HECKMAN-OPDAM HYPERGEOMETRIC FUNCTIONS AND THEIR SPECIALIZATIONS

TOSHIO OSHIMA
(JOINT WORK WITH NOBUKAZU SHIMENO)

§0 Introduction. Heckman-Opdam hypergeometric function is defined by the second order differential operator

$$L(k) := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{\alpha\in\Sigma^+} k_\alpha \coth\langle\alpha, x\rangle \cdot \partial_\alpha.$$

Here Σ^+ is the set of positive roots of a root system Σ, $\partial_\alpha \phi(x) = \frac{d}{dt}\phi(x+t\alpha)|_{t=0}$ and the complex numbers k_α satisfy $k_\alpha = k_\beta$ if $|\alpha| = |\beta|$.

For a generic $\lambda \in \mathbb{C}^n$ we have a unique local solution

$$\Phi(\lambda, k; x) = e^{\langle\lambda-\rho, x\rangle} + \cdots \quad \text{(a series expansion at } \langle\alpha, x\rangle \to 0 \quad (\alpha \in \Sigma^+))$$

of the differential equation

$$L(k)u = \big(\langle\lambda, \lambda\rangle - \langle\rho(k), \rho(k)\rangle\big)u$$

and define Heckman-Opdam hypergeometric function

$$F(\lambda, k; x) := \sum_{w\in W} c(w\lambda)\Phi(\lambda, k; x)$$

as a generalization of the zonal spherical function of a Riemannian symmetric space. Here $\rho = \rho(k) = \sum_{\alpha\in\Sigma^+} k_\alpha \alpha$, W is the Weyl group of Σ and $c(\lambda)$ is a generalization of Harish-Chandra's c-function given by

$$c(\lambda) := \frac{\tilde{c}(\lambda)}{\tilde{c}(\rho(k))}, \quad \tilde{c}(\lambda) := \prod_{\alpha\in\Sigma^+} \frac{\Gamma\big(\frac{\langle\lambda,\check{\alpha}\rangle + k_{\alpha/2}}{2}\big)}{\Gamma\big(\frac{\langle\lambda,\check{\alpha}\rangle + k_{\alpha/2} + 2k_\alpha}{2}\big)} \quad \text{and} \quad \check{\alpha} := \frac{2\alpha}{\langle\alpha, \alpha\rangle}.$$

Put $\delta(k)^{\frac{1}{2}} = \prod_{\alpha\in\Sigma^+} \big(\sinh\langle\alpha, x\rangle\big)^{k_\alpha}$. Then the Schrödinger operator

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - \sum_{\alpha\in\Sigma^+} \frac{k_\alpha(k_\alpha + 2k_{2\alpha} - 1)\langle\alpha, \alpha\rangle}{\sinh^2\langle\alpha, x\rangle} = \delta(k)^{\frac{1}{2}} \circ (L(k) + \langle\rho(k), \rho(k)\rangle) \circ \delta(k)^{-\frac{1}{2}}$$

is completely integrable and hence $L(k)$ is in a commuting system of differential operators with n algebraically independent operators.

Then we have the following fundamental result (cf. [1]).

Theorem [Heckman, Opdam]. *When k_α are generic, the function $F(\lambda, k; x)$ has an analytic extension on $\mathbb{R}^n$ and defines a unique simultaneous eigenfunction of the commuting system of differential operators with the eigenvalue parametrized by λ so normalized that the eigenfunction takes the value 1 at the origin.*

Heckman-Opdam hypergeometric system of differential equations characterizing $F(\lambda, k; x)$, which will be denoted by (HO), is a multi-variable analogue of a "rigid local system" among completely integrable quantum systems and we study three types of specializations of the system and the function $F(\lambda, k; x)$ as follows.

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§1 Confluence. We examine its confluent limits such as Toda finite lattice (cf. [2] for the limiting procedure and the limits). For example, if Σ is of type A_{n-1}, under the correspondence $x_j \mapsto x_j + jt$ with $t \to \infty$ we remark that

$$\sum_{1\le i<j\le n} \frac{Ce^{2t}}{\sinh^2\bigl((x_i+it)-(x_j+jt)\bigr)} = \sum_{1\le i<j\le n} \frac{4Ce^{2(1-j+i)t}e^{2(x_i-x_j)}}{(1-e^{2(x_i-x_j)}e^{-2(j-i)t})^2} \to \sum_{i=1}^{n-1} 4Ce^{2(x_j-x_{i+1})},$$

which holomorphically depends on $s = e^{-t}$.

Theorem 1. i) *For $v \in \mathbb{R}^n \setminus \{0\}$ the commuting system* (HO) *holomorphically continued to a confluent commuting system* $(\mathrm{HO})_{\mathrm{conf}}$ *by $x \mapsto x + tv$ with $t \to \infty$ and suitable $k_\alpha = k_\alpha(t)$. When Σ is of type BC_n (resp. F_4 or G_2), there exist three (resp. two) kinds of irreducible confluent limits.*

ii) *A suitably normalized Heckman-Opdam hypergeometric function has a non-zero holomorphic limit $W(x)$ with its expansion at an infinite point corresponding to a Weyl chamber $\mathcal{C}$. The limit has the moderate growth property:*

$$\exists C>0,\ \exists m>0 \text{ such that } |W(x)| \le Ce^{m|x|}.$$

iii) *The dimension of the solutions of the holomorphic family of the commuting systems including* $(\mathrm{HO})_{\mathrm{conf}}$ *with the moderate growth property is always one.*

iv) *For example, in the case of Toda finite lattice the limit $W(x)$ satisfies*

$$\exists C>0,\ \exists m>0,\ \exists K>0 \text{ such that } |W(x)| \le C\exp(mx - e^{K\,\mathrm{dist}(x,\mathcal{C})}).$$

§2 Restriction. Let Ψ denote the fundamental system of Σ^+. For a subset Ψ' of Ψ let $H_{\Psi'}$ be the intersection of the walls defined by the elements of Ψ'. For a local solution u of (HO) at a generic point of $H_{\Psi'}$ we examine the differential equations satisfied by $u|_{H_{\Psi'}}$. Note that if $\#\Psi' = \#\Psi - 1$, the differential equations are ordinary differential equations. For example we have the following.

Theorem 2. *When (Ψ, Ψ') is of type (A_n, A_{n-1}) (resp. (BC_n, BC_{n-1})), the ordinary differential equations coincide with those satisfied by hypergeometric family ${}_{n+1}F_n$ of order $n+1$ (resp. even family of order $2n$). These are rigid local systems classified by Deligne-Simpson problem (cf.* [5]*).*

This theorem reduces the Gauss summation formula for (HO) given by [4] to the connection formula of the solutions of the ordinary differential equations.

§3 Real forms. For a signature

$$\epsilon : \Sigma \to \{\pm 1\} \qquad (\epsilon(\alpha+\beta) = \epsilon(\alpha)\epsilon(\beta) \text{ for } \forall\alpha,\ \beta,\ \alpha+\beta \in \Sigma)$$

of the root system Σ introduced by [3] we put

$$L(k)_\epsilon := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + \sum_{\substack{\alpha\in\Sigma^+\\ \epsilon(\alpha)>0}} k_\alpha \coth\langle\alpha,x\rangle\cdot\partial_\alpha + \sum_{\substack{\alpha\in\Sigma^+\\ \epsilon(\alpha)<0}} k_\alpha \tanh\langle\alpha,x\rangle\cdot\partial_\alpha.$$

Note that $L(k)_\epsilon$ is obtained from $L(k)$ by the coordinate transformation $x \mapsto x + \sqrt{-1}v_\epsilon$ with a suitable $v_\epsilon \in \mathbb{R}^n$. We denote by $(\mathrm{HO})_\epsilon$ the corresponding commuting system of differential equations. Let W_ϵ be the Weyl group generated by the reflections with respect to the roots α satisfying $\epsilon(\alpha) = 1$.

Theorem 3. i) *If k_α are generic, the dimension of the solutions of* $(\mathrm{HO})_\epsilon$ *is $\#W/W_\epsilon$ and the vector $F_\epsilon(\lambda,k;x)$ of the independent solutions can be*

$$\bigl(F_\epsilon(\lambda,k;vx)\bigr)_{v\in W_\epsilon\backslash W} \sim \sum_{w\in W} A^\epsilon_w(\lambda,k)c(w\lambda,k)(e^{\langle w\lambda-\rho,x\rangle}+\cdots).$$

Here $A^{\epsilon}_w(\lambda,k)$ are intertwining matrices of size $\#W/W_\epsilon$ which satisfy

$$A^{\epsilon}_{wv}(\lambda,k)=A^{\epsilon}_w(v\lambda,k)A^{\epsilon}_v(\lambda,k)\qquad(w,\,v\in W).$$

If s_α is a simple reflection with respect to $\alpha\in\Psi$, $A^{\epsilon}_{s_\alpha}(\lambda,k)$ is a suitable direct product of the following matrices and scalars

$$A(\lambda,k):=\begin{pmatrix}\frac{\sin\pi k}{\sin\pi(\lambda+k)} & \frac{\sin\pi\lambda}{\sin\pi(\lambda+k)}\\ \frac{\sin\pi\lambda}{\sin\pi(\lambda+k)} & \frac{\sin\pi k}{\sin\pi(\lambda+k)}\end{pmatrix},\quad \frac{\cos\frac{1}{2}\pi(\lambda-k)}{\cos\frac{1}{2}\pi(\lambda+k)}\ \text{ and }\ 1.$$

ii) *We have a functional equation of the spherical functions:*

$$F_\epsilon(\lambda,k;x)=F_\epsilon(w\lambda,k;x)A^{\epsilon}_w(\lambda,k).$$

References

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