

Fractional analysis of linear differential equations on the Riemann sphere

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Web-seminar on Painlevé Equations and related topics

Kobe University (Zoom)

14 January, 2022

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- Generalized Riemann scheme (GRS) and spectral type
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Most results are realized in a library [Om] of computer algebra

§ Generalized Riemann scheme

$$\mathcal{M} : \boxed{Pu = 0} \quad (P = a_n(x)\partial^n + \cdots + a_1(x)\partial + a_0(x))$$

$$P \in W[x] = \mathbb{C}[x, \partial], \quad \partial = \frac{d}{dx}, \quad \vartheta = x\partial$$

$c_0 = \infty, c_1, \dots, c_p$: the singular points of $\mathcal{M}$ ($a_n(c_j) = 0$)

Under the coordinate $c_1 = 0$, we assume $\mathcal{M}$ has (formal) local solutions

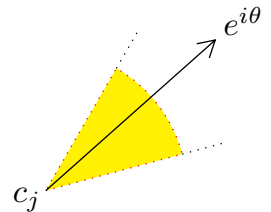
$$(E) \quad \boxed{u_\nu(x) = \phi(x)x^{\lambda_{\nu,0}^o} \exp\left(-\frac{\lambda_{\nu,r}^o}{rx^r} - \frac{\lambda_{\nu,r-1}^o}{(r-1)x^{r-1}} - \cdots - \frac{\lambda_{\nu,1}^o}{x}\right)} \quad (\nu = 1, \dots, n)$$

$$\phi(x) = 1 + a_1x + a_2x^2 + \cdots \in \mathbb{C}[[x]]$$

$$\lambda_\nu(x) := \lambda_{\nu,r}^o x^r + \cdots + \lambda_{\nu,1}^o x + \lambda_{\nu,0}^o \in \mathbb{C}[x], \quad \lambda_\nu \neq \lambda_{\nu'} \quad (\Leftarrow 1 \leq \nu < \nu' \leq n)$$

Theorem [Hu,...] $\theta \in \mathbb{R} \Rightarrow \exists R > 0, \theta_1, \theta_2$ and solutions u_ν to $\mathcal{M}$ such that $\theta \in (\theta_1, \theta_2)$ and u_ν have asymptotic (E) for $x \rightarrow c_j = 0$ on

$$V_{c_j, R, (\theta_1, \theta_2)} := \{x \in \mathbb{C} \mid |x - c_j| < R, \theta_1 < \arg(x - c_j) < \theta_2\}$$



Characteristic exponents $\lambda_{j,\nu} \in \mathbb{C}[x]$ of $\mathcal{M}$ at $x = c_j$ are defined through $c_j \mapsto 0$

- No logarithmic term ($\Leftarrow$ for simplicity)
- No ramified irregular singularity ($\Leftarrow$ essential)

$\{[\lambda_{j,1}]_{(m_1)}, \dots, [\lambda_{j,n_j}]_{(m_{n_j})}\} : \text{characteristic exponents of } \mathcal{M} \text{ at } x = c_j$

$$\lambda_{j,\nu} \in \mathbb{C}[x], \quad n = m_{j,1} + \dots + m_{j,n_j} \quad (j = 0, \dots, p),$$

$$[\lambda]_{(m)} := \{\lambda, \lambda + 1, \dots, \lambda + m - 1\} \quad (\lambda \in \mathbb{C}[x])$$

Generalized Riemann scheme (GRS)

$$(\star) : \left\{ \begin{array}{c} x = c_j \quad (j = 0, \dots, p) \\ [\lambda_{j,1}]_{(m_{j,1})} \\ \vdots \\ [\lambda_{j,n_j}]_{(m_{j,n_j})} \end{array} \right\}, \quad \left\{ \begin{array}{ccc} x = 0 & x = 1 & x = \infty \\ 0 & [0]_{(n-1)} & \alpha_1 \\ 1 - \beta_1 & & \alpha_2 \\ \vdots & & \vdots \\ 1 - \beta_{n-1} & -\beta_n & \alpha_n \end{array} \right\} \quad \begin{array}{l} 1^n, (n-1)1, 1^n \\ {}_nF_{n-1}(\alpha; \beta; x) \\ \sum \alpha_\nu = \sum \beta_\nu \end{array}$$

- $\deg \lambda_{j,\nu} = 0 \quad (\nu = 1, \dots, n_j) \Rightarrow c_j \text{ is a regular singular point}$
- Versal unfolding of (GRS) $\hookrightarrow$ (GRS) of a Fuchsian equation (cf. [Ov])
 $\Rightarrow$ spectral type (a tuple of partitions of $n = \text{ord } P$)

Example : spectral type 211|22|22, 31

$Pu = 0$: $\text{ord } P = 4$ with $x = \infty$ (Poincaré rank 2) and 0 (regular singularity)

$$u(x) \approx x^{-a_0} (1 + o(x^{-1})) e^{-a_1 x - \frac{1}{2} a_2 x^2}, \quad x^{-a_0-1} e^{-a_1 x - \frac{1}{2} a_2 x^2} \quad (x \rightarrow +\infty)$$

$$\approx x^{-b_0} e^{-b_1 x}, \quad x^{-c_0} e^{-b_1 x} \quad (x \rightarrow +\infty)$$

$$\approx (1 + o(x^2)) x^{c_1}, \quad (x + o(x))^{c_1+1}, \quad x^{c_1+2}, \quad x^{c_2} \quad (x \rightarrow +0)$$

Fuchs-Hukuhara relation : $2a_0 + b_0 + c_0 + 2c_1 + c_2 + c_3 = 4$

$$\left\{ \begin{array}{cc} x = \infty & x = 0 \\ [a_0 + a_1 x + a_2 x^2]_{(2)} & [c_1]_{(3)} \\ b_0 + b_1 x & c_2 \\ c_0 + b_1 x & \end{array} \right\} = \left\{ \begin{array}{cccc} x = \infty & (1) & (2) & x = 0 \\ [a_0]_{(2)} & [a_1]_2 & [a_2]_2 & [c_1]_{(3)} \\ b_0 & [b_1]_2 & [0]_2 & c_2 \\ c_0 & & & \end{array} \right\}$$

$\text{idx}(211|22|22, 31) = 0 \Rightarrow 1 (= 1 - \frac{\text{idx}}{2})$ accessory parameter $\rightsquigarrow$ versal Heun, P_{IV}

$$\left\{ \begin{array}{cccc} x = \infty & x = \frac{1}{t_1} & x = \frac{1}{t_2} & x = 0 \\ [a_0 - \frac{a_1}{t_1} + \frac{a_2}{t_1 t_2}]_{(2)} & [\frac{a_1}{t_1} + \frac{a_2}{t_1(t_1-t_2)}]_{(2)} & [\frac{a_2}{t_2(t_2-t_1)}]_{(2)} & [c_1]_{(3)} \\ b_0 - \frac{b_1}{t_1} & [\frac{b_1}{t_1}]_{(2)} & [0]_{(2)} & c_2 \\ c_0 - \frac{b_1}{t_1} & & & \end{array} \right\}$$

§ Index of rigidity and two operations (cf. [Of, Ov])

$$\begin{aligned}\text{idx } \mathbf{m} &= 2n^2 - \sum (n^2 - m_{j,\nu}^2) - \sum \deg(\lambda_{j,\nu} - \lambda_{j,\nu'}) \cdot m_{j,\nu} m_{j,\nu'} \\ &= 2n^2 - \sum (n^2 - (m_{j,\nu}^{(r)})^2)\end{aligned}$$

$$\begin{aligned}d_1(\mathbf{m}) &= 2n - \sum (n - m_{j,1}) - \sum \deg(\lambda_{j,\nu} - \lambda_{j,1}) \cdot m_{j,\nu} \\ &= 2n - \sum (n - m_{j,1}^{(r)})\end{aligned}$$

$$(\text{FC}) \quad \sum m_{j,\nu} \lambda_{j,\nu}(0) - \text{ord } \mathbf{m} + \frac{1}{2} \text{idx } \mathbf{m} := |\{[\lambda_{j,\nu}^{(r)}]_{(m_{j,1}^{(r)})}\}| = 0$$

Accessory parameters = $1 - \frac{1}{2} \text{idx } \mathbf{m}$ ($\# = 0 \Leftrightarrow \text{rigid}$, Pfaffian Eq. $\Rightarrow 2 - \text{idx } \mathbf{m}$)

Addition

$$\begin{aligned}u(x) \mapsto v(x) = \varphi(x)u(x) &\Rightarrow \partial \mapsto \text{Ad}(\varphi)\partial = \varphi \circ \partial \circ \varphi^{-1} = \partial - \frac{\varphi'}{\varphi} \\ \text{Ad}((x-c)^\lambda)\partial &= \partial - \frac{\lambda}{x-c}, \quad \vartheta \mapsto \text{Ad}(x^\lambda)\vartheta = \vartheta - \lambda, \quad \text{Ad}(\varphi)^{-1} = \text{Ad}(\varphi^{-1})\end{aligned}$$

Middle convolution : a microlocal operator

$$u(x) \mapsto v(x) = \partial^{-\mu} u = \frac{1}{\Gamma(\mu)} \int_c^x u(t)(x-t)^{\mu-1} dt \quad (c : \text{singular point}, \mu \in \mathbb{C})$$

$$Pu(x) = 0 \Rightarrow \text{mc}_\mu(P)v(x) = 0, \quad \text{mc}_\mu^{-1} = \text{mc}_{-\mu} \quad (\text{mc}_\mu(\partial) = \partial, \text{mc}_\mu(\vartheta) = \vartheta - \mu)$$

$$\text{mc}_\mu(P) := \partial^{-L} \partial^{-\mu} (\partial^N P) \partial^\mu \in W[x] \quad (\partial^N P \in \mathbb{C}[\vartheta, \partial], \text{largest } L \in \mathbb{Z})$$

$$(\star) : \left\{ \begin{array}{cccc} x = c_0 = \infty & x = c_1 & \cdots & x = c_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \cdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{array} \right\}$$

$$\text{mc}_\mu : \{[\lambda_{j,\nu}]_{(m_{j,\nu})}\} \mapsto \{[\lambda'_{j,\nu}]_{(m_{j,\nu})}\}, \quad Pu = 0 \mapsto P'v = 0 \quad (P' = \text{mc}_\mu(P))$$

Theorem. Suppose $\lambda_{0,1} = 1 + \mu$ ($\deg \lambda_{0,1} = 0$), $\lambda_{1,1} = \cdots = \lambda_{p,1} = 0$

and other exponents are generic ($m_{j,1}$ may be 0). Then ([DR, Of, Hi1])

$$m'_{j,\nu} = m_{j,\nu} - \delta_{\nu,1} \cdot d_1(\mathbf{m}) \quad (1 \leq \nu \leq n_j, \ 0 \leq j \leq p),$$

$$\lambda'_{j,1} = \delta_{j,0} \cdot (1 - \mu) \quad (j = 0, \dots, p),$$

$$\lambda'_{j,\nu} = \lambda_{j,\nu} + (\deg \lambda_{j,\nu} + (-1)^{\delta_{j,0}}) \cdot \mu \quad (2 \leq \nu \leq n_j, \ 0 \leq j \leq p)$$

and the **index of rigidity** and the **irreducibility** are kept.

By suitable (versal) additions and transpositions of the indices, we may assume

($\star$) with ($\sharp$) $\sum_r m_{j,1}^{(r)} \geq \sum_r m_{j,\nu}^{(r)} \quad (j = 0, \dots, p, \ \nu = 2, \dots, n_j)$. Then

Theorem [Kz, Of,...]. If $\text{idx } \mathbf{m} > 0 \Rightarrow d_1(\mathbf{m}) > 0 \Rightarrow$ **rigid Eq.** is reduced to $u' = 0$

Theorem [CB, Of, Hi2, ?]. If $d_1(\mathbf{m}) \leq 0 \Rightarrow \mathbf{m}$ is **irreducibly realizable** (**basic**)

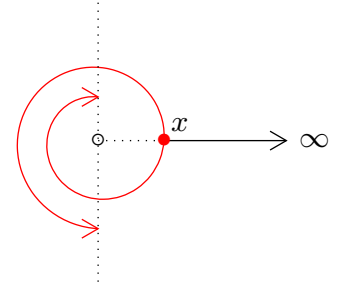
Theorem [Of, HO]. $\#\{\mathbf{m} \mid d_1(\mathbf{m}) \leq 0, \text{idx } \mathbf{m} = K\} < \infty \quad (\forall K \in \mathbb{Z})$

$$(\partial^{-\mu} u)(x) = (I_a^\mu u)(x) := \frac{1}{\Gamma(\mu)} \int_a^x u(t)(x-t)^{\mu-1} dt$$

(a : a singular point of $u(x)$)

$$I_0^\mu(x^\lambda) = \frac{1}{\Gamma(\mu)} \int_0^x t^\lambda (x-t)^{\mu-1} dt = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} x^{\lambda+\mu}$$

$$\tilde{I}_\infty^\mu(x^\lambda e^{-x}) \sim x^\lambda e^{-x} {}_2F_0(-\lambda, \mu; -\frac{1}{x}) \quad (|x| \rightarrow \infty, |\arg x| < \frac{3\pi}{2})$$



$${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; x) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n n!} x^n$$

$$e^x = {}_0F_0(x), \quad (1-x)^{-\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} x^n = {}_1F_0(\lambda; x),$$

$$\begin{aligned} & I_0^\mu(x^\lambda \cdot {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; \pm x)) \\ &= x^{\lambda+\mu} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} \cdot {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, \lambda+1; \beta_1, \dots, \beta_q, \lambda+\mu+1; \pm x) \end{aligned}$$

$$I_0^\mu(x^\lambda e^{-x}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} x^{\lambda+\mu} \cdot {}_1F_1(\lambda+1; \lambda+\mu+1; -x) \quad (\text{Kummer})$$

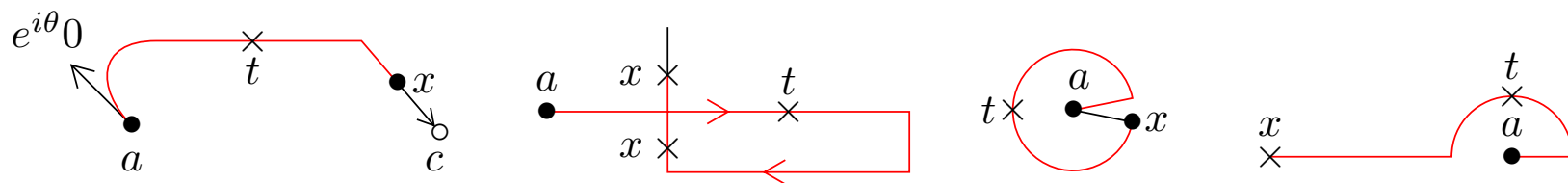
$$I_0^\mu(x^\lambda (1-x)^{\lambda'}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} x^{\lambda+\mu} \cdot {}_2F_1(\lambda+1, -\lambda'; \lambda+\mu+1; x) \quad (\text{Gauss})$$

§ Riemann-Liouville transform

$$(I_{a+e^{i\theta}0}^\mu u)(x) := \frac{1}{\Gamma(\mu)} \int_{a+e^{i\theta}0}^x u(t)(x-t)^{\mu-1} dt = \frac{1}{\Gamma(\mu)} \int_L u(t)(x-t)^{\mu-1} dt$$

$$L : [\alpha, \beta] \ni t \mapsto L(t) \in \mathbb{C}, \quad L(\alpha) = a, \quad L(\beta) = x, \quad \theta = \arg L'(\alpha),$$

$$L(s) \neq L(t) \quad \text{for } \alpha \leq s < t \leq \beta$$



$$\tilde{L} : [\alpha, \gamma] \ni t \mapsto \tilde{L}(t) \in \mathbb{C}, \quad a = \tilde{L}(\alpha) \text{ and } c = \tilde{L}(\gamma),$$

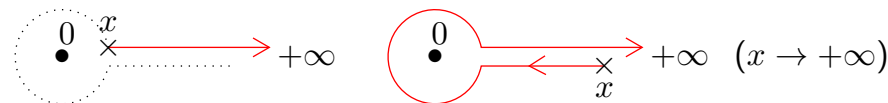
$$\tilde{L}(s) \neq \tilde{L}(t) \text{ for } \alpha \leq s < t < \gamma.$$

- $u(x)$ is holomorphic for $x = \tilde{L}(t_0)$ ($t_0 \in (\alpha, \gamma)$), $L := \tilde{L}|_{[\alpha, t_0]}$

Want to know the asymptotics of $(I_a^\mu u)(x)$ for $t_0 \rightarrow \alpha$ and $t_0 \rightarrow \gamma$

in terms of those of $u(x)$.

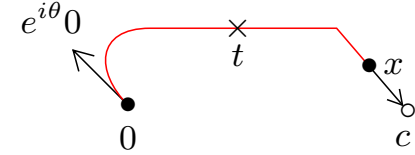
$$(\tilde{I}_a^\mu u)(x) := \frac{1}{\Gamma(\mu)} \int_x^a u(t)(t-x)^{\mu-1} dt$$



$$(\tilde{I}_a^\mu u)(x) = x^{\mu-1} \cdot \left(I_{\frac{1}{a}}^\mu x^{-\mu-1} u\left(\frac{1}{x}\right) \right) \left(\frac{1}{x}\right) \quad (a = e^{i\theta} \infty \rightarrow \frac{1}{a} = e^{-i\theta} 0)$$

§ Riemann-Liouville transform (Main Theorem) (cf. [Of, Ov, Or])

$$(I_{e^{i\theta}0}^\mu u)(x) = \frac{1}{\Gamma(\mu)} \int_{e^{i\theta}0}^x u(t)(x-t)^{\mu-1} dt$$



$$(1.1) \quad u(x) \sim \left(\sum_{n=0}^{\infty} a_n x^n \right) x^\lambda \quad (x \rightarrow e^{i\theta}0) \quad (\phi(x) := a_0 + a_1 x + \dots \in \mathbb{C}[[x]])$$

$$\Rightarrow (I_{e^{i\theta}0}^\mu u)(x) \sim \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)} \left(\sum_{n=0}^{\infty} \frac{(\lambda+1)_n a_n}{(\lambda+\mu+1)_n} x^n \right) x^{\lambda+\mu}$$

$$(1.2) \quad u(x) \sim \phi(x) x^\lambda \exp\left(-\frac{C_0}{x^{m_0}} - \frac{C_1}{x^{m_1}} - \dots\right) \quad (x \rightarrow e^{i\theta}0) \quad \text{and} \quad \operatorname{Re} C_0 e^{-im_0\theta} > 0$$

$$\Rightarrow (I_{e^{i\theta}0}^\mu u)(x) \sim (m_0 C_0)^{-\mu} \psi(x) x^{\lambda+(m_0+1)\mu} \exp\left(-\frac{C_0}{x^{m_0}} - \frac{C_1}{x^{m_1}} - \dots\right)$$

$$\phi, \psi \in \mathbb{C}[[x]], \phi(0) = \psi(0), m_0 > m_1 > \dots > m_k > 0 \text{ and } m_j \in \mathbb{Z}_{>0}$$

$$(2.1) \quad u(x) \approx (c-x)^{\lambda'} \quad (x \rightarrow c - e^{i\theta'}0) \quad \text{and} \quad \operatorname{Re}(\lambda' + \mu) < 0$$

$$\Rightarrow (I_{e^{i\theta}0}^\mu u)(x) \approx \frac{\Gamma(-\lambda' - \mu)}{\Gamma(-\lambda')} (c-x)^{\lambda'+\mu} \Leftrightarrow (\operatorname{Re}(\lambda - \lambda') > 0 \Leftrightarrow c = 0, (1.1))$$

$$(2.2) \quad u(x) \approx (c-x)^{\lambda'} \exp\left(\frac{C'_0}{(c-x)^{m'_0}} + \frac{C'_1}{(c-x)^{m'_1}} + \dots\right) \quad (x \rightarrow c - e^{i\theta'}0) \quad \text{and}$$

$$\operatorname{Re} C'_0 e^{-im'_0\theta'} > 0$$

$$\Rightarrow (I_{e^{i\theta}0}^\mu u)(x) \approx (m'_0 C'_0)^{-\mu} (c-x)^{\lambda'+(m'_0+1)\mu} \exp\left(\frac{C'_0}{(c-x)^{m'_0}} + \frac{C'_1}{(c-x)^{m'_1}} + \dots\right)$$

$$(2.3) \quad \lim_{x \rightarrow c - e^{i\theta'}0} u(x)(c-x)^{-\operatorname{Re} \mu} = 0 \Rightarrow \lim_{x \rightarrow c - e^{i\theta'}0} u(x) = (I_{e^{i\theta}0}^\mu u)(c - e^{i\theta'}0) \in \mathbb{C}$$

§ Example (${}_{n-1}F_{n-1}$: a confluence of ${}_nF_{n-1}$)

$1^n | (n-1)1, 1^n$ ($n=2 \Rightarrow$ Kummer's Eq.)

$$G^\pm := \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} (1 \leq \nu < n) & \lambda_{0,\nu} (1 \leq \nu \leq n) \\ \pm x - \lambda_{1,n} & \end{array} \right\} \quad \left(\sum_{\nu=1}^n \lambda_{0,\nu} = \sum_{\nu=1}^n \lambda_{1,\nu} \right)$$

$$\xrightarrow{\text{Ad}(x^{-\lambda_{0,1}})} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} + \lambda_{0,1} (1 \leq \nu < n) & 0 \\ \pm x - \lambda_{1,n} + \lambda_{0,1} & \lambda_{0,\nu} - \lambda_{0,1} (2 \leq \nu \leq n) \end{array} \right\}$$

$$\xrightarrow{\text{mc} \lambda_{0,1} - \lambda_{1,1}} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} + \lambda_{1,1} (2 \leq \nu < n) & \lambda_{0,\nu} - \lambda_{1,1} (2 \leq \nu \leq n) \\ \pm x - \lambda_{1,n} + \lambda_{0,1} & \end{array} \right\}$$

$$\xrightarrow{\text{Ad}(x^{\lambda_{1,1} - \lambda_{0,2}})} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} + \lambda_{0,2} (3 \leq \nu < n) & 0 \\ \pm x - \lambda_{1,n} + \lambda_{0,1} + \lambda_{0,2} - \lambda_{1,1} & \lambda_{0,\nu} - \lambda_{0,2} (3 \leq \nu \leq n) \end{array} \right\}$$

$$\dots \xrightarrow{\text{Ad}(x^{\lambda_{0,n-1} - \lambda_{1,n-2}})}$$

$$\left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \pm x - \lambda_{1,n} + \lambda_{0,1} + \sum_{\nu=2}^{n-1} (\lambda_{0,\nu} - \lambda_{1,\nu-1}) & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\}$$

$$= G_2^\pm := \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \pm x + \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\} \quad (\text{Kummer})$$

$$G_2^\pm = \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \pm x + \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\} \quad (\text{Kummer})$$

$$\xrightarrow{\text{mc} \lambda_{0,n-1} - \lambda_{1,n-1}} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ \pm x + \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{1,n-1} \end{array} \right\} \ni x^{\lambda_{0,n} - \lambda_{1,n-1}} e^{\mp x}$$

$$G_2^\pm \xrightarrow{x \mapsto -x} G_2^\mp = \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \mp x + \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\}$$

$$\xrightarrow{\text{Ad}(e^{\mp x})} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ \pm x + 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & 0 \\ \lambda_{1,n-1} - \lambda_{0,n} & \lambda_{0,n} - \lambda_{0,n-1} \end{array} \right\}$$

$$\xrightarrow{\text{mc} \lambda_{1,n-1} - \lambda_{0,n-1}} \left\{ \begin{array}{cc} x = \infty & x = 0 \\ \pm x + 1 - \lambda_{1,n-1} + \lambda_{0,n-1} & \lambda_{1,n-1} - \lambda_{0,n-1} - 1 \end{array} \right\}$$

$$G^\pm := \left\{ \begin{array}{cc} x = \infty & x = 0 \\ 1 - \lambda_{1,\nu} (1 \leq \nu < n) & \lambda_{0,\nu} (1 \leq \nu \leq n) \\ \pm x - \lambda_{1,n} & \end{array} \right\}$$

$$G^+ : u_{\infty, x - \lambda_{1,n}}(x) = \text{Ad}(x^{\lambda_{0,1}}) \tilde{I}_\infty^{\lambda_{1,1} - \lambda_{0,1}} \text{Ad}(x^{\lambda_{0,2} - \lambda_{1,1}}) \tilde{I}_\infty^{\lambda_{1,2} - \lambda_{0,2}} \dots$$

$$\dots \text{Ad}(x^{\lambda_{0,n-1} - \lambda_{1,n-2}}) \tilde{I}_\infty^{\lambda_{1,n-1} - \lambda_{0,n-1}} (x^{\lambda_{0,n} - \lambda_{1,n-1}} e^{-x})$$

$$\approx x^{\lambda_{1,n}} e^{-x} \quad (|x| \rightarrow \infty, \quad |\arg x| < \frac{3\pi}{2}),$$

$$u_{\infty, 1 - \lambda_{1,\nu}}^\pm(x) \approx (e^{\pm \pi i} x)^{\lambda_{1,\nu} - 1} \quad (|x| \rightarrow \infty, \quad |\arg x \pm \pi| < \frac{3\pi}{2}),$$

$$u_{0, \lambda_{0,\nu}}(x) \approx x^{\lambda_{0,\nu}} \quad (x \rightarrow 0) \quad (\Leftarrow \tilde{I}_\infty^\mu \mapsto I_0^\mu)$$

Connection formula: $I_n = \{1, \dots, n\}$, $J_{n-1, m} = \{1, \dots, n-1\} \setminus \{m\}$

$$\begin{aligned}
u_{\infty, 1-\lambda_{1, m}}^{\pm}(x) &= \sum_{k=1}^n c_{\Gamma}(\infty: 1-\lambda_{1, m} \rightsquigarrow 0: \lambda_{0, k}) e^{\pm \lambda_{0, k} \pi i} u_{0, \lambda_{0, k}}(x) \\
&\sim (e^{\pm \pi i} x)^{\lambda_{1, m}-1} \cdot {}_n F_{n-2}(\{\lambda_{0, \nu} - \lambda_{1, m} + 1\}_{\nu \in I_n}; \{\lambda_{1, \nu} - \lambda_{1, m} + 1\}_{\nu \in J_{n-1, m}}; -\frac{1}{e^{\pm \pi i} x}) \approx (e^{\pm \pi i} x)^{\lambda_{1, m}-1} \quad (|x| \rightarrow \infty, |\arg x \pm \pi| < \frac{3\pi}{2}) \\
u_{\infty, x-\lambda_{1, n}}(x) &= \sum_{k=1}^n c_{\Gamma}(\infty: x-\lambda_{1, n} \rightsquigarrow 0: \lambda_{0, k}) u_{0, \lambda_{0, k}}(x) \approx x^{\lambda_{1, n}} e^{-x} \quad (|x| \rightarrow \infty, |\arg x| < \frac{3\pi}{2}) \\
u_{0, \lambda_{0, k}}(x) &= \sum_{m=1}^n c_{\Gamma}(0: \lambda_{0, k} \rightsquigarrow \infty: 1-\lambda_{1, m}) e^{\pm (1-\lambda_{1, m}) \pi i} u_{\infty, 1-\lambda_{1, m}}^{\pm}(x) \\
&\quad + c_{\Gamma}(0: \lambda_{0, k} \rightsquigarrow \infty: x-\lambda_{1, n}) e^{\pm (\lambda_{1, n} - \lambda_{0, k}) \pi i} u_{\infty, x-\lambda_{1, n}}(x) \\
&= x^{\lambda_{0, k}} {}_{n-1} F_{n-1}(\{\lambda_{0, k} - \lambda_{1, \nu} + 1\}_{\nu \in J_{n-1}}; \{\lambda_{0, k} - \lambda_{0, \nu} + 1\}_{\nu \in I_{n, k}}; -x)
\end{aligned}$$

Stokes relations:

$$\begin{aligned}
&e^{\mp (\lambda_{1, m}-1) \pi i} \cdot u_{\infty, 1-\lambda_{1, m}}^{\pm}(e^{\mp 2\pi i} x) - e^{\pm (\lambda_{1, m}-1) \pi i} \cdot u_{\infty, 1-\lambda_{1, m}}^{\pm}(x) \\
&= \pm 2\pi i \cdot c_{\Gamma}(\infty: 1-\lambda_{1, m} \rightsquigarrow \infty: x-\lambda_{1, n}) \cdot u_{\infty, x-\lambda_{1, n}}(x), \\
&e^{-\lambda_{1, n} \pi i} \cdot u_{\infty, x-\lambda_{1, n}}(e^{\pi i} x) - e^{\lambda_{1, n} \pi i} \cdot u_{\infty, x-\lambda_{1, n}}(e^{-\pi i} x) \\
&= -2\pi i \sum_{m=1}^{n-1} c_{\Gamma}(\infty: x-\lambda_{1, n} \rightsquigarrow \infty: 1-\lambda_{1, m}) \cdot u_{\infty, 1-\lambda_{1, m}}^{\pm}(e^{\mp \pi i} x)
\end{aligned}$$

$$n=2 \Rightarrow {}_2F_0(\alpha, \beta; -\frac{1}{e^{\pm \pi i} x}) - {}_2F_0(\alpha, \beta; -\frac{1}{e^{\mp \pi i} x}) = \frac{2\pi i x^{\alpha+\beta-1} e^{-x}}{\Gamma(1-\alpha)\Gamma(1-\beta)} {}_2F_0(1-\alpha, 1-\beta; -\frac{1}{x})$$

Γ -factors:

$$c_{\Gamma}(\infty : 1 - \lambda_{1,m} \rightsquigarrow 0 : \lambda_{0,k}) = \frac{\prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,\nu} - \lambda_{0,k}) \prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{1,\nu} - \lambda_{1,m} + 1)}{\prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,\nu} - \lambda_{1,m} + 1) \prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{1,\nu} - \lambda_{0,k})}$$

$$c_{\Gamma}(\infty : x - \lambda_{1,n} \rightsquigarrow 0 : \lambda_{0,k}) = \frac{\prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,\nu} - \lambda_{0,k})}{\prod_{\nu \in J_{n-1}} \Gamma(\lambda_{1,\nu} - \lambda_{0,k})}$$

$$c_{\Gamma}(0 : \lambda_{0,k} \rightsquigarrow \infty : 1 - \lambda_{1,m}) = \frac{\prod_{\nu \in I_{n-1,m}} \Gamma(\lambda_{1,m} - \lambda_{1,\nu}) \prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,k} - \lambda_{0,\nu} + 1)}{\prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{0,k} - \lambda_{1,\nu} + 1) \prod_{\nu \in I_{n,k}} \Gamma(\lambda_{1,m} - \lambda_{0,\nu})}$$

$$c_{\Gamma}(0 : \lambda_{0,k} \rightsquigarrow \infty : x - \lambda_{1,n}) = \frac{\prod_{\nu \in I_{n,k}} \Gamma(\lambda_{0,n} - \lambda_{0,\nu} + 1)}{\prod_{\nu \in J_{n-1}} \Gamma(\lambda_{0,k} - \lambda_{1,\nu} + 1)}$$

$$c_{\Gamma}(\infty : x - \lambda_{1,n} \rightsquigarrow \infty : 1 - \lambda_{1,m}) = \frac{\prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{1,\nu} - \lambda_{1,m} + 1)}{\prod_{\nu \in I_n} \Gamma(\lambda_{0,\nu} - \lambda_{1,m} + 1)}$$

$$c_{\Gamma}(\infty : 1 - \lambda_{1,m} \rightsquigarrow \infty : x - \lambda_{1,n}) = \frac{\prod_{\nu \in J_{n-1,m}} \Gamma(\lambda_{1,m} - \lambda_{1,\nu})}{\prod_{\nu \in I_n} \Gamma(\lambda_{1,m} - \lambda_{0,\nu})}$$

Irreducible $\Leftrightarrow \lambda_{0,k} - \lambda_{1,m} \notin \mathbb{Z} \quad (1 \leq k \leq n, \ 1 \leq m < n)$

$$\left\{ \begin{array}{cc} x = \infty & 0 \\ 1 - \lambda_{1,1} & \lambda_{0,1} \\ x - \lambda_{1,2} & \lambda_{0,2} \end{array} \right\} \xrightarrow{\times x^{-\lambda_{0,1}} e^x} \overset{\text{Kummer}}{\left\{ \begin{array}{cc} x = \infty & 0 \\ -x + \gamma - \alpha & 0 \\ \alpha & 1 - \gamma \end{array} \right\}} \xrightarrow{\times x^{\frac{\gamma}{2}} e^{-\frac{x}{2}}} \overset{\text{Whittaker}}{\left\{ \begin{array}{cc} x = \infty & 0 \\ -\frac{x}{2} - k & \frac{1}{2} - m \\ \frac{x}{2} + k & \frac{1}{2} + m \end{array} \right\}}$$

§ Middle convolution of versal unfolding (Pfaffian form)

$$\frac{du}{dx} = \left(\sum_{j=1}^p \sum_{i=1}^{r_j} \frac{A_{j,i}}{(x - a_j)^i} - \sum_{i=2}^{r_0} A_{0,i} x^{i-2} \right) u \quad (A_{j,i} \in M_N(\mathbb{C})) \quad (*)$$

Definition. Versal unfolding of $(*)$:

$$\begin{aligned} \frac{d\tilde{u}}{dx} = & \left(\sum_{j=1}^p \sum_{i=1}^{r_j} \frac{\tilde{A}_{j,i}(a)}{(x - a_{j,1})(x - a_{j,2}) \cdots (x - a_{j,i})} \right. \\ & \left. - \sum_{i=2}^{r_0} \frac{\tilde{A}_{0,i}(a) x^{i-2}}{(1 - a_{0,2}x)(1 - a_{0,3}x) \cdots (1 - a_{0,i}x)} \right) \tilde{u} \end{aligned} \quad (**)$$

$$(*) \Leftarrow [a_{j,i} \mapsto a_j \quad (a_0 = 0)]$$

index of rigidity is stable under the parameter (keeps rigidity)

$$\text{index} : 2N^2 - \sum_{j \geq 0} (N^2 - \dim Z_{M_N(\mathbb{C})} C_j) \quad \text{for} \quad \frac{du}{dx} = \sum_{j \geq 1} \frac{C_j}{x - a_j} u, \quad C_0 := - \sum_{j \geq 1} C_j$$

Assumption. $(*)$ is transformed into Hukuhara-Turrittin-Levelt's normal form by $GL(N, \mathbb{C}[[x]]_{a_j})$ ($\Rightarrow$ without ramified irregular singularities)

Conjecture. $(*) : \text{irreducible} \Rightarrow \exists! \text{ versal unfolding}$

(irreducible $\Leftrightarrow$ no non-trivial subspace V of $\mathbb{C}^N$ with $A_{j,i}V \subset V$)

Theorem [Oc, Ov]. $\exists$ middle convolution of versal unfolding

$\Rightarrow$ **Conjecture** is stable under middle convolutions

Corollary. $(*) : \text{rigid} \Rightarrow \text{Conjecture is true}$ (True in general by Hiroe)

Example:

$$\begin{aligned} \frac{du}{dx} &= \frac{A_1 u}{x - a_1} + \frac{A_2 u}{(x - a_1)(x - a_2)} + \frac{A_3 u}{(x - a_1)(x - a_2)(x - a_3)} \\ \xrightarrow{\text{MC}_\mu} \frac{d\hat{u}}{dx} &= \frac{\begin{pmatrix} A_1 + \mu & A_2 & A_3 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}}{x - a_1} \hat{u} + \frac{\begin{pmatrix} 0 & 0 & 0 \\ A_1 + \mu & A_2 + (a_2 - a_1)\mu & A_3 \\ 0 & \mu & (a_3 - a_1)\mu \end{pmatrix}}{(x - a_1)(x - a_2)} \hat{u} \\ &\quad + \frac{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_1 + \mu & A_2 + (a_3 - a_1)\mu & A_3 + (a_3 - a_1)(a_3 - a_2)\mu \end{pmatrix}}{(x - a_1)(x - a_2)(x - a_3)} \hat{u}, \end{aligned}$$

$$\mathcal{K} := \text{Ker } K, \quad K := \begin{pmatrix} A_1 & A_2 & A_3 \\ A_2 & A_3 + (a_2 - a_1)A_2 & (a_3 - a_1)A_3 \\ A_3 & (a_3 - a_1)A_3 & (a_3 - a_1)(a_3 - a_2)A_3 \end{pmatrix},$$

$$\mathcal{L} := \{^t(v, 0, 0) \in \mathbb{C}^{3N} \mid (A_1 + \mu)v = 0\}$$

$$\Rightarrow \text{mc}_\mu := \text{MC}_\mu \big|_{\mathbb{C}^{3N}/(\mathcal{K} + \mathcal{L})}$$

Theorem [Oc].

$$q_i(x) := \frac{1}{\prod_{\nu=1}^i (x - a_\nu)}, \quad \mathbf{q} := \begin{pmatrix} q_1 I_N \\ \vdots \\ q_r I_N \end{pmatrix} \in M(rN, N),$$

$$\frac{du}{dx} = \sum_{i=1}^r \frac{A_i}{(x - a_1) \cdots (x - a_i)} u = \sum_{i=1}^r A_i q_i u, \quad A_i \in M(N, N; \mathbb{C}(\mathbf{a}))$$

$$\frac{d\hat{u}}{dx} = \mathbf{q}(A_1, \dots, A_r) \hat{u} + \mu \mathbf{D}_r(\mathbf{a}) \hat{u} = \sum_{i=1}^r \hat{A}_i q_i \hat{u}, \quad \hat{A}_i \in M(rN, rN; \mathbb{C}(\mathbf{a}) + \mu \mathbb{C}(\mathbf{a}))$$

$$\mathbf{D}_r(\mathbf{a}) = \sum_{k=1}^r \mathbf{D}_{r,k}(\mathbf{a}) q_k(x), \quad \mathbf{D}_{r,k}(\mathbf{a}) = \left(\mathbf{D}_{r,k,i,j}(\mathbf{a}) I_N \right)_{\substack{1 \leq i \leq r, \\ 1 \leq j \leq r}},$$

$$\mathbf{D}_{r,k,i,j}(\mathbf{a}) = \begin{cases} 0 & (L < 0 \text{ or } i < j \text{ or } i < k), \\ 1 & (L = 0), \\ \sum_{0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_L \leq i-k} \prod_{p=1}^L (a_{i-\nu_p} - a_{\nu_p+p}) & (L > 0, \ i \geq j, \ i \geq k) \end{cases}$$

$$L := k + j - i - 1.$$

$$\left(q_i(x), \hat{u} = \begin{pmatrix} I_c^\mu q_1(x)u \\ \vdots \\ I_c^\mu q_r(x)u \end{pmatrix}, D_{r,k,i,j}(\mathbf{a}) \right) \leftrightarrow [\text{DR}] : \left(\frac{1}{x-a_i}, \hat{u} = \begin{pmatrix} I_c^\mu \frac{u}{x-x_1} \\ \vdots \\ I_c^\mu \frac{u}{x-x_r} \end{pmatrix}, \delta_{k,i} \delta_{i,j} \right)$$

§ Versal unfolding of systems with several variables

$$\frac{du}{dx} = \sum_{j=1}^p \frac{A_j}{x - a_j} u \quad (A_j \in M_N(\mathbb{C})) \quad (\star)$$

$(\star) : \text{rigid} \Rightarrow (\star) \rightarrow \text{extended to KZ (Knizhnik-Zamolodchikov) equation:}$

$$\frac{\partial u}{\partial x_i} = \sum_{\substack{\nu \neq i \\ 0 \leq \nu \leq p}} \frac{A_{i,\nu}}{x_i - x_\nu} u \quad (x_0 = x, x_j = a_j, A_{0,\nu} = A_\nu, i = 0, \dots, p) \quad (\star\star)$$

$$A_{i,i} = 0, A_{i,j} = A_{j,i}, [A_{i,j}, A_{k,\ell}] = [A_{i,j}, A_{i,k} + A_{j,k}] = 0 \quad (\#\{i, j, k, \ell\} = 4)$$

$\Leftarrow$ middle convolution mc_μ of $(\star)$ is extended to $(\star\star)$ (Haraoka [Ha])

$$\frac{du}{dx} = \left(\sum_{j=1}^p \sum_{i=1}^{r_j} \frac{A_{j,i}}{(x - a_j)^i} - \sum_{i=2}^{r_0} A_{0,i} x^{i-2} \right) u \quad (x_0 = x, x_j = a_j) \quad (*)$$

Rigid system $(*)$ with unramified irregular singularities

$\Rightarrow$ versal unfolding $(**)$ is rigid Fuchsian system if singular points $a_{j,\nu}$ are generic

Theorem [Oc]. KZ eq. with generic singular points of versal unfolding (Fuchsian)

$\Rightarrow$ KZ eq. is holomorphically extended to confluent points $(a_{j,\nu} \rightarrow x_j = a_j)$

$\Rightarrow$ rigid system $(*)$ is extended to confluent KZ Eq.

middle convolution of $(\star\star)_j$: $\mathrm{mc}_\mu(A_{i,j}) := \tilde{A}_{i,j} \mid_{\mathbb{C}^{pN}/(\mathcal{K}+\mathcal{L})}$

$$\tilde{A}_{0,j=j} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ A_{0,1} & \cdots & A_{0,j} + \mu & \cdots & A_{0,p} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \in M(pN, \mathbb{C}) \quad (1 \leq j \leq p)$$

$$\mathcal{K} := \text{Ker } K, \quad K := \text{Diag}(A_{0,1}, \dots, A_{0,p}),$$

$$\mathcal{L} = \text{Ker } L, \quad L := -(\tilde{A}_{0,1} + \cdots + \tilde{A}_{0,p})$$

by Dettweiler-Reiter [DR],

$$\tilde{A}_{i,j} = \begin{pmatrix} & & i & & j \\ A_{i,j} & & & & \\ & \ddots & & & \\ i & & A_{i,j} + A_{0,j} & & -A_{0,j} \\ & & & \ddots & \\ j & & -A_{0,i} & & A_{i,j} + A_{0,i} \\ & & & & \ddots \\ & & & & & A_{i,j} \end{pmatrix} \in M(pN, \mathbb{C})$$

$(1 \leq i < j \leq p)$

by Haraoka [Ha]

Example (versal unfolding/confluence of KZ Eq. (cf. [Oc]))

$$\begin{cases} \frac{\partial u}{\partial x_0} = \left(\frac{A_{01}}{x_0 - x_1} + \frac{A_{02}}{(x_0 - x_1)(x_0 - x_1 - \mathbf{e}_2)} + \frac{A_{03}}{x_0 - x_3} \right) u, \\ \frac{\partial u}{\partial x_1} = \left(\frac{A_{10}}{x_1 - x_0} - \frac{A_{20}}{(x_1 - x_0)(x_1 - x_0 + \mathbf{e}_2)} \right. \\ \quad \left. + \frac{A_{13}}{x_1 - x_3} - \frac{A_{23}}{(x_1 - x_3)(x_1 - x_3 + \mathbf{e}_2)} \right) u, \\ \frac{\partial u}{\partial x_3} = \left(\frac{A_{30}}{x_3 - x_0} + \frac{A_{31}}{x_3 - x_1} + \frac{A_{32}}{(x_3 - x_1)(x_3 - x_1 - \mathbf{e}_2)} \right) u, \end{cases}$$

$$A_{i,j} = A_{j,i} \quad (0 \leq i < j \leq 3), \quad (x_0, x_1, x_1 + \mathbf{e}_2, x_3) = (x, a_1, a_2, a_3)$$

$\xrightarrow{\text{MC}_\mu}$

$$\begin{aligned} \hat{A}_{01} &= \begin{pmatrix} A_{01} + \mu & A_{02} & A_{03} \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{A}_{02} = \begin{pmatrix} 0 & 0 & 0 \\ A_{01} + \mu & A_{02} + \mathbf{e}_2 \mu & A_{03} \\ 0 & 0 & 0 \end{pmatrix}, \\ \hat{A}_{03} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{01} & A_{02} & A_{03} + \mu \end{pmatrix}, \quad \hat{A}_{13} = \begin{pmatrix} A_{13} + A_{03} & 0 & -A_{03} \\ 0 & A_{13} + A_{03} & 0 \\ -A_{01} & -A_{02} & A_{13} + A_{01} \end{pmatrix}, \\ \hat{A}_{23} &= \begin{pmatrix} A_{23} & 0 & 0 \\ A_{03} & A_{23} + \mathbf{e}_2 A_{03} & -A_{03} \\ -A_{02} & -\mathbf{e}_2 A_{02} & A_{23} + A_{02} \end{pmatrix} \end{aligned}$$

$$\left\{ \begin{array}{l} A_{i,j} \in \mathbb{C} \Rightarrow \text{Appell's } F_1 \\ 21, 21, 21, 21 \rightarrow 21|21, 21, 21 \end{array} \right.$$

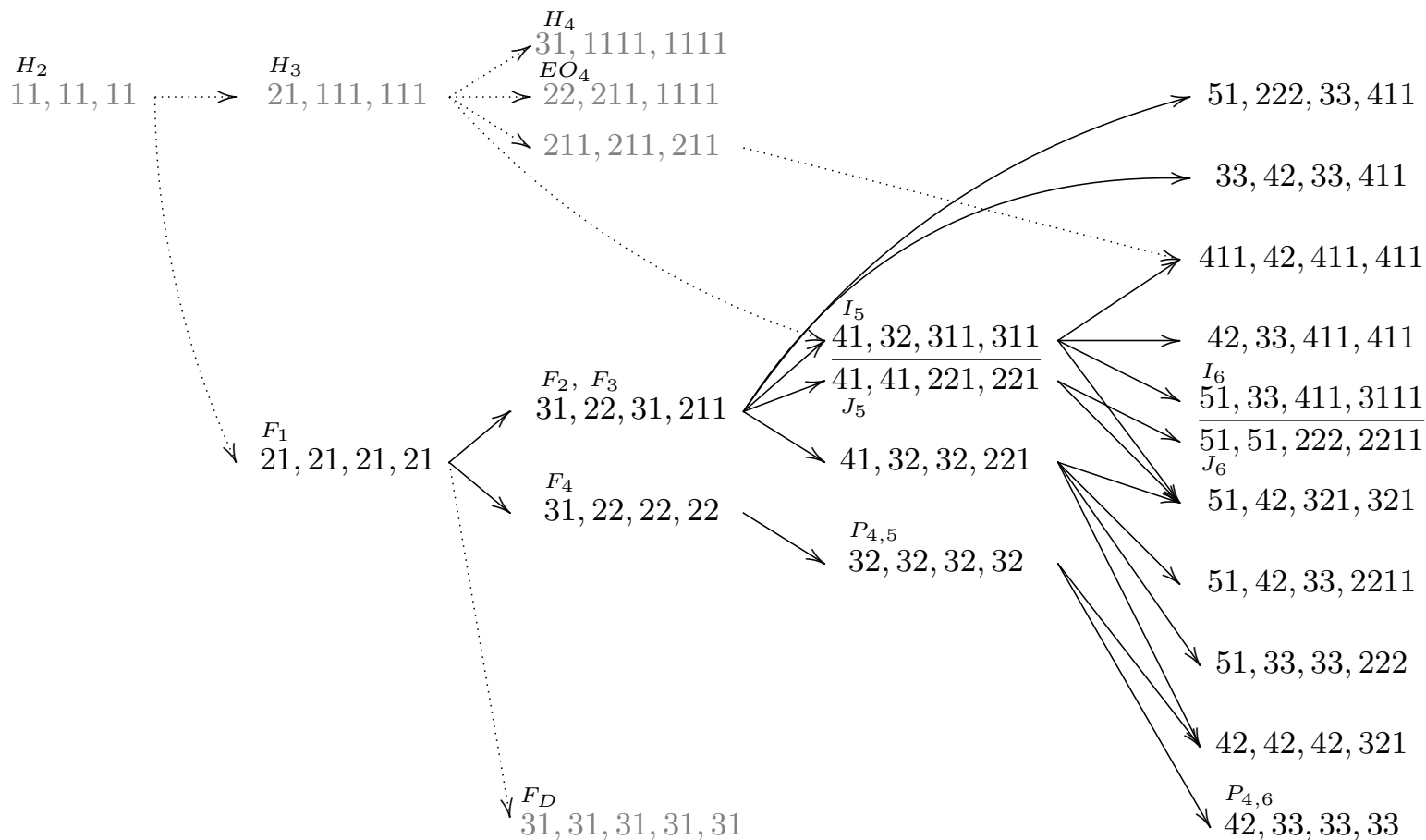
§ Hierarchy of rigid quartets (cf. [Of, Ot])

$\{\text{Rigid Fuchsian ODE with } p+1 \text{ singular points } \{0, 1, \infty, y_1, \dots, y_{p-2}\}\}$

$\hookrightarrow \{\text{HG Eq. with } (p-1)\text{-variables } (x, y_1, \dots, y_{p-2})\} \quad (p=3)$

$\hookrightarrow \{\text{KZ Eq. on } \{(x_0, \dots, x_{p+1}) \mid x_i \in \mathbb{P}^1\}\} \quad (x_0, x_1, \dots, x_{p+1}) = (x, y_1, \dots, y_{p-2}, 0, 1, \infty)$

$(m_{0,1} \cdots m_{0,n_0}, \dots, m_{p,1} \cdots m_{p,n_p}) : \text{spectre type } (p+1 \text{ tuples of partitions of ord } \mathbf{m})$



$$\# \text{parameters} = \sum (\# \text{ blocks at sing. point} - 1)$$

§ HG with two variables corresponding to KZ Eq's (cf. [Ot])

$$A_{i,p+1} := - \sum_{\nu=0}^p A_{i,\nu} \quad (\text{at } \infty), \quad A_{i_1, \dots, i_k} := \sum_{1 \leq \mu < \nu \leq k} A_{i_\mu, i_\nu}$$

$$\begin{aligned} [A_I, A_J] &= 0 \quad (I \subset J \text{ or } I \cap J = \emptyset) \\ A_{0, \dots, p+1} &= A_K - A_{\{0, \dots, p+1\} \setminus K} \quad (K \subset \{0, \dots, p\}) \\ &= \kappa \quad (: \text{ scalar} \Leftarrow \text{Irred.}) \end{aligned}$$

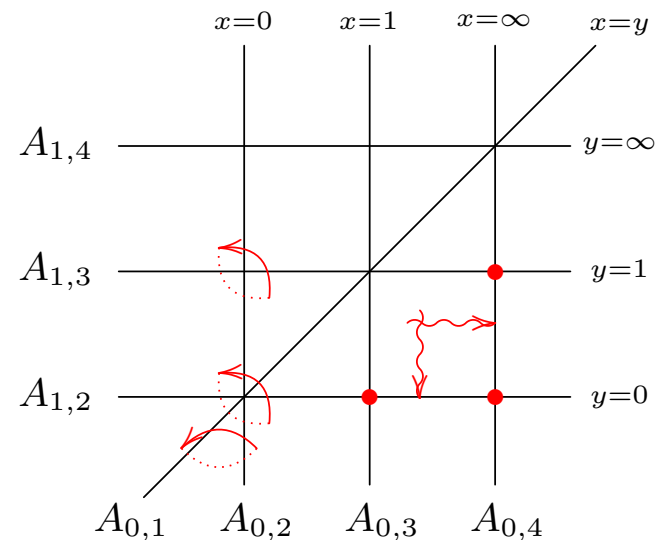
$$p = 3 : (x_0, x_1, x_2, x_3, x_4) \leftrightarrow (x, y, 0, 1, \infty)$$

$$[A_{0,2} + A_{1,3}] \Leftarrow [A_{0,2} : A_{1,3}] \text{ (conj. class)}$$

$$[A_{0,1} + A_{0,2} + A_{1,2}] = [A_{0,1,2}] = [\kappa + A_{3,4}]$$

$$[A_{0,1} + A_{0,2}] = [A_{0,1,2} - A_{1,2}] = [\kappa + A_{3,4} - A_{1,2}]$$

$$[A_{0,2}, A_{1,3}] = [A_{3,4}, A_{1,2}] = 0$$



In general, the Fuchsian ODE satisfied by x_i ($i \geq 1$) is **not rigid** (rigid $\rightsquigarrow$ non-rigid)
 $\exists$ **Series expansion** of (x_i, x_j) around $\{x_i = x_{i'}, x_j = x_{j'}\}$ ($\#\{i, i', j, j'\} = 4$, cf. [KO])

#Rigid spectral types corresponding to the same KZ Eq. with $p = 3$ and order N

$N \rightarrow$	3	4	5	6	7	8	9	10	11	12	13	14
rigid ODE	1	2	4	11	16	35	58	109	156	299	402	658
same KZ	0	0	1	1	2	4	9	12	18	31	47	68
KZ Eq.	1	2	3	10	14	31	49	97	138	268	355	590

§ Example (Appell's F_4) (cf. [Oh, Ot])

31, 22, 22, 22 : rank = 4, 4 parameters,

$$(1^8 \cdot 2^1) \rightarrow F_1 : 21, 12, 12, 12$$

$$= 10, 01, 01, 01 \oplus 21, 21, 21, 21 \quad (8)$$

$$= 2(20, 11, 11, 11) \oplus (-1)1, 00, 00, 00 \quad (1)$$

$$31, 22, 22|22 \quad 31, 22|22|22 \text{ (confluence)}$$

$$a + b + c + d + e = 0 (= \frac{3}{2}) \text{ (Fuchs condition)}$$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		211	22	211	22	-4
t_0	211		22	211	22	-4
t_y	22	22		22	31	2
t_1	211	211	22		22	-4
t_x	22	22	31	22		2

$$\left\{ \begin{array}{ccccccccc} A_{01} & A_{02} & A_{03} & A_{04} & A_{12} & A_{13} & A_{14} & A_{23} & A_{24} & A_{34} \\ x = y & x = 0 & x = 1 & x = \infty & y = 0 & y = 1 & y = \infty & t_0 = t_1 & t_0 = \infty & t_1 = \infty \\ [0]_3 & [0]_2 & [0]_2 & [d]_2 & [0]_2 & [0]_2 & [d]_2 & [-b-c]_2 & [a+c]_2 & [a+b]_2 \\ 2a & [b]_2 & [c]_2 & [e]_2 & [b]_2 & [c]_2 & [e]_2 & 2d & 2c & 2b \\ & & & & & & & 2e & 0 & 0 \end{array} \right\}$$

$$(x_0, x_1, x_2, x_3, x_4) = (t_x, t_y, t_0, t_1, t_\infty) = (x, y, 0, 1, \infty)$$

$$H_1 + \textcolor{red}{DF} : [01] \quad (\text{Dotsenko-Fateev's Eq. } 1^3, 1^3, 1^3)$$

$$2H_2 : [02], [03], [04], [12], [13], [14]$$

$$2H_1 + H_2 : [23], [24], [34]$$

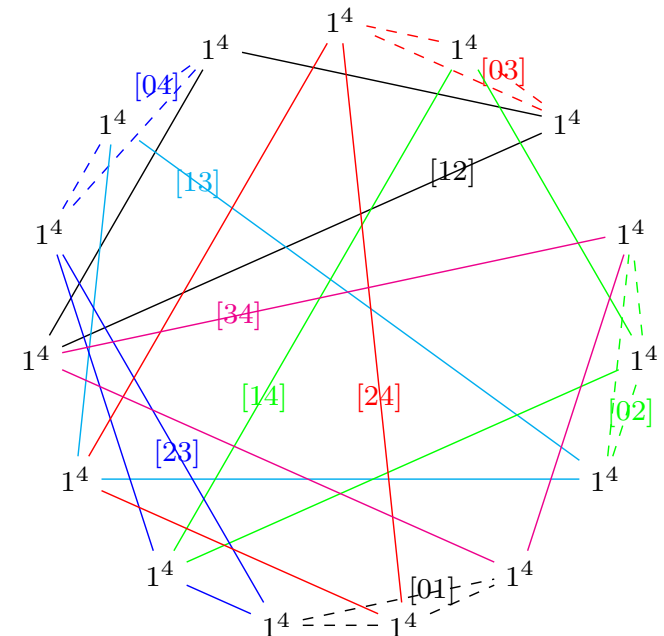
$$u(x, y) = F_4(\alpha, \beta; \gamma, \gamma'; xy, (1-x)(1-y)) \quad [\text{Ka}],$$

$$b = 1 - \gamma, \quad c = -\gamma', \quad d = \alpha, \quad e = \beta$$

$$\{d, e, b+d, b+e, c+d, c+e, b+c+d, b+c+e\}$$

$$(8) \quad \cap \mathbb{Z} = \emptyset \quad (\Leftrightarrow \text{irred.})$$

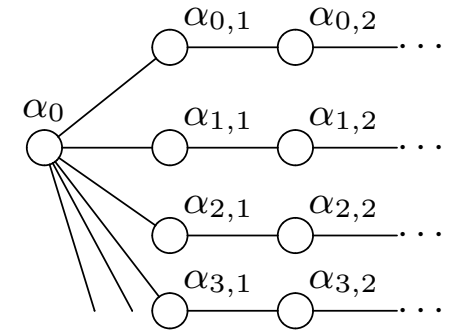
$$(1) \textcolor{teal}{2a} \in \mathbb{Z} : u(x, y) = F(\alpha, \beta, \gamma; x) \cdot F(\alpha, \beta, \gamma; y)$$



§ A Kac-Moody root system (Fuchsian case) (cf. [CB, Of])

$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi), \quad (\alpha_0|\alpha_{j,\nu}) = -\delta_{\nu,1},$$

$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \neq j \text{ or } |\mu - \nu| > 1) \\ -1 & (i = j \text{ and } |\mu - \nu| = 1) \end{cases}$$



$W := \langle s_\alpha : x \mapsto x - (\alpha|x)\alpha \mid \alpha \in \Pi \rangle$: Weyl group

Δ_+^{re} : positive real roots $\Delta_+ = \Delta_+^{re} \cup \Delta_+^{im}$

Δ_+^{im} : positive imaginary roots

$$\begin{array}{ccc} \{Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\}, (\text{FC})\} & \xrightarrow{\pi} & \{(\Lambda(\lambda), \alpha_{\mathbf{m}}) \in \mathfrak{h}^\vee \times \overline{\Delta}_+, (\text{FC})\} \\ \downarrow u \mapsto \begin{cases} \partial^{-\mu} u \\ (x - c_j)^{\lambda_j} u \end{cases} & \circlearrowright & \downarrow \begin{cases} s_\alpha \ (\alpha \in \Pi) : \text{reflections} \\ +\lambda_j \Lambda_{0,j}^0 \end{cases} \\ \{Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\}, (\text{FC})\} & \xrightarrow{\pi} & \{(\Lambda(\lambda), \alpha_{\mathbf{m}}) \in \mathfrak{h}^\vee \times \overline{\Delta}_+, (\text{FC})\} \end{array} \quad \begin{array}{l} \{\lambda_{\mathbf{m}}\} = \left\{ \begin{array}{l} x = c_j \\ [\lambda_{j,\nu}]_{(m_{j,\nu})} \end{array} \right\} \\ \{(m_{j,\nu})\} \xrightarrow{\sim} \sum_{\alpha \in \Pi} \mathbb{Z}\alpha \end{array}$$

$\alpha_{\mathbf{m}} := (\text{ord } \mathbf{m})\alpha_0 + \sum_{j \geq 0, k \geq 1} \sum_{\nu > k} m_{j,\nu} \alpha_{j,k}$ (Crawley-Boevey [CB])

$\overline{\Delta}_+ := \{\alpha \mid \alpha \in \Delta_+, \text{ supp } \alpha \ni \alpha_0\} \cup \{k\alpha \mid \alpha \in \Delta_+^{im}, (\alpha|\alpha) < 0, k = 2, 3, \dots\}$

$\Lambda(\lambda) = -\Lambda_0 - \sum_{j \geq 0} \sum_{\nu \geq 1} (\sum_{1 \leq i \leq \nu} \lambda_{j,i}) \alpha_{j,\nu} \in \mathfrak{h}^\vee := \prod_{\alpha \in \Pi} \mathbb{C}\alpha / \mathbb{C}\Lambda^0$

$\Lambda_0 := \frac{1}{2}\alpha_0 + \frac{1}{2} \sum_{j \geq 0} \sum_{\nu \geq 1} (1 - \nu) \alpha_{j,\nu}$

$\Lambda_{j,\nu} := \sum_{i > \nu} (\nu - i) \alpha_{j,i} \quad (j = 0, \dots, p, \nu = 0, 1, 2, \dots)$

$\Lambda^0 := 2\Lambda_0 - 2\Lambda_{0,0}, \quad \Lambda_{j,k}^0 := \Lambda_{j,0} - \Lambda_{k,0}$

$\text{FC}: |\{\lambda_{\mathbf{m}}\}| = (\Lambda(\lambda) + \frac{1}{2}\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}) = 0, \quad \text{idx } \mathbf{m} = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}})$

Theorem [CB, Of]. $\{\lambda_{\mathbf{m}}\}$ with (FC) is irreducibly realizable $\Leftrightarrow \alpha_{\mathbf{m}} \in \overline{\Delta}_+$

Suppose $\mathbf{m}$ is a **rigid Fuchsian** spectral type

$\exists! w_{\mathbf{m}} \in W$ such that $w_{\mathbf{m}} \alpha_{\mathbf{m}} = \alpha_0$ with the minimal length

$$\Delta(\mathbf{m}) := \Delta_+^{re} \cap w_{\mathbf{m}}^{-1} \Delta_-^{re}$$

$$\left\{ \begin{array}{l} x = c_j \\ [\lambda_{j,\nu}]_{(m_{j,\nu})} \end{array} \right\}$$

Theorem [CB, Of]. $Pu = 0$ is **irred.** $\Leftrightarrow \sum \lambda_{j,\nu} m'_{j,\nu} \notin \mathbb{Z} \quad (\forall \alpha_{\mathbf{m}'} \in \Delta(\mathbf{m}))$

Theorem [Of]. Suppose $m_{0,n_0} = m_{1,n_1} = 1, c_0 = 0, c_1 = 1$.

$$c(0 : \lambda_{0,n_0} \rightsquigarrow 1 : \lambda_{1,n_1}) = \frac{\prod_{\nu=1}^{n_0-1} \Gamma(\lambda_{0,n_0} - \lambda_{0,\nu} + 1) \cdot \prod_{\nu=1}^{n_1-1} \Gamma(\lambda_{1,\nu} - \lambda_{1,n_1})}{\prod_{\substack{\alpha_{\mathbf{m}'} \in \Delta(\mathbf{m}) \\ m'_{0,n_0}=1 \\ m'_{1,n_1}=0}} \Gamma(|\{\lambda_{\mathbf{m}'}\}|) \cdot \prod_{\substack{\alpha_{\mathbf{m}'} \in \Delta(\mathbf{m}) \\ m'_{0,n_0}=0 \\ m'_{1,n_1}=1}} \Gamma(1 - |\{\lambda_{\mathbf{m}'}\}|) \cdot \prod_{j=2}^p (1 - \frac{1}{c_j})^{-L_j}$$

Theorem [Of]. Any contiguous relation of $Pu = 0$ is constructed ($\exists$ algorithm).

Basic spectral types with $\text{idx} = 0$ ($(\mathfrak{h}), d_1(\mathbf{m}) \leq 0, (\alpha_{\mathbf{m}} | \alpha_{\mathbf{m}}) = 0$)

$$\tilde{D}_4 : 11, 11, 11, 11 \quad 11|11, 11, 11 \quad 11|11, 11|11 \quad 11|11|11, 11 \quad 11|11|11|11$$

$$\tilde{E}_6 : 111, 111, 111 \quad 111|111, 111 \quad 111|111|111$$

$$\tilde{E}_7 : 1111, 1111, 22 \quad 1111|1111, 22 \quad 1111, 1111|22 \quad 1111|1111|22$$

$$\tilde{E}_8 : 111111, 222, 33 \quad 111111|222, 33 \quad 111111|33, 222$$

Example with $\text{idx} = 6 - 2p$: $11|11|\cdots|11$ ($(p+1)$ copies of 11) Gauss, Heun, ... [Ov]

$$\tilde{P} = \prod_{j=1}^p (1 - t_j x) \partial^2 + \left(\sum_{j=1}^p \lambda_j x^{j-1} \right) \partial + \mu \left(\lambda_p - (-1)^p (\mu + 1) \prod_{j=1}^{p-1} t_j \right) x^{p-2} + \sum_{j=0}^{p-3} r_j x^j$$

References

- [Ar] D. Arinkin, *Rigid irregular connections on $\mathbb{P}^1$* , Compos. Math., **146** (2010), 1323–1338
- [CB] W. Crawley-Boevey, *On matrices in prescribed conjugacy classes with no common invariant subspaces and sum zero*, Duke Math. J. **118** (2003), 339–352
- [DR] M. Dettweiler and S. Reiter, *An algorithm of Katz and its applications to the inverse Galois problems*, J. Symbolic Comput. **30** (2000), 761–798
- [Ha] Y. Haraoka, *Middle convolution for completely integrable systems with logarithmic singularities along hyperplane arrangements*, Adv. Studies in Pure Math. **62** (2012), 109–136
- [Hi1] K. Hiroe, *Linear differential equations on $\mathbb{P}^1$ and root systems*, J. Algebra **382** (2013), 1–38
- [Hi2] K. Hiroe, *Linear differential equations on the Riemann sphere and representations of quivers*, Duke Mathematical Journal **166** (2017), 855–935.
- [HO] K. Hiroe and T. Oshima, *A classification of roots of symmetric Kac-Moody root systems and its application*, Symmetries, Integrable Systems and Representations, Springer Proceedings in Mathematics and Statistics **40** (2012), 195–241
- [Hu] M. Hukuhara, *Sur les points singuliers des équations différentielles linéaires, III*, Mem. Fac. Sci. Kyûsyû Imp. Univ. Ser. A **2** (1941), 125–137
- [KO] M. Kashiwara and T. Oshima, *Systems of differential equations with regular singularities and their boundary value problems*, Ann. of Math. **106** (1977), 145—200
- [Ka] M. Kato, *Connection formulas for Appell’s system F_4 and some applications*, Funkcial. Ekvac. **38** (1995), 243–266.

[Kz] N. M. Katz, *Rigid Local Systems*, Annals of Mathematics Studies, vol. 139, Princeton University Press, 1995

[Of] T. Oshima, Fractional calculus of Weyl algebra and Fuchsian differential equations, MSJ Memoirs **58**, Mathematical Society of Japan, 2012

[Oh] T. Oshima, *Reducibility of hypergeometric equations*, Analytic, Algebraic and Geometric Aspects of Differential Equations, Trends in Mathematics, Springer, 2017, 429–453

[Ot] T. Oshima, *Transformations of KZ type equations*, RIMS Kôkyûroku Bessatsu **B61**(2017), 141–161

[Os] T. Oshima, *Semilocal monodromy of rigid local systems*, Formal and Analytic Solutions of Diff. Equations, Springer Proceedings in Mathematics and Statistics 256 (2018) 189–199

[Oc] T. Oshima, *Confluence and versal unfolding of Pfaffian systems*, Josai Mathematical Monographs **19** (2019), 1–32

[Ov] T. Oshima, *Versal unfolding of irregular singularities of a linear differential equation on the Riemann sphere*, Publ. RIMS Kyoto Univ. **57** (2021), 893–920

[Or] T. Oshima, *Riemann-Liouville transform and linear differential equations on the Riemann sphere*, preprint, 34pp

[Om] T. Oshima, A library `os_muldif.rr` of Computer Algebra Risa/Asir
<https://www.ms.u-tokyo.ac.jp/~oshima/index.html>

Thank you for your attention!