Kac-Moody ルート系 と 線型微分方程式

Toshio OSHIMA（大島利雄）
The University of Tokyo

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Ikaho (伊香保)

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Contents

Transformation of linear Pfaffian system

$$\frac{\partial u}{\partial x_i} = \sum_{\substack{1\le\nu\le m\\ \nu\ne i}} \frac{A_{i\nu}}{x_i - x_\nu} u \quad (1 \le i \le n) \qquad (A_{ij} = A_{ji},\ A_{ii} = 0,\ n \le m)$$

Transformations

(1) addition (gauge transformation) : $A_j \mapsto A_j + \lambda_j \quad (u \mapsto f^\lambda u)$

(2) middle convolution (fractional derivation): $\mathrm{mc}_{v,\mu} = \partial_v^{-\mu}$

(3) coordinate transformation (incl. restriction to a generic hypersurface)

(4) boundary value map and and extension to a system with more variables

(5) confluence and unfolding

(6) Fourier-Laplace transform (Harnad dual) $\quad (x, \frac{d}{dx}) \mapsto (-\frac{d}{dx}, x)$

(7) shift operator (difference equations for parameters)

(1)+(2)+(3)+(5)+(6), (1)+(2)+(4)+(5), (1)+(2)+(4)+(5)+(7)

Fuchsian ordinary differential equation

$$\mathcal{M} : \frac{du}{dx} = \sum_{1\le j<p} \frac{A_j}{x-a_j} u \quad (A_j \in \mathrm{Mat}(n,\mathbb{C}))$$

$A_p = A_\infty := -(A_1 + \cdots + A_{p-1})$: residue matrix at ∞

A_j : eigenvalue $\lambda_{j,\nu}$ with multiplicity $m_{j,\nu}$ $(\nu = 1,\ldots,n_j)$

$[A_j] := \{[\lambda_{j,\nu}]_{m_{j,\mu}} \mid \nu = 1,\ldots,n_j\}$: spectrum of A_j

$$n = m_{j,1} + \cdots + m_{j,n_j}$$: spectral type of A_j

$$\mathrm{rank} \prod_{\nu=1}^{k} (A_j - \lambda_{j,\nu}) = N - m_{j,1} - \cdots - m_{j,k} \quad (k = 1,\ldots,n_j)$$

when $m_{j,1} \ge m_{j,2} \ge \cdots \ge m_{j,n_j}$

$$\begin{Bmatrix} x = a_j \ (j=1,\ldots,p-1) & x = \infty \\ [\lambda_{j,1}]_{m_{j,1}} & [\lambda_{p,1}]_{m_{p,1}} \\ \vdots & \vdots \\ [\lambda_{j,n_j}]_{m_{j,n_j}} & [\lambda_{p,n_p}]_{m_{p,n_p}} \end{Bmatrix}$$: (generalized) Riemann scheme

$\exists$sol. $u \sim C_{j,\nu,i}(x-a_j)^{\lambda_{j,\nu}} \quad (x \to a_j, \quad i = 1,\ldots,m_{j,\nu}, \ \nu = 1,\ldots,n_j)$

$\sim C_{p,\nu,i}(\frac{1}{x})^{\lambda_{p,\nu}} \quad (x \to \infty, \quad i = 1,\ldots,m_{p,\nu}, \ \nu = 1,\ldots,n_p)$

Spectral type

Spectral type $\mathbf{m}$ of $\mathcal{M}$:

$$\mathbf{m} := \big(m_{j,\nu}\big)_{\substack{1\le\nu\le n_j\\ 1\le j\le p}}$$
$$= m_{1,1}\cdots m_{1,n_1},\ \ldots,\ m_{p-1,1}\cdots m_{p-1,n_{p-1}},\ m_{p,1}\cdots m_{p,n_p}$$
$$= \big[[m_{1,1},\ldots,m_{1,n_1}],\ldots,[m_{p-1,1},\ldots,m_{p-1,n_{p-1}}],[m_{p,1},\ldots,m_{p,n_p}]\big]$$

$n = m_{j,1}+\cdots+m_{j,n_j}\quad (j=1,\ldots,p)$: tuple of p partitions of n

Ex. ${}_nF_{n-1} : n = \overbrace{1+\cdots+1}^{n} = (n-1)+1 = \overbrace{1+\cdots+1}^{n}$

$$1\cdots1,(n-1)1,1\cdots1 = 1^n,(n-1)1,1^n$$
$$\big[[1,\ldots,1],[n-1,1],[1,\ldots,1]\big]$$

Gauss : $11,11,11$ $\quad(\mathrm{ord}=2,$ singular points : $\{0,1,\infty\})$

${}_3F_2$: $111,21,111$ $\quad(\mathrm{ord}=3,$ $\{0,1,\infty\}$, 2-dim local hol. sol. around 1)

Heun: $11,11,11,11$ $(\mathrm{ord}=2,$ 4 singular points $\Rightarrow$ Painlevé VI)

Jordan-Pochhammer : $21,21,21,21$ $(\Rightarrow$ Appell's $F_1)$

Middle convolution

Middle convolution $\mathrm{mc}_\sigma\,\mathbf{m}$ of a tuple (spectral type) $\mathbf{m}$:

$$\mathrm{mc}_\sigma\,\mathbf{m} := \Big(m_{j,\nu} - \delta_{\nu,\sigma_j} d_\sigma(\mathbf{m})\Big)_{\substack{1\le\nu\le n_j\\ 0\le j<p}} \quad (\sigma\in\mathbb{Z}^p_{\ge 0} \leftarrow \text{addition})$$

$$\boxed{\operatorname{ord}\mathrm{mc}_\sigma\,\mathbf{m} = \operatorname{ord}\mathbf{m} - d_\sigma(\mathbf{m}), \quad \mathrm{mc}_\sigma^2 = \mathrm{id}}$$

$$\operatorname{ord}\mathbf{m} := n = m_{j,1} + \cdots + m_{j,n_j}$$

$$d_\sigma(\mathbf{m}) := m_{1,\sigma_1} + \cdots + m_{p,\sigma_p} - (p-2)\cdot\operatorname{ord}\mathbf{m} \quad (\nu > n_j \Rightarrow m_{j,\nu} = 0)$$

Ex. $\mathrm{mc}_{(1,1,1)} : \underline{1}11, \underline{2}1, \underline{1}11 \xrightarrow{1+2+1-1\cdot 3} 011, 11, 011 \qquad \underline{1}1, \underline{1}1, \underline{1}1 \xrightarrow{3-2} 1, 1, 1$

$\underline{2}11, \underline{2}11, \underline{1}111 \xrightarrow{5-4=1} \underline{1}11, \underline{1}11, 0\underline{1}11 \xrightarrow{3-3=0} 111, 111, 111$

$\underline{4}31, \underline{3}311, \underline{4}1111 \xrightarrow{11-8=3} \underline{3}11, \underline{3}11, \underline{1}1111 \xrightarrow{7-5=2} \times$

Remark. $m_{j,\nu}$ is allowed to be 0 or n

$$\operatorname{idx}\mathbf{m} := \sum_{\substack{1\le\nu\le n_j\\ 1\le j\le p}} m_{j,\nu}^2 - (p-2)(\operatorname{ord}\mathbf{m})^2 \;:\; \text{index of rigidity of } \mathbf{m}$$

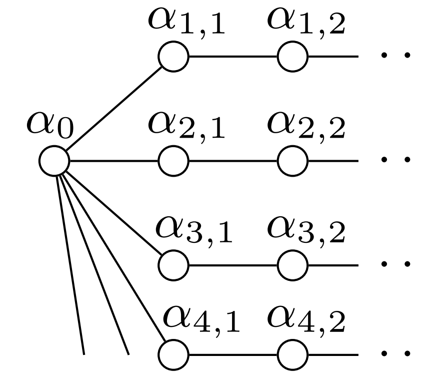
$$\boxed{\operatorname{idx}\mathrm{mc}_\sigma\,\mathbf{m} = \operatorname{idx}\mathbf{m}}$$

Star-shaped Kac-Moody root system

$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi := \{\alpha_0,\, \alpha_{j,\nu} \mid j \ge 1,\, \nu \ge 1\})$$

$$(\alpha_0|\alpha_{j,\nu}) = -\delta_{\nu,1}$$

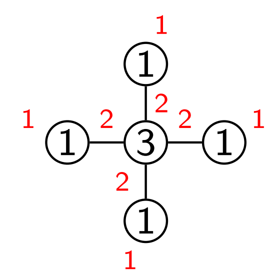
$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \ne j \quad \text{or} \quad |\mu - \nu| > 1) \\ -1 & (i = j \quad \text{and} \quad |\mu - \nu| = 1) \end{cases}$$



$$s_\alpha \;:\; x \mapsto x - (x|\alpha)\alpha \quad (\alpha \in \Pi), \qquad W := \langle s_\alpha | \alpha \in \Pi \rangle$$

$$\alpha_{\mathbf{m}} := n\alpha_0 + \sum_{j\ge1}\sum_{i\ge1}\Bigl(\sum_{\nu>i} m_{j,\nu}\Bigr)\alpha_{j,i} \qquad (n = \operatorname{ord}\mathbf{m})$$

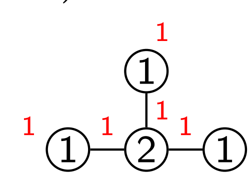
$$= n\alpha_0 + \sum_{j\ge1}\sum_{\nu\ge1} n_{j,\nu}\alpha_{j,\nu}$$

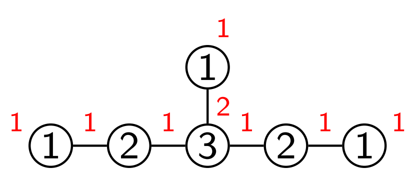


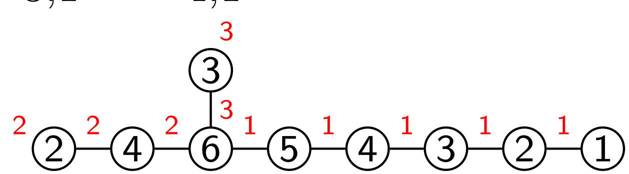
$$m_{j,\nu} = n_{j,\nu-1} - n_{j,\nu} \quad (j = 1, 2, \dots,\ \nu = 1, 2, \dots,\ \ n_{j,0} = n)$$

$$11,11,11 \leftrightarrow \alpha = 2\alpha_0 + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1}$$

$$21,21,21,21 \leftrightarrow \alpha = 3\alpha_0 + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1}$$



111,21,111 : 

33,222,111111 : 

Deligne-Simpson problem and Fundamental tuples

Th. $\alpha_{\mathrm{mc}_{(1,\dots,1)}\,\mathbf{m}} = s_{\alpha_0}\alpha_{\mathbf{m}}$

$\alpha_{\mathbf{m}'} = s_{\alpha_{j,\nu}}\alpha_{\mathbf{m}} \quad (\mathbf{m}' = \cdots m_{j,\nu+1} m_{j,\nu} \cdots)$

$\operatorname{idx}\mathbf{m} = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}), \quad d_{(1,\dots,1)}\mathbf{m} = (\alpha_0|\alpha_{\mathbf{m}})$

Deligne-Simpson Prob (additive). For given $\bar{A}_1, \dots, \bar{A}_p \in \mathrm{Mat}(n, \mathbb{C})$,

$\exists$ irred. $A_1, \dots, A_p$ with $A_j \sim \bar{A}_j$ and $\sum_j A_j \in \mathbb{C}$?

Th [Kostov, Katz, Crawley-Boevey, Oshima].

$\exists$ irred. $A_1, \dots, A_p$ in the above ($\overset{\text{def}}{\Leftrightarrow}$ $\mathbf{m}$: irreducibly realizable)

$\Leftrightarrow$ $\alpha_{\mathbf{m}}$ is a positive root and moreover $\mathrm{GCD}\{m_{j,\nu}\} = 1$ when $\operatorname{idx}\mathbf{m} = 0$

Th [Katz]. $\Rightarrow$ $\dim\{(A_1, \dots, A_p) \mid A_j \sim \bar{A}_j\}/GL(n, \mathbb{C}) = 2 - \operatorname{idx}\mathbf{m}$

Th [Katz]. $\mathbf{m}$ is irreducibly realizable

$\Leftrightarrow$ $\mathrm{mc}_\sigma(\mathbf{m})$ is irreducibly realizable or $\operatorname{ord}\mathbf{m} = 1$

$\mathbf{m}$: rigid $\overset{\text{def}}{\Leftrightarrow}$ $\mathbf{m}$: irred. realizable and $\operatorname{idx}\mathbf{m} = 2$ $\Leftrightarrow$ $\alpha_{\mathbf{m}}$: real positive root

$\mathbf{m}$: ordered $\overset{\text{def}}{\Leftrightarrow}$ $\begin{cases} m_{j,1} \ge m_{j,2} \ge \cdots \ge m_{j,n_j} > 0 & (1 \le j \le p) \\ m_{j,\nu} = m_{j+1,\nu}\ (1 \le \nu < k) \text{ and } m_{j,k} > m_{j+1,k} \text{ or } k = n_j+1 \end{cases}$

$$\sum_{j=1}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}(m_{j,1} - m_{j,\nu}) = d_{(1,\dots,1)}(\mathbf{m}) \cdot \operatorname{ord}\mathbf{m} - \operatorname{idx}\mathbf{m}$$

- $\mathbf{m}$: $\operatorname{idx}\mathbf{m} = 2$ and ordered $\Rightarrow$ $d_{(1,\dots,1)}\mathbf{m} > 0$ $\Rightarrow \operatorname{ord}\operatorname{mc}_{(1,\dots,1)}\mathbf{m} < \operatorname{ord}\mathbf{m}$

  $\Rightarrow$ rigid $\mathcal{M}$ is reduced to trivial equation by additions and middle convolutions

  $\mathbf{m}$: rigid $\Leftrightarrow$ $\alpha_{\mathbf{m}} \in W\alpha_0$ $\Rightarrow$ $\alpha_{\mathbf{m}}$: positive real root

  $\Rightarrow \exists_1 w_{\mathbf{m}} \in W$ with minimal length such that $w_{\mathbf{m}}\alpha_{\mathbf{m}} = \alpha_0$

  $\Rightarrow$ $\mathcal{M}$ has irred. monodromy $\overset{\text{Th.}}{\Leftrightarrow}$ $(\boldsymbol{\lambda}|\alpha) \notin \mathbb{Z}$ $(\forall\alpha > 0 : \text{root with } w_{\mathbf{m}}\alpha < 0)$

- $\mathbf{m}$: $\operatorname{idx}\mathbf{m} \le 0$, irreducibly realizable $\Rightarrow$ $\alpha_{\mathbf{m}}$: positive imaginary root

$\mathcal{F}_\ell := \{\mathbf{m} : \text{ordered} \mid d_{(1,\dots,1)}\mathbf{m} \le 0,\ \operatorname{idx}\mathbf{m} = -\ell,\ \operatorname{GCD}\{m_{j,\nu}\} = 1 \text{ if } \ell = 0\}$

$\mathbf{m}$: fundamental tuple $\in \mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_{2k}$ $(\mathcal{F}_\ell \ne \emptyset \Leftrightarrow \ell \in 2\mathbb{Z}_{\ge 0})$: irred. realizable

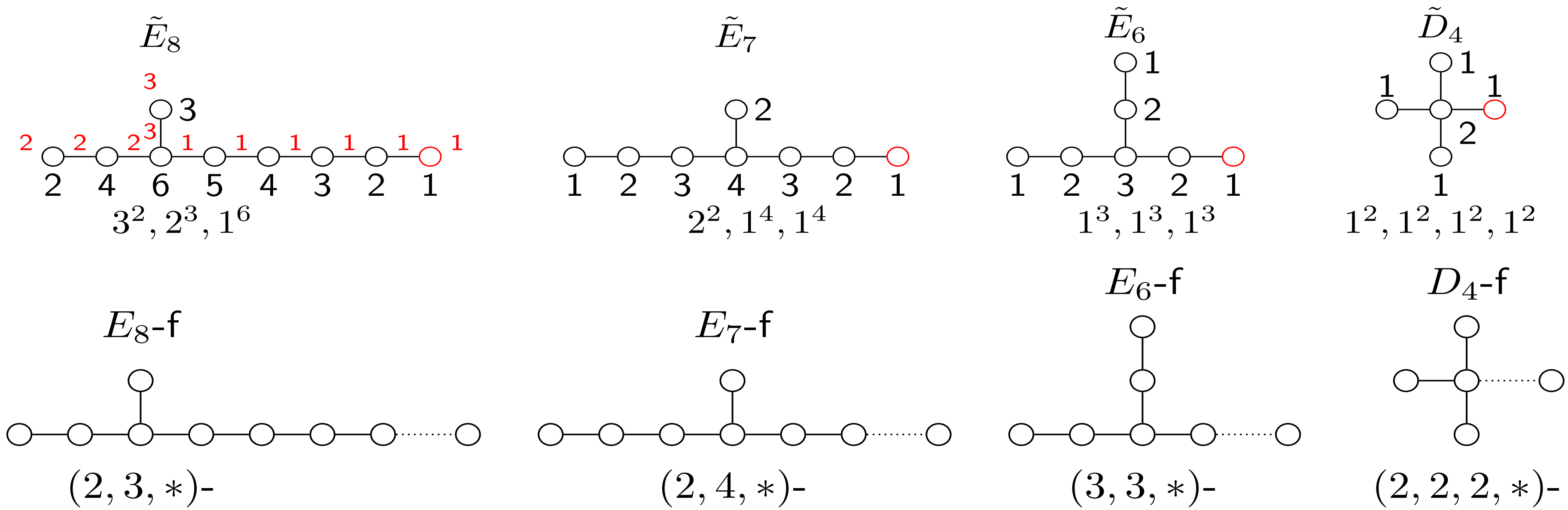
Th [Kostov]. $\mathcal{F}_0 = \{33, 222, 111111\ \ 22, 1111, 1111\ \ 111, 111, 111\ \ 11, 11, 11, 11\}$

Th [O, 2012]. $\#\mathcal{F}_\ell < \infty$

Def. A fundamental tuple of partitions $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{1\le\nu\le n_j\\ 1\le j<p}}$ is

E_k-fundamental for $k = 8,\ 7,\ 6$ if $p = 3$ and $(n_1, n_2) = (2,3),\ (2,4),\ (3,3)$, resp.

D_4-fundamental if $p = 4$ and $(n_0, n_1, n_2) = (2,2,2)$.



$$E_8\text{-f}:\ n = a_1 + a_2 = b_1 + b_2 + b_3 = c_1 + c_2 + \cdots + c_k,$$

$$a_1 \ge a_2 > 0,\ \ b_1 \ge b_2 \ge b_3 > 0,\ c_1 \ge c_2 \ge \cdots \ge c_k > 0$$

$$\text{fundamental} \Leftrightarrow n \ge a_1 + b_1 + c_1 \Leftrightarrow \boxed{a_2 \ge b_1 + c_1}$$

$$\Rightarrow\ \ \tfrac{n}{3} \le b_1 < a_2 = n - a_1 \le \tfrac{n}{2} \le a_1 < \tfrac{2n}{3},\quad c_1 \le \tfrac{n}{6},\quad k \ge 6$$

$$b_3 - c_1 \ge (n - b_1 - b_2) - (n - a_1 - b_1) = a_1 - b_2 > 0$$

Numbers of fundamental of all, E_8-f, 3, 4, 5 partitions with a given index

idx	0	−2	−4	−6	−8	−10	−12	−14	−16	−18
all	4	13	37	69	113	198	291	415	647	884
E_8	1	2	5	7	10	15	18	21	33	36
3	3	9	25	46	73	127	182	249	395	522
4	1	3	9	17	29	50	76	115	172	243
5	0	1	2	4	8	15	24	36	55	83

idx	−20	−22	−24	−26	−28	−30	−32	−34	−36	−38
all	1186	1682	2183	2889	3841	4831	6203	8012	9789	12290
E_8	40	55	62	70	92	96	114	142	148	168
3	680	963	1230	1577	2105	2570	3257	4187	5009	6144
4	345	478	625	861	1126	1430	1890	2411	2982	3832
5	111	164	229	310	416	570	716	940	1214	1546

idx	−40	−42	−44	−46	−48	−50	−52	−54	−56	−58
all	15379	18683	22910	28096	33387	40617	48703	57481	68622	81728
E_8	212	220	246	298	314	351	412	429	486	570
3	7731	9153	11072	13493	15768	18878	22594	26040	30819	36495
4	4721	5785	7227	8763	10484	12953	15392	18259	22056	26041
5	1939	2491	3046	3805	4675	5700	6916	8521	10114	12203

Confluence and Unfolding

Irregular singularity at ∞ with Poincaré rank 1:

$$\left\{\begin{matrix} x=\infty & x=a_j\ (j=2,\dots,p-1) \\ [\lambda_{1,1}x+\lambda_{1,1}^{(0)}]_{m_{1,1}^{(0)}} & [\lambda_{j,1}]_{m_{j,1}} \\ \vdots & \vdots \\ [\lambda_{1,n_1}x+\lambda_{n_1,R_{n_1}}^{(0)}]_{m_{n_0,R_{n_0}}^{(0)}} & [\lambda_{j,n_j}]_{m_{j,n_j}} \end{matrix}\right\} : \text{Riemann scheme}$$

$\exists$ sol $u\sim C_{\nu,k,\ell}(\frac{1}{x})^{\lambda_{\nu,k}^{(0)}}e^{-\lambda_{1,\nu}x}\quad (x\to\infty,\ 1\le\ell\le m_{\nu,k}^{(0)},\ 1\le k\le R_\nu)$

Spectrum at ∞ : $\{[\lambda_{1,\nu}x+\lambda_{\nu,k}^{(0)}]_{m_{\nu,k}^{(0)}}\mid 1\le k\le R_\nu,\ 1\le\nu\le n_1\}$

$m_{1,\nu}:=\sum_{k=1}^{R_\nu}m_{\nu,k}^{(0)}\ \Rightarrow\ \{m_{\nu,k}^{(0)}\}$ is a refinement of $n=m_{1,1}+\cdots+m_{m,n_1}$

Spectral type $\mathbf{m}$: $(m_{1,1}^{(0)}\cdots m_{1,R_1}^{(0)})\cdots(m_{n_1,1}^{(0)}\cdots m_{n_1,R_{n_1}}^{(0)}),m_{2,1}\cdots,\cdots m_{p-1,n_{p-1}}$

$=m_{1,1}^{(0)}m_{1,2}^{(0)}\cdots m_{n_1,R_{n_1}}^{(0)}\,|\,m_{1,1}\cdots m_{1,n_1},m_{2,1}\cdots,\cdots,m_{p-1,1}\cdots m_{p-1,n_{p-1}}$

Unfolding of $\mathbf{m}$:

$m_{1,1}^{(0)}m_{1,2}^{(0)}\cdots m_{n_1,R_{n_1}}^{(0)},m_{1,1}\cdots m_{1,n_1},m_{2,1}\cdots,\cdots,m_{p-1,1}\cdots m_{p-1,n_{p-1}}$

Ex. $33, 222, 111111 \xrightarrow{\text{confluence}}$

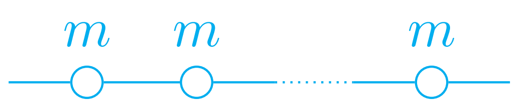
$111111|222, 33 = (11)(11)(11), 33 \qquad 111111|33, 22 = (111)(111), 222$

$42, 21111, 21111 \xrightarrow{\text{confluence}}$

$11112|42, 21111 = (1111)(2), 21111 \qquad 21111|42, 21111 = (211)(11), 21111$

$21111|21111, 42 = (2)(1)(1)(1), 42 \qquad 11112|11112|42 = ((1)(1)(1)(1))((2))$

$21111|21111|42 = ((2)(1)(1))((1)(1))$

Th [O-Hiroe]. The number of fundamental spectral types of linear Pfaffians system with unramified irregular/regular singularities and a fixed index of rigidity is finite. ($\Leftarrow$ (certain) finiteness of the Weyl group orbits of imaginary roots of symmetric Kac-Moody root system : 

).

Th [O, Kawakami, Hiroe]. $\exists$versal unfolding of a linear Pfaffian system with unramified irregular singularities.

$$\mathcal{M} : \frac{du}{dx} = \frac{A_1}{1 - bx}u + \sum_{j=2}^{p-1} \frac{A_j}{x - a_j}u \qquad \text{(versal unfolding, } |b| \ll 1)$$

Harnad dual

Harnad dual Hd_σ $\quad$ ($\longleftarrow (x, \frac{d}{dx}) \mapsto (-\frac{d}{dx}, x)$ and addition)

$$(m^{(0)}_{1,1}\cdots m^{(0)}_{1,R_1})\cdots(m^{(0)}_{n_1,1}\cdots m^{(0)}_{n_1,R_{n_1}}), \underline{m_{2,1}}m_{2,2}\cdots m_{2,n_2}, \ \ldots, \underline{m_{q,1}}m_{q,2}\cdots m_{q,n_q}$$

$$= m^{(0)}_{1,1}\cdots m^{(0)}_{1,R_1}\cdots m^{(0)}_{n_1,1}\cdots m^{(0)}_{n_1,R_{n_1}} \,|\, m_{1,1}\cdots m_{1,n_1}, m_{2,1}\cdots m_{2,n_2}, m_{q,1}\cdots m_{q,n_q}$$

$$\xrightarrow{\mathrm{Hd}_{(1,\ldots,1)}}$$

$$(m_{2,2}\cdots m_{2,n_2})\cdots(m_{q,2}\cdots m_{q,n_q}), \underline{m^{(0)}_{1,0}}m^{(0)}_{1,1}\cdots m^{(0)}_{1,R_1}, \ \ldots, \underline{m^{(0)}_{n_1,0}}m^{(0)}_{n_1,1}\cdots m^{(0)}_{n_1,R_{n_1}}$$

$$\boxed{\mathrm{idx}\,\mathrm{Hd}_\sigma\,\mathbf{m} = \mathrm{idx}\,\mathbf{m}, \quad \mathrm{Hd}_\sigma^2 = \mathrm{id}}$$

Transformation : $\mathbf{m}^{(0)} \xrightarrow{\text{confluence}} \mathbf{m}^{(1)} \xrightarrow{\text{Harnad dual}} \mathbf{m}^{(2)} \xrightarrow{\text{unfolding}} \mathbf{m}^{(3)}$

$$33, 222, 111111 \xrightarrow{\text{conf}} (11)(11)(11), \underline{3}3 \xrightarrow{\mathrm{Hd}} (3), \underline{1}11, \underline{1}11, \underline{1}11 \xrightarrow{\text{unf}} 111, 111, 111$$

$$:\ 33, 222, 111111 \xrightarrow{(1^2)(1^2)(1^2)} 111, 111, 111$$

$$33, 222, 111111 \xrightarrow{\text{conf}} (111)(111), \underline{2}22 \xrightarrow{\mathrm{Hd}} (22), \underline{1}111, \underline{1}111 \xrightarrow{\text{unf}} 22, 1111, 1111$$

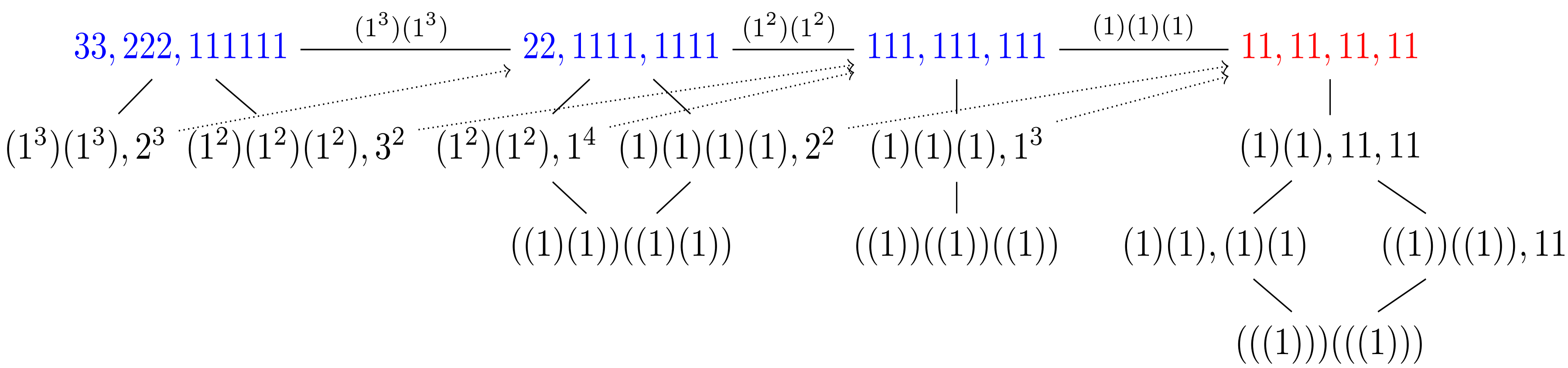
$$:\ 33, 222, 111111 \xrightarrow{(1^3)(1^3)} 22, 1111, 1111$$

Def [Kawakami]. $\mathbf{m}$, $\mathbf{m}'$: irreducibly realizable tuples of partitions

$\boxed{\mathbf{m} \sim \mathbf{m}'}$ $\overset{\text{def}}{\Leftrightarrow}$ $\mathbf{m}$ is transformed to $\mathbf{m}'$ by applications of these transformations:

the above transformations, mc_σ and permutations of partitions (=singular points) `idx=0 (Pidx=1), 4 basic (fundamental) tuples`

`6:` `33,222,111111` $E_8^{(1)}$	`4:` `22,1111,1111` $E_7^{(1)}$	`3:` `111,111,111` $E_6^{(1)}$	`2:` `11,11,11,11` $D_4^{(1)}$



index : $2-2m\ (m\ge 1)$

$$[11]_{m+3} : \overbrace{11,11,\dots,11}^{m+3} = [11]_{m+3} \qquad [11]_{m+3}^T : (m+2)^2,(m+1)^2 2,1^{2m+4}$$

$$FS_m : m1,m1,1^{m+1},1^{m+1} \qquad Sa_m : m^2,m^2,(2m-1)1,1^{2m}$$

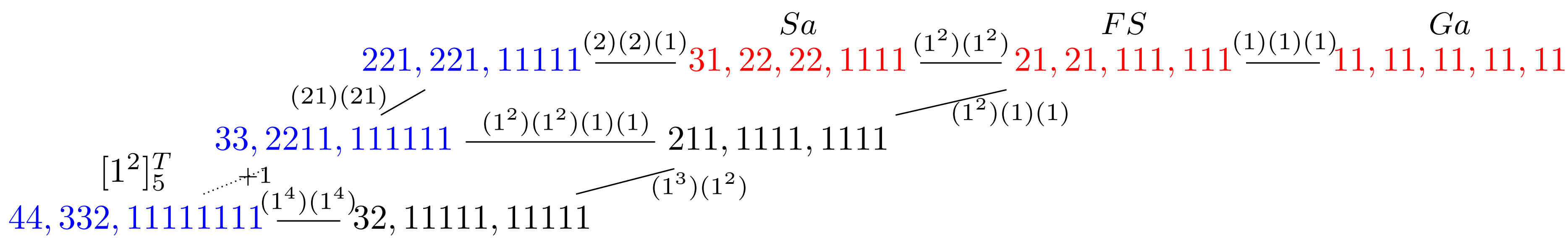
$$D_4^{(m)} : m^2,m^2,m^2,m(m-1)1 \qquad E_6^{(m)} : m^3,m^3,m^2(m-1)1$$

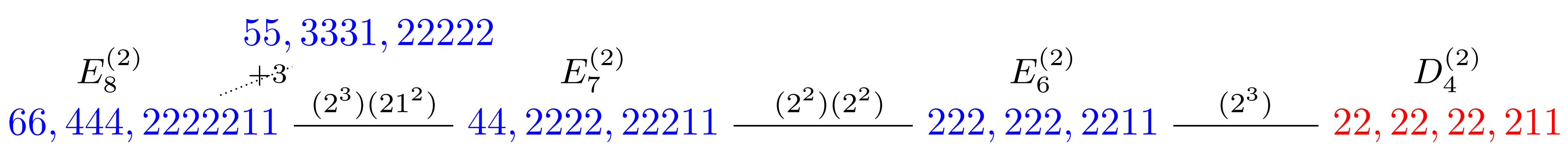
$$E_7^{(m)} : (2m)^2,m^4,m^3(m-1)1 \qquad E_8^{(m)} : (3m)^2,(2m)^3,m^5(m-1)1$$

$$\text{idx} = 0\ \ (m=1)\ :\quad D_4^{(1)} = [11]_4 = Ga_1 = FS_1 = Sa_1$$

idx=−2 (Pidx=2), 13 basic (fundamental) tuples

8: 44,332,11111111 $[11]_5^T$	6: 33,2211,111111	5: 32,11111,11111	5: 221,221,11111
4: 211,1111,1111	4: 31,22,22,1111 Sa	3: 21,21,111,111 FS	2: 11,11,11,11,11 $[11]_5$
12: 66,444,2222211 $E_8^{(2)}$	10: 55,3331,22222	8: 44,2222,22211 $E_7^{(2)}$	6: 222,222,2211 $E_6^{(2)}$
4: 22,22,22,211 $D_4^{(2)}$			





$$44, 332, 11111111 \underset{+1}{\xrightarrow{\text{mc}}} \begin{matrix} 11111 \cdot 211 | 54, \underline{3}33 \\ 54, 333, 21111111 \end{matrix} \xrightarrow{\text{Hd}} 33, \underline{1}11111, \underline{2}211$$

$$66, 444, 2222211 \underset{+3}{\xrightarrow{\text{mc}}} 2^4 \cdot 3^2 1 | 87, 5^3 \xrightarrow{\text{Hd}} 5^2, \underline{2}2^4, \underline{3}3^2 1 = 55, 3331, 22222$$

$$66, 444, 2^5 1^2 \xleftarrow{\text{mc}} 76, 544, 2^6 1 \xleftarrow{\text{mc}} 77, 554, 32^5 1 \xleftarrow{\text{mc}} 87, 555, 3^2 2^4 1$$

Results and Conjecture (2026.3.1)

Def. $\mathcal{F}^T := \{\mathbf{m} \in \mathcal{F} \mid \mathbf{m}$ is T-fundamental$\}$ $\quad (T: E_8, E_7, E_6$ or $D_4)$

$\mathcal{F}_\ell := \{\mathbf{m} \in \mathcal{F} \mid \mathrm{idx}\,\mathbf{m} = -\ell\}, \quad \mathcal{F}_\ell^T := \mathcal{F}^T \cap \mathcal{F}_\ell$

$\mathcal{F}_\mathbf{n}^T := \mathcal{F}^T \cap \mathcal{F}_\mathbf{n} \quad (\mathcal{F}_\mathbf{n} \subset \mathcal{F})$

Results. For $\mathbf{n} \in \mathcal{F}^{E_8}$ with $\mathrm{idx}\,\mathbf{n} \geq -47$, $\exists \mathcal{F}_\mathbf{n} \subset \mathcal{F}$ such that

1. $\mathcal{F}_\mathbf{n}^{E_8} = \{\mathbf{n}\}$ and $\mathrm{ord}\,\mathbf{m} < \mathrm{ord}\,\mathbf{n} \quad (\forall \mathbf{m} \in \mathcal{F}_\mathbf{n} \setminus \{\mathbf{n}\})$
2. $\mathcal{F}_\ell = \bigsqcup_{\mathbf{n} \in \mathcal{F}_\ell^{E_8}} \mathcal{F}_\mathbf{n} \quad (\ell \leq 47)$
3. $\mathbf{m} \sim \mathbf{n} \quad (\forall \mathbf{m} \in \mathcal{F}_\mathbf{n})$
4. $\exists \mathbf{n}' \in \mathcal{F}_\mathbf{n}^{E_7}$ such that $\mathrm{ord}\,\mathbf{m} \leq \mathrm{ord}\,\mathbf{n}' \quad (\forall \mathbf{m} \in \mathcal{F}_\mathbf{n} \setminus \{\mathbf{n}\})$
5. $\#\mathcal{F}_\mathbf{n}^{E_6} \geq 1$ and $\#\mathcal{F}_\mathbf{n}^{D_4} \leq 1$

Conj. $\mathbf{m} \sim \mathbf{m}' \in \mathcal{F} \Leftrightarrow \exists \mathcal{F}_\mathbf{n} \ni \mathbf{m}, \mathbf{m}'$ without the condition on ℓ. Moreover,

$\mathcal{F}^{E_8} \xrightarrow{\sim} \{\mathbf{m} : \text{non-rigid irreducibly realizable}\}/\sim \;\simeq \mathcal{F}/\sim\; \simeq \{\mathcal{F}_\mathbf{n} \mid \mathbf{n} \in \mathcal{F}^{E_8}\}$

Ex. $\#\mathcal{F}_{40} = 15379,\ \max_{\mathbf{m} \in \mathcal{F}_{40}} \mathrm{ord}\,\mathbf{m} = 126,\ \max_{\mathbf{n} \in \mathcal{F}_{40}^{E_8}} \#\mathcal{F}_\mathbf{n} = 465,$

$\#\mathcal{F}_{40}^{E_8} = 212,\ \#\mathcal{F}_{40}^{E_7} = 512,\ \#\mathcal{F}_{40}^{E_6} = 410,\ \#\mathcal{F}_{40}^{D_4} = 85,\ \mathcal{F}_{40} \ni [11]_{24}$

Results and Conjecture (2026.3.31)

Def. $\mathcal{F}^T := \{\mathbf{m} \in \mathcal{F} \mid \mathbf{m} \text{ is } T\text{-fundamental}\}$ $\quad (T:\ E_8,\ E_7,\ E_6 \text{ or } D_4)$

$\mathcal{F}_\ell := \{\mathbf{m} \in \mathcal{F} \mid \mathrm{idx}\,\mathbf{m} = -\ell\}, \quad \mathcal{F}_\ell^T := \mathcal{F}^T \cap \mathcal{F}_\ell.$

$\mathcal{F}_\mathbf{n} := \{\mathbf{m} \in \mathcal{F} \mid \mathbf{m} \sim \mathbf{n}\}, \qquad \mathcal{F}_\mathbf{n}^T := \mathcal{F}^T \cap \mathcal{F}_\mathbf{n} \quad (\mathbf{n} \in \mathcal{F}^{E_8}).$

Theorem. $\mathcal{F} = \bigsqcup_{\mathbf{n} \in \mathcal{F}^{E_8}} \mathcal{F}_\mathbf{n}$

For any $\mathbf{n} \in \mathcal{F}^{E_8}$ with $\mathrm{idx}\,\mathbf{n} \geq -100$,

1. $\mathrm{ord}\,\mathbf{m} < \mathrm{ord}\,\mathbf{n} \quad (\forall \mathbf{m} \in \mathcal{F}_\mathbf{n} \setminus \{\mathbf{n}\})$
4. $\exists \mathbf{n}' \in \mathcal{F}_\mathbf{n}^{E_7}$ such that $\mathrm{ord}\,\mathbf{m} \leq \mathrm{ord}\,\mathbf{n}' \quad (\forall \mathbf{m} \in \mathcal{F}_\mathbf{n} \setminus \{\mathbf{n}\})$
5. $\#\mathcal{F}_\mathbf{n}^{E_6} \geq 1$ and $\#\mathcal{F}_\mathbf{n}^{D_4} \leq 1$

Example. $\#\mathcal{F}_2 = 13, \quad \max_{\mathbf{m} \in \mathcal{F}_2} \mathrm{ord}\,\mathbf{m} = 12, \quad \max_{\mathbf{n} \in \mathcal{F}_2^{E_8}} \#\mathcal{F}_\mathbf{n} = 8,$

$\#\mathcal{F}_2^{E_8} = 2, \quad \#\mathcal{F}_2^{E_7} = 3, \quad \#\mathcal{F}_2^{E_6} = 2, \quad \#\mathcal{F}_2^{D_4} = 2, \quad \mathcal{F}_2 \ni [11]_5$

$\#\mathcal{F}_{100} = 1461132, \quad \max_{\mathbf{m} \in \mathcal{F}_{100}} \mathrm{ord}\,\mathbf{m} = 306, \quad \max_{\mathbf{n} \in \mathcal{F}_{100}^{E_8}} \#\mathcal{F}_\mathbf{n} = 7613,$

$\#\mathcal{F}_{100}^{E_8} = 3181, \quad \#\mathcal{F}_{100}^{E_7} = 10389, \quad \#\mathcal{F}_{100}^{E_6} = 7500, \quad \#\mathcal{F}_{100}^{D_4} = 743, \quad \mathcal{F}_{100} \ni [11]_{54}$

Results (2026.5.4)

Def. $\mathcal{F}^T := \{\mathbf{m} \in \mathcal{F} \mid \mathbf{m} \text{ is } T\text{-fundamental}\}$ $(T:\ E_8,\ E_7,\ E_6 \text{ or } D_4)$

$\mathcal{F}_\ell := \{\mathbf{m} \in \mathcal{F} \mid \operatorname{idx}\mathbf{m} = -\ell\}$, $\mathcal{F}^T_\ell := \mathcal{F}^T \cap \mathcal{F}_\ell$.

$\mathcal{F}_\mathbf{n} := \{\mathbf{m} \in \mathcal{F} \mid \mathbf{m} \sim \mathbf{n}\}$, $\mathcal{F}^T_\mathbf{n} := \mathcal{F}^T \cap \mathcal{F}_\mathbf{n}$ $(\mathbf{n} \in \mathcal{F}^{E_8})$.

Theorem (1) $\mathcal{F} = \bigsqcup_{\mathbf{n}\in\mathcal{F}^{E_8}} \mathcal{F}_\mathbf{n}$

$$\operatorname{ord}\mathbf{m} < \operatorname{ord}\mathbf{n} \quad (\forall\mathbf{m} \in \mathcal{F}_\mathbf{n} \setminus \{\mathbf{n}\},\ \mathbf{n} \in \mathcal{F}^{E_8}).$$

(2) For $\mathbf{m} \in \mathcal{F}^T$ with $T = D_4,\ E_6,\ E_7$, we explicitly get $\mathbf{n} \in \mathcal{F}^{E_8}$ with $\mathbf{n} \sim \mathbf{m}$.

In particular, for ordered $\bar{m}_{11}\bar{m}_{12},\ \bar{m}_{21}\bar{m}_{22}\bar{m}_{23},\ \bar{m}_{31}\bar{m}_{32}\bar{m}_{33}\cdots\bar{m}_{3n_3} \in \mathcal{F}^{E_8}$

- $\mathcal{F}^{D_4}_\mathbf{n} \neq \emptyset \ \Leftrightarrow\ \#\mathcal{F}^{D_4}_\mathbf{n} = 1$ and $\bar{m}_{11} = \bar{m}_{12},\ \bar{m}_{21} = \bar{m}_{22},\ \bar{m}_{23} = 2\bar{m}_{31} = 2\bar{m}_{32}$
- $\exists\bar{\mathbf{n}} \in \mathcal{F}^{E_8}$ such that $\mathcal{F}^{E_7}_{\bar{\mathbf{n}}} = \mathcal{F}^{E_6}_{\bar{\mathbf{n}}} = \emptyset$

We sometimes express $\mathbf{m} = m_{11}\cdots m_{1n_1}, m_{21}\cdots m_{2n_2}, \cdots, m_{p1}\cdots m_{pn_p}$ by

$$\big[[m_{11},\ldots,m_{1n_1}],[m_{21},\ldots,m_{2n_2}],\ldots,[m_{p1},\ldots,m_{pn_p}]\big]$$

Example. $\mathcal{F}^{E_7}_{\bar{\mathbf{n}}} = \mathcal{F}^{E_6}_{\bar{\mathbf{n}}} = \mathcal{F}^{D_4}_{\bar{\mathbf{n}}} = \emptyset$ and $\operatorname{idx}\bar{\mathbf{n}} = -630$ for

$$\bar{\mathbf{n}} = \big[[50,49],[34,33,32],[14,13,12,11,10,9,8,7,6,5,4]\big]$$

In Theorem (2), for example, when $T = E_7$, we have the following:

For an ordered tuple

$$\big[[m_{11}, m_{12}], [m_{21}, m_{22}, m_{23}, m_{24}], [m_{31}, \dots, m_{3n_3}]\big] \in \mathcal{F}^{E_7}$$

with $n = \operatorname{ord}\mathbf{m}$, the tuple $\mathbf{n} \in \mathcal{F}^{E_8}$ satisfying $\mathbf{n} \sim \mathbf{m}$ is

$$\begin{aligned}
&\big[[m_{11}+m_{24}, m_{12}+m_{24}], [m_{21}+m_{24}, m_{22}+m_{24}, m_{23}+m_{24}],\\
&\quad [m_{24}, m_{24}, m_{31}, \dots, m_{3n_3}]\big] \qquad\qquad \text{if } \ m_{11} < m_{22}+m_{23},\\
&\big[[m_{12}+m_{23}, m_{12}+m_{24}], [m_{12}, m_{12}, m_{23}+m_{24}],\\
&\quad [m_{12}-m_{22}, m_{12}-m_{21}, m_{31}, \dots, m_{3n_3}]\big] \quad \text{if } \ m_{11} > m_{21}+m_{23},\\
&\big[[n-m_{21}, n-m_{21}], [m_{22}+m_{23}, m_{22}+m_{24}, m_{23}+m_{24}],\\
&\quad [m_{11}-m_{21}, m_{12}-m_{21}, m_{31}, \dots, m_{3n_3}]\big] \quad \text{if } \ m_{11} < m_{21}+m_{24},\\
&\big[[n-m_{21}, m_{12}+m_{24}], [m_{12}, m_{22}+m_{24}, m_{23}+m_{24}],\\
&\quad [m_{24}, m_{12}-m_{21}, m_{31}, \dots, m_{3n_3}]\big]\\
&\quad \text{if } \ m_{11} \geq m_{22}+m_{23}, \ m_{11} \leq m_{21}+m_{23} \ \text{ and } \ m_{11} \geq m_{21}+m_{24}.
\end{aligned}$$

Third and the last tuples are ordered. The last tuple satisfies $d(\mathbf{n}) = 0$.

Proof and Examples

Def. $\mathbf{m} = m_{11}\cdots m_{1n_1}, m_{21}\cdots m_{2n_2}, \ldots, m_{p1}\cdots m_{pn_p}$: irreducibly realizable

$\mathbf{m} \underset{\mathrm{mc}}{\sim} \mathbf{m}' \overset{\text{def.}}{\Leftrightarrow} \exists\, \mathrm{mc}_{\sigma(1)}, \ldots, \mathrm{mc}_{\sigma(k)}$ such that $\mathbf{m}' = \mathrm{mc}_{\sigma(1)}\cdots \mathrm{mc}_{\sigma(k)}\,\mathbf{m}$

$\mathcal{R}\mathbf{m} \in \mathcal{F}$ satisfying $\mathcal{R}\mathbf{m} \underset{\mathrm{mc}}{\sim} \mathbf{m}$

$$\mathcal{S}\mathbf{m} := \big[[(p-1)n, n], [\overbrace{n, \ldots, n}^{p}], [m_{11}, \ldots, m_{p1}, \{m_{j\nu}\}_{\nu=2,\ldots,n_j,\ j=1,\ldots,p}]\big]$$

: unfolding of Harnad dual of $n, \underline{0}m_{11}\cdots, \underline{0}m_{1n_1}, \ldots, \underline{0}m_{p1}\cdots m_{pn_p}$

: unfolding of $(m_{11}\cdots m_{1n_1})\cdots(m_{p1}\cdots m_{pn_p}), \underline{((p-1)n)}n$

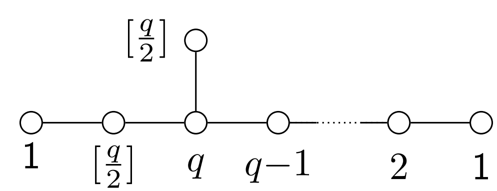
$$\mathcal{S}^2\mathbf{m} := \big[[2pn, pn], [pn, pn, pn], [(p-1)n, \overbrace{n, \ldots, n}^{p+1}, \{m_{j,\nu}\}]\big]$$

$\mathrm{mc}_\sigma\,\mathbf{m} = \mathrm{Hd}_2\mathrm{Hd}_{(1,\ldots,1)}(n, \mathbf{m})$

Lemma. (1) $\mathbf{m}$: $(2,3,*)$-type $\Rightarrow$ $\mathcal{S}\mathbf{m} \underset{\mathrm{mc}}{\sim} \mathbf{m}$

$\mathbf{m}$: $(*,*,*)$-type (i.e. $p=3$) $\Rightarrow$ $\mathcal{S}^2\mathbf{m} \underset{\mathrm{mc}}{\sim} \mathcal{S}\mathbf{m}$

β_q :



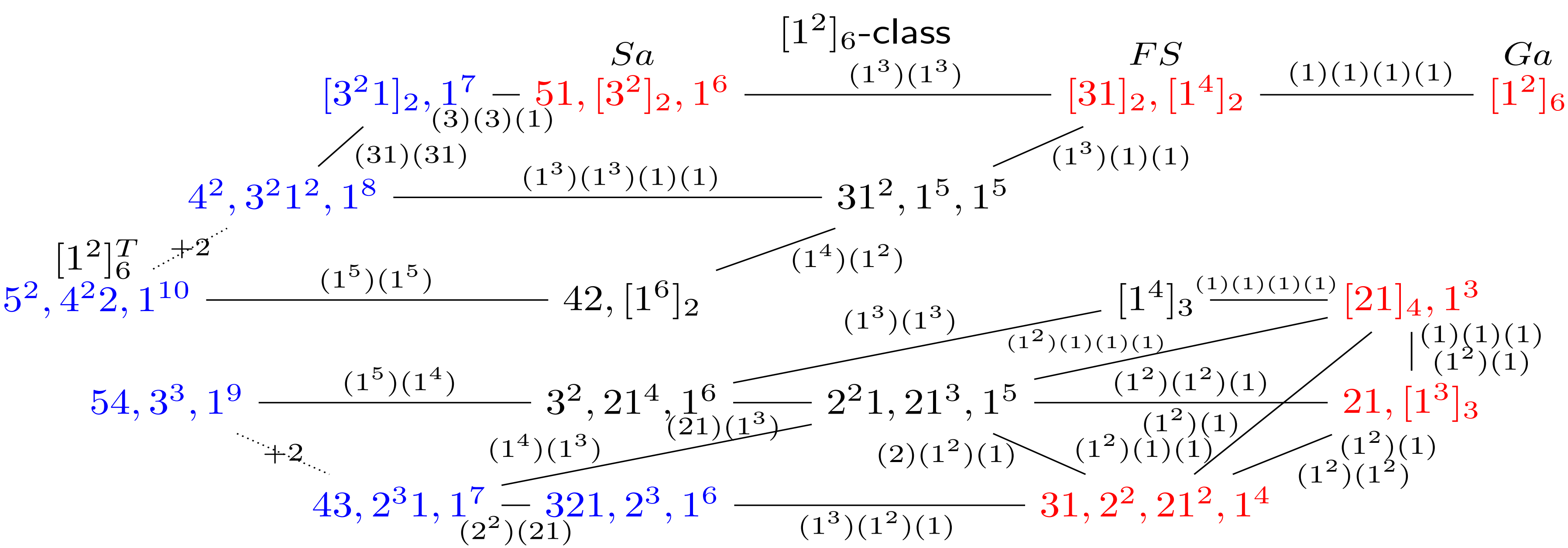
(2) $\mathbf{m}'$: an unfolding of Harnad dual of a confluence of $\mathbf{m}$ $\Rightarrow$ $\mathcal{S}^2\mathbf{m}' \underset{\mathrm{mc}}{\sim} \mathcal{S}^2\mathbf{m}$

In particular, $\mathcal{S}^2\mathrm{mc}_\sigma\mathbf{m} = s_{\beta_\sigma}\mathcal{S}^2\mathbf{m}$ $\quad(\exists\beta_\sigma$: a real root$)$

(3) $\operatorname{ord}\mathcal{R}\mathcal{S}^2\mathbf{m} > \operatorname{ord}\mathbf{m}$ $\quad(\mathbf{m} \in \mathcal{F}\setminus\mathcal{F}^{E_8})$

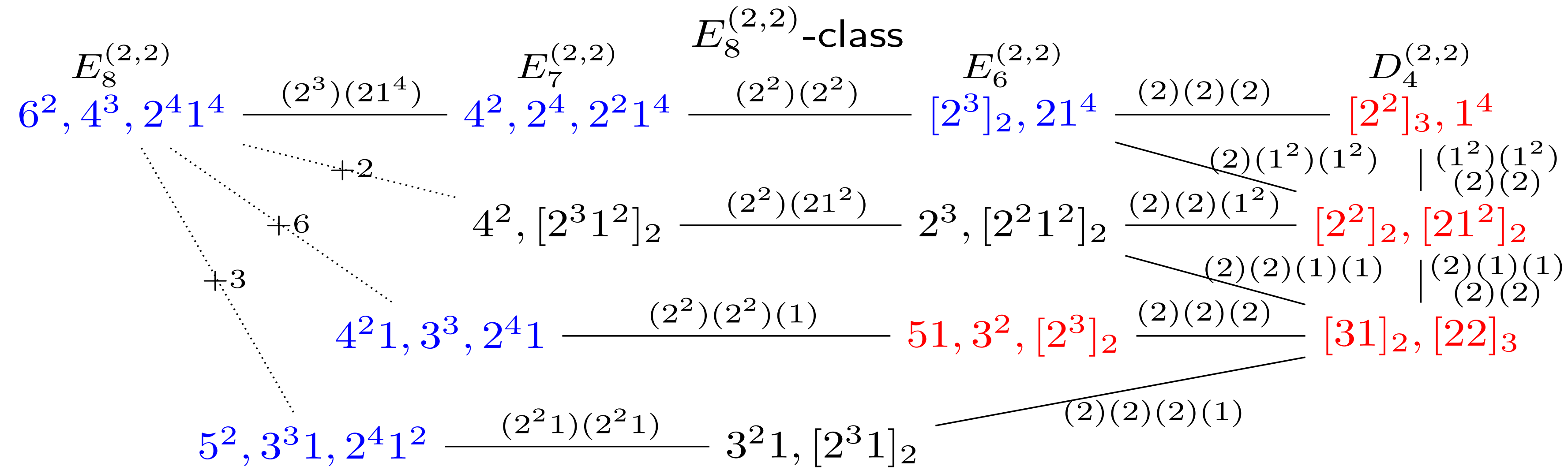
idx$=-4$ (Pidx=3), 37 basic (fundamental) tuples

10: 55,442,1111111111 $[11]_6^T$	8: 44,3311,11111111	7: 331,331,1111111	6: 42,111111,111111
6: 51,33,33,111111 Sa	5: 311,11111,11111	4: 31,31,1111,1111 FS	2: 11,11,11,11,11,11 $[11]_6$
18: 99,666,3333321 $E_8^{(3)}$	12: 66,3333,33321 $E_7^{(3)}$	9: 333,333,3321 $E_6^{(3)}$	6: 33,33,33,321 $D_4^{(3)}$
9: 54,333,111111111	7: 43,2221,11111111	6: 321,222,111111	6: 33,21111,11111
5: 221,2111,11111	4: 1111,1111,1111	4: 31,22,211,1111	3: 21,111,111,111
3: 21,21,21,21,111			
12: 66,444,22221111 $E_8^{(2,2)}$	10: 55,3331,222211	9: 441,333,22221	8: 44,22211,22211
8: 44,2222,221111 $E_7^{(2,2)}$	7: 331,2221,2221	6: 222,2211,2211	6: 222,222,21111 $E_6^{(2,2)}$
6: 51,33,222,222	4: 22,22,211,211	4: 31,31,22,22,22	4: 22,22,22,1111 $D_4^{(2,2)}$
14: 77,554,2222222	10: 55,3322,22222	8: 332,332,2222	6: 42,33,33,222



$$55,442,1111111111 \xrightarrow[+2]{\text{mc}} 1111111\cdot 311|75,444 \xrightarrow{\text{Hd}} 44,\underline{1}1111111,\underline{3}311$$

$$54,333,111111111 \xrightarrow[+2]{\text{mc}} 1^6\cdot 221|65,443 \xrightarrow{\text{Hd}} 43,\underline{1}111111,\underline{2}221$$



$$66,444,22221111 \xrightarrow[+2]{\text{mc}} 2221\cdot 2221|77,644 \xrightarrow{\text{Hd}} 44,\underline{1}2221,\underline{1}2221$$

$$66,444,22221111 \xrightarrow[+6]{\text{mc}} 2221\cdot 33\cdot 41|765,99 \xrightarrow{\text{Hd}} \underline{2}2221,\underline{3}33,\underline{4}41$$

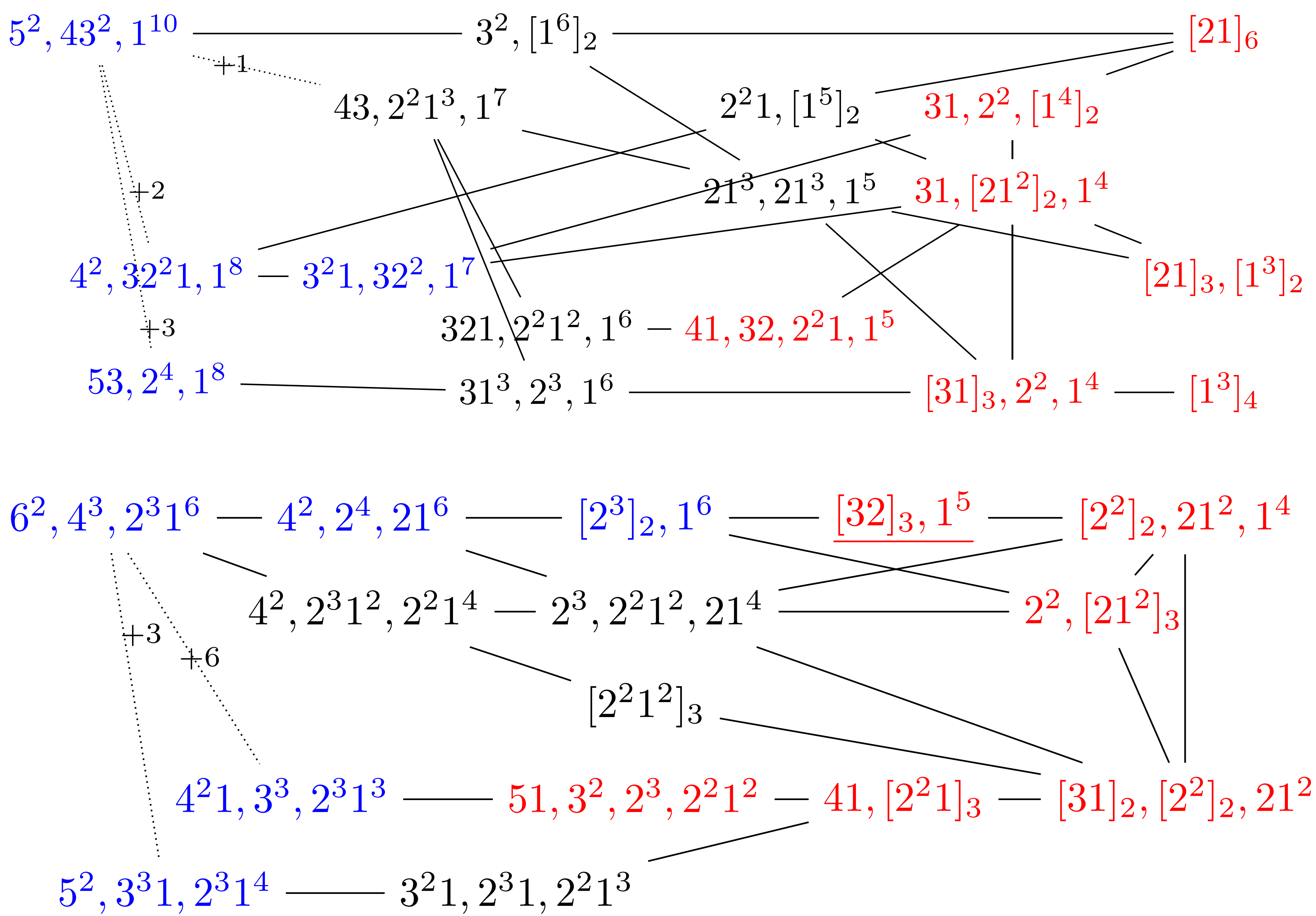
$$66,444,22221111 \xrightarrow[+3]{\text{mc}} 22211\cdot 331|87,555 \xrightarrow{\text{Hd}} 55,\underline{2}22211,\underline{3}331$$

$$77,554,2222222 \xrightarrow[+1]{\text{mc}} 2^4\cdot 32^2|87,5^3 \xrightarrow{\text{Hd}} 5^2,\underline{2}2^4,\underline{3}32^2 = 55,3322,22222$$

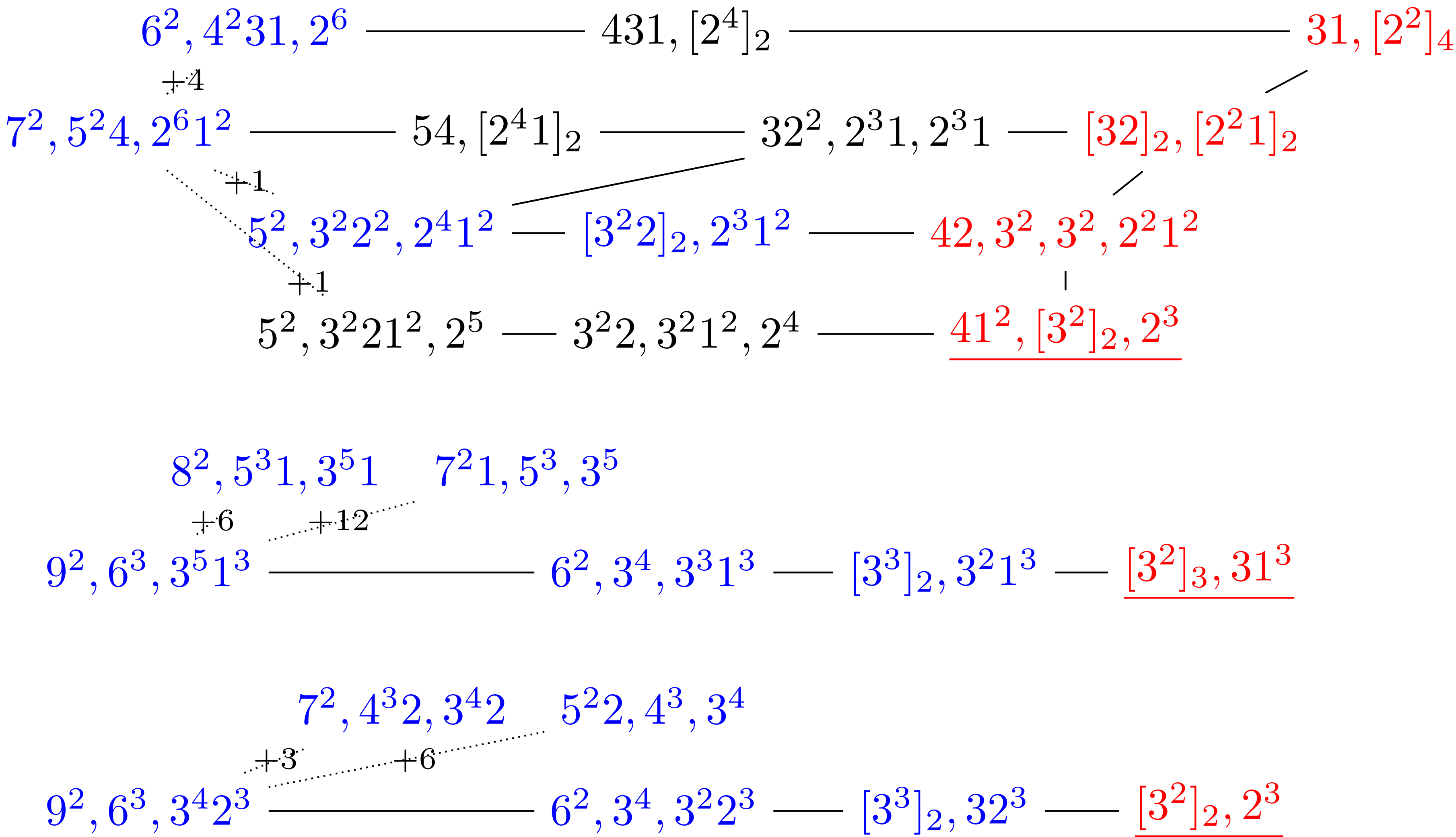
$$\to 32\cdot 32|55,\underline{2}2222 \xrightarrow{\text{Hd}} 2222,\underline{3}32,\underline{3}32 \to 3\cdot 3\cdot 2|332,\underline{2}222 \xrightarrow{\text{Hd}} 222,\underline{3}3,\underline{3}3,\underline{4}2$$

Fundamental class of 69 tuples with idx=-6 (Pidx=4)

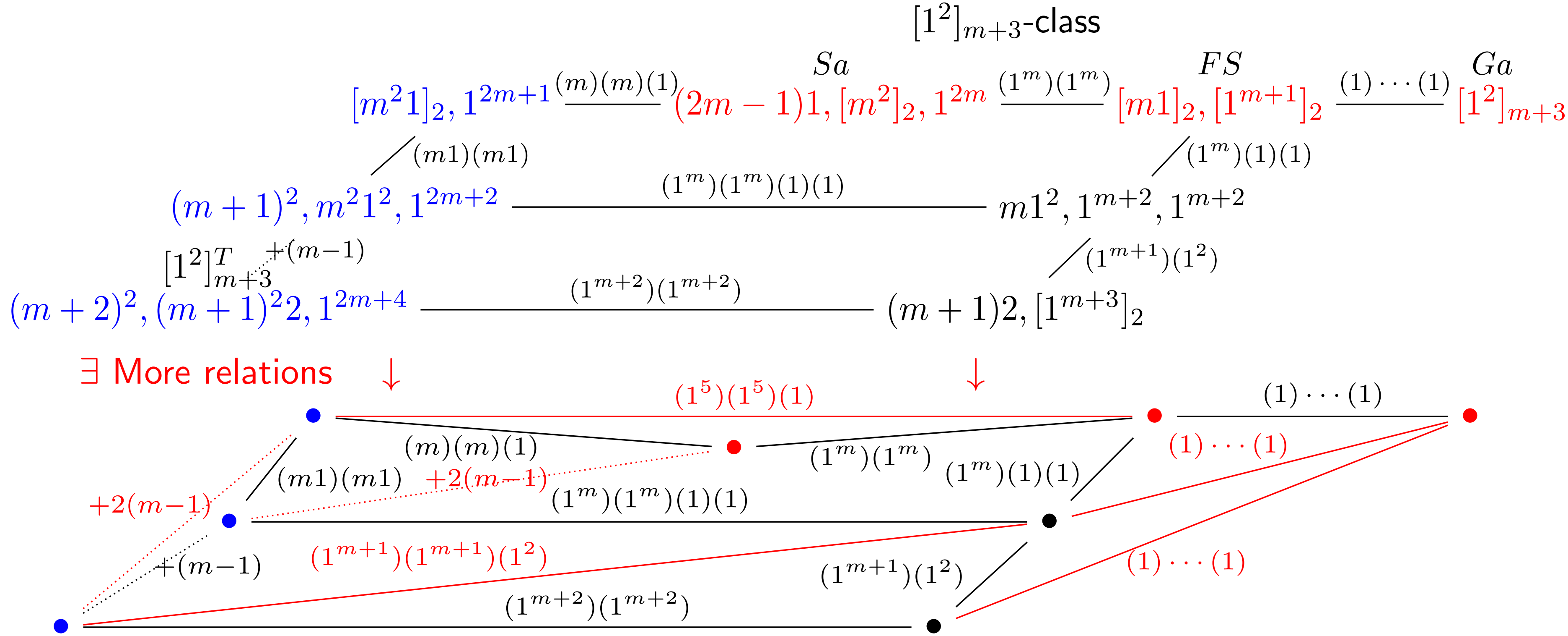
12: 66,552,111111111111 $[11]_7^T$	10: 55,4411,1111111111	9: 441,441,111111111	8: 71,44,44,11111111 Sa
7: 52,1111111,1111111	6: 411,111111,111111	5: 41,41,11111,11111 FS	2: 11,11,11,11,11,11,11
24: cc,888,4444431 $E_8^{(4)}$	16: 88,4444,44431 $E_7^{(4)}$	12: 444,444,4431 $E_6^{(4)}$	8: 44,44,44,431 $D_4^{(4)}$
10: 55,433,1111111111	8: 53,2222,11111111	8: 44,3221,11111111	7: 43,22111,1111111
7: 331,322,1111111	6: 33,111111,111111	6: 321,2211,111111	6: 3111,222,111111
5: 221,11111,11111	5: 2111,2111,11111	5: 41,32,221,11111	4: 31,22,1111,1111
4: 31,211,211,1111	4: 31,31,31,22,1111	3: 111,111,111,111	3: 21,21,21,111,111
3: 21,21,21,21,21,21			
12: 66,444,222111111 $E_8^{(2,3)}$	10: 55,3331,2221111	9: 441,333,222111	8: 44,2222,2111111 $E_7^{(2,3)}$
8: 44,22211,221111	7: 331,2221,22111	6: 222,222,111111 $E_6^{(2,3)}$	6: 222,2211,21111
6: 2211,2211,2211	6: 51,33,222,2211	5: 41,221,221,221	5: 32,32,32,11111
4: 22,22,211,1111 $D_4^{(2,3)}$	4: 22,211,211,211	4: 31,31,22,22,211	
14: 77,554,22222211	12: 66,4431,222222	10: 55,3322,222211	10: 55,33211,22222
9: 54,22221,22221	8: 431,2222,2222	8: 332,332,22211	8: 332,3311,2222
7: 322,2221,2221	6: 42,33,33,2211	6: 411,33,33,222	5: 32,32,221,221
4: 31,22,22,22,22			
18: 99,666,33333111 $E_8^{(3,1)}$	16: 88,5551,333331	15: 771,555,33333	12: 66,3333,333111 $E_7^{(3,1)}$
9: 333,333,33111 $E_6^{(3,1)}$	6: 33,33,33,3111 $D_4^{(3,1)}$		
18: 99,666,3333222 $E_8^{(3,2)}$	14: 77,4442,33332	12: 66,3333,33222 $E_7^{(3,2)}$	12: 552,444,3333
9: 333,333,3222 $E_6^{(3,2)}$	6: 33,33,33,222 $D_4^{(3,2)}$		



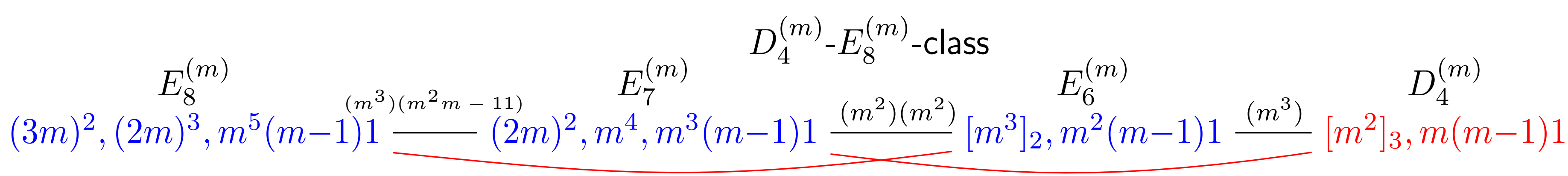
$5^2, 43^2, 1^{10}$
$3^2, [1^6]_2$
$[21]_6$
+1
$43, 2^21^3, 1^7$
$2^21, [1^5]_2$
$31, 2^2, [1^4]_2$
$21^3, 21^3, 1^5$
$31, [21^2]_2, 1^4$
+2
$4^2, 32^21, 1^8$
$3^21, 32^2, 1^7$
$[21]_3, [1^3]_2$
+3
$321, 2^21^2, 1^6$
$41, 32, 2^21, 1^5$
$53, 2^4, 1^8$
$31^3, 2^3, 1^6$
$[31]_3, 2^2, 1^4$
$[1^3]_4$
$6^2, 4^3, 2^31^6$
$4^2, 2^4, 21^6$
$[2^3]_2, 1^6$
$[32]_3, 1^5$
$[2^2]_2, 21^2, 1^4$
$4^2, 2^31^2, 2^21^4$
$2^3, 2^21^2, 21^4$
$2^2, [21^2]_3$
+3
+6
$[2^21^2]_3$
$4^21, 3^3, 2^31^3$
$51, 3^2, 2^3, 2^21^2$
$41, [2^21]_3$
$[31]_2, [2^2]_2, 21^2$
$5^2, 3^31, 2^31^4$
$3^21, 2^31, 2^21^3$


$6^2, 4^2 31, 2^6$
$431, [2^4]_2$
$31, [2^2]_4$
$+4$
$7^2, 5^2 4, 2^6 1^2$
$54, [2^4 1]_2$
$32^2, 2^3 1, 2^3 1$
$[32]_2, [2^2 1]_2$
$+1$
$5^2, 3^2 2^2, 2^4 1^2$
$[3^2 2]_2, 2^3 1^2$
$42, 3^2, 3^2, 2^2 1^2$
$+1$
$5^2, 3^2 21^2, 2^5$
$3^2 2, 3^2 1^2, 2^4$
$41^2, [3^2]_2, 2^3$
$8^2, 5^3 1, 3^5 1$
$7^2 1, 5^3, 3^5$
$+6$
$+12$
$9^2, 6^3, 3^5 1^3$
$6^2, 3^4, 3^3 1^3$
$[3^3]_2, 3^2 1^3$
$[3^2]_3, 31^3$
$7^2, 4^3 2, 3^4 2$
$5^2 2, 4^3, 3^4$
$+3$
$+6$
$9^2, 6^3, 3^4 2^3$
$6^2, 3^4, 3^2 2^3$
$[3^3]_2, 32^3$
$[3^2]_2, 2^3$

idx $= 2(m-1)$



$$1^{2m+4}, (m+2)^2, (m+1)^2 2 \xleftarrow[m-1]{\text{mc}} 1^{2m+1} \cdot m1^2 | (2m+1)(m+2), (m+1)^3 \xrightarrow{\text{Hd}} (m+1)^2, \underline{1}1^{2m+1}, \underline{m}m1^2$$



idx $= -2$: $D_4^{(2)}$-$E_8^{(2)}$-class contains $5^3, 3^31, 2^5$

Fundamental class of 113 tuples with idx=−8 (Pidx=5)

14: 77,662,11111111111111	12: 66,5511,111111111111	11: 551,551,11111111111	10: 91,55,55,1111111111
8: 62,11111111,11111111	7: 511,1111111,1111111	6: 51,51,111111,111111	2: 11,11,11,11,11,11,11,11
30: ff,aaa,5555541	20: aa,5555,55541	15: 555,555,5541	10: 55,55,55,541
12: 66,444,2211111111	10: 55,3331,22111111	9: 441,333,2211111	8: 44,2222,11111111
8: 44,22211,2111111	8: 44,221111,221111	7: 331,2221,211111	7: 331,22111,22111
7: 322,322,1111111	6: 222,2211,111111	6: 222,21111,21111	6: 2211,2211,21111
6: 51,33,222,21111	6: 51,33,2211,2211	6: 42,42,33,111111	5: 41,221,221,2111
5: 32,32,311,11111	4: 22,22,1111,1111	4: 22,211,211,1111	4: 211,211,211,211
4: 31,31,22,22,1111	4: 31,31,22,211,211	4: 31,31,31,31,22,22	
14: 77,554,222221111	12: 66,4431,2222211	11: 551,443,222221	10: 55,3322,2221111
10: 55,33211,222211	10: 55,331111,22222	9: 54,22221,222111	9: 441,3321,22221
8: 431,2222,22211	8: 332,332,221111	8: 332,3311,22211	8: 3311,3311,2222
8: 71,44,332,2222	7: 322,2221,22111	7: 3211,2221,2221	6: 51,321,222,222
6: 42,33,33,21111	6: 411,33,33,2211	6: 51,51,33,33,222	5: 32,32,221,2111
5: 32,311,221,221	4: 31,22,22,22,211		
11: 65,443,11111111111	9: 54,3321,111111111	8: 53,22211,11111111	8: 44,32111,11111111
8: 431,332,11111111	7: 43,211111,1111111	7: 421,2221,1111111	7: 331,3211,1111111
6: 321,21111,111111	6: 3111,2211,111111	6: 51,42,222,111111	6: 51,33,321,111111
5: 2111,11111,11111	5: 41,32,2111,11111	5: 41,311,221,11111	4: 31,211,1111,1111
4: 31,31,31,211,1111	3: 21,21,111,111,111	3: 21,21,21,21,21,111	
18: 99,666,33332211	16: 88,5551,333322	14: 77,4442,333311	14: 77,44411,33332
12: 66,3333,332211	12: 66,33321,33321	12: 5511,444,3333	11: 551,3332,3332
10: 442,3331,3331	9: 333,333,32211	9: 333,3321,3321	6: 33,33,33,2211
6: 33,33,321,321	6: 51,42,33,33,33		
15: 87,555,22222221	11: 65,3332,222221	10: 55,32221,22222	9: 432,333,22221
8: 332,3221,2222	6: 42,33,321,222		
20: aa,776,3333332	14: 77,4433,33332	11: 443,443,3332	8: 53,44,44,332
24: cc,888,4444422	20: aa,6662,44444	16: 88,4444,44422	12: 444,444,4422
8: 44,44,44,422			
16: 88,664,22222222	12: 66,4422,222222	10: 64,22222,22222	10: 442,442,22222
8: 422,2222,2222	8: 62,44,44,2222	6: 42,42,222,222	4: 22,22,22,22,22

The tables of the classification are calculated for $\mathrm{idx} \ge -100$ by `spfundclass( )` in [O3], 2026.2.8.

`spbasic()` : get required irreducibly realizable tuples (idx, ord, $\#$partitions)

`spharnadcv()` : calculate mc_σ, Hd_σ

`spdiagram()` : draw a diagram of tuples with relations defined by mc_σ, Hd_σ

`spfundauto()` : get the automorphism group of a tuple defined by mc_σ

`sptab()` : draw table of classified tuples

`refinements()` : get all confluences of a tuple

Classification of fundamental tuples using `Risa/Asir`

idx	0	−2	−4	−6	−8	−10	−12	−14	−16	−18	−20
run time(sec)	0.02	0.03	0.05	0.09	0.13	0.25	0.44	0.72	1.89	2.77	3.00
# fund	4	13	37	69	113	198	291	415	647	884	1186
# class	1	2	5	7	10	15	18	21	33	36	40

Thank you for your attention!

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