, R$. We extend (5.22) to an action of $W(x, y)$ on $\mathbb{C}(x, y)^R$ by linearity. By abuse of notation, we write \mathcal{M} for the $W(x, y)$ -module $\mathbb{C}(x, y)^R$. On the other hand, we write $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ for the $W(x, y)$ -module $W(x, y)/(W(x, y)P_1 + W(x, y)P_2 + W(x, y)P_{12})$.

Theorem 5.8. *Suppose that none of $\alpha_i + \alpha'_{i'}$, $\beta_j + \beta'_{j'}$, $\gamma_k + \gamma'_{k'}$, $\alpha_i + \beta_j + \gamma'_k$, $\alpha'_i + \beta'_j + \gamma_k$ is integral. Then, $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ is isomorphic to \mathcal{M} as a $W(x, y)$ -module through the correspondence $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \ni [1] \mapsto v_R \in \mathcal{M}$.*

Proof. The assumption ensures that $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ is irreducible (cf. Theorem 6.9). We consider a partial derivative $\partial_{gen} := a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$ for generic complex numbers a, b . Let $u(x, y) = (u_1(x, y), \dots, u_R(x, y))$ be a non-trivial solution vector to the system \mathcal{M} . By the construction of \mathcal{M} , the last entry $u_R(x, y)$ is a solution to $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ (see Remark 5.2). There exist R^2 rational functions $a_{ij}(x, y)$ ($1 \leq i \leq j \leq R$) such that the following equations hold true:

$$\begin{cases} \partial_{gen} u_1(x, y) &= a_{11}(x, y)u_1(x, y) + \dots + a_{1R}(x, y)u_R(x, y) \\ &\vdots \\ \partial_{gen} u_R(x, y) &= a_{R1}(x, y)u_1(x, y) + \dots + a_{RR}(x, y)u_R(x, y). \end{cases} \quad (5.23)$$

Suppose that $a_{R1}(x, y), \dots, a_{R(R-1)}(x, y)$ are all equal to zero. Then, the restriction of $u_R(x, y)$ to a line $\{bx - ay + c = 0\}$ for some generic c is subject to a first order ODE. This contradicts the irreducibility of $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ as $u_R(x, y)$ is a solution to the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$. Without loss of generality, we may assume that $a_{R(R-1)}(x, y) \neq 0$. We can express $u_{R-1}(x, y)$ as a linear combination of $u_1, u_2, \dots, u_{R-2}, u_R, \partial_{gen} u_R$ and we obtain a system of equations of the follow-

ing form

$$\left\{ \begin{array}{l} \partial_{gen} u_1(x, y) = b_{11}(x, y)u_1(x, y) + \cdots + b_{1(R-2)}u_{R-2}(x, y) \\ \quad + b_{1(R-1)}(x, y)\partial_{gen}u_R(x, y) + b_{1R}(x, y)u_R(x, y) \\ \quad \vdots \\ \partial_{gen} u_{R-2}(x, y) = b_{(R-2)1}(x, y)u_1(x, y) + \cdots + b_{(R-2)(R-2)}u_{R-2}(x, y) \\ \quad + b_{(R-2)(R-1)}(x, y)\partial_{gen}u_R(x, y) + b_{(R-2)R}(x, y)u_R(x, y) \\ \partial_{gen}^2 u_R(x, y) = b_{(R-1)1}(x, y)u_1(x, y) + \cdots + b_{(R-1)(R-2)}(x, y)u_{R-2}(x, y) \\ \quad + b_{(R-1)(R-1)}(x, y)\partial_{gen}u_R(x, y) + b_{(R-1)R}(x, y)u_R(x, y). \end{array} \right. \quad (5.24)$$

Again by the irreducibility of $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$, we may assume that $b_{(R-1)(R-2)}(x, y) \neq 0$. Repeating this argument, we can conclude that $u_1(x, y), \dots, u_{R-1}(x, y)$ are linear combinations of partial derivatives of $u_R(x, y)$. It is readily seen that \mathcal{M} is isomorphic to the $\mathbb{C}(x, y)$ -span of $u_1(x, y), \dots, u_R(x, y)$ in the space of local meromorphic functions through the correspondence $v_i \mapsto u_i(x, y)$. Now the correspondence $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} \ni [1] \mapsto u_R(x, y) \in \mathcal{M}$ is well-defined and surjective. Since $\dim_{\mathbb{C}(x, y)} \mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma} = \dim_{\mathbb{C}(x, y)} \mathcal{M} = R$, we are done. \square

Remark 5.9. i) We constructed a $W(x, y)$ homomorphism between $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ and \mathcal{M} , which gives an isomorphism when the parameters are generic. Since \mathcal{M} and $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ have the same rank, it follows from [33, Lemma 2.1] that the condition for the irreducibility of \mathcal{M} equals that of $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$.

ii) Suppose, for example, $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ is reducible when $\alpha_i + \alpha'_i = c$ with a suitable $c \in \mathbb{C}$. Then it also follows from [33, Lemma 2.1] and Proposition 2.2 that the system is reducible when $\alpha_i + \alpha'_i - c \in \mathbb{Z}$.

Owing to Theorem 5.5, we get the Riemann scheme and the spectral type of the corresponding KZ equation \mathcal{M} (cf. [36]) as follows.

$$\left\{ \begin{array}{cccccc} A_{01} & A_{02} & A_{03} & A_{04} & A_{12} & \\ [0]_{pq+(p+q-1)r} & [0]_{pr+(p+r-1)q} & [\alpha'_i]_{q+r} & [\alpha_i]_{q+r} & [0]_{qr+(q+r-1)p} & \\ [-\alpha'' - \beta'']_r & [-\alpha'' - \gamma'']_q & \beta_j + \gamma'_k & \beta'_j + \gamma_k & [-\beta'' - \gamma'']_p & \\ A_{13} & A_{23} & A_{14} & A_{24} & A_{34} & \\ [\beta'_j]_{p+r} & [\gamma_k]_{p+q} & [\beta_j]_{p+r} & [\gamma'_k]_{p+q} & [0]_{pq+qr+rp-(p+q+r)+1} & \\ \alpha_i + \gamma'_k & \alpha_i + \beta_j & \alpha'_i + \gamma_k & \alpha'_i + \beta'_j & [-\alpha'' - \beta'' - \gamma'']_2 & \\ & & & & [-\alpha'' - \beta'']_{r-1} & \\ & & & & [-\beta'' - \gamma'']_{p-1} & \\ & & & & [-\alpha'' - \gamma'']_{q-1} & \end{array} \right\} \quad (5.25)$$

Spectral type of the KZ equation satisfied by $F_{p,q,r}(x,y)$

	$x_0 = x$	$x_1 = y$	$x_2 = 1$	$x_3 = 0$	$x_4 = \infty$
x_0		$(R-r)r$	$(R-q)q$	$(q+r)^p 1^{qr}$	$(q+r)^p 1^{qr}$
x_1	$(R-r)r$		$(R-p)p$	$(p+r)^q 1^{pr}$	$(p+r)^q 1^{pr}$
x_2	$(R-q)q$	$(R-p)p$		$(p+q)^r 1^{pq}$	$(p+q)^r 1^{pq}$
x_3	$(q+r)^p 1^{qr}$	$(p+r)^q 1^{pr}$	$(p+q)^r 1^{pq}$		S
x_4	$(q+r)^p 1^{qr}$	$(p+r)^q 1^{pr}$	$(p+q)^r 1^{pq}$	S	

rank : $R := pq + qr + rp$, $S := (R - p - q - r + 1)(p - 1)(q - 1)(r - 1)2$

This table shows that, for example, the spectral type of the residue matrix

$A_{0,3}$ is $(q+r)^p 1^{qr}$, which represents the partition of $R : \overbrace{q+r, \dots, q+r}^p, \overbrace{1, \dots, 1}^{qr}$, namely, $A_{0,3}$ has p eigenvalues with multiplicity $q+r$ and qr multiplicity free eigenvalues. The symmetry $S_{x,y,0} \times S_{0,\infty}$ acts on this table. For example, the spectral type at $x_1 = x_4$ is obtained from that at $x_0 = x_4$ by exchanging p and q . This symmetry corresponds to $x_0 \leftrightarrow x_1$.

The next table shows the spectral type of the equation when $p = q = r = 2$ together with the index of rigidity for each variable.

$p = q = r = 2$ (rank: 12)

	x_0	x_1	x_2	x_3	x_4	idx
x_0		2	(10)2	$4^2 1^4$	$4^2 1^4$	-8
x_1	(10)2		(10)2	$4^2 1^4$	$4^2 1^4$	-8
x_2	(10)2	(10)2		$4^2 1^4$	$4^2 1^4$	-8
x_3	$4^2 1^4$	$4^2 1^4$	$4^2 1^4$		$7 2 1^3$	-124
x_4	$4^2 1^4$	$4^2 1^4$	$4^2 1^4$	$7 2 1^3$		-124

Focusing on one variable, for example x_0 , we have an ordinary differential equation (5.5) of rank R which has $n+1 = 4$ singular points including ∞ in \mathbb{P}^1 . The rigidity index of the equation equals the sum of squares of multiplicities of the eigenvalues of the residue matrices minus $(n-1)R^2$. It is

$$\begin{aligned} \text{idx}_{x_0} \mathcal{M} &= (R-q)^2 + q^2 + (R-r)^2 + r^2 + 2(p(q+r)^2 + qr) - 2R^2 \\ &= 2 - 2(q-1)(r-1)(q+r+1). \end{aligned} \quad (5.26)$$

Then the number $2 - \text{idx}_{x_0} \mathcal{M}$ coincides with the maximal possible number of the accessory parameters of the equation which has the same local structure as the equation (5.5). An ordinary differential equation is rigid if and only if the rigidity index equals 2. Then the KZ equation satisfied by $F_{p,q,r}$ has a rigid variable if and only if

$$(p-1)(q-1)(r-1) = 0. \quad (5.27)$$

If the KZ equation has a rigid variable, the condition of the irreducibility of the monodromy group of its solution space is obtained by [33] and a precise analysis is possible for the reducibility. In general, the necessary and sufficient condition for the irreducibility is given by Theorem 6.9.

The transformation given by additions and middle convolutions are invertible. $d(\mathbf{m}_x)$ gives the maximal number of the change of the rank of the equation

by one middle convolution with additions for the variable x . Namely, the rank decreases by $d(\mathbf{m}_x)$, which equals the sum of maximal multiplicities of eigenvalues of residue matrices minus $(n-1)R$ (see $d_{max}(\mathbf{m})$ in [32, (5.36)]).

$$d(\mathbf{m}_x) = (R-r) + (R-q) + (q+r) + (q+r) - 2R = q+r. \quad (5.28)$$

The number $d(\mathbf{m}_x)$ is always positive if \mathbf{m}_x is a rigid spectral type. Moreover these transformations do not change the rigidity index. Hence there is an invertible algorithm due to Katz, Dettweiler-Reiter and Haraoka transforming the rigid system to the trivial system (cf. [6, 14, 20]). In fact, the change of the spectral type is described by decreasing the maximal multiplicities of the residue matrices by $d(\mathbf{m}_x)$.

$$\mathbf{m}_x : \underline{(R-r)r}, \underline{(R-q)r}, \underline{(q+r)^p 1^{qr}}, \underline{(q+r)^p 1^{qr}}$$

In the above, the maximal multiplicities are indicated by underlines and they decrease by $d(\mathbf{m}_x)$, which leads to the change $(p, q, r) \mapsto (p-1, q, r)$ since $R = (q+r)p + qr$. We can repeat this algorithm until (p, q, r) changes into $(1, p, q)$. Note that $\text{idx}_x(\mathcal{M})$ does not depend on p . If $p=1$, the equation is rigid with respect to y variable and we have the above reduction until we get the trivial equation. Inverting these algorithm, we get the original equation \mathbf{m} from the trivial equation.

Owing to [34, Theorem 7.1], the above procedure can be constructed only by the data describing the simultaneous eigenvalues with their multiplicities for commuting pairs of residue matrices of the equation.

5.3. Appell's F_1

Appell's F_1 series is defined by

$$F_1(a; b, c; d; x, y) := F_{1,1,1}^{1,1,1} \left(\begin{matrix} \alpha & \beta & \gamma \\ 0 & 0 & \gamma' \end{matrix}; x, y \right)$$

with $\alpha = b, \beta = c, \gamma = a, \gamma' = 1 - d$.

The Riemann scheme and the spectral type of the corresponding KZ equation are

$$\left\{ \begin{array}{ccccc} A_{01} & A_{02} & A_{03} & A_{04} & A_{12} \\ [0]_2 & [0]_2 & [0]_2 & [\alpha]_2 & [0]_2 \\ -\alpha - \beta & \alpha - \gamma - \gamma' & \beta + \gamma' & \gamma & -\beta - \gamma - \gamma' \\ A_{13} & A_{23} & A_{14} & A_{24} & A_{34} \\ [0]_2 & [\gamma]_2 & [\beta]_2 & [\gamma']_2 & [-\alpha - \beta - \gamma - \gamma']_2 \\ \alpha + \gamma' & \alpha + \beta & \gamma & 0 & 0 \end{array} \right\}, \quad (5.29)$$

$F_1 : p = q = r = 1$ (rank: 3)

	x_0	x_1	x_2	x_3	x_4	idx
x_0		21	21	21	21	2
x_1	21		21	21	21	2
x_2	21	21		21	21	2
x_3	21	21	21		21	2
x_4	21	21	21	21		2

It follows from Remark. 2.1 that $F_1(\gamma; \alpha, \beta; 1 - \gamma'; x, y)$ is a solution to

$$\bar{\mathcal{M}}_{0,0,\gamma'}^{\alpha,\beta,\gamma} : \begin{cases} \partial_x(\vartheta_x + \vartheta_y - \gamma')u &= (\vartheta_x + \alpha)(\vartheta_x + \vartheta_y + \gamma)u, \\ \partial_y(\vartheta_x + \vartheta_y - \gamma')u &= (\vartheta_y + \beta)(\vartheta_x + \vartheta_y + \gamma)u, \\ (\vartheta_x + \alpha)\partial_y u &= (\vartheta_y + \beta)\partial_x u. \end{cases}$$

Put $(X, Y) = (1 - x, 1 - y)$. Then

$$(\vartheta_X + \alpha)\partial_Y u = (\vartheta_Y + \beta)\partial_X u$$

and the first line of the equations $\bar{\mathcal{M}}_{0,0,\gamma'}^{\alpha,\beta,\gamma}$ with this equation says

$$\begin{aligned} 0 &= -\partial_X(\vartheta_X + \vartheta_Y - \partial_X - \partial_Y - \gamma')u \\ &\quad + (\vartheta_X - \partial_X + \alpha)(\vartheta_X + \vartheta_Y - \partial_X - \partial_Y + \gamma)u \\ &= (\vartheta_X + \alpha + \gamma + \gamma')\partial_X u + (\vartheta_X + \alpha)\partial_Y u - (\vartheta_X + \alpha)(\vartheta_X + \vartheta_Y + \gamma)u \\ &= \partial_X(\vartheta_X + \alpha + \beta + \gamma + \gamma' - 1)u - (\vartheta_X + \alpha)(\vartheta_X + \vartheta_Y + \gamma)u. \end{aligned}$$

Hence we have

$$T_{(x,y) \rightarrow (1-x,1-y)} \mathcal{M}_{0,0,\gamma'}^{\alpha,\beta,\gamma} = \mathcal{M}_{0,0,1-\alpha-\beta-\gamma-\gamma'}^{\alpha,\beta,\gamma}$$

and

$$\begin{aligned} \mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma} &\xrightarrow{\text{Ad}(x^{-\alpha'}y^{-\beta'})} \mathcal{M}_{0,0,\gamma'-\alpha'-\beta'}^{\alpha+\alpha',\beta+\beta',\gamma+\alpha'+\beta'} \xrightarrow{T_{(x,y) \rightarrow (1-x,1-y)}} \\ &\mathcal{M}_{0,0,1-\alpha-\alpha'-\beta-\beta'-\gamma-\gamma'}^{\alpha+\alpha',\beta+\beta',\gamma+\alpha'+\beta'} \xrightarrow{\text{Ad}(x^{\alpha'}y^{\beta'})} \mathcal{M}_{\alpha',\beta',1-\alpha-\beta-\gamma-\gamma'}^{\alpha,\beta,\gamma}. \end{aligned}$$

Proposition 5.10. *One has*

$$T_{0 \leftrightarrow 1}^{F_1} \mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma} = \mathcal{M}_{\alpha',\beta',1-\alpha-\beta-\gamma-\gamma'}^{\alpha,\beta,\gamma}$$

by putting $T_{0 \leftrightarrow 1}^{F_1} = \text{Ad}(x^{\alpha'}y^{\beta'}) \circ T_{(x,y) \rightarrow (1-x,1-y)} \circ \text{Ad}(x^{-\alpha'}y^{-\beta'})$.

Remark 5.11 (Symmetry of F_1). i) If $u(x,y)$ is a solution to $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$, then $(\frac{x}{1-x})^{\alpha'}(\frac{y}{1-y})^{\beta'}u(1-x, 1-y)$ is a solution to $T_{0 \leftrightarrow 1}^{F_1} \mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma}$.

ii) In the above, we may replace γ and γ' by operators which commute with x, y, ∂_x and ∂_y .

iii) The Riemann scheme of the ordinary differential equation of the variable x satisfied by $F_{1,1,1}^{1,1,1} \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 0 & \gamma' \end{smallmatrix}; x, y \right)$ is

$$\left\{ \begin{array}{cccc} x = y & x = 1 & x = 0 & x = \infty \\ [0]_{(2)} & [0]_{(2)} & [0]_{(2)} & [\alpha]_{(2)} \\ -\alpha - \beta + 1 & \alpha - \gamma - \gamma' + 1 & \beta + \gamma' & \gamma \end{array} \right\},$$

which is rigid. Hence, we can recover Proposition 5.10 by using this.

As is shown in §5.1, the group $S_{\{x,y,0,1,\infty\}} \simeq S_{\{x_0,x_1,x_2,x_3,x_4\}} \simeq S_{\{0,1,2,3,4\}} \simeq S_5$ acts on KZ equation (5.2) with $n = 2$ as a group of coordinate transformations, which corresponds to the group of coordinate transformations under the (x, y) coordinate given in (3.1). The transformations of $\mathcal{M}_{\alpha',\beta,\gamma'}^{\alpha,\beta,\gamma}, T_{(x,y)\rightarrow(y,x)}, T_{(x,y)\rightarrow(\frac{y}{x},\frac{1}{y})}, T_{(x,y)\rightarrow(\frac{1}{x},\frac{1}{y})}$ and $T_{0\leftrightarrow 1}^{F_1}$ correspond to the transposition $(0, 1)$, $(1, 2)$, $(3, 4)$ and $(2, 3)$ in $S_{\{0,1,2,3,4\}}$, respectively. They generate $S_{\{0,1,2,3,4\}}$.

In this section, \tilde{X} denotes the space obtained by resolving the singularities $\text{Sing}(\mathcal{M}_{\alpha',\beta,\gamma'}^{\alpha,\beta,\gamma}) \subset \mathbb{P}^1 \times \mathbb{P}^1$ at $(0, 0)$, $(1, 1)$ and (∞, ∞) . In \tilde{X} there are 10 singular lines parametrized by $\{i, j\}$ ($0 \leq i < j \leq 4$) which corresponds to $x_i = x_j$. Here $\{2, 4\}$ corresponds to the exceptional fiber at $(0, 0)$. There are 15 normally crossing singular points parametrized by $\{\{i, j\}, \{k, \ell\}\}$ with $\{i, j, k, \ell\} \subset \{0, 1, 2, 3, 4\}$ and $\#\{i, j, k, \ell\} = 4$. In each singular line $\{i, j\}$ there are three normally crossing singular points by the intersection of lines $\{k, \ell\}$ with $\{k, \ell\} \subset \{0, 1, 2, 3, 4\} \setminus \{i, j\}$.

The connection formula from $(0^2, 0)$ to $(0, \infty)$ given in Theorem 3.6 corresponds to the ordered pair $(\{0, 3\}, \{2, 4\}), \{\{0, 3\}, \{1, 4\}\}$. The group $S_{\{0,1,2,3,4\}}$ simply and transitively acts on the set of singular lines with ordered two singular points in the lines. Hence applying suitable coordinate transformations corresponding to elements in $S_{\{0,1,2,3,4\}}$, we get the connection formula between normally crossing points belonging to any singular line in \tilde{X} .

The subgroup of $S_{\{0,1,2,3,4\}}$ which keeps the exceptional line at $(0, 0)$ corresponding to $x_2 = x_4$ is generated by transpositions $(0, 1)$, $(1, 3)$ and $(2, 4)$, which correspond to $(x, y) \mapsto (y, x)$, $(\frac{y-x}{y-1}, \frac{y}{y-1})$ and $(\frac{x}{x-1}, \frac{y}{y-1})$, respectively. The subgroup is isomorphic to $S_3 \times S_2$. Applying this subgroup to $F\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 0 & \gamma' \end{smallmatrix}; x, y\right)$ and $F_1(a; b, c; d; x, y)$, we obtain the identity between transformed 12 functions as in the case of Kummer's formula of Gauss hypergeometric function.

For example, since $(2\ 4) = (3\ 4)(2\ 3)(3\ 4)$, the corresponding transformation of the system is given by $T_{(x,y)\rightarrow(\frac{1}{x},\frac{1}{y})} \circ T_{0\leftrightarrow 1}^{F_1} \circ T_{(x,y)\rightarrow(\frac{1}{x},\frac{1}{y})}$ and therefore it follows from Theorem 3.1 and Proposition 5.10 that $\mathcal{M}_{\alpha',\gamma',\beta'}^{\alpha,\beta,\gamma}$ and its solution $u(x, y)$ are transformed into $\mathcal{M}_{\alpha',\gamma',\beta'}^{\alpha,\beta,1-\alpha'-\beta'-\gamma-\gamma'}$ and $(1-x)^{-\alpha}(1-y)^{-\beta}u(\frac{x}{x-1}, \frac{y}{y-1})$. Hence we have

$$F\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 0 & \gamma' \end{smallmatrix}; x, y\right) = (1-x)^{-\alpha}(1-y)^{-\beta}F\left(\begin{smallmatrix} \alpha & \beta & 1-\gamma-\gamma' \\ 0 & 0 & \gamma' \end{smallmatrix}; \frac{x}{x-1}, \frac{y}{y-1}\right). \quad (5.30)$$

Using the transformation $T_{(x,y)\rightarrow(\frac{x}{y},\frac{1}{y})} \circ T_{0\leftrightarrow 1}^{F_1} \circ T_{(x,y)\rightarrow(\frac{x}{y},\frac{1}{y})}$ corresponding to $(1\ 3) = (1\ 2)(2\ 3)(1\ 2)$, we have

$$F\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 0 & \gamma' \end{smallmatrix}; x, y\right) = (1-y)^{-\alpha}F\left(\begin{smallmatrix} \alpha & 1-\alpha-\beta-\gamma' & \gamma \\ 0 & 0 & \gamma' \end{smallmatrix}; \frac{y-x}{y-1}, \frac{y}{y-1}\right). \quad (5.31)$$

The subgroup of $S_{\{0,1,2,3,4\}}$ fixing the point $(0, \infty)$ in \tilde{X} , which corresponds to $\{\{0, 3\}, \{1, 4\}\}$, is generated by $(0\ 3)$, $(1\ 4)$ and $(0\ 1)(3\ 4)$. The subgroup is isomorphic to W_{B_2} , the reflection group of the root space of type B_2 . Hence we have identities among 8 functions obtained by transforming $G_2(a, a'; b, b'; x, y)$ by the corresponding coordinate transformations (the formula is in [27, §1.3]).

5.4. *Example: Generalization of Appell's F_2 and F_3*

In this section we study the hypergeometric functions under the condition (F2) or (F3). We put

$$\begin{aligned} I_{p,q} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma \end{matrix} ; x, y \right) &:= F_{p,q,0}^{p-1,q-1,1} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma \end{matrix} ; x, y \right), \\ J_{p,q} \left(\begin{matrix} \alpha & \beta \\ \alpha' & \beta' \end{matrix} ; x, y \right) &:= F_{p-1,q-1,1}^{p,q,0} \left(\begin{matrix} \alpha & \beta \\ \alpha' & \beta' \end{matrix} ; x, y \right). \end{aligned}$$

Note that the transformation $(x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$ induces $\mathcal{M}_{\alpha',\beta',\gamma'}^{\alpha,\beta,\gamma} \mapsto \mathcal{M}_{\alpha,\beta,\gamma}^{\alpha',\beta',\gamma'}$.

A KZ equation satisfied by $I_{p,q} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha', 0 & \beta', 0 & \emptyset \end{matrix} ; x, 1-y \right)$ is constructed by transformations of KZ equations according to the integral transformation of $(y-x)^{-\gamma_{q-1}}$ in a neighborhood of $(0, 1)$. Then the residue matrices $A_{i,j}$ of the KZ equation have the following property.

Theorem 5.12 (F2). *The simultaneous eigenvalues with their multiplicities of the pair of commuting residue matrices at the 15 normally crossing points are give by*

$$\begin{aligned} [A_{01} : A_{23}] &= \{[-\alpha'' - \beta'' - \gamma : \gamma], [0 : \gamma]_{p+q-1}, [0 : \alpha_i + \beta_j]\}, \\ [A_{01} : A_{24}] &= \{[-\alpha'' - \beta'' - \gamma : -\beta'' - \gamma], [0 : -\beta'' - \gamma]_{p-1}, [0 : \alpha'_i]_{q-1}\}, \\ [A_{01} : A_{34}] &= \{[-\alpha'' - \beta'' - \gamma : -\alpha'' - \gamma], [0 : \beta'_j]_{p-1}, [0 : -\alpha'' - \gamma]_{q-1}\}, \\ [A_{02} : A_{13}] &= \{[0 : \alpha_i - \beta'' - \gamma], [0 : 0]_{pq-p-q+2}, [\beta_j - \alpha'' - \gamma : 0]\}, \\ [A_{02} : A_{14}] &= \{[0 : \alpha'_i + \gamma], [0 : \beta_j]_{p-1}, [\beta_j - \alpha'' - \gamma : \beta_j]\}, \\ [A_{02} : A_{34}] &= \{[0 : -\alpha'' - \gamma], [0 : \beta'_j]_{p-1}, [\beta_j - \alpha'' - \gamma : -\alpha'' - \gamma]\}, \\ [A_{03} : A_{12}] &= \{[\alpha'_i : \beta'_j]\}, \\ [A_{03} : A_{14}] &= \{[\alpha'_i : \alpha'_i + \gamma], [\alpha'_i : \beta_j]\}, \\ [A_{03} : A_{24}] &= \{[\alpha'_i : -\beta'' - \gamma], [\alpha'_i : \alpha'_i]_{q-1}\}, \\ [A_{04} : A_{12}] &= \{[[\alpha_i : \beta'_j], [\beta'_j + \gamma : \beta'_j]]\}, \\ [A_{04} : A_{13}] &= \{[\alpha_i : \alpha_i - \beta'' - \gamma], [\alpha_i : 0]_{q-1}, [\beta'_j + \gamma : 0]\}, \\ [A_{04} : A_{23}] &= \{[\alpha_i : \gamma], [\beta'_j + \gamma : \gamma], [\alpha_i : \alpha_i + \beta_j]\}, \\ [A_{12} : A_{34}] &= \{[\beta'_j : -\alpha'' - \gamma], [\beta'_j : \beta'_j]_{p-1}\}, \\ [A_{13} : A_{24}] &= \{[\alpha_i - \beta'' - \gamma : -\beta'' - \gamma], [0 : -\beta'' - \gamma], [0 : \alpha'_i]_{q-1}\}, \\ [A_{14} : A_{23}] &= \{[\alpha'_i + \gamma : \gamma], [\beta_j : \gamma], [\beta_j : \alpha_i + \beta_j]\}. \end{aligned}$$

The proof is omitted since it is an induction on (p, q) which is similar to the proof of Theorem 5.5. We only remark that in one step of the induction we increase p by one by applying Theorem 5.3 to our setting with $\dim \mathcal{L}_1 = pq - 1$ and $\dim \mathcal{L}_2 = pq - q + 1$.

The following Riemann scheme and spectral type of the corresponding KZ equation are obtained from Theorem 5.12.

Spectral type of $I_{p,q}$ (rank: pq)

	$x_0 = x$	$x_1 = 1 - y$	$x_2 = 1$	$x_3 = 0$	$x_4 = \infty$	idx
x_0		$(pq - 1)1$	S_2	q^p	$q^{p-1}1^q$	2
x_1	$(pq - 1)1$		p^q	S_1	$p^{q-1}1^p$	2
x_2	S_2	p^q		S	$(q - 1)^p p$	
x_3	q^p	S_1	S		$(p - 1)^q q$	
x_4	$q^{p-1}1^q$	$q^{p-1}1^q$	$(q - 1)^p p$	$(p - 1)^q q$		

$S = (p + q - 1)1^{(p-1)(q-1)}$, $S_1 = (pq - p + 1)1^{p-1}$, $S_2 = (pq - q + 1)1^{q-1}$

$$\left\{ \begin{array}{ccccc} A_{01} & A_{02} & A_{03} & A_{04} & A_{12} \\ [0]_{pq-1} & [0]_{pq-q+1} & [\alpha'_i]_q & [\alpha_i]_q & [\beta'_j]_p \\ -\alpha'' - \beta'' - \gamma & \beta_j - \alpha'' - \gamma & & \beta'_j + \gamma & \\ \\ A_{13} & A_{23} & A_{14} & A_{24} & A_{34} \\ [0]_{pq-p+1} & [\gamma]_{p+q-1} & [\beta_j]_p & [\alpha'_i]_{q-1} & [\beta'_j]_{p-1} \\ \alpha_i - \beta'' - \gamma & \alpha_i + \beta_j & \alpha'_i + \gamma & [-\beta'' - \gamma]_p & [-\alpha'' - \gamma]_q \end{array} \right\}$$

 $F_2 : p = q = 2$ (rank: 4)

	x_0	x_1	x_2	x_3	x_4	idx
x_0		31	31	22	211	2
x_1	31		22	31	211	2
x_2	31	22		31	211	2
x_3	22	31	31		211	2
x_4	211	211	211	211		-8

 $I_{4,3} : p = 4, q = 3$ (rank: 12)

	$x_0 = x$	$x_1 = 1 - y$	$x_2 = 1$	$x_3 = 0$	$x_4 = \infty$	idx
x_0		$(11)1$	$(10)1^2$	3^4	3^31^3	2
x_1	$(11)1$		4^3	91^3	4^21^3	2
x_2	$(10)1^2$	4^3		61^6	42^4	-64
x_3	3^4	91^3	61^6		3^4	-90
x_4	3^31^3	4^21^4	42^4	3^4		-154

We note that the the transformation $(x, y) \mapsto (y, x)$ of the coordinate of $I_{p,q} \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \end{smallmatrix}; x, y \right)$ corresponds to the element $(0 \ 1)(2 \ 3) \in S_{0,1,2,3,4}$ because of the correspondence $(x_0, x_1, x_2, x_3, x_4) = (x, 1 - y, 1, 0, \infty)$.

The ordinary differential equation for the variable x_0 is rigid and its rigid spectral type with direct decompositions are

$$\begin{aligned} & (pq - 1)1, ((p - 1)q + 1)1^{q-1}, q^p, q^{p-1}1^q \\ & = (p(q - 1) - 1)1, ((p - 1)(q - 1) + 1)1^{q-1}, (q - 1)^p, (q - 1)^{p-1}1^{q-1} \\ & \oplus p0, (p - 1)1, 1^p, 1^{p-1}1 \\ & = 10, 10, 1, 01 \oplus (pq - 2)1, ((p - 1)q)1^{q-1}, (q - 1)q^{p-1}, q^{p-2}1^{q-1} \\ & = q(10, 10, 1, 10) \oplus ((p - 1)q - 1)1, ((p - 2)q + 1)1^{q-1}, q^{p-1}, q^{p-2}1^q \end{aligned}$$

The direct decomposition is a certain decomposition of spectral type into two parts (cf. [32, §10.1]) which corresponds to $\Delta(\mathbf{m})$ introduced in Preface of [32].

From the decomposition we get the connection formula and the reducible condition decomposing into the corresponding subsystems etc. These are explained in [32] and [33]. In particular, the algorithm to get the direct decomposition and the condition for the irreducibility of the corresponding rigid KZ equation are explained in [33].

For example, the second part of the first decomposition in the above is $p0, (p-1)1, 1^p, 1^{p-1}1$ and it corresponds to the following $q(q-1)$ subschemes of the original Riemann scheme:

$$\left\{ \begin{array}{cccc} x_0 = x_1 & x_0 = x_2 & x_0 = x_3 & x_0 = x_4 \\ [0]_p & [0]_{p-1} & \alpha'_{i'} & \alpha_i \\ \emptyset & \beta_\nu - \alpha'' - \gamma & & \beta_{\nu'} + \gamma \end{array} \right\} \quad (1 \leq \nu \leq q, 1 \leq \nu' \leq q-1).$$

Then, in the subscheme, if the sum of all the exponents counting with their multiplicities is an integer, the system is reducible. Namely, the condition

$$\begin{aligned} p \cdot 0 + (p-1) \cdot 0 + (\beta_\nu - \alpha'' - \gamma) + \sum \alpha'_{i'} + \sum \alpha_i + \beta_{\nu'} + \gamma \\ = \beta_\nu + \beta'_{\nu'} \in \mathbb{Z} \end{aligned}$$

means the reducibility of the system. We get the same condition from the first part of the first decomposition. Then the necessary and sufficient condition for the reducibility is obtained from the direct decompositions. Here we choose a part without multiplicity such as $10, 10, 1, 10$ in the above $q(10, 10, 1, 10)$. Thus we have the condition for the irreducibility:

$$\alpha_i + \alpha'_{i'}, \beta_j + \beta'_{j'}, \alpha'_i + \beta'_j + \gamma \notin \mathbb{Z},$$

which coincides with Theorem 6.9.

Since the spectral type of the system for the variable x is

$$\mathbf{m}_x : (pq-1)1, (pq-q+1)1^{q-1}, q^p, q^{p-1}1^q,$$

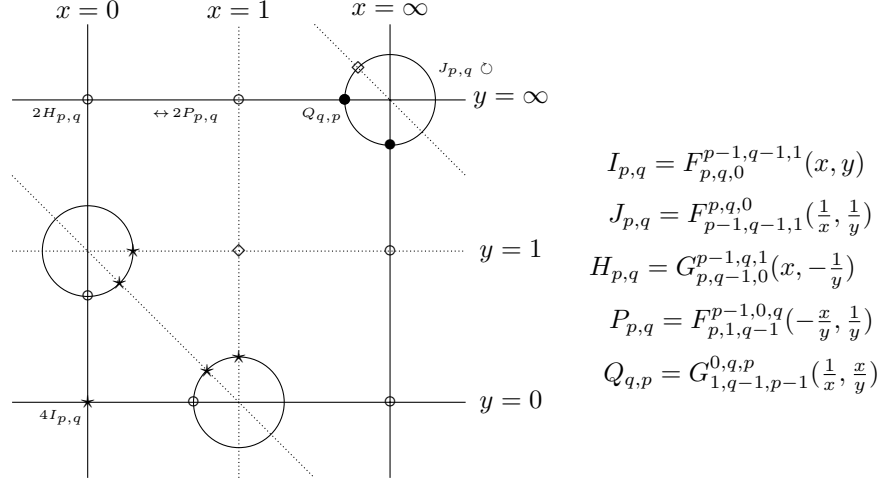
one step of Katz-Haraoka's reduction algorithm by an addition and a middle convolution reduces the rank by

$$d(\mathbf{m}_x) = (pq-1) + (pq-q+1) + q + q - 2pq = q,$$

which means $p \mapsto p-1$. When $p=1$, the spectral type of the differential equation for x -variable is

$$\underline{(q-1)}1, \underline{1}1^{q-1}, \underline{q}, \underline{0}1^q$$

which equals the spectral type of the equation satisfied by ${}_qF_{q-1}(x)$



5.5. Appell's F_2 and F_3

Appell's F_2 is given by

$$F_2(a; b, b'; c, c'; x, y) = F_{2,2,0}^{1,1,1}\left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha', 0 & \beta', 0 & \emptyset \end{matrix}; x, y\right),$$

$$\alpha = b, \quad \beta = b', \quad \gamma = a, \quad \alpha' = 1 - c, \quad \beta' = 1 - c'$$

and it satisfies $\mathcal{M}_{(\alpha',0),(\beta',0),\emptyset}^{\alpha,\beta,\gamma}$. The solution $F_{1,1,1}^{2,2,0}\left(\begin{matrix} \alpha', 0 & \beta', 0 & \emptyset \\ \alpha & \beta & \gamma \end{matrix}; \alpha, \beta; \frac{1}{x}, \frac{1}{y}\right)$ to this system at (∞, ∞) corresponds to Appell's F_3 . Note that

$$T_{(x,y) \rightarrow (-\frac{x}{y}, \frac{1}{y})} \mathcal{M}_{\alpha', \beta', \emptyset}^{\alpha, \beta, \gamma} = \mathcal{M}_{\alpha', \gamma, \beta}^{\alpha, \emptyset, \beta'}. \quad (5.32)$$

Lemma 5.13. $T_{(x,y) \rightarrow (-x, 1-y)} \bar{\mathcal{M}}_{(\alpha',0),0,\gamma'}^{\alpha,\emptyset,(\gamma_1,\gamma_2)} = \bar{\mathcal{M}}_{(\alpha',0),0,1-\gamma_1-\gamma_2-\gamma'}^{\alpha,\emptyset,(\gamma_1,\gamma_2)}$

Proof. Note that $\bar{\mathcal{M}}_{(\alpha',0),0,\gamma'}^{\alpha,\emptyset,(\gamma_1,\gamma_2)}$ is

$$\begin{cases} \partial_x(\vartheta_x - \alpha')(\vartheta_x + \vartheta_y - \gamma')u = (\vartheta_x + \alpha)(\vartheta_x + \vartheta_y + \gamma_1)(\vartheta_x + \vartheta_y + \gamma_2)u, \\ \partial_y(\vartheta_x + \vartheta_y - \gamma')u = (\vartheta_x + \vartheta_y + \gamma_1)(\vartheta_x + \vartheta_y + \gamma_2)u, \\ (\vartheta_x + \alpha)\partial_y u = \partial_x(\vartheta_x - \alpha')u. \end{cases}$$

Putting $(x, y) = (-X, 1 - Y)$, we have $(\vartheta_X + \alpha)\partial_Y u = \partial_X(\vartheta_X - \alpha')u$ and

$$\begin{aligned} & (\vartheta_X + \alpha)(\vartheta_X + \vartheta_Y + \gamma_1)(\vartheta_X + \vartheta_Y + \gamma_2)u + \partial_X(\vartheta_X - \alpha')(\vartheta_X + \vartheta_Y - \gamma')u \\ &= (\vartheta_X + \alpha)((\vartheta_X + \vartheta_Y + \gamma_2 + 1)\partial_Y + (\vartheta_X + \vartheta_Y + \gamma_1)\partial_X)u \\ &= (2\vartheta_X + 2\vartheta_Y + \gamma_1 + \gamma_2 + 1)\partial_X(\vartheta_X - \alpha')u \\ &= \partial_X(\vartheta_X - \alpha')(2\vartheta_X + 2\vartheta_Y + \gamma_1 + \gamma_2 - 1)u, \\ & (\vartheta_X + \vartheta_Y + \gamma_1)(\vartheta_X + \vartheta_Y + \gamma_2)u + \partial_Y(\vartheta_X + \vartheta_Y - \gamma')u \\ &= (\vartheta_X + \vartheta_Y + \gamma_1)\partial_Y u + \partial_Y(\vartheta_X + \vartheta_Y + \gamma_2)u \\ &= \partial_Y(2\vartheta_X + 2\vartheta_Y + \gamma_1 + \gamma_2 - 1)u, \end{aligned}$$

which implies the lemma. \square

Remark 5.14. *The above lemma is valid in the case $\alpha, \alpha' \in \mathbb{C}^{p-1}$ for $p \geq 2$. Moreover γ, γ_1 and γ_2 can be operators which commute with x, ∂_x, y and ∂_y .*

Proposition 5.15. *One has a relation*

$$T_{y:0 \rightarrow 1}^{F_2} \mathcal{M}_{\alpha', \beta', \emptyset}^{\alpha, \beta, \gamma} = \mathcal{M}_{\alpha', \gamma, 1-\beta-\beta'_1-\beta'_2}^{\alpha, \emptyset, \beta'}$$

where

$$T_{y:0 \rightarrow 1}^{F_2} = \text{Ad}(y^\gamma) \circ T_{(x,y) \rightarrow (-x, 1-y)} \circ \text{Ad}(y^{-\gamma}) \circ T_{(x,y) \rightarrow (-\frac{x}{y}, \frac{1}{y})}.$$

Proof. The proposition follows from the following sequence of transformations.

$$\begin{aligned} \mathcal{M}_{\alpha', \beta', \emptyset}^{\alpha, \beta, \gamma} &\xrightarrow{T_{(x,y) \rightarrow (-\frac{x}{y}, \frac{1}{y})}} \mathcal{M}_{\alpha', \gamma, \beta}^{\alpha, \emptyset, \beta'} \xrightarrow{\text{Ad}(x^{-\alpha'_1} y^{-\gamma})} \mathcal{M}_{\alpha' - \alpha'_1, 0, \beta - \alpha'_1 - \gamma}^{\alpha + \alpha'_1, \emptyset, \beta' + \alpha'_1 + \gamma} \\ &\xrightarrow{T_{(x,y) \rightarrow (-x, 1-y)}} \mathcal{M}_{\alpha' - \alpha'_1, 0, 1 - \alpha'_1 - \beta - \beta'_1 - \beta'_2 - \gamma}^{\alpha + \alpha'_1, \emptyset, \beta' + \alpha'_1 + \gamma} \\ &\xrightarrow{\text{Ad}(x^{\alpha'_1} y^\gamma)} \mathcal{M}_{\alpha', \gamma, 1 - \beta - \beta'_1 - \beta'_2}^{\alpha, \emptyset, \beta'}. \end{aligned}$$

\square

Remark 5.16. i) *Note that $T_{y:0 \rightarrow 1}^{F_2}$ acts only on the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$ satisfied by $F_{p,2,0}^{p-1,1,1} \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x, y \right)$.*

ii) $T_{(x,y) \rightarrow (-x, 1-y)} \circ T_{(x,y) \rightarrow (-\frac{x}{y}, \frac{1}{y})} = T_{(x,y) \rightarrow (\frac{x}{y}, 1 - \frac{1}{y})}$. Hence $y^{-\gamma} u(\frac{x}{y}, 1 - \frac{1}{y})$ is a solution to $\mathcal{M}_{(\alpha', 0), (\beta', 0), \emptyset}^{\alpha, \beta, \gamma}$ if $u(x, y)$ is a solution to $\mathcal{M}_{(\alpha', 0), 0, 1 - \beta - \beta' - \gamma}^{\alpha, \emptyset, (\beta' + \gamma)}$. In particular, putting $u(x, y) = F_{2,0,1}^{1,0,2} \left(\begin{smallmatrix} \alpha & \emptyset & \beta' + \gamma, \gamma \\ \alpha', 0 & 0 & 1 - \beta - \beta' - \gamma \end{smallmatrix} ; x, y \right)$, we have a solution to $\mathcal{M}_{(\alpha', 0), (\beta', 0), \emptyset}^{\alpha, \beta, \gamma}$ around $(0, 1)$.

Olsson [29] gives the solution to $\mathcal{M}_{(\alpha', 0), (\beta', 0), \emptyset}^{\alpha, \beta, \gamma}$ around $(0, 1)$:

$$F_P(a, b, b', c, c'; x, y) := \sum \frac{(a)_{m+n} (a - c' + 1)_m (b)_m (b')_n}{(a + b' - c' + 1)_{m+n} (c)_m m! n!} x^m (1 - y)^n. \quad (5.33)$$

Owing to the above remark, we have the identity

$$F_P(a, b, b', c, c'; x, y) := y^{-a} F_{2,1,1}^{1,0,2} \left(\begin{smallmatrix} b_1 & \emptyset & a - c' + 1, b' \\ 1 - c, 0 & 0 & c' - a - b' \end{smallmatrix} ; \frac{x}{y}, 1 - \frac{1}{y} \right), \quad (5.34)$$

which is also proved by the following direct calculation

$$\begin{aligned} \sum \frac{(a)_{m+n} (f)_m (g)_n}{(f + g)_{m+n} m! n!} x^m y^n &= \sum_m \frac{(a)_m (f)_m}{(f + g)_m m!} x^m \sum_n \frac{(a + m)_n (g)_n}{(f + g + m)_n n!} y^n \\ &= \sum_m \frac{(a)_m (f)_m}{(f + g)_m m!} x^m (1 - y)^{-a-m} \sum_n \frac{(a + m)_n (f + m)_n}{(f + g + m)_n n!} \left(\frac{y}{y - 1} \right)^n \\ &= (1 - y)^{-a} \sum \frac{(a)_{m+n} (f)_{m+n}}{(f + g)_{m+n} m! n!} \left(\frac{x}{1 - y} \right)^m \left(\frac{y}{y - 1} \right)^n. \end{aligned}$$

Here the second equality in the above follows from Kummer's formula for Gauss' hypergeometric series. Applying the integral transformation $K_x^{1-\mu-\lambda,\lambda}$, we have

$$F_{p,1,1}^{p,1,1} \left(\begin{matrix} \alpha, a & \beta & \gamma \\ \alpha', 0 & 0 & 1-a-\beta \end{matrix}; x, y \right) = (1-y)^{-\gamma} F_{p,1,1}^{p-1,0,2} \left(\begin{matrix} \alpha & \emptyset & a, \gamma \\ \alpha', 0 & 0 & 1-a-\beta \end{matrix}; \frac{x}{1-y}, \frac{y}{y-1} \right). \quad (5.35)$$

Remark 5.17. i) [Symmetry of $I_{p,2}$] *Since*

$$T_{(x,y) \rightarrow (-\frac{x}{y}, \frac{1}{y})} \circ T_{(x,y) \rightarrow (-x, 1-y)} \circ T_{(x,y) \rightarrow (-\frac{x}{y}, \frac{1}{y})} = T_{(x,y) \rightarrow (\frac{x}{1-y}, \frac{y}{y-1})}$$

and $T_{(x,y) \rightarrow (-\frac{x}{y}, \frac{1}{y})} \circ T_{y:0 \leftrightarrow 1}^{F_2} \circ T_{(x,y) \rightarrow (-\frac{x}{y}, \frac{1}{y})} \mathcal{M}_{(\alpha',0),(\beta',0),\emptyset}^{\alpha,\beta,\gamma} = \mathcal{M}_{(\alpha',0),(\beta',0),\emptyset}^{\alpha,1-\beta-\gamma,\gamma}$, we have

$$F_{p,2,0}^{p-1,1,1} \left(\begin{matrix} \alpha & \beta & \gamma \\ \alpha', 0 & \beta', 0 & \emptyset \end{matrix}; x, y \right) = (1-y)^{-\gamma} F_{p,2,0}^{p-1,1,1} \left(\begin{matrix} \alpha & 1-\beta-\gamma & \gamma \\ \alpha', 0 & \beta', 0 & \emptyset \end{matrix}; \frac{x}{1-y}, \frac{y}{y-1} \right).$$

ii) [Symmetry of $I_{2,2}$] *When $p = q = 2$, $T_{(x,y) \rightarrow (y,x)}$ naturally acts on $\mathcal{M}_{(\alpha',0),(\beta',0),\emptyset}^{\alpha,\beta,\gamma}$ and the function $F_2(a; b, b'; c, c'; x, y)$ has a symmetry of the reflection group W_{B_2} of type B_2 since the coordinate transformations $T_{(x,y) \rightarrow (\frac{x}{1-y}, \frac{y}{y-1})}$ and $T_{(x,y) \rightarrow (y,x)}$ generate W_{B_2} . For example,*

$$\begin{aligned} T_{(x,y) \rightarrow (y,x)} \circ T_{(x,y) \rightarrow (\frac{x}{1-y}, \frac{y}{y-1})} \circ T_{(x,y) \rightarrow (x,y)} \circ T_{(x,y) \rightarrow (\frac{x}{1-y}, \frac{y}{y-1})} \\ = T_{(x,y) \rightarrow (\frac{x}{x+y-1}, \frac{y}{x+y-1})} \end{aligned}$$

and we have (cf. [7, §5.11])

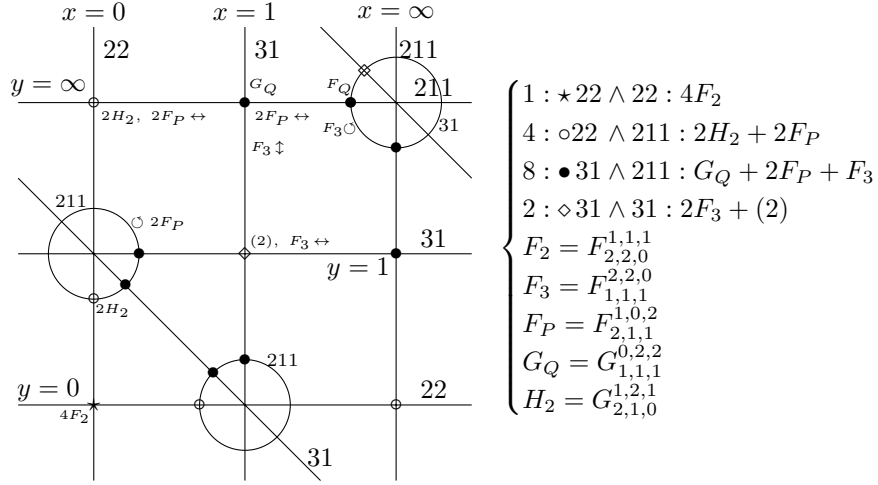
$$\begin{aligned} F_2(a; b, b'; c, c'; x, y) &= (1-y)^{-a} F_2(a; b, c' - b'; c, c'; \frac{x}{1-y}, \frac{y}{y-1}) \\ &= (1-x)^{-a} F_2(a; c-b, b, b'; c, c'; \frac{x}{x-1}, \frac{y}{1-x}) \\ &= (1-x-y)^{-a} F_2(a; c-b; c-b'; c, c'; \frac{x}{x+y-1}, \frac{y}{x+y-1}). \end{aligned}$$

Resolving the singularities $(0, 1)$, $(1, 0)$ and (∞, ∞) of $\mathcal{M}_{(\alpha',0),(\beta',0),\emptyset}^{\alpha,\beta,\gamma}$ in $\mathbb{P}^1 \times \mathbb{P}^1$, we have a manifold \tilde{X} , which is isomorphic to the manifold constructed in §5.2 by the correspondence $(x, y) \mapsto (x, 1-y)$. We denote the exceptional lines in \tilde{X} corresponding to $(0, 1)$, $(1, 0)$ and (∞, ∞) by $L_{(0,1)}$, $L_{(1,0)}$ and $L_{(\infty,\infty)}$, respectively.

Under the coordinate $(X, Y) = (\frac{x}{y}, 1 - \frac{1}{y})$, the original system has a solution $F_{p,1,1}^{p-1,0,2}(X, Y)$ and we apply the result in §3.3. Then we have solutions at the vertices of the pentagon with the edges $L_{(0,1)}$, $L_{y=1}$, $L_{x=\infty}$, $L_{y=0}$ and $L_{x=0}$ and solve the connection problem between them. We have already solved the connection problem between the vertices of the pentagon with the edges $L_{y=0}$, $L_{x=\infty}$, $L_{(\infty,\infty)}$, $L_{y=\infty}$ and $L_{x=\infty}$. Here $L_{x=0}$ means the line in \tilde{X} and the pentagon is obtained by shrinking the edge $L_{(0,0)}$ of a hexagon to a vertex.

Hence we can solve the connection problem among 8 points when $p > 2$ and $q = 2$.

If $p = q = 2$, $T_{(x,y) \rightarrow (y,x)}$ can be applied and we can add other three points, which include $(1, \infty)$. The transformation $T_{(x,y) \rightarrow (\frac{x}{1-y}, \frac{y}{y-1})}$ maps $y = \infty$ to $(0, 1)$ and $(1, \infty)$ to $L_{(0,1)} \cap L_{x+y=1}$. Using the isomorphism corresponding to this transformation, we can analyze the local solutions at $L_{(0,1)} \cap L_{x+y=1}$. Thus, we have local solutions and connection formula among them at 13 normally crossing singular points in \tilde{X} . The remaining 2 points are $(1, 1)$ and $L_{(\infty, \infty)} \cap L_{x+y=1}$.



Remark 5.18. As a consequence we have a base of local solutions expressed by our hypergeometric functions at every normally crossing singular point in \tilde{X} whose multiplicities of simultaneous eigenvalues are free. Moreover the calculation of the connection coefficients between two such points along a singular line is given. Note that all such points are connected by singular lines.

Some connection formulas for F_2 are obtained in [26].

6. Many variables and GKZ system

In this section, we introduce a multivariate extension $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ of the system $\mathcal{M}_{\alpha', \beta', \gamma'}^{\alpha, \beta, \gamma}$. To analyze this system, the viewpoint of GKZ system will be useful.

6.1. The system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$

Let n be a positive integer, $x = (x_1, \dots, x_n)$ be n complex variables. We consider non-negative integers $p_1, \dots, p_n, r, p'_1, \dots, p'_n, r'$ and parameters $\alpha_i \in \mathbb{C}^{p_i}, \gamma \in \mathbb{C}^r, \alpha'_i \in \mathbb{C}^{p'_i}$, and $\gamma' \in \mathbb{C}^{r'}$. Then, for $i = 1, \dots, n$ and $1 \leq i < j \leq n$, we set

$$\begin{aligned} P_i &:= (\vartheta_{x_i} - \alpha'_i)(\vartheta_{x_1} + \dots + \vartheta_{x_n} - \gamma') - x_i(\vartheta_{x_i} + \alpha_i)(\vartheta_{x_1} + \dots + \vartheta_{x_n} + \gamma) \\ P_{ij} &:= x_i(\vartheta_{x_i} + \alpha_i)(\vartheta_{x_j} - \alpha'_j) - x_j(\vartheta_{x_j} + \alpha_j)(\vartheta_{x_i} - \alpha'_i), \end{aligned} \quad (6.1)$$

where we use the convention as (2.8) and $\theta_{x_i} := x_i \frac{\partial}{\partial x_i}$. For an unknown function $u(x)$, the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}$ is defined by

$$\begin{cases} P_i \cdot u(x) = 0 & (i = 1, \dots, n) \\ P_{ij} \cdot u(x) = 0 & (1 \leq i < j \leq n). \end{cases} \quad (6.2)$$

For a multi-index $m = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{-1, +1\}^n$, we set $|m| := m_1 + \dots + m_n$ and $\epsilon \cdot m := \epsilon_1 m_1 + \dots + \epsilon_n m_n$. We write $F_{p'_1, \dots, p'_n, r'}^{\alpha_1, \dots, \alpha_n, \gamma'}(\alpha'_1, \dots, \alpha'_n, \gamma'; x)$ for the following multi-variate series:

$$\sum_{m=(m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n} \frac{(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} (\gamma)_{|m|}}{(\mathbf{1} - \alpha'_1)_{m_1} \cdots (\mathbf{1} - \alpha'_n)_{m_n} (\mathbf{1} - \gamma')_{|m|}} x^m. \quad (6.3)$$

Instead of $F_{p'_1, \dots, p'_n, r'}^{\alpha_1, \dots, \alpha_n, \gamma'}(\alpha'_1, \dots, \alpha'_n, \gamma'; x)$ we often use a symbol $F(\alpha'_1, \dots, \alpha'_n, \gamma'; x)$ in the following.

Proposition 3.1 is generalized as follows.

Proposition 6.1. *One has the following transformations of the system $\mathcal{M}_{\alpha', \beta', \gamma'}$.*

- (1) $T_{(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})} \mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma} = \mathcal{M}_{\alpha'_{\sigma(1)}, \dots, \alpha'_{\sigma(n)}, \gamma'}^{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}, \gamma}$.
- (2) $T_{(x_1, \dots, x_n) \mapsto (\frac{1}{x_1}, \dots, \frac{1}{x_n})} \mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma} = \mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha'_1, \dots, \alpha'_n, \gamma'}$.
- (3) $T_{(x_1, \dots, x_n) \mapsto (\epsilon \frac{x_1}{x_n}, \dots, \epsilon \frac{x_{n-1}}{x_n}, \frac{1}{x_n})} \mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma} = \mathcal{M}_{\alpha'_1, \dots, \alpha'_{n-1}, \gamma', \alpha'_n}^{\alpha_1, \dots, \alpha_{n-1}, \gamma', \alpha'_n}$.

The proof of Proposition 6.1 is parallel to that of Proposition 3.1 and is omitted.

As in §2.3, we always assume the following condition

$$p_i + r = p'_i + r' \quad (i = 1, \dots, n) \quad (\text{F})$$

for the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}$. We call the number

$$L := r - r' \quad (6.4)$$

the level of the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}$ and set

$$\epsilon := (-1)^L.$$

Moreover, we assume

$$(\alpha'_1)_1 = (\alpha'_2)_1 = \dots = (\alpha'_n)_1 = 0, \quad (\text{T})$$

$p_1 + p'_1, \dots, p_n + p'_n > 1$ and $r + r' > 0$ in the following discussion. It is easy to see that $F_{p'_1, \dots, p'_n, r'}^{\alpha_1, \dots, \alpha_n, \gamma'}(\alpha'_1, \dots, \alpha'_n, \gamma'; x)$ is a solution to the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}$ under the condition (T).

As in §2.5, the series (6.3) admits an integral representation as follows.

Proposition 6.2. *Assume $\alpha'_{ip'_i} = 0$ for any $i = 1, \dots, n$. Then, the following identity holds.*

$$\begin{aligned}
& F_{p'_1, \dots, p'_n, r'}^{p_1, \dots, p_n, r}(\alpha_1, \dots, \alpha_n, \gamma; \alpha'_1, \dots, \alpha'_n, \gamma'; x) \\
&= \prod_{i=1}^n \prod_{\nu=1}^{p'_i-1} K_{x_i}^{1-\alpha'_{i,\nu}-\alpha_{i,\nu}, \alpha_{i,\nu}} \prod_{k=1}^r \tilde{K}_{\mathbf{x}}^{1-\gamma'_k-\gamma_k, \gamma_k} \\
&\quad \prod_{k=1}^{r'-r} K_{\mathbf{x}}^{1-\gamma'_{r+k}-\sum_{i=1}^n \alpha_{p'_i+k-1}, (\alpha_{1,p'_1+k-1}, \dots, \alpha_{n,p'_n+k-1})} \prod_{i=1}^n (1-x_i)^{-\alpha_{p_i}} \\
&\quad \text{if } r \leq r', \\
&= \prod_{i=1}^n \prod_{\nu=1}^{p_i} K_{x_i}^{1-\alpha'_{i,\nu}-\alpha_{i,\nu}, \alpha_{i,\nu}} \prod_{k=1}^{r'} \tilde{K}_{\mathbf{x}}^{1-\gamma'_k-\gamma_k, \gamma_k} \\
&\quad \prod_{k=1}^{r-r'-1} L_{\mathbf{x}}^{1-\gamma_{r+k}-\sum_{i=1}^n \alpha'_{i,p+k}, (\alpha'_{1,p+k}, \dots, \alpha'_{n,q+k})} \left(1 - \sum_{i=1}^n x_i\right)^{-\gamma_r} \quad \text{if } r > r'.
\end{aligned}$$

6.2. The singular set

We describe the singular set of the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$. As in §4.2, we define the singular set $\text{Sing}(\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma})$ of the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ by

$$\begin{aligned}
& \{x \in \mathbb{C}^n \mid \exists \xi \in \mathbb{C}^n \setminus \{0\} \text{ such that} \\
& \quad \sigma(P_i)(x, \xi) = 0 \ (i = 1, \dots, n) \text{ and } \sigma(P_{ij})(x, \xi) = 0 \ (1 \leq i < j \leq n)\}.
\end{aligned}$$

Note that $\text{Sing}(\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma})$ is a closed subvariety of \mathbb{C}^n . To state the theorem, we prepare a notation. Let $f(x)$ be a Laurent polynomial in the variables x_1, \dots, x_n . When $f(x)$ contains a negative power in x_i , we define $m_{f,i}$ as the smallest integer such that $x_i^{m_{f,i}} f(x)$ is a polynomial in x_i . When $f(x)$ is a polynomial in x_i , we set $m_{f,i} := 0$. We define the polynomial part of f by

$$\text{p.p.}f(x) := \prod_{i=1}^n x_i^{m_{f,i}} f(x).$$

Theorem 6.3. *The singular set $\text{Sing}(\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma})$ is given by the union of the following irreducible subvarieties:*

- (1) *Suppose $L \neq 0$. Then, the subvarieties are as follows*

$$x_i = 0 \quad (i = 1, \dots, n), \tag{6.5}$$

$$x_i = 1 \quad \left(i = 1, \dots, n, \prod_{j \neq i} p_j > 0 \right), \tag{6.6}$$

$$\text{p.p.} \prod_{i \in I} \prod_{\omega_i \in U_L} \left(\sum_{i \in I} \omega_i x_i^{\frac{1}{L}} \right) = 0 \quad \left(\text{if } r' > 0, I \subset \{1, \dots, n\}, \prod_{i \notin I} p_i > 0, |I| \geq 2 \right), \quad (6.7)$$

$$\text{p.p.} \prod_{i \in I} \prod_{\omega_i \in U_L} \left(1 - \sum_{i \in I} \omega_i x_i^{\frac{1}{L}} \right) = 0 \quad \left(I \subset \{1, \dots, n\}, \prod_{i \notin I} p_i > 0, |I| \geq 2 \right). \quad (6.8)$$

Here, U_L is the set of L -th roots of unity. Note that the symbol p.p. can be omitted when $L > 0$.

(2) Suppose $L = 0$. Then, the subvarieties are all linear:

$$x_i = 0, \quad x_i = 1, \quad x_i = x_j \quad (i = 1, \dots, n, i < j). \quad (6.9)$$

Proof. In view of Proposition 6.1, we may assume $L \geq 0$. The singular set $\text{Sing}(\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'})$ is contained in the vanishing locus of the symbols of the operators (6.1). The equations are given by

$$(x_i \xi_i)^{p_i} (x_1 \xi_1 + \dots + x_n \xi_n)^{r'} ((x_i \xi_i)^L - x_i (x_1 \xi_1 + \dots + x_n \xi_n)^L) = 0 \quad (6.10)$$

for $i = 1, \dots, n$ and

$$(x_i \xi_i)^{p_i} (x_j \xi_j)^{p_j} (x_i (x_j \xi_j)^L - x_j (x_i \xi_i)^L) \quad (6.11)$$

for $1 \leq i < j \leq n$. A point $x \in \mathbb{C}^n$ lies on the singular locus if there exists a vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n \setminus \{0\}$ such that (x, ξ) satisfies (6.10) and (6.11). Since $x_i = 0$ satisfies (6.10) and (6.11) with $\xi_i = 1$ and $\xi_j = 0$ for any $j \neq i$, we assume $x_1 \cdots x_n \neq 0$ below. Let (x, ξ) satisfy (6.10) and (6.11). We may assume that there is an index set $\emptyset \subsetneq I \subset \{1, \dots, n\}$ so that $\xi_i \neq 0$ if and only if $i \in I$. Such a condition is consistent if and only if $p_j > 0$ for any $j \notin I$ because otherwise (6.11) implies that $\xi_i = 0$ for any $i \in I$. To simplify the notation, we assume that $1 \in I$. Since the conditions (6.10) and (6.11) are homogeneous with respect to ξ , we may assume $\xi_1 = 1$.

(1) We consider the case $L > 0$. The condition (6.11) is equivalent to

$$\xi_i^L = \left(\frac{x_1}{x_i} \right)^{L-1} \quad (i \in I) \quad (6.12)$$

or equivalently

$$\xi_i = \left(\frac{x_1}{x_i} \right)^{\frac{L-1}{L}} \quad (i \in I) \quad (6.13)$$

for a choice of an L -th root. When $I = \{1\}$, (6.10) is reduced to $x_1 = 1$. Thus, the variety (6.6) appears.

Now, we consider the case when $|I| \geq 2$. Assume $r' = 0$ and $\sum_{i \in I} x_i \xi_i = 0$. Then, it is inconsistent with (6.10). Assume $r' > 0$ and $\sum_{i \in I} x_i \xi_i = 0$. Then,

it follows from (6.13) that

$$x_1 + \sum_{i \in I, i \neq 1} x_i \left(\frac{x_1}{x_i} \right)^{\frac{L-1}{L}} = 0 \quad (6.14)$$

or equivalently

$$\sum_{i \in I} x_i^{\frac{1}{L}} = 0. \quad (6.15)$$

Thus, the variety (6.7) appears.

Finally, we assume $\sum_{i \in I} x_i \xi_i \neq 0$. By (6.10) and (6.12), we obtain

$$(x_1)^{L-1} = x_i \left(\sum_{i \in I} x_i \xi_i \right)^L \quad (i \in I). \quad (6.16)$$

By taking L -th root of (6.16) and substituting (6.13), we obtain a relation

$$\sum_{i \in I} x_i^{\frac{1}{L}} = 1. \quad (6.17)$$

Thus, the variety (6.8) appears.

(2) We consider the case $L = 0$. The condition (6.11) implies that

$$x_i = x_j \quad i, j \in I. \quad (6.18)$$

Assume $I = \{1\}$. Then, the condition (6.18) is an empty condition and (6.10) implies that $x_1 = 1$. Assume $|I| = 2$ and set $I = \{1, j\}$. Then, the condition (6.18) is just $x_1 = x_j$. Setting $\xi_j = -\frac{x_1}{x_j}$, we see that the condition (6.10) is also satisfied. Finally, assume $|I| > 2$. Then, the condition (6.18) defines a subvariety of codimension higher or equal to 2, which does not contribute to the singular locus. □

Remark 6.4. *As in Theorem 4.7, the varieties (6.7) and (6.8) admit rational parametrizations. Let $I \subset \{1, \dots, n\}$ be a subset such that $\prod_{i \notin I} p_i > 0$ and $|I| \geq 2$.*

The varieties (6.7) and (6.8) are regarded as subvarieties of \mathbb{C}^I . Let us take an element $i_0 \in I$. When $r' > 0$, a rational parametrization of (6.7) is given by

$$\mathbb{C}^{I \setminus \{i_0\}} \ni (t_i)_{i \in I \setminus \{i_0\}} \mapsto \left(\epsilon \left(\sum_{i \in I, i \neq i_0} t_i \right)^L, (t_i^L)_{i \in I \setminus \{i_0\}} \right) \in \mathbb{C}^I. \quad (6.19)$$

On the other hand, a rational parametrization of (6.8) is given by

$$\mathbb{C}^{I \setminus \{i_0\}} \ni (t_i)_{i \in I \setminus \{i_0\}} \mapsto \left(\frac{1}{(1 + \sum_{i \in I, i \neq i_0} t_i)^L}, \left(\left(\frac{t_i}{1 + \sum_{i \in I, i \neq i_0} t_i} \right)^L \right)_{i \in I \setminus \{i_0\}} \right) \in \mathbb{C}^I. \quad (6.20)$$

Note that the parametrization (6.20) takes a symmetric form

$$[t_i : i \in I] \mapsto \left(\left(\frac{t_i}{\sum_{i \in I} t_i} \right)^L \right)_{i \in I} \quad (6.21)$$

where $[t_i : i \in I]$ is a homogeneous coordinate.

6.3. GKZ system

We first recall the definition of GKZ system ([11]). Our description follows that of [10]. We fix a natural number N and set $M_{\geq 0} := \mathbb{Z}_{\geq 0}^N$ viewed as a free abelian monoid. Let M be the associated abelian group \mathbb{Z}^N . Let L be an abelian subgroup of M such that the quotient M/L has no torsion. We write M^\vee for the dual lattice of M . If $\{e_1, \dots, e_N\}$ is a free generator of $M_{\geq 0}$, the coordinate ring of $M_{\mathbb{C}}^\vee := M^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ is a polynomial ring $\mathbb{C}[M_{\geq 0}]$ and each element e_i defines a differential operator: for a pair of elements $v \in M$ and $\psi \in M^\vee$, we set $\langle \psi, v \rangle := \psi(v)$. For a differentiable function f on $M_{\mathbb{C}}^\vee$, we set

$$(\partial_{e_i} f)(z) := \frac{d}{dt} f(e^{t\langle e_i, z \rangle} \cdot z) |_{t=0}. \quad (6.22)$$

In general, an element $v = \sum_{i=1}^N v_i e_i \in M_{\geq 0}$ defines a differential operator $\partial^v := \prod_{i=1}^N \partial_{e_i}^{v_i}$. In the same way, we define the Euler operator: for an element $z \in M_{\mathbb{C}}^\vee$, $\psi \in M^\vee$, and $t \in \mathbb{C}^\times$, we define $t^\psi \cdot z$ by

$$\langle t^\psi \cdot z, v \rangle = t^{\langle \psi, v \rangle} \langle z, v \rangle \quad (v \in M). \quad (6.23)$$

Then, for a differentiable function f on $M_{\mathbb{C}}^\vee$, we set

$$(\vartheta_\psi f)(z) := \frac{d}{dt} f(t^\psi \cdot z) |_{t=1}. \quad (6.24)$$

Let $\varphi : M_{\geq 0} \rightarrow M/L$ be the composition of the natural inclusion $M_{\geq 0} \rightarrow M$ and the quotient map $A : M \rightarrow M/L$. The map φ induces a ring homomorphism $\mathbb{C}[M_{\geq 0}] \rightarrow \mathbb{C}[M/L]$ which is denoted by the same symbol φ . We set $I_A := \text{Ker } \varphi$ and $L^\perp := \{\psi \in M^\vee \mid \psi \equiv 0 \text{ on } L\}$. For any complex vector $v \in M_{\mathbb{C}}$, we set $c := -Av$. The GKZ system $M_A(c)$ is defined by

$$M_A(c) : \begin{cases} P \cdot u(z) = 0 & P \in I_A, \\ \vartheta_\psi u(z) + \langle \psi, c \rangle = 0 & \psi \in L^\perp. \end{cases} \quad (6.25)$$

We identify the projection $A : M \rightarrow M/L$ with the image of a set $\{e_1, \dots, e_N\}$ by A . We set $a_i := Ae_i$ for each i . Let $\text{Cone}(A)$ be the cone spanned by A . Given a subset $I \subset \{1, \dots, N\}$, the cone $C(I)$ spanned by $\{a_i \mid i \in I\}$ defines a face of $\text{Cone}(A)$ if there is a dual vector $\phi \in (M/L)^\vee$ such that

$$\begin{cases} \langle \phi, a_i \rangle = 0 & (i \in I), \\ \langle \phi, a_i \rangle < 0 & (i \notin I). \end{cases} \quad (6.26)$$

Conversely, any face F of $\text{Cone}(A)$ has a form $C(I)$ for some $I \subset \{1, \dots, N\}$. We say $c \in (M/L) \otimes_{\mathbb{Z}} \mathbb{C}$ is non-resonant if it does not lie on a set $\text{span}_{\mathbb{C}}(F) + \mathbb{Z}^n$ for any face F of $\text{Cone}(A)$. Here, $\text{span}_{\mathbb{C}}$ is the spanning subspace over \mathbb{C} . Note that $c \in (M/L) \otimes_{\mathbb{Z}} \mathbb{C}$ does not lie on a set $\text{Cone}(I) + \mathbb{Z}^n$ if and only if the condition

$$\langle \phi, c \rangle \notin \mathbb{Z} \quad (6.27)$$

holds for any $\phi \in (M/L)^{\vee}$ with (6.26).

We briefly recall the definition of a regular polyhedral subdivision ([12, Chapter CHAPTER 7], [42, CHAPTER8]). A collection S of subsets of $\{1, \dots, N\}$ is called a polyhedral subdivision if $\{\text{Cone}(I) \mid I \in S\}$ is a set of cones in a polyhedral fan whose support equals $\text{Cone}(A)$. For any choice of a vector $\omega \in \mathbb{R}^N$ we define a polyhedral subdivision $S(\omega)$ as follows: a subset $I \subset \{1, \dots, N\}$ belongs to $S(\omega)$ if there exists a dual vector $\phi \in (M/L)_{\mathbb{R}}^{\vee}$ such that $\langle \phi, a_i \rangle = \omega_i$ if $i \in I$ and $\langle \phi, a_i \rangle < \omega_j$ if $j \notin I$. A polyhedral subdivision S is called a regular polyhedral subdivision if $S = S(\omega)$ for some ω . Given a regular polyhedral subdivision S , we write $C_S \subset \mathbb{R}^N$ for the cone consisting of vectors ω such that $S(\omega) = S$. If any maximal (with respect to inclusion) element I of a regular polyhedral subdivision T is a simplex, we call T a regular triangulation. A collection of polyhedral cones $\{C_S \mid S \text{ is a regular subdivision}\}$ is a complete fan in \mathbb{R}^N , which we call the secondary fan. By identifying \mathbb{R}^N with $M_{\mathbb{R}}^{\vee}$ via dot product with respect to $\{e_1, \dots, e_N\}$, it is readily seen that each cone C_S contains a linear subspace $L_{\mathbb{R}}^{\perp} := L^{\perp} \otimes_{\mathbb{Z}} \mathbb{R}$. Therefore, we often regard secondary fan as a fan in $M_{\mathbb{R}}^{\vee}/L_{\mathbb{R}}^{\perp} \simeq L_{\mathbb{R}}^{\vee}$.

6.4. The system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}$ from the view point of GKZ system

Next, we consider GKZ extension of the system $\mathcal{M}_{\alpha_1, \dots, \alpha_n, \gamma}$. We set

$$M_{\geq 0} := \mathbb{Z}_{\geq 0}^{p_1} \oplus \dots \oplus \mathbb{Z}_{\geq 0}^{p_n} \oplus \mathbb{Z}_{\geq 0}^r \oplus \mathbb{Z}_{\geq 0}^{p'_1} \oplus \dots \oplus \mathbb{Z}_{\geq 0}^{p'_n} \oplus \mathbb{Z}_{\geq 0}^{r'}.$$

An element of $M_{\geq 0}$ has $p_1 + \dots + p_n + r + p'_1 + \dots + p'_n + r'$ entries. As an index set, we use a set

$$\{\alpha_{ij}\}_{\substack{i=1, \dots, n \\ j=1, \dots, p_i}} \cup \{\gamma_k\}_{k=1, \dots, r} \cup \{\alpha'_{ij}\}_{\substack{i=1, \dots, n \\ j=1, \dots, p'_i}} \cup \{\gamma'_k\}_{k=1, \dots, r'}. \quad (6.28)$$

Namely, any element $u \in M$ is specified by its entries

$$\begin{aligned} u(\alpha_{ij}) & \quad (i = 1, \dots, n, j = 1, \dots, p_i), \\ u(\gamma_k) & \quad (k = 1, \dots, r), \\ u(\alpha'_{ij}) & \quad (i = 1, \dots, n, j = 1, \dots, p'_i), \\ u(\gamma'_k) & \quad (k = 1, \dots, r'). \end{aligned}$$

We also set

$$u(\alpha_i) := (u(\alpha_{i1}), \dots, u(\alpha_{ip_i}))$$

and use analogous notation for $u(\boldsymbol{\gamma}), u(\boldsymbol{\alpha}'_i), u(\boldsymbol{\gamma}')$. The index set (6.28) is denoted by $\{\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\alpha}', \boldsymbol{\gamma}'\}$. We define a sub-lattice $L \subset M$ by

$$L = \left\{ u \in M \mid \begin{array}{l} u(\alpha_{i1}) = \dots = u(\alpha_{ip_i}) = -u(\alpha'_{i1}) = \dots = -u(\alpha'_{ip'_i}) \quad (1 \leq i \leq n), \\ u(\gamma_1) = \dots = u(\gamma_r) = -u(\gamma'_1) = \dots = -u(\gamma'_{r'}) = \sum_{i=1}^n u(\alpha_{i1}) \end{array} \right\}. \quad (6.29)$$

Let v be an element of $M_{\mathbb{C}}$ and write it as $v = (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n, \boldsymbol{\gamma}, \boldsymbol{\alpha}'_1, \dots, \boldsymbol{\alpha}'_n, \boldsymbol{\gamma}')$. Then, the GKZ system with these data $M, L, c := -Av$ corresponds to the system $\mathcal{M}_{\boldsymbol{\alpha}'_1, \dots, \boldsymbol{\alpha}'_n, \boldsymbol{\gamma}'}$ as we will see below.

Let us first find a set of generators of the toric ideal I_A . For any $1 \leq i \leq n$ and $1 \leq j \leq p_i$, we consider a vector u of which all the entries are zero but $u(\alpha_{ij}) = 1$. We write $z(\alpha_{ij})$ for the corresponding element of $\mathbb{C}[M_{\geq 0}]$. Similarly, we define elements $z(\gamma_k), z(\alpha'_{ij}), z(\gamma'_k) \in \mathbb{C}[M_{\geq 0}]$. We For $i = 1, \dots, n$, we set

$$g_i(z) := \frac{\left(\prod_{j=1}^{p'_i} z(\alpha'_{ij}) \right) \left(\prod_{j=1}^{r'} z(\gamma'_j) \right)}{\left(\prod_{j=1}^{p_i} z(\alpha_{ij}) \right) \left(\prod_{j=1}^r z(\gamma_j) \right)}. \quad (6.30)$$

For $1 \leq i < j \leq n$, we set

$$g_{ij}(z) := \frac{\left(\prod_{k=1}^{p'_i} z(\alpha'_{ik}) \right) \left(\prod_{k=1}^{p_j} z(\alpha_{jk}) \right)}{\left(\prod_{k=1}^{p'_j} z(\alpha'_{jk}) \right) \left(\prod_{k=1}^{p_i} z(\alpha_{ik}) \right)}. \quad (6.31)$$

Let $<$ be any term order on the polynomial ring $\mathbb{C}[M_{\geq 0}]$ such that the monomials with underlines in the equations (6.30) and (6.31) are leading terms of g_i and g_{ij} with respect to $<$. Then, we obtain the following description of the toric ideal.

Proposition 6.5. *The followings are true.*

- (1) $\mathcal{G} = \{g_i\}_{i=1}^n \cup \{g_{ij}\}_{1 \leq i < j \leq n}$ is a reduced $<$ -Gröbner basis of I_A .
- (2) Suppose $p_i = p'_i$ for any $i = 1, \dots, n$ and $r = r'$. Then, \mathcal{G} is a universal Gröbner basis of I_A .

Proof. The second claim (2) follows from (1), the fact that A is a Lawrence lifting by (6.29) and [42, Theorem 7.1]. In the following, we prove (1). In view of [3, Lemma 1.1], it is enough to prove the following claim: if $u_1, u_2 \in M_{\geq 0}$ satisfy conditions $z^{u_1}, z^{u_2} \notin \text{in}_{<}(\mathcal{G})$ and $Au_1 = Au_2$, then, $u_1 = u_2$.

The condition that $u \notin \text{in}_{<}(\mathcal{G})$ implies that for any indices $i < j$, either $u(\alpha'_i)$ or $u(\alpha_j)$ has at least one zero-entry. Without loss of generality, we may assume that there is an index $1 \leq i_0 \leq n+1$ such that $u_1(\alpha'_1), \dots, u_1(\alpha'_{(i_0-1)})$ all have a zero entry and $u_1(\alpha'_{i_0k}) > 0$ for $k = 1, \dots, p'_{i_0}$. It follows that $u_1(\alpha_{(i_0+1)}), \dots, u_1(\alpha_n)$ all have at least one zero-entry because otherwise z^{u_1} is divisible by some $\text{in}_{<}(g_{i_0j})$. Similarly, we may assume that there is an index

$i_0 \leq j_0 \leq n+1$ such that $u_2(\alpha'_1), \dots, u_2(\alpha'_{(j_0-1)})$ all have at least one zero-entry and $u_2(\alpha'_{j_0 k}) > 0$ for $k = 1, \dots, p'_{j_0}$. It follows that $u_1(\alpha_{(j_0+1)}), \dots, u_1(\alpha_n)$ all have at least one zero-entry.

We assume $u_1 \neq u_2$ and derive a contradiction. Let us first consider the case $i_0 < j_0$. For any $j < j_0$, there is an index $k = 1, \dots, p'_j$ such that $u_2(\alpha'_{jk}) = 0$. Since $Au_1 = Au_2$ and $u_1 \in M_{\geq 0}$, we obtain $u_1(\alpha'_{j1}) - u_2(\alpha'_{j1}) = u_1(\alpha'_{jk}) - u_2(\alpha'_{jk}) \geq 0$. In the same way, for any $i > i_0$, there is an index $k = 1, \dots, p_i$ such that $u_1(\alpha_{ik}) = 0$ and $u_1(\alpha'_{i1}) - u_2(\alpha'_{i1}) = u_2(\alpha_{ik}) - u_1(\alpha_{ik}) \geq 0$. We conclude that $u_1(\gamma'_k) - u_2(\gamma'_k) = \sum_{i=1}^n (u_1(\alpha'_{i1}) - u_2(\alpha'_{i1})) \geq 0$ for any $k = 1, \dots, r'$. Since $u_1 \neq u_2$, there is an index $i = 1, \dots, n$ such that $u_1(\alpha'_{i1}) - u_2(\alpha'_{i1}) > 0$, which implies that z^{u_1} is divisible by $\text{in}_{<}(g_i)$. This is a contradiction.

We consider the case $i_0 = j_0$. We may assume that $u_1(\alpha'_{i_0 1}) - u_2(\alpha'_{i_0 1}) \geq 0$. As in the previous case, we can prove that $u_1(\alpha'_{i1}) - u_2(\alpha'_{i1}) \geq 0$ for any $i = 1, \dots, n$, which implies the same contradiction. \square

For any subset $I \subset \{\alpha, \gamma, \alpha', \gamma'\}$, we write I^c for its complement. The primary decomposition of the initial ideal $\text{in}_{<}(I_A)$ determines a regular triangulation of A (cf. [42, Corollary 8.4]). For the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma', \alpha_n, \gamma}$, we obtain an explicit description.

Corollary 6.6. *The term order $<$ defines a regular triangulation*

$$\begin{aligned} & \left\{ \{\alpha'_{1i_1}, \dots, \alpha'_{ni_n}\}^c \right\}_{\substack{i_j=1, \dots, p'_j \\ j=1, \dots, n}} \cup \\ & \bigcup_{\ell=1}^n \left\{ \{\alpha_{1i_1}, \dots, \alpha_{(\ell-1)i_{\ell-1}}, \gamma'_k, \alpha'_{(\ell+1)i_{\ell+1}}, \dots, \alpha'_{ni_n}\}^c \right\}_{\substack{k=1, \dots, r' \\ i_j=1, \dots, p'_j \\ j=1, \dots, n, j \neq \ell}} \end{aligned} \quad (6.32)$$

Proof. The corollary follows from the primary decomposition

$$\begin{aligned} \text{in}_{<}(\mathcal{G}) = & \bigcap_{\substack{i_j=1, \dots, p'_j \\ j=1, \dots, n}} \langle z(\alpha'_{1i_1}), \dots, z(\alpha'_{ni_n}) \rangle \cap \bigcap_{\substack{i_1=1, \dots, r' \\ i_j=1, \dots, p'_j \\ j=2, \dots, n}} \langle z(\gamma'_{i_1}), z(\alpha'_{2i_2}), \dots, z(\alpha'_{ni_n}) \rangle \\ & \cap \dots \cap \bigcap_{\substack{i_j=1, \dots, p'_j \\ j=1, \dots, n-1 \\ i_n=1, \dots, r'}} \langle z(\alpha_{1i_1}), \dots, z(\alpha_{(n-1)i_{n-1}}), z(\gamma'_{i_n}) \rangle. \end{aligned} \quad (6.33)$$

\square

The second equation of the GKZ system (6.25) can be used to reduce the number of variables. Namely, it is equivalent to an identity

$$u(z) = t^{-\langle \psi, c \rangle} u(t^\psi \cdot z) \quad (t \in \mathbb{C}^\times, \psi \in L^\perp). \quad (6.34)$$

The identity (6.34) shows that the value of a solution u to (6.25) on $T := \text{Spec } \mathbb{C}[M] \subset M_{\mathbb{C}}^\vee$ is determined by that on a suitable subspace of T . Let us

assume $\alpha'_{i1} = 0$ for any $i = 1, \dots, n$. For a function $u(z)$ on T , we define a function u_{red} as a function depending only on n -complex variables $z(\alpha'_{11}), \dots, z(\alpha'_{n1})$ obtained from u by its restriction to

$$\begin{cases} z(\alpha'_{ij}) = 1 & (i = 1, \dots, n, j = 2, \dots, p'_i) \\ z(\gamma'_k) = 1 & (k = 1, \dots, r') \\ z(\alpha_{ij}) = 1 & (i = 1, \dots, n, j = 1, \dots, p_i) \\ z(\gamma_k) = 1 & (k = 1, \dots, r). \end{cases} \quad (6.35)$$

Then, (6.34) is equivalent to

$$u(z) = z(\alpha)^\alpha z(\gamma')^\gamma z(\alpha')^{\alpha'} z(\gamma')^{\gamma'} u_{\text{red}} \left(\frac{z(\alpha'_1)z(\gamma')}{z(\alpha_1)z(\gamma)}, \dots, \frac{z(\alpha'_n)z(\gamma')}{z(\alpha_n)z(\gamma)} \right), \quad (6.36)$$

where we use notation

$$z(\alpha)^\alpha := \prod_{i=1}^n \prod_{j=1}^{p_i} z(\alpha_{ij})^{\alpha_{ij}}, \quad z(\alpha'_i) := \prod_{j=1}^{p_i} z(\alpha_{ij})^{\alpha_{ij}} \quad (6.37)$$

and so on. Using the identity (6.36), it can be shown that a holomorphic function $u(z)$ on T is annihilated by $g_i(\partial), g_{ij}(\partial)$ if and only if its restriction to (6.35) and

$$z(\alpha'_{i1}) = (-1)^{p_i+r} x_i \quad i = 1, \dots, n \quad (6.38)$$

is a solution to the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$. Thus, we obtain the following proposition

Proposition 6.7. *Let $u(x_1, \dots, x_n)$ be a holomorphic function defined on a domain in \mathbb{C}^n . Then, u is a solution to the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ if and only if*

$$z(\alpha)^\alpha z(\gamma')^\gamma z(\alpha')^{\alpha'} z(\gamma')^{\gamma'} u \left((-1)^{p_1+r} \frac{z(\alpha'_1)z(\gamma')}{z(\alpha_1)z(\gamma)}, \dots, (-1)^{p_n+r} \frac{z(\alpha'_n)z(\gamma')}{z(\alpha_n)z(\gamma)} \right)$$

is a solution to the GKZ system $M_A(c)$.

Proposition 6.8. *The rank of the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ is*

$$\begin{aligned} & p'_1 \cdots p'_n + r' \sum_{k=1}^n p_1 \cdots p_{k-1} p_{k+1} \cdots p'_n \quad (6.39) \\ & = \begin{cases} p_1 \cdots p_n + r \sum_{k=1}^n p_1 \cdots p_{k-1} p_{k+1} \cdots p_n & (L = 0), \\ \frac{1}{L} (p'_1 \cdots p'_n r - p_1 \cdots p_n r') & (L \neq 0). \end{cases} \end{aligned}$$

Proof. By (6.29) and [5, Theorem 8.1.5], A is totally unimodular, i.e., any simplex has a normalized volume one. In view of [42, Proposition 13.5], the toric ring $\mathbb{C}[M_{\geq 0}]/I_A$ is normal and hence Cohen-Macaulay by [18, Theorem 1]. Thus, [1, Corollary 5.21] and Corollary 6.6 of this paper proves that the holonomic rank of the GKZ system $M_A(c)$ is given by (6.39). The Proposition follows from Proposition 6.7. \square

6.5. The structure of the secondary fan and irreducibility

In this section, we provide a combinatorial description of the secondary fan of the GKZ system $M_A(c)$. We define the following elements of \mathbb{Z}^n :

$$\begin{cases} \mathbf{b}(\alpha'_{ij}) = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) & (j = 1, \dots, p'_i), \\ \mathbf{b}(\gamma'_j) = (1, \dots, 1) & (j = 1, \dots, r'), \\ \mathbf{b}(\alpha_{ij}) = (0, \dots, 0, \overset{i}{-1}, 0, \dots, 0) & (j = 1, \dots, p_i), \\ \mathbf{b}(\gamma_j) = (-1, \dots, -1) & (j = 1, \dots, r). \end{cases} \quad (6.40)$$

The vectors (6.40) define a Gale dual of the lattice configuration A . It determines the secondary fan. Below, we only consider the case $p_i p'_i \neq 0$ for any $i = 1, \dots, n$.

Case 1: $rr' \neq 0$

The secondary fan consists of $(n+1)!$ simplicial cones. We set $y_0 := 0$. Then, for a permutation σ of $\{0, \dots, n\}$, we set

$$C_\sigma := \{y = (y_1, \dots, y_n) \in \mathbb{R}^n \mid y_{\sigma(0)} \leq y_{\sigma(1)} \leq \dots \leq y_{\sigma(n)}\}. \quad (6.41)$$

Each C_σ defines an affine space U_σ near a torus fixed point. We set $x_0 := 1$. Then, the local coordinate ring of U_σ specified by σ is given by

$$\mathbb{C} \left[\frac{x_{\sigma(1)}}{x_{\sigma(0)}}, \frac{x_{\sigma(2)}}{x_{\sigma(1)}}, \dots, \frac{x_{\sigma(n)}}{x_{\sigma(n-1)}} \right]. \quad (6.42)$$

The secondary fan is given by $\{C_\sigma\}_{\sigma \in S_{n+1}}$. We introduce a convention $[0^0 : 0^1 : \dots : 0^n]$ as the unique torus fixed point contained in U_{id} . We allow to scale it by 0 as $[0^0 : 0^1 : \dots : 0^n] = [0^{-1} : 0^0 : \dots : 0^{n-1}] = [\infty : 0^0 : \dots, 0^{n-1}]$. Similarly, we denote by $p_\sigma := [0^{\sigma^{-1}(0)} : \dots : 0^{\sigma^{-1}(n)}]$ the unique torus fixed point in U_σ . Since $\{C_\sigma\}_\sigma$ is a polyhedral fan, it defines a toric variety X . It turns out that X is isomorphic to an iterated blowing-up of a product of projective lines: for a subset $I \subset \{1, \dots, n\}$, we set $D_I := (\cap_{i \in I} \{x_i = 0\}) \cup (\cap_{i \in I} \{x_i = \infty\})$. We write Bl_{D_I} for the blowing-up along D_I .

$$X = \prod_{\substack{I \subset \{1, \dots, n\} \\ |I|=2}} \text{Bl}_{D_I} \cdots \prod_{\substack{I \subset \{1, \dots, n\} \\ |I|=n}} \text{Bl}_{D_I} (\mathbb{P}^1 \times \dots \times \mathbb{P}^1).$$

The proof of the above identity is a direct comparison of local coordinates.

Given a pair of torus fixed points p_1, p_2 on X , we say p_1 is adjacent to p_2 if there is a one-dimensional torus orbit on X . In our set-up, p_{σ_1} is adjacent to p_{σ_2} if $\sigma_1^{-1} \sigma_2$ is a consecutive transposition. Then, for a pair of cones C_{σ_1} and C_{σ_2} , the intersection $C_{\sigma_1} \cap C_{\sigma_2}$ is a common facet if and only if σ_1 is adjacent to σ_2 .

One can also recognize a polytope embedded in X which is a generalization of the hexagon appeared in **Fig 1** in §3.1. Indeed, by a well-known theorem

on toric moment map [8, §4.2], the Euclidean closure $X_{\geq 0}$ of the first quadrant $(0, \infty)^n$ in X is homeomorphic to a polytope which is dual to the secondary fan $\{C_\sigma\}_{\sigma \in S_{n+1}}$. This polytope is called a permutohedron ([9, §7.3.C]), which is a polytope whose edge graph is isomorphic to the Cayley graph of the permutation group S_{n+1} with respect to the set of generators given by consecutive transpositions.

Case 2: $rr' = 0$

Since the argument is symmetric, we may assume $r = 0$. Let $I \subset \{1, \dots, n\}$ be a subset. We write S_I for the set of bijections from I to itself. S_I is a group whose product is given by the composition of maps. For a given subset $I = \{i_1, \dots, i_a\} \subset \{1, \dots, n\}$ and $\sigma \in S_I$, we set

$$C_{I,\sigma} := \{y = (y_1, \dots, y_n) \in \mathbb{R}^n \mid y_i \leq 0 \ (i \notin I), \ y_{\sigma(i_1)} \leq \dots \leq y_{\sigma(i_a)}\}. \quad (6.43)$$

Each $C_{I,\sigma}$ defines an affine space $U_{I,\sigma}$ near a torus fixed point. Then, the local coordinate ring of $U_{I,\sigma}$ specified by (I, σ) is given by

$$\mathbb{C} \left[\frac{1}{x_i} \ (i \notin I), \ x_{\sigma(i_1)}, \ \frac{x_{\sigma(i_2)}}{x_{\sigma(i_1)}}, \dots, \ \frac{x_{\sigma(i_a)}}{x_{\sigma(i_{a-1})}} \right]. \quad (6.44)$$

The secondary fan is given by $\{C_{I,\sigma}\}_{I \subset \{1, \dots, n\}, \sigma \in S_I}$. The number of maximal cones is $\sum_{i=0}^n \frac{n!}{i!}$. The first few terms are 2, 5, 16, 65, 326, Since $\{C_{I,\sigma}\}_{I,\sigma}$ is a polyhedral fan, it defines a toric variety X . It turns out that X is isomorphic to an iterated blow-up of a product of projective lines: for a subset $I \subset \{1, \dots, n\}$, we set $D_I := (\cap_{i \in I} \{x_i = 0\})$.

$$X = \prod_{\substack{I \subset \{1, \dots, n\} \\ |I|=2}} \text{Bl}_{D_I} \cdots \prod_{\substack{I \subset \{1, \dots, n\} \\ |I|=n}} \text{Bl}_{D_I} (\mathbb{P}^1 \times \cdots \times \mathbb{P}^1).$$

Now, let us determine the irreducibility.

Theorem 6.9. *$M_A(c)$ is irreducible if and only if none of $\alpha_{1i_1} + \dots + \alpha_{ni_n} + \gamma'_k$, $\alpha'_{1i_1} + \dots + \alpha'_{ni_n} + \gamma_k$, $\alpha_{ij_1} + \alpha'_{ij_2}$, $\gamma_{k_1} + \gamma'_{k_2}$ is an integer.*

Remark 6.10. *Some of these conditions may be empty. For example, if $r' = 0$, the condition is that none of $\alpha'_{1i_1} + \dots + \alpha'_{ni_n} + \gamma_k$, $\alpha_{ij_1} + \alpha'_{ij_2}$ is an integer.*

Proof. In view of [41], we need to write down the condition (6.27). To do this, we note that (6.40) is a dual configuration of A ([44, §6.4]). By [13, p88, item 1], for a subset $I \subset \{\alpha, \gamma, \alpha', \gamma'\}$, $C(I)$ defines a face of $\text{Cone}(A)$ if and only if $I = \{\alpha, \gamma, \alpha', \gamma'\}$ or $0 \in \text{rel.int.}(\mathbf{b}(i) \mid i \notin I)$ where rel.int. is the relative interior. Therefore, an index set $I \subset \{\alpha, \gamma, \alpha', \gamma'\}$ defines a facet of $\text{Cone}(A)$ if and only if it is minimal with respect to inclusion. The complements of such I are listed as follows:

$$\{\alpha_{1i_1}, \dots, \alpha_{ni_n}, \gamma'_k\}, \{\alpha'_{1i_1}, \dots, \alpha'_{ni_n}, \gamma_k\}, \{\alpha_{ij_1}, \alpha'_{ij_2}\}, \{\gamma_{k_1}, \gamma'_{k_2}\}. \quad (6.45)$$

Let us identify the dual lattice M^\vee as a set of integral vectors

$$\mathbb{Z}^{p_1} \oplus \dots \oplus \mathbb{Z}^{p_n} \oplus \mathbb{Z}^r \oplus \mathbb{Z}^{p'_1} \oplus \dots \oplus \mathbb{Z}^{p'_n} \oplus \mathbb{Z}^{r'}$$

so that the duality pairing $\langle \psi, v \rangle$ for $\psi \in M^\vee$ and $v \in M$ is given by the dot product of integral vectors. In this sense, we write $\psi(\delta) \in \mathbb{Z}$ for the δ -th entry of ψ for any $\delta \in \{\alpha, \gamma, \alpha', \gamma'\}$. In view of the identification $(M/L)^\vee = L^\perp$, a dual vector $\phi \in (M/L)^\vee$ defining a face I is represented by a vector $\psi \in M^\vee$ such that

$$\psi|_L \equiv 0 \quad \text{and} \quad \begin{cases} \psi(\delta) = 0 & (\delta \notin I) \\ \psi(\delta) < 0 & (\delta \in I) \end{cases}. \quad (6.46)$$

To the complement of each of (6.45), we solve (6.46) to obtain the condition of Theorem 6.9 from (6.27). For example, let us derive the condition $\alpha_{1i_1} + \dots + \alpha_{ni_n} + \gamma'_k \notin \mathbb{Z}$. We take $I = \{\alpha_{1i_1}, \dots, \alpha_{ni_n}, \gamma'_k\}$. The condition (6.46) reads

$$\psi(\gamma'_k) = \psi(\alpha_{ip_i}) \quad (i = 1, \dots, n) \quad \text{and} \quad \psi(\delta) = 0 \quad (\delta \notin I).$$

It follows that the primitive vector ψ defining the facet $\{\alpha, \gamma, \alpha', \gamma'\} \setminus I$ is given by

$$\psi(\delta) = \begin{cases} -1 & (\delta = \alpha_{1i_1}, \dots, \alpha_{ni_n}, \gamma'_k) \\ 0 & (\text{otherwise}). \end{cases}$$

The condition (6.27) for the facet $\{\alpha, \gamma, \alpha', \gamma'\} \setminus I$ reads $\alpha_{1i_1} + \dots + \alpha_{ni_n} + \gamma'_k \notin \mathbb{Z}$. \square

6.6. Local solutions and the connection problem

In this section, we provide a general formula for the analytic continuation of the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}$. Let (ij) denotes the transposition of i and j . We take the base point b from the region $\{|x_n| \ll |x_{n-1}| \ll \dots \ll |x_1| \ll 1\}$. We first consider a path

$$C_0 : (0, \infty) \ni t \mapsto (-t, c_2(t), \dots, c_n(t)) \in (-\infty, 0) \times (0, \infty)^{n-1} \quad (6.47)$$

with small $c_i(t)$ so that $c_n(t) \ll c_{n-1}(t) \ll \dots \ll c_2(t)$ and the system $\mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}$ has no singularities along the path. Let p_{σ_1} be adjacent to p_{σ_2} . The coordinate transformations

$$(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)}), \left(\frac{1}{x_1}, \dots, \frac{1}{x_n} \right), \left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n} \right)$$

all induce automorphisms on X . These automorphisms induce an action of $S_{n+1} \times S_2$ onto X . For any element $g \in S_{n+1} \times S_2$, we write

$$\varphi_g : X \rightarrow X \quad (6.48)$$

for the corresponding automorphism. Then, there exists an element $g \in S_{n+1} \times S_2$ so that $\varphi_g(p_{\sigma_1}) = p_{\text{id}}$ and $\varphi_g(p_{\sigma_2}) = p_{(01)}$. Then, we write $C(\sigma_2\sigma_1)$ for the

path $\varphi_g \circ C_0$. In view of Proposition 6.1, the connection problem between p_{σ_1} and p_{σ_2} is reduced to that between p_{id} and $p_{(01)}$.

Therefore, we discuss the connection problem between p_{id} and $p_{(01)}$ below. For any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, +1\}^n$ such that $p_j + \varepsilon_j r = p'_j + \varepsilon_j r'$, we write $G_{p'_1, \dots, p'_n, r', \varepsilon}^{p_1, \dots, p_n, r}(\alpha'_1, \dots, \alpha'_n, \gamma'; x)$ for a series

$$\sum_{m=(m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n} \frac{(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} (\gamma)_{\varepsilon \cdot m}}{(\mathbf{1} - \alpha'_1)_{m_1} \cdots (\mathbf{1} - \alpha'_n)_{m_n} (\mathbf{1} - \gamma')_{\varepsilon \cdot m}} x^m. \quad (6.49)$$

Note that (6.3) is a special case of (6.49) with $\varepsilon = (+1, \dots, +1)$. For simplicity, we write $G_\varepsilon(\alpha'_1, \dots, \alpha'_n, \gamma'; x)$ for $G_{p'_1, \dots, p'_n, r', \varepsilon}^{p_1, \dots, p_n, r}(\alpha'_1, \dots, \alpha'_n, \gamma'; x)$.

A basis of solutions at p_{id} : it consists of the following two kinds of series. The first $p'_1 \cdots p'_n$ solutions to the system are

$$x^{\alpha'_I} F_{p'_1, \dots, p'_n, r'}^{p_1, \dots, p_n, r} \left(\begin{matrix} \alpha_1 + \alpha'_{1i_1} & \cdots & \alpha_n + \alpha'_{ni_n} & \gamma + |\alpha'_I| \\ \alpha'_1 - \alpha'_{1i_1} & \cdots & \alpha'_n - \alpha'_{ni_n} & \gamma' - |\alpha'_I| \end{matrix}; x_1, \dots, x_n \right) \quad (6.50)$$

with $\alpha'_I := (\alpha'_{1i_1}, \dots, \alpha'_{ni_n})$ and $|\alpha'_I| := \alpha'_{1i_1} + \cdots + \alpha'_{ni_n}$. To simplify the notation below, for any $\beta \in \mathbb{C}^n$, we set

$$F \left(\begin{matrix} \alpha_1 & \cdots & \alpha_n & \gamma \\ \alpha'_1 & \cdots & \alpha'_n & \gamma' \end{matrix}; \beta; x \right) := x^\beta F \left(\begin{matrix} \alpha_1 + \beta_1 & \cdots & \alpha_n + \beta_n & \gamma + |\beta| \\ \alpha'_1 - \beta_1 & \cdots & \alpha'_n - \beta_n & \gamma' - |\beta| \end{matrix}; x \right) \quad (6.51)$$

and

$$G_\varepsilon \left(\begin{matrix} \alpha_1 & \cdots & \alpha_n & \gamma \\ \alpha'_1 & \cdots & \alpha'_n & \gamma' \end{matrix}; \beta; x \right) := x^\beta G_\varepsilon \left(\begin{matrix} \alpha_1 + \beta_1 & \cdots & \alpha_n + \beta_n & \gamma + \varepsilon \cdot \beta \\ \alpha'_1 - \beta_1 & \cdots & \alpha'_n - \beta_n & \gamma' - \varepsilon \cdot \beta \end{matrix}; x \right). \quad (6.52)$$

For any $k = 1, \dots, n$, the $r'p_1 \cdots p_{k-1}p'_{k+1} \cdots p'_n$ solutions are

$$G_{\varepsilon(k)} \left(\begin{matrix} \alpha'_1 & \cdots & \alpha'_{k-1} & \gamma & \alpha_{k+1} & \cdots & \alpha_n & \alpha_k \\ \alpha_1 & \cdots & \alpha_{k-1} & \gamma' & \alpha'_{k+1} & \cdots & \alpha'_n & \alpha'_k \end{matrix}; \beta(k); \xi(k) \right), \quad (6.53)$$

where we set

$$\varepsilon(k) := \left(\overbrace{+1, \dots, +1}^k, \overbrace{-1, \dots, -1}^{n-k} \right), \quad (6.54)$$

$$\beta(k) := (\alpha_{1i_1}, \dots, \alpha_{(k-1)i_{k-1}}, \gamma'_{i_k}, \alpha'_{(k+1)i_{k+1}}, \dots, \alpha'_{ni_n}), \quad (6.55)$$

$$\xi(k) := (\varepsilon x_1^{-1} x_k, \dots, \varepsilon x_{k-1}^{-1} x_k, x_k, x_k^{-1} x_{k+1}, \dots, x_k^{-1} x_n). \quad (6.56)$$

A basis of solutions at $p_{(01)}$:

In view of Proposition 6.1, we obtain a relation

$$T_{(x_1, \dots, x_n) \rightarrow \left(\frac{1}{x_1}, \varepsilon \frac{x_2}{x_1}, \dots, \varepsilon \frac{x_n}{x_1} \right)} \mathcal{M}_{\alpha'_1, \dots, \alpha'_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma} = \mathcal{M}_{\gamma, \alpha'_2, \dots, \alpha'_n, \alpha'_1}^{\gamma', \alpha_2, \dots, \alpha_n, \alpha'_1}.$$

Thus, a basis of local solutions at $p_{(01)}$ is a transformations of that at p_{id} via the transformation $(x_1, \dots, x_n) \mapsto \left(\frac{1}{x_1}, \varepsilon \frac{x_2}{x_1}, \dots, \varepsilon \frac{x_n}{x_1} \right)$. The first $rp'_2 \cdots p'_n$ solutions are

$$F \left(\begin{matrix} \gamma', \alpha_2 \dots \alpha_n & \alpha'_1 \\ \gamma, \alpha'_2 \dots \alpha'_n & \alpha_1 \end{matrix}; (\gamma_k, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); \frac{1}{x_1}, \epsilon \frac{x_2}{x_1}, \dots, \epsilon \frac{x_n}{x_1} \right). \quad (6.57)$$

In the same manner, we obtain $p_1 p'_2 \dots p'_n$ solutions

$$G_{\varepsilon(1)} \left(\begin{matrix} \alpha'_1 & \alpha_2 \dots \alpha_n & \gamma' \\ \alpha_1 & \alpha'_2 \dots \alpha'_n & \gamma \end{matrix}; (\alpha_{1i_1}, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); \eta(1) \right) \quad (6.58)$$

with

$$\eta(1) = \left(\frac{1}{x_1}, \epsilon x_2, \dots, \epsilon x_n \right).$$

The other $r' \sum_{k=2}^n p_1 \dots p_{k-1} p'_{k+1} \dots p'_n$ solutions are given by (6.53) for $k = 2, \dots, n$. These solutions are holomorphic along the path C_0 of analytic continuation.

To state the formula, for any $\varepsilon, \tilde{\varepsilon} = (\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_n) \in \{-1, +1\}^n$ and $\beta \in \mathbb{C}^n$, we put

$$F \left(\begin{matrix} \alpha_1 \dots \alpha_n & \gamma \\ \alpha'_1 \dots \alpha'_n & \gamma' \end{matrix}; \beta; \tilde{\varepsilon}; x \right) := (\tilde{\varepsilon}_1 x_1)^{\beta_1} \dots (\tilde{\varepsilon}_n x_n)^{\beta_n} F \left(\begin{matrix} \alpha_1 + \beta_1 \dots \alpha_n + \beta_n & \gamma + |\beta| \\ \alpha'_1 - \beta_1 \dots \alpha'_n - \beta_n & \gamma' - |\beta| \end{matrix}; x \right), \quad (6.59)$$

$$G_{\varepsilon} \left(\begin{matrix} \alpha_1 \dots \alpha_n & \gamma \\ \alpha'_1 \dots \alpha'_n & \gamma' \end{matrix}; \beta; \tilde{\varepsilon}; x \right) := (\tilde{\varepsilon}_1 x_1)^{\beta_1} \dots (\tilde{\varepsilon}_n x_n)^{\beta_n} G_{\varepsilon} \left(\begin{matrix} \alpha_1 + \beta_1 \dots \alpha_n + \beta_n & \gamma + \varepsilon \cdot \beta \\ \alpha'_1 - \beta_1 \dots \alpha'_n - \beta_n & \gamma' - \varepsilon \cdot \beta \end{matrix}; x \right). \quad (6.60)$$

and

$$\delta_{i_2 \dots i_n} := \sum_{j=2}^n \alpha'_{ji_j}.$$

Theorem 6.11. *One has the following connection formula along the path C_0 .*

$$\begin{aligned} & F \left(\begin{matrix} \alpha_1 \dots \alpha_n & \gamma \\ \alpha'_1 \dots \alpha'_n & \gamma' \end{matrix}; (\alpha'_{1i_1}, \dots, \alpha'_{ni_n}); -\varepsilon(1); x \right) \\ &= \sum_{k=1}^r a_{i_1 k}^{i_2 \dots i_n} F \left(\begin{matrix} \gamma', \alpha_2 \dots \alpha_n & \alpha'_1 \\ \gamma, \alpha'_2 \dots \alpha'_n & \alpha_1 \end{matrix}; (\gamma_k, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); (-1, \epsilon, \dots, \epsilon); \frac{1}{x_1}, \epsilon \frac{x_2}{x_1}, \dots, \right. \\ & \quad \left. \epsilon \frac{x_n}{x_1} \right) + \sum_{\ell=1}^{p_1} b_{i_1 \ell}^{i_2 \dots i_n} G_{\varepsilon(1)} \left(\begin{matrix} \alpha'_1 & \alpha_2 \dots \alpha_n & \gamma' \\ \alpha_1 & \alpha'_2 \dots \alpha'_n & \gamma \end{matrix}; (\alpha_{1\ell}, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); -\varepsilon(1); \eta(1) \right), \\ & a_{i_1 k}^{i_2 \dots i_n} = \frac{\prod_{\nu \neq i_1} \Gamma(1 + \alpha'_{1i_1} - \alpha'_{1\nu}) \prod_{\nu=1}^{r'} \Gamma(1 + \alpha'_{1i_1} + \gamma'_{\nu} + \delta_{i_2 \dots i_n})}{\prod_{\nu \neq i_1} \Gamma(1 - \gamma_k - \delta_{i_2 \dots i_n} - \alpha'_{1\nu}) \prod_{\nu=1}^{r'} \Gamma(1 - \gamma_k - \gamma'_{\nu})} \\ & \quad \times \frac{\prod_{\nu \neq k} \Gamma(\gamma_{\nu} - \gamma_k) \prod_{\nu=1}^{p_1} \Gamma(\alpha_{1\nu} - \delta_{i_2 \dots i_n} - \gamma_k)}{\prod_{\nu \neq k} \Gamma(\alpha'_{1i_1} + \gamma_{\nu} + \delta_{i_2 \dots i_n}) \prod_{\nu=1}^{p_1} \Gamma(\alpha'_{1i_1} + \alpha_{1\nu})}, \end{aligned}$$

$$b_{i_1 \ell}^{i_2 \dots i_n} = \frac{\prod_{\nu \neq i_1} \Gamma(1 + \alpha'_{1i_1} - \alpha'_{1\nu}) \prod_{\nu=1}^{r'} \Gamma(1 + \alpha'_{1i_1} - \gamma'_\nu + \delta_{i_2 \dots i_n})}{\prod_{\nu \neq i_1} \Gamma(1 - \alpha'_{1\nu} - \alpha_{1\ell}) \prod_{\nu=1}^{r'} \Gamma(1 - \gamma'_\nu + \delta_{i_2 \dots i_n} - \alpha_{1\ell})} \\ \times \frac{\prod_{\nu \neq \ell} \Gamma(\alpha_{1\nu} - \alpha_{1\ell}) \prod_{\nu=1}^r \Gamma(\gamma_\nu + \delta_{i_2 \dots i_n} - \alpha_{1\ell})}{\prod_{\nu \neq \ell} \Gamma(\alpha'_{1i_1} + \alpha_{1\nu}) \prod_{\nu=1}^r \Gamma(\alpha'_{1i_1} + \gamma_\nu + \delta_{i_2 \dots i_n})}$$

and

$$G_{\varepsilon(1)} \left(\begin{matrix} \gamma & \alpha_2 & \dots & \alpha_n & \alpha_1 \\ \gamma' & \alpha_2' & \dots & \alpha_n' & \alpha_1' \end{matrix}; (\gamma'_{i_1}, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); -\varepsilon(n); \xi(1) \right) \\ = \sum_{\ell=1}^r c_{i_1 \ell}^{i_2 \dots i_n} F \left(\begin{matrix} \gamma', \alpha_2 & \dots & \alpha_n & \alpha_1' \\ \gamma, \alpha_2' & \dots & \alpha_n' & \alpha_1 \end{matrix}; (\gamma_\ell, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); (-1, \epsilon, \dots, \epsilon); \frac{1}{x_1}, \epsilon \frac{x_2}{x_1}, \dots, \right. \\ \left. \epsilon \frac{x_n}{x_1} \right) + \sum_{k=1}^{p_1} d_{i_1 k}^{i_2 \dots i_n} G_{\varepsilon(1)} \left(\begin{matrix} \alpha_1' & \alpha_2 & \dots & \alpha_n & \gamma' \\ \alpha_1 & \alpha_2' & \dots & \alpha_n' & \gamma \end{matrix}; (\alpha_{1k}, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); -\varepsilon(1); \eta(1) \right), \\ c_{i_1 \ell}^{i_2 \dots i_n} = \frac{\prod_{\nu \neq i_1} \Gamma(1 + \gamma'_{i_1} - \gamma'_\nu) \prod_{\nu=1}^{p_1'} \Gamma(1 + \gamma'_{i_1} - \delta_{i_2 \dots i_n} - \alpha'_{1\nu})}{\prod_{\nu \neq i_1} \Gamma(1 - \gamma_\ell - \gamma'_\nu) \prod_{\nu=1}^{p_1'} \Gamma(1 - \gamma_\ell - \delta_{i_2 \dots i_n} - \alpha'_{1\nu})} \\ \times \frac{\prod_{\nu \neq \ell} \Gamma(\gamma_\nu - \gamma_\ell) \prod_{\nu=1}^{p_1} \Gamma(\alpha_{1\nu} - \gamma_\ell - \delta_{i_2 \dots i_n})}{\prod_{\nu \neq \ell} \Gamma(\gamma'_{i_1} + \gamma_\nu) \prod_{\nu=1}^{p_1} \Gamma(\gamma'_{i_1} - \delta_{i_2 \dots i_n} + \alpha_{1\nu})}, \\ d_{i_1 k}^{i_2 \dots i_n} = \frac{\prod_{\nu \neq i_1} \Gamma(1 + \gamma'_{i_1} - \gamma'_\nu) \prod_{\nu=1}^{p_1'} \Gamma(1 + \gamma'_{i_1} - \delta_{i_2 \dots i_n} - \alpha'_{1\nu})}{\prod_{\nu \neq i_1} \Gamma(1 - \alpha_{1k} - \gamma'_\nu + \delta_{i_2 \dots i_n}) \prod_{\nu=1}^{p_1'} \Gamma(1 - \alpha_{1k} - \alpha'_{1\nu})} \\ \times \frac{\prod_{\nu \neq k} \Gamma(\alpha_{1\nu} - \alpha_{1k}) \prod_{\nu=1}^r \Gamma(\gamma_\nu + \delta_{i_2 \dots i_n} - \alpha_{1k})}{\prod_{\nu \neq k} \Gamma(\gamma'_{i_1} - \delta_{i_2 \dots i_n} + \alpha_{1\nu}) \prod_{\nu=1}^r \Gamma(\gamma'_{i_1} + \gamma_\nu)}.$$

The proof of Theorem 6.11 is same as that of Theorem 3.6. Indeed, local behavior of solutions are given as follows.

$$F \left(\begin{matrix} \alpha_1 & \dots & \alpha_n & \gamma' \\ \alpha_1' & \dots & \alpha_n' & \gamma' \end{matrix}; (\alpha'_{1i_1}, \dots, \alpha'_{ni_n}); -\varepsilon(1); x \right) \\ \underset{x_2, \dots, x_n \rightarrow 0}{\sim} x_2^{\alpha'_{2i_2}} \dots x_n^{\alpha'_{ni_n}} F_{p_1+r} \left(\begin{matrix} \alpha_1, \gamma + \delta_{i_2 \dots i_n} \\ \alpha_1', \gamma' - \delta_{i_2 \dots i_n} \end{matrix}; \alpha'_{1i_1}; -; x_1 \right),$$

$$\begin{aligned}
& G_{\varepsilon(1)} \left(\begin{array}{c} \gamma \quad \alpha_2 \dots \alpha_n \quad \alpha_1 \\ \gamma' \quad \alpha_2' \dots \alpha_n' \quad \alpha_1' \end{array}; (\gamma'_{i_1}, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); -\varepsilon(n); \xi(1) \right) \\
& \quad x_2, \dots, x_n \rightarrow 0 \quad x_2^{\alpha'_{2i_2}} \dots x_n^{\alpha'_{ni_n}} F_{p_1+r} \left(\begin{array}{c} \gamma + \delta_{i_2 \dots i_n}, \alpha_1 \\ \gamma' - \delta_{i_2 \dots i_n}, \alpha_1' \end{array}; \gamma'_{i_1}; -; x_1 \right), \\
& F \left(\begin{array}{c} \gamma' \quad \alpha_2 \dots \alpha_n \quad \alpha_1' \\ \gamma \quad \alpha_2' \dots \alpha_n' \quad \alpha_1 \end{array}; (\gamma_k, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); (-1, \epsilon, \dots, \epsilon); \frac{1}{x_1}, \epsilon \frac{x_2}{x_1}, \dots, \epsilon \frac{x_n}{x_1} \right) \\
& \quad x_2, \dots, x_n \rightarrow 0 \quad x_2^{\alpha'_{2i_2}} \dots x_n^{\alpha'_{ni_n}} F_{p_1+r} \left(\begin{array}{c} \gamma' - \delta_{i_2 \dots i_n}, \alpha_1' \\ \gamma + \delta_{i_2 \dots i_n}, \alpha_1 \end{array}; \gamma_k + \delta_{i_2 \dots i_n}; -; \frac{1}{x_1} \right), \\
& G_{\varepsilon(1)} \left(\begin{array}{c} \alpha_1' \quad \alpha_2 \dots \alpha_n \quad \gamma' \\ \alpha_1 \quad \alpha_2' \dots \alpha_n' \quad \gamma \end{array}; (\alpha_{1k}, \alpha'_{2i_2}, \dots, \alpha'_{ni_n}); -\varepsilon(1); \eta(1) \right) \\
& \quad x_2, \dots, x_n \rightarrow 0 \quad x_2^{\alpha'_{2i_2}} \dots x_n^{\alpha'_{ni_n}} F_{p_1+r} \left(\begin{array}{c} \gamma' - \delta_{i_2 \dots i_n}, \alpha_1' \\ \gamma + \delta_{i_2 \dots i_n}, \alpha_1 \end{array}; \alpha_{1k}; -; \frac{1}{x_1} \right).
\end{aligned}$$

6.7. Braid monodromy

In this section, we focus on the case $p_i = p'_i$ for any $i = 1, \dots, n$ and $r = r'$. By Theorem 6.3, the singular locus of the system $\mathcal{M}_{\alpha_1, \dots, \alpha_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ is the braid arrangement

$$\mathcal{A} := \left\{ \prod_{i=1}^n x_i(x_i - 1) \prod_{1 \leq i < j \leq n} (x_i - x_j) \right\} \subset \mathbb{C}^n.$$

We show that the connection problem among adjacent points generate the global monodromy. We identify $\mathbb{C}^n \setminus \mathcal{A}$ with the configuration space $\mathcal{M}_{0, n+3}$ of $n+3$ points on \mathbb{P}^1 . Any point $(x_1, \dots, x_n) \in \mathbb{C}^n \setminus \mathcal{A}$ corresponds to a configuration $[1, x_1, \dots, x_n, 0, \infty] \in \mathcal{M}_{0, n+3}$. We cite the following well-known description of its fundamental group. Let $b = [1, x_1, \dots, x_n, 0, \infty] \in \mathcal{M}_{0, n+3}$ be a base point. For any $i, j = 0, \dots, n+3$ with $i < j$ and $(i, j) \neq (0, n+1), (0, n+2), (n+1, n+2)$, we define a loop γ_{ij} on $\mathcal{M}_{0, n+2}$ as a move which starts from b , goes only around $x_i = x_j$ once and comes back to b . Here, we used a convention $x_0 = 1, x_{n+1} = 1, x_{n+2} = \infty$. The fundamental group $P_{n+3} := \pi_1(b, \mathcal{M}_{0, n+3})$ is known as the pure braid group. The following lemma can be found in [4, Lemma 1.8.2].

Lemma 6.12. γ_{ij} generates P_{n+3} .

Let $\sigma_1, \sigma_2 \in S_{n+1}$ be given and assume that σ_1 is adjacent to σ_2 . Given a pair of indices $0 \leq i < j \leq n$ we take a composition of paths of the form $C(\sigma_2\sigma_1)$ from p_{id} to some p_σ with $\sigma(0) = j$ and $\sigma(1) = i$. The composed path is denoted by C . Then, the path $C^{-1} \circ \varphi_\sigma(\gamma_{(01)}) \circ C$ is homotopic to γ_{ij} . In the same way, one can construct γ_{ij} for any $i, j = 0, \dots, n+2$ with $i < j$ and $(i, j) \neq (0, n+1), (0, n+2), (n+1, n+2)$ using paths of the form $C(\sigma_2\sigma_1)$ and $\varphi_g(\gamma_{01})$ for some $g \in S_{n+1} \times S_2$. By the discussion of §6.6 and transformations in Proposition 6.1, the analytic continuations along these paths are explicitly computable. We summarize the discussion of this section as a theorem.

Theorem 6.13. Suppose $p_i = p'_i$ for any $i = 1, \dots, n$ and $r = r'$. Then, the monodromy representation of the system $\mathcal{M}_{\alpha_1, \dots, \alpha_n, \gamma'}^{\alpha_1, \dots, \alpha_n, \gamma}$ is a representation of the

pure braid group P_{n+3} . By taking a basis of local solutions as in §6.6, one can compute monodromy matrices in terms of Gamma functions and trigonometric functions.

Acknowledgements

The first author was supported by JSPS KAKENHI Grant Number 22K13930, and partially supported by JST CREST Grant Number JP19209317. The second author was supported by JSPS KAKENHI Grant Number 18K03341. The second author expresses his sincere gratitude to Jiro Sekiguchi who explains his study on Appell's hypergeometric functions to the author, which helps the author to write this paper.

References

- [1] A. Adolphson. Hypergeometric functions and rings generated by monomials. 1994.
- [2] K. Appell and J. Kampé de Fériet. *Fonctions hypergéométriques et hypersphériques polynomes d'Hermite*. Gauthier-Villars, 1926.
- [3] A. Aramova, J. Herzog, and T. Hibi. Finite lattices and lexicographic Gröbner bases. *European Journal of Combinatorics*, 21(4):431–439, 2000.
- [4] J. S. Birman. *Braids, links, and mapping class groups*. Number 82. Princeton University Press, 1974.
- [5] J. De Loera, J. Rambau, and F. Santos. *Triangulations: structures for algorithms and applications*, volume 25. Springer Science & Business Media, 2010.
- [6] M. Dettweiler and S. Reiter. An algorithm of Katz and its application to the inverse Galois problems. *J. Symbolic Comput.*, 30:761–798, 2000.
- [7] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricom. *Higher Transcendental Functions, 3 volumes*. McGraw-Hill Book Co., 1953.
- [8] W. Fulton. *Introduction to toric varieties*. Number 131. Princeton university press, 1993.
- [9] I. Gelfand, M. Kapranov, and A. Z. Discriminants. Resultants and multidimensional determinants. *Mathematics: Theory and Applications Birkhäuser*, 1994.
- [10] I. M. Gel'fand, M. I. Graev, and V. S. Retakh. General hypergeometric systems of equations and series of hypergeometric type. *Russian Mathematical Surveys*, 47(4):1, 1992.

- [11] I. M. Gel'fand, A. V. Zelevinskii, and M. M. Kapranov. Hypergeometric functions and toral manifolds. *Functional Analysis and its applications*, 23(2):94–106, 1989.
- [12] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.
- [13] B. Grünbaum, V. Klee, M. A. Perles, and G. C. Shephard. *Convex polytopes*, volume 16. Springer, 1967.
- [14] Y. Haraoka. Middle convolution for completely integrable systems with logarithmic singularities along hyperplane arrangements. *Adv. Studies in Pure Math.*, 62:109–136, 2012.
- [15] Y. Haraoka. Multiplicative middle convolution for KZ equations. *Math. Z.*, 294:1787–1839, 2020.
- [16] R. Hattori and N. Takayama. The singular locus of lauricella's FC. *Journal of the Mathematical Society of Japan*, 66(3):981–995, 2014.
- [17] T. Hibi. Gröbner bases. *Statistics and software systems, Japanese edn. (Springer, Tokyo, 2013)*, 2003.
- [18] M. Hochster. Rings of invariants of tori, cohen-macaulay rings generated by monomials, and polytopes. *Annals of Mathematics*, 96(2):318–337, 1972.
- [19] M. Kashiwara and T. Oshima. Systems of differential equations with regular singularities and their boundary value problems. *Ann. of Math.*, 106:145–200, 1977.
- [20] N. M. Katz. *Rigid local systems*. Number 139. Princeton University Press, 1996.
- [21] K. Knizhnik and A. Zamolodchikov. Current algebra and Wess-Zumino model in 2 dimensions. *Nucl. Phys. B*, 247:83–103, 1984.
- [22] T. Kohno. Quantum representations of braid groups and holonomy lie algebras. In *Geometry, Dynamics, and Foliations 2013: In honor of Steven Hurder and Takashi Tsuboi on the occasion of their 60th birthdays*, volume 72, pages 117–145. Mathematical Society of Japan, 2017.
- [23] G. Lauricella. Sulle funzioni ipergeometriche a piu variabili. *Rendiconti del Circolo Matematico di Palermo*, 7:111–158, 1983.
- [24] S.-J. Matsubara-Heo. Global analysis of GG systems. *International Mathematics Research Notices*, 2022(19):14923–14963, 2022.
- [25] K. Matsumoto and M. Yoshida. Monodromy of lauricella's hypergeometric f_a -system. *ANNALI SCUOLA NORMALE SUPERIORE-CLASSE DI SCIENZE*, pages 551–577, 2014.

- [26] K. Mimachi. Connection formulas related with Appell's F_2 , Horn's H_2 and Olsson's F_p functions. *Kyushu J. Math.*, 74:14–42, 2020.
- [27] K. Mimachi. Connection formulas related with Appell's hypergeometric function F_1 . *PoS MA2019*, pages 1–73, 2022 010.
- [28] M. Noro and T. Takeshima. Risa/Asir — a computer algebra system. In *Papers from the international symposium on Symbolic and algebraic computation*, pages 387–396, 1992.
- [29] P. Olsson. On the integration of the differential equations of five-parametric double-hypergeometric functions of second order. *J. Math. Phys.*, pages 1285–1294, 1977.
- [30] T. Oshima. Boundary value problems for systems of linear partial differential equations with regular singularities. *Adv. Studies in Math.*, 4:391–432, 1984.
- [31] T. Oshima. Annihilators of generalized verma modules of the scalar type for classical lie algebras. In *Harmonic Analysis, Group Representations, Automorphic forms and Invariant Theor, in honor of Roger Howe*, volume 12 of *Lecture Notes Series*, pages 277–319. National University of Singapore, 2007.
- [32] T. Oshima. *Fractional calculus of Weyl algebra and Fuchsian differential equations*. Number 28 in MSJ Memoirs. Mathematical Society of Japan, 2012.
- [33] T. Oshima. Reducibility of hypergeometric equations. In *Analytic, Algebraic and Geometric Aspects of Differential Equations, Trends in Mathematics*, pages 429–453. Birkhäuser, 2017.
- [34] T. Oshima. Transformation of KZ type equations. In *Microlocal Analysis and Singular Perturbation Theory*, volume B61 of *RIMS Kôkyûroku Bessatsu*, pages 141–162. 2017.
- [35] T. Oshima. Confluence and versal unfolding of Pfaffian systems. *Josai Mathematical Monographs*, 12:117–151, 2020.
- [36] T. Oshima. Integral transformations of hypergeometric functions with several variables. 2023. Arxiv:2311.08947.
- [37] T. Oshima. os_muldif.rr, a library of computer algebra Risa/Asir, 2008–2023, <http://www.ms.u-tokyo.ac.jp/~oshima/>. 2023.
- [38] T. Oshima and J. Sekiguchi. Eigenspaces of invariant differential operators on an affine symmetric space. *Invent. Math.*, 57:1–81, 1980.
- [39] T. Oshima and N. Shimeno. Heckman-Opdam hypergeometric functions and their specializations. *RIMS Kôkyûroku Bessatsu*, B20:129–162, 2010.

- [40] M. Saito and N. Takayama. Restrictions of A-hypergeometric systems and connection formulas of the $\Delta_1 \times \Delta_{n-1}$ -hypergeometric function. *International Journal of Mathematics*, 5(04):537–560, 1994.
- [41] M. Schulze and U. Walther. Resonance equals reducibility for a-hypergeometric systems. *Algebra & Number Theory*, 6(3):527–537, 2012.
- [42] B. Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.
- [43] T. Tsuda. On a fundamental system of solutions of a certain hypergeometric equation. *The Ramanujan Journal*, 38:597–618, 2015.
- [44] G. M. Ziegler. *Lectures on polytopes*, volume 152. Springer Science & Business Media, 2012.