Second Main Theorem, Degeneracy of Entire Curves, and Applications

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§1 Introduction

1. *abc*-Conjecture and Nevanlinna's Second Main Theorem.
2. Higher dimensional generalization of First Main Theorem for coherent ideal sheaves (N.).
3. Higher dimensional generalization due to Bloch, Cartan, Weyls, Ahlfors, Stoll.
4. Eremenko-Sodin's SMT.
5. Nochka and Corvaja-Zannier-Min Ru's SMT.
6. Equidimensional SMT due to Griffiths et al.
7. Conjectures for entire curves.
8. Geometric proof of Cartan's SMT. o
9. An inequality of SMT type and Log Bloch-Ochiai's Theorem.
10. Yamanoi's *abc*-Theorem.
11. SMT (abc) for a semi-abelian variety due to N.-Winkelmann-Yamanoi.
12. Applications for degeneracy, unicity, and intersections.
13. Analogue for arithmetic recurrences.

§2 abc-Conjecture

First recall the unit equation with variables a, b, c:

$$a + b = c. \tag{2.1}$$

This was studied by C.L. Siegel, K. Mahler, and ....

Why is this equation interesting?

There might be several answers, but one should be that

(2.1) gives a *hyperbolic space* X.

In fact, since a, b, c are units, equation (2.1) is transformed by $x = a/c,\ y = b/c$ to

$$x + y = 1.$$

Then the triple (a, b, c) is mapped to x which is unit and different to 1:

$$x \in X \cong \mathbf{P}^1 \setminus \{0, 1, \infty\}.$$

abc-Conjecture

(Masser-Oesterlé (1985)). Let $a, b, c \in \mathbf{Z}$ be co-prime integers satisfying

$$a + b = c.$$

Then for ${}^{\forall}\epsilon > 0$, ${}^{\exists}C_\epsilon > 0$ such that

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \left(\prod_{\text{prime } p|(abc)} p \right)^{1+\epsilon}. \tag{2.2}$$

N.B. The order of *abc* at every prime *p* is counted only by "1" (truncation), when it is positive.

We put $x = [a, -b] \in \mathbf{P}^1(\mathbf{Q})$, and set

$$\text{(2.3)} \qquad \mathrm{h}(x) = \log \max\{|a|, |b|\} \geq 0 \quad \text{(height)},$$

$$\text{(2.4)} \qquad N_1(x; \infty) = \sum_{p|a} \log p \quad \text{(counting function truncated to level 1)},$$

$$N_1(x; 0) = \sum_{p|b} \log p \qquad (\quad '' \quad),$$

$$N_1(x; 1) = \sum_{p|c} \log p \qquad (\quad '' \quad).$$

Then *abc*-Conjecture (2.2) is rewritten as

$$\text{(2.5)} \qquad (1-\epsilon)\mathrm{h}(x) \leq N_1(x; 0) + N_1(x; \infty) + N_1(x; 1) + C_\epsilon, \ x \in \mathbf{P}^1(\mathbf{Q}).$$

More generally, for q distinct points $a_i \in \mathbf{P}^1(\mathbf{Q}), 1 \leq i \leq q$,

$$(q-2-\epsilon)\mathrm{h}(x) \leq \sum_{i=1}^{q} N_1(x; a_i) + C_\epsilon. \tag{2.6}$$

(N. '96, Vojta '98) $^\exists abc \cdots$ Conjecture.

Without trunction,

$$(q-2-\epsilon)\mathrm{h}(x) \leq \sum_{i=1}^{q} N(x; a_i) + C_\epsilon; \tag{2.7}$$

this is **Roth's Theorem** and the corresponding higher dimensional version is **Schmidt's Subspace Theorem**.

§3 R. Nevanlinna's theory

In complex function theory, the unit equation (2.1) was studied earlier by E. Picard and E. Borel for *units of entire functions*.

Picard's Theorem (1879). *A meromorphic function f on $\mathbf{C}$ omitting three distinct values of $\mathbf{P}^1$ must be constant.*

How are they related?

If f omits $0, 1, \infty$, then $f, (1-f)$ and 1 are units in the ring of entire functions, and satisfy

$$f + (1-f) = 1;$$

this observation is due to E. Borel, who dealt with more variables.

R. Nevanlinna (Acta 1925): Quantitative theory to measure the frequencies for non-constant f to take those three values.

Second Main Theorem $\Longleftrightarrow$ abc Conjecture.

Let $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ be a meromorphic function;
Ω a Fubini-Study metric form on $\mathbf{P}^1(\mathbf{C})$;
$a \in \mathbf{P}^1(\mathbf{C})$.
Set

$$T_f(r) = T_f(r, \Omega) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\Omega \quad \text{(Shimizu (1929)-Ahlfors)},$$

$$N_k(r, f^*a) = \int_1^r \frac{dt}{t} \int_{|z|<t} \min\{\mathrm{ord}_z f^*a, k\}, \quad k \leq \infty,$$

$$N(r, f^*a) = N_\infty(r, f^*a),$$

$$m_f(r, a) = \int_{|z|=r} \log^+ \frac{1}{|f(z) - a|} \frac{d\theta}{2\pi}, \quad \left(\tfrac{1}{|f(z)-\infty|} = |f(z)| \right).$$

These are called the order (characteristic , height) function,
the counting function truncated to level k;
the proximity (approximation) function.

Theorem 3.1

(Nevanlinna's First Main Theorem (1925))

$$T_f(r) = N(r, f^*a) + m_f(r, a) + O(1);$$
$$N(r, f^*a) \leqq T_f(r) + O(1).$$

Theorem 3.2

(Nevanlinna's Second Main Theorem (1925)) *For q distinct points $a_j \in \mathbf{P}^1(\mathbf{C}), 1 \leq j \leq q$*

$$(3.3) \qquad (q-2)T_f(r) \leq \sum_{j=1}^{q} N_1(r, f^*a_j) + S_f(r),$$

where $S_f(r) = O(\log rT_f(r))||_E$ $(E \subset [1, \infty), meas(E) < \infty, r \notin E)$.

N.B. (i) The truncation of level 1 in the counting function $N_1(r, f^*a_j)$ is important in the theory and applications.

N.B. (ii) The arithmetic analogue of (3.3) with non-truncated counting functions $N_\infty(*, *)$ holds (Diophantine approx.; Roth's Theorem (1955)).

The Mandala of the analogues:

"Value Dist."		"Dist. of Rational Points"
$\boxed{f : \mathbf{C} \to X}$	$\Longleftrightarrow$	$\boxed{\text{Infin. Family of Rat'l Pts}}$
$\boxed{\text{Kobay. Hyperbolic.}}$	$\Longleftrightarrow$ $\boxed{\text{Lang's Conj.}}$	$\boxed{\text{Finiteness of Rat'l Pts}}$
$\Uparrow$		$\Uparrow$
$\boxed{\text{Nevan. Theory}}$	$\Longleftrightarrow$ $\boxed{\text{Vojta's Dict.}}$	$\boxed{\text{Dioph. Approx.}}$

§4 Higher dimensional generalizations

FMT for coherent ideal sheaves (N. Book, 2003)

Theorem 4.1

(K. Oka VII 1948) *For a complex space X, the structure sheaf $\mathcal{O}_X$ is coherent.*

Let $f : \mathbf{C}^m \to X$ be a mer. map into a compact complex space X.
Let $\mathcal{I}$ be a coherent ideal sheaf of $\mathcal{O}_X$.
Let $\{U_j\}$ be a finite open covering of X such that

1. $^{\exists}$partition of unity $\{c_j\}$ associated with $\{U_j\}$,
2. $^{\exists}$finitely many sections $\sigma_{jk} \in \Gamma(U_j, \mathcal{I})$, $k = 1, 2, \ldots$, generating every fiber $\mathcal{I}_x$ over $^{\forall}x \in U_j$.

Setting $\rho_{\mathcal{I}}(x) = C\left(\sum_j c_j(x) \sum_k |\sigma_{jk}(x)|^2\right)$ with constant $C > 0$, we may have

$$\rho_{\mathcal{I}}(x) \leqq 1, \qquad x \in X.$$

Set $Y = \mathcal{O}_X/\mathcal{I}$ (possibly non-reduced), and

$$\omega_Y = \omega_{\mathcal{I}} = -dd^c \log \rho_{\mathcal{I}}, \qquad d^c = \frac{i}{4\pi}(\bar{\partial} - \partial).$$

Suppose that $f(\mathbf{C}^m) \not\subset \mathrm{Supp}\, Y$. Set

$$T_f(r, \omega_{\mathcal{I}}) = \int_1^r \frac{dt}{t^{2m-1}} \int_{\|z\|<t} f^*\omega_{\mathcal{I}} \wedge (dd^c\|z\|^2)^{m-1},$$

$$N(r, f^*\mathcal{I}) = \int_1^r \frac{dt}{t^{2m-1}} \int_{\{\|z\|<t\}\cap f^*Y} (dd^c\|z\|^2)^{m-1},$$

$$N_k(r, f^*\mathcal{I}) = \int_1^r \frac{dt}{t^{2m-1}} \int_{\{\|z\|<t\}\cap \underline{f^*Y}_k} (dd^c\|z\|^2)^{m-1},$$

$$m_f(r, \mathcal{I}) = \int_{\|z\|=r} \frac{1}{2} \log \frac{1}{\rho_{\mathcal{I}}\circ f}\, \eta. \quad \left(\int_{\|z\|=r} \eta = 1.\right)$$

$T_f(r, \omega_{\mathcal{I}})$ is well-defined up to $O(1)$-term.

Theorem 4.2

(FMT, N. 2003)

$$T_f(r, \omega_{\mathcal{I}}) = N(r, f^*\mathcal{I}) + m_f(r, \mathcal{I}) - m_f(1, \mathcal{I}).$$
$$N(r, f^*\mathcal{I}) < T_f(r, \omega_{\mathcal{I}}) + O(1).$$

N.B. If Y is a Cartier divisor D, $\omega_D \in c_1(L(D))$ and

$$T_f(r, \omega_{\mathcal{I}}) = T_f(r, \omega_D) = T_f(r, L(D)).$$

SMT: The following SMT is a typical case due to H. Cartan, A.L. Ahlfors, W. Stoll, ...

Theorem 4.3

Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate meromorphic map. Let $H_j, 1 \le j \le q$, be q hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. Then

$$(q - n - 1) T_f(r) \le \sum_{j=1}^{q} N_n(r, f^* H_j) + O(\log^+(r T_f(r)))||$$

The next is due to Carlson, Griffiths, King, Shiffman, N. ... ($Y \to \mathbf{C}^m$, a finite cover, $f : Y \to V$)

Theorem 4.4

Let $f : \mathbf{C}^m \to V$ be a diff. non-deg. $(\max_z \mathrm{rank}\, df_z = \dim V)$ *meromorphic map into a projective algebraic manifold V, and $D = \sum_j D_j$ a s.n.c. (simple noraml crossing) divisor on V.*

$$T_f(r, L(D)) + T_f(r, K_V) < \sum_j N_1(r, f^* D_j) + O(\log r T_f(r))||.$$

Corollary 4.5

If V is of general type, then $f : \mathbf{C}^m \to V$ is diff. deg.

N.B. The above SMT in the diff. non-deg. case is quite complete, and the case of entire curves remains open.

§5 Conjectures for entire curves

Referring to the above mentioned results, one may pose the following:

Fund. Conj. for entire curves.

Let X be a smooth compact algebraic variety, and let $D = \sum_j D_j$ be a s.n.c. divisor on X with irreducible components D_j.

Then, for an alg. non-deg. $f : \mathbf{C} \to X$ we have

$$(5.1) \qquad T_f(r; L(D)) + T_f(r; K_X) \leq \sum_j N_k(r; f^* D_j) + \epsilon T_f(r)||, \ {}^\forall \epsilon > 0,$$
$$1 \leq {}^\exists k \leq \dim X.$$

N.B. If X is an abelian variety, $K_X = 0$.

This implies

Green-Griffiths' Conjecture (1980). If X is a variety of (log) general type, then ${}^\forall f : \mathbf{C} \to X$ is algebraically degenerate.

For $T_f(r) \leq \epsilon T_f(r)||$.

Then this implies

Kobayashi Conjecture (1970). If $X \subset \mathbf{P}^n(\mathbf{C})$ is a general hypersurface of $\deg X \geq 2n - 1$, then X is Kobayashi hyperbolic.

The implication is supported by

Theorem 5.2

(C. Voisin (1996/98))

Let $X \subset \mathbf{P}^n(\mathbf{C})$ be a general hypersurface of deg $\geq 2n - 1$ for $n \geq 3$. Then every subvariety of X is of general type.

§6 Lemma on Log. Jet Differential

Let X be a complex manifold,
$\mathcal{M}^*_X$ be the sheaf of non-zero meromophic functions,
Ω^1_X be the sheaf of holomorphic 1-fomrs, and set

$$\mathcal{A}^1_X = d \log \mathcal{M}^*_X + \Omega^1_X.$$

This is the sheaf of log. differentials (Abelian differentials of 3'rd kind).

Lemma 6.1

(Lemma on Log. Jet Diff., N. 1977)
Let X be compact, and $\eta \in H^0(X, \mathcal{A}^1_X)$.
Let $f : \mathbf{C} \to X$ such that $f(\mathbf{C}) \not\subset \mathrm{Supp}\,(\eta)_\infty$ (polar set), and set $f^*\eta = \xi(z)dz$. Then

$$m(r, \xi^{(k)}) = O(\log rT_f(r))||, \quad k = 1, 2, \ldots.$$

Lemma on Log. Der.: $\eta = d \log w = \frac{dw}{w}$, $f^*\eta = \frac{f'}{f}dz$, so that $\xi = \frac{f'}{f}$.

§7 Geometric Proof of Cartan's SMT

Recall the point of the proof of Griffiths' SMT (another proof): Let D be a s.n.c. divisor on V and $\sigma_D \in H^0(V, L(D))$ defining D. Let Ω be a C^∞ volume form on V and set a singular volume form

$$\Psi = \frac{\Omega}{\|\sigma_D\|^2} = \frac{v(x)}{\|\sigma_D(x)\|^2} |dx_1 \wedge \cdots \wedge dx_n|^2.$$

Let $\mathbf{C}^n \to V$ be diff. non-deg. and set

$$f^*\Psi = \frac{v \circ f \cdot |\mathrm{Jac}(f)|^2}{\|\sigma_D \circ f\|^2} |dz_1 \wedge \cdots \wedge dz_n|^2 \not\equiv 0.$$

Then

$$dd^c \log f^*\Psi = f^*c_1(K_V) + f^*c_1(L(D)) + (\mathrm{Jac}(f))_0 - f^*D.$$

Lemma on Log. Jet Diff. implies

$$\int dd^c \log f^*\Psi = O(\log rT_f(r))||.$$

Then, Theorem 4.4 (Griffiths et ál.) is deduced.

For an entire curve $f : \mathbf{C} \to V$ we do not have $f^*\Psi \not\equiv 0$ a priori. But we try to form an infinitesimal area in the dual K_V^* by means of a C^∞ connection ∇ in the hol. tangent bundle $\mathbf{T}(V)$. Set

$$f^{(1)}(z) = f'(z) \text{ (hol.)}, \quad f^{(k+1)}(z) = \nabla_{f'} f^{(k)}(z) \text{ (not hol.)}, k = 1, 2, \ldots.$$

Then $f^{(k)}(z) \in \mathbf{T}(V)$. Define the *Wronskian* of f w.r.t. ∇:

$$W(f; \nabla)(z) = f^{(1)}(z) \wedge \cdots \wedge f^{(n)}(z) \in K^*_{V, f(z)}$$

Then the paring " $|W(f; \nabla)|^2 \cdot \Omega \circ f$ " makes sense, and we have

$$dd^c \log \frac{|W(f; \nabla)|^2 \cdot \Omega \circ f}{\|\sigma_D \circ f\|^2} = f^* c_1(K_V) + f^* c_1(L(D)) - f^* D + dd^c \log |W(f; \nabla)|^2.$$

Lemma on Log. Jet Diff. works for $dd^c \log \frac{v \circ f \cdot |W(f; \nabla)|^2}{\|\sigma_D \circ f\|^2}$, provided that D is ∇-totally geodesic, but

Problem: What is $dd^c \log |W(f; \nabla)|^2$?

Proposition 7.1

(N. to appear in J. Math. Sci. U-Tokyo, 2011?) *Assume that* $\log|W(f;\nabla)| \not\equiv -\infty$ *is subharmonic, and that* $D = \sum D_j$ *is s.n.c. and* ∇*-totally geodesic. Then,*

$$T_f(r; L(D)) + T_f(r; K_V) \leq \sum_j N_n(r; f^* D_j) + O(\log r T_f(r))||.$$

Theorem 7.2

(ibid.) *Let* $V = \mathbf{P}^n(\mathbf{C})$ *and* ∇ *be the Fubini-Study metric connection. Then,*
(i) $W(f;\nabla)$ *is holomorphic.*
(ii) $W(f;\nabla) \equiv 0$ *iff* f *is linearly degenerate.*

N.B. (i) $n = 2$ due to Siu (1987). (Needs to check one tensor $\bar{\partial}\Gamma_{ij}^k$.)
(ii) $f^{(k)}, k \geq 2$, are *not* holomorphic at all.
(iii) See Preprint UTMS 2011 for details.

Fact: (i) $K^n_{\mathbf{P}}(\mathbf{C}) = O(-n-1) = (-n-1)O(1)$.
(ii) Hyperplanes are ∇-totally geodesic.

These together $\Longrightarrow$ Cartan's SMT, Theorem 4.3.

N.B. Y. Tiba is working on $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$ from this approach (Preprints UTMS 2010/11).

§8 SMT & Log. Bloch-Ochiai's Theorem

By Lemma on Log. Jet Diff. we have

- **Inequality of SMT type** (N. (1977∼'81)): Let D be a reduced divisor on V such that the quasi-Albanese mapping $\alpha : V \setminus D \to A_{V\setminus D}$ satisfies the following conditions:
  1. α is dominant;
  2. $\mathrm{St}(W) = \{a \in A_{V\setminus D}; a + W = W\}$ is finite, where $W = \overline{\alpha(V \setminus D)}^{\mathrm{Zar}}$.

  Then ${}^{\exists}\kappa > 0$ such that for algebraically non-degenerate $f : \mathbf{C} \to V$,

$$\kappa T_f(r) \leq N_1(r, f^*D) + O(\log rT_f(r))||.$$

- **Log Bloch-Ochiai Theorem** (N. 1981): If log. irreg. $\bar{q}(V \setminus D) = \dim H^0(V, \Omega^1(\log D)) > \dim V$, then ${}^{\forall}f : \mathbf{C} \to V \setminus D$ is alg. deg.

  N.B. (i) This unifies Picard's Theorem. Borel's Theorem (1897) and Bloch-Ochiai's Theorem ($D = \emptyset$) (1926/77) for projective V.
  (ii) This holds for compact Kähler V (N.-Winkelmann 2002).

§9 SMT (abc) for semi-abelian varieties

Definitions:

Let A be a semi-abelian variety:

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \text{ (abelian variety)} \to 0.$$

The universal covering $\tilde{A} \cong \mathbf{C}^n$, $n = \dim A$.

N.B. Borel's case: $\mathbf{P}^n(\mathbf{C}) \setminus n+1$ hyperplanes in gen. pos. $= (\mathbf{C}^*)^n$.

Let $f : \mathbf{C} \to A$ be an entire curve.

Set

- $J_k(A)$: the k-jet bundle over A; $J_k(A) \cong A \times \mathbf{C}^{nk}$;
- $J_k(f) : \mathbf{C} \to J_k(A)$: the k-jet lift of f;
- $X_k(f)$: the Zariski closure of the image $J_k(f)(\mathbf{C})$.

Set the jet projection

$$I_k : J_k(A) \cong A \times \mathbf{C}^{nk} \to \mathbf{C}^{nk}.$$

Lemma 9.1

(Lemma on Log. Jet Diff., N. '77) *For $f : \mathbf{C} \to \bar{A}$ (compactification),*

$$m(r; I_k \circ J_k(f)) \stackrel{\text{def}}{=} \int_{|z|=r} \log^+ \|I_k \circ J_k(f)(z)\| \frac{d\theta}{2\pi} = O(\log^+(rT_f(r))) \,||.$$

Theorem 9.2

(N.-Winkelmann-Yamanoi, Acta 2002 & Forum Math. 2008, Yamanoi Forum Math. 2004)

Let $f : \mathbf{C} \to A$ *be alg. non-deg.*

(i) *Let* Z *be an algebraic reduced subvariety of* $X_k(f)$ $(k \geqq 0)$. *Then* ${}^{\exists}\bar{X}_k(f)$, *compactification of* $X_k(f)$ *such that*

(9.3) $$T_{J_k(f)}(r; \omega_{\bar{Z}}) = N_1(r; J_k(f)^* Z) + o(T_f(r))||.$$

(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geqq 2$, *then*

(9.4) $$T_{J_k(f)}(r; \omega_{\bar{Z}}) = o(T_f(r))||.$$

(iii) *If* $k = 0$ *and* Z *is an effective divisor* D *on* A, *then* $\bar{A}$ *is smooth, equivariant, and independent of* f; *furthermore,* (9.5) *takes the form*

(9.5) $$T_f(r; L(\bar{D})) = N_1(r; f^* D) + o(T_f(r; L(\bar{D})))||.$$

N.B. For the truncation of level 1, the error term "$o(T_f(r))$" cannot be replaced by "$O(\log^+(rT_f(r)))$".

Corollary 9.6

Let A be a semi-abelian variety and D a non-zero divisor. If $f : \mathbf{C} \to A$ omits D, then f is alg. deg.

This was called Lang's Conjeture: The abelian case due to Siu-Yeung (1996); the semi-abelian case due to N. (1998).

The trunction level 1 in Theorem 9.2 implies a number of applications.

Theorem 9.7

(Conjectured by M. Green, 1974) *Assume that $f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C})$ omits two lines $\{x_i = 0\}, i = 1, 2$, and the conic $\{x_0^2 + x_1^2 + x_2^2 = 0\}$. Then f is algebraically degenerate.*

N.B. This holds in an even more general setting as seen in the next section.

N.B. Theorem 9.7 is optimal in two senses:

1. the number of irreducible components is 3;
2. the total degree is 4.
3. The case of 4 lines is due to E. Borel (1897).
4. For deg $D \leq 3$, $^{\exists}$counter-example of an alg. non-deg. $f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C}) \setminus D$ ($^{\exists}$diff. non-deg. $f : \mathbf{C}^2 \to \mathbf{P}^2(\mathbf{C}) \setminus D$ (Buzzard-Lu 2000).

§10 Application I: Degeneracy theorems

Theorem 10.1

(N.-Winkelmann-Yamanoi, J. Math. Pure. Appl. '07)
Let X be an algebraic variety such that

(i) $\bar{q}(X) \geqq \dim X$ *(log irregularity);*

(ii) $\bar{\kappa}(X) > 0$ *(log Kodaira dimension);*

(iii) *the (quasi-)Albanese map $X \to A$ is proper.*

Then $^{\forall} f : \mathbf{C} \to X$ is algebraically degenerate.
Moreover, the normalization of $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a semi-abelian variety which is finite étale over a translate of a proper semi-abelian subvariety of A.

N.B. The case "$\bar{q}(X) > \dim X$" was "Log Bloch-Ochiai's Theorem" (N. '77-'81). The proof for the case "$\bar{q}(X) = \dim X$" requires the new S. M. Theorem 9.2.

Corollary 10.2

Let $\pi : X \to A$ be a finite cover over a simple abelian variety. If X is not isomorphic to an abelian variety, then X is Kobayashi hyperbolic.

Specializing $X = \mathbf{P}^n(\mathbf{C})$, we have

Theorem 10.3

Let $D = \sum_{j=1}^{q} D_j \subset \mathbf{P}^n(\mathbf{C})$ be an s.n.c. divisor.
Assume that $q > n$ and $\deg D > n+1$.
Then ${}^\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus D$ is algebraically degenerate.

M. Green's Conjecture: $n = 2$, $q = 3$, $\deg D = 4 > 2 + 1$.

Question 1. Let $D = \sum_{i=1}^{q} D_i \subset \mathbf{P}^n(\mathbf{C})$ be a divisor in general positionAssume that $q > n$ and $\deg D > n+1$. Then, is $\bar{\kappa}(\mathbf{P}^n(\mathbf{C}) \setminus D) > 0$? ($f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus D$ alg. deg.?)

Question 2. Let $D = D_1 + D_2 \subset \mathbf{P}^2(\mathbf{C})$ be an s.n.c. divisor with two conics D_j. Then, is ${}^\forall f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C}) \setminus D$ algebraically degenerate?

§11 Application II: A new unicity theorem

The followings are famous applications of Nevnlinna-Cartan's SMT.

Theorem 11.1

Let $f, g : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ be non-constant meromorphic functions.

1. (Unicity Theorem, Nevanlinna (1926))
   Asuume that there are 5 distinct points $\{a_j\}_{j=1}^5 \subset \mathbf{P}^1(\mathbf{C})$ such that $\mathrm{Supp}\, f^* a_j = \mathrm{Supp}\, g^* a_j, 1 \leq j \leq 5$. Then $f \equiv g$.
2. (Unicity Theorem, H. Cartan (1927))
   Asuume that there are given 3 distinct points $\{a_j\}_{j=1}^3 \subset \mathbf{P}^1(\mathbf{C})$, Then there exist at most 2 meromorphic functions f and g such that $f^* a_j = g^* a_j$ for all $j = 1, 2, 3$.

Question. Does $\mathrm{Supp}\, f^* a_j = \mathrm{Supp}\, g^* a_j$ suffice for Cartan's unicity theorem?

Erdös' Problem (1988) (Unicity problem for arithmetic recurrences). *Let x, y be positive integers. Is it true that*

$$\{p; prime, p|(x^n - 1)\} = \{p; prime, p|(y^n - 1)\}, \forall n \in \mathbf{N}$$

$$\Longleftrightarrow x = y \quad ?$$

The answer is Yes by Corrales-Rodorigáñez and R. Schoof, JNT 1997, with the elliptic analogue.

In complex analysis, K. Yamanoi proved the following striking unicity theorem in Forum Math. 2004:

Theorem 11.2

(Yamanoi's Unicity Theorem) *Let*
$A_i, i = 1, 2,$ be abelian varieties;
$D_i \subset A_i$ be irreducible divisors such that $\mathrm{St}(D_i) = \{0\}$;
$f_i : \mathbf{C} \to A_i$ be algebraically nondegenerate.
Assume that $f_1^{-1} D_1 = f_2^{-1} D_2$ as sets.
Then $\exists$ isomorphism $\phi : A_1 \to A_2$ such that

$$f_2 = \phi \circ f_1, \quad D_1 = \phi^* D_2.$$

N.B. The new point is that we can determine not only f, but the moduli point of a polarized abelian variety (A, D) through the distribution of $f^{-1}D$ by an algebraically nondegenerate $f : \mathbf{C} \to A$.

We want to generalize this to semi-abelian varieties to have a uniformized theory.

Let $A_i, i = 1, 2$ be semi-abelian varieties:

$$0 \to (\mathbf{C}^*)^{t_i} \to A_i \to A_{0i} \to 0.$$

Let D_i be an irreducible divisor on A_i such that

$$\mathrm{St}(D_i) = \{0\}$$

just for simplicity; this is not restrictive.

Theorem 11.3

(Corvaja-N., preprint 2009, to appear in Math. Ann.) *Let* $f_i : \mathbf{C} \to A_i$ $(i = 1, 2)$ *be algebraically non-degenerate entire curves.*
Assume that

$$\text{(11.4)} \qquad \underline{\mathrm{Supp}\, f_1^* D_1}_\infty \subset \underline{\mathrm{Supp}\, f_2^* D_2}_\infty \text{ (germs at } \infty),$$

$$\text{(11.5)} \qquad {}^\exists C > 0 \text{ such that },\ C^{-1} N_1(r, f_1^* D_1) \leq N_1(r, f_2^* D_2)||.$$

Then ${}^\exists$*a finite étale morphism* $\phi : A_1 \to A_2$ *such that*

$$\phi \circ f_1 = f_2, \quad D_1 \subset \phi^* D_2.$$

If equality holds in (11.4)*, then* ϕ *is an isomorphism and* $D_1 = \phi^* D_2$*.*

N.B. (i) Assumption (11.5) is necessary by example.
(ii) There is some analogous results in arithmetic recurrence (dynamics).

The following is immediate from Theorem 11.3.

Corollary 11.6

1. *There are at most 2 entire functions* $f, g : \mathbf{C} \to \mathbf{C}^*$ *such that* $f^{-1}1 = g^{-1}1$.
2. *Let* $f : \mathbf{C} \to \mathbf{C}^*$ *and* $g : \mathbf{C} \to E$ *with an elliptic curve* E *be holomorphic and non-constant. Then*
$$\underline{f^{-1}\{1\}}_{\infty} \neq \underline{g^{-1}\{0\}}_{\infty}.$$
3. *If* $\dim A_1 \neq \dim A_2$ *in Theorem 11.3, then*
$$\underline{f_1^{-1}D_1}_{\infty} \neq \underline{f_2^{-1}D_2}_{\infty}.$$

N.B.

$$\begin{tikzcd} \mathbf{C} \arrow[r, "f"] \arrow[dr, "g"'] & \mathbf{C}^* \arrow[d, "/\langle\tau\rangle"] \\ & E \end{tikzcd}$$

§12 Application III: Intersections on the image space.

In his book, “Introduction to Transcendental Numbers”, Addison-Wesley, 1966, S. Lang wrote at the last paragraph of Chap. 3:

“Independently of transcendental problem one can raise an interesting question of algebraic-analytic nature, namely given a 1-parameter subgroup of an abelian variety (say Zariski dense),

(i) is its intersection with a hyperplane section necessarily non-empty? ,

(ii) and infinite unless this subgroup is algebraic?”

In 6 years later, J. Ax (Amer. J. Math. (1972)) essentially solved the two questions affirmatively.

The first (i) extended to enitre curves was called Lang's Conjecture and aslo solved affirmatively as we have seen already.

Question. Does (ii) hold for entire curves?

By applying our SMT, Theorem 9.2 (N.-W.-Y.) we have

Theorem 12.1

(Corvaja-N., ibid.) *Let A be a semi-abelian variety of $\dim A \geq 2$, $f : \mathbf{C} \to A$ be alg. non-deg., D be a reduced divisor on A.*

1. *If $|\mathrm{St}(D)| < \infty$, then there exists an irreducible component $D' \subset D$ such that $f(\mathbf{C}) \cap D'$ is Zariski dense in D'; in particular, $|f(\mathbf{C}) \cap D| = \infty$.*
2. *If f is a 1-parameter subgroup and A is abelian, then $|f(\mathbf{C}) \cap D| = \infty$.*

N.B. "$|\mathrm{St}(D)| < \infty$" is necessary. Set

$A = (\mathbf{C}/\mathbf{Z}[i])^2, \quad f : z \in \mathbf{C} \to ([z], [z^2]) \in A, \quad D = \{0\} \times \mathbf{C}/\mathbf{Z}[i].$

Then $|f(\mathbf{C}) \cap D| = \{0\}$.

Corollary-Example.

Let $A = (\mathbf{C}^*)^n$ be $\mathbf{P}^n(\mathbf{C})$ minus $n+1$ hyperplanes $H_j, 0 \leq j \leq n$, in gen. pos., and take one more hyperplane or hypersurface D in gen. pos.
Then for ${}^\forall f : \mathbf{C} \to A$ alg. non-deg., $|f(\mathbf{C}) \cap D| = \infty$ (Zariski dense in D).

Question. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be alg. non-deg. Then is $|f(\mathbf{C}) \cap (\cup_j H_j \cup D)| = \infty$? (Zariski dense?)

`Thank you for your attention!`