Nevanlinna theory in higher dimensions and related Diophantine problems

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§1 Introduction

1. *abc*-Conjecture and Nevanlinna's Second Main Theorem.
2. Higher dimensional generalization of First Main Theorem for coherent ideal sheaves (N.).
3. Higher dimensional generalization due to Bloch, Cartan, Weyls, Ahlfors, Stoll.
4. Nochka and Corvaja-Zannier-Min Ru's improvement.
5. Equidimensional SMT due to Griffiths et al.
6. An inequality of SMT type and Log Bloch-Ochiai's Theorem.
7. Conjectures for holomorphic curves.
8. Yamanoi's *abc*-Theorem.
9. SMT (abc) for a semi-abelian variety due to N.-Winkelmann-Yamanoi.
10. Applications for degeneracy and unicity for holomorphic.
11. Analogue for arithmetic recurrences.

§2 abc Conjecture

First recall the unit equation with variables a, b, c:

$$a + b = c. \tag{2.1}$$

This was studied by C.L. Siegel, K. Mahler, and ....

Why is this equation interesting?

There might be several answers, but one should be that

(2.1) gives a *hyperbolic space* X.

In fact, since a, b, c are units, equation (2.1) is transformed by $x = a/c,\ y = b/c$ to

$$x + y = 1.$$

Then the triple (a, b, c) is mapped to x which is unit and different to 1:

$$x \in X \cong \mathbf{P}^1 \setminus \{0, 1, \infty\}.$$

abc-**Conjecture** (Masser-Oesterlé (1985)). Let $a, b, c \in \mathbf{Z}$ be co-prime integers satisfying

$$a + b = c.$$

Then for ${}^{\forall}\epsilon > 0$, ${}^{\exists}C_\epsilon > 0$ such that

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \left(\prod_{\text{prime } p|(abc)} p \right)^{1+\epsilon}. \tag{2.2}$$

N.B. The order of *abc* at every prime p is counted only by "1" (truncation), when it is positive.

We put $x = [a, -b] \in \mathbf{P}^1(\mathbf{Q})$, and set

$$(2.3) \qquad \mathrm{h}(x) = \log \max\{|a|, |b|\} \geq 0 \quad \text{(height)},$$

$$(2.4) \qquad N_1(x; \infty) = \sum_{p|a} \log p \quad \text{(counting function truncated to level 1)},$$

$$N_1(x; 0) = \sum_{p|b} \log p \qquad (\quad '' \quad),$$

$$N_1(x; 1) = \sum_{p|c} \log p \qquad (\quad '' \quad).$$

Then *abc*-Conjecture (2.2) is rewritten as

$$(2.5) \qquad (1-\epsilon)\mathrm{h}(x) \leq N_1(x; 0) + N_1(x; \infty) + N_1(x; 1) + C_\epsilon, \ x \in \mathbf{P}^1(\mathbf{Q}).$$

More generally, for q distinct points $a_i \in \mathbf{P}^1(\mathbf{Q}), 1 \leq i \leq q$,

(2.6) $$(q-2-\epsilon)\mathrm{h}(x) \leq \sum_{i=1}^{q} N_1(x; a_i) + C_\epsilon.$$

(N. '96, Vojta '98) ${}^{\exists}abc \cdots$ Conjecture.

Without trunction,

(2.7) $$(q-2-\epsilon)\mathrm{h}(x) \leq \sum_{i=1}^{q} N(x; a_i) + C_\epsilon;$$

this is Roth's Theorem and the corresponding higher dimensional version is Schmidt's Subspace Theorem.

§3 R. Nevanlinna's theory

In complex function theory, the unit equation (2.1) was studied earlier by E. Picard and E. Borel for *units of entire functions*.

Picard's Theorem (1879). *A meromorphic function f on $\mathbf{C}$ omitting three distinct values of $\mathbf{P}^1$ must be constant.*

How are they related?

If f omits $0, 1, \infty$, then $f, (1-f)$ and 1 are units in the ring of entire functions, and satisfy

$$f + (1-f) = 1;$$

this observation is due to E. Borel, who dealt with more variables.

R. Nevanlinna (Acta 1925): Quantitative theory to measure the frequencies for non-constant f to take those three values.

Second Main Theorem $\Longleftrightarrow$ abc Conjecture.

Let $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ be a meromorphic function;
Ω a Fubini-Study metric form on $\mathbf{P}^1(\mathbf{C})$;
$a \in \mathbf{P}^1(\mathbf{C})$.
Set

$$T_f(r) = T_f(r, \Omega) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\Omega \quad \text{(T. Shimizu (1929))},$$

$$N_k(r, f^*a) = \int_1^r \frac{dt}{t} \int_{|z|<t} \min\{\mathrm{ord}_z f^*a, k\}, \quad k \leq \infty,$$

$$N(r, f^*a) = N_\infty(r, f^*a),$$

$$m_f(r, a) = \int_{|z|=r} \log^+ \frac{1}{|f(z) - a|} \frac{d\theta}{2\pi}, \quad \left(\frac{1}{|f(z)-\infty|} = |f(z)| \right).$$

These are called the order (characteristic , height) function,
the counting function truncated to level k;
the proximity (approximation) function.

Theorem 3.1

(Nevanlinna's First Main Theorem (1925))

$$T_f(r) = N(r, f^*a) + m_f(r, a) + O(1);$$
$$N(r, f^*a) \leqq T_f(r) + O(1).$$

Theorem 3.2

(Nevanlinna's Second Main Theorem (1925)) *For q distinct points $a_j \in \mathbf{P}^1(\mathbf{C}), 1 \leq j \leq q$*

$$(q-2)T_f(r) \leq \sum_{j=1}^{q} N_1(r, f^*a_j) + S_f(r), \tag{3.3}$$

where $S_f(r) = O(\log rT_f(r))||_E$ $(E \subset [1, \infty), meas(E) < \infty, r \notin E)$.

N.B. (i) The truncation of level 1 in the counting function $N_1(r, f^*a_j)$ is important in the theory and applications.

N.B. (ii) The arithmetic analogue of (3.3) with non-truncated counting functions $N_\infty(*,*)$ holds (Diophantine approx.; Roth's Theorem (1955)).

The Mandala of the analogues:

$$
\begin{array}{ccc}
\text{“Value Dist.”} & & \text{“Dist. of Rational Points”} \\[2em]
\boxed{f : \mathbf{C} \to X} & \Longleftrightarrow & \boxed{\text{Infin. Family of Rat'l Pts}} \\[2em]
\boxed{\text{Kobay. Hyperbolic.}} & \begin{array}{c} \Longleftrightarrow \\ \boxed{\text{Lang's Conj.}} \end{array} & \boxed{\text{Finiteness of Rat'l Pts}} \\[2em]
\Uparrow & & \Uparrow \\[2em]
\boxed{\text{Nevan. Theory}} & \begin{array}{c} \Longleftrightarrow \\ \boxed{\text{Vojta's Dict.}} \end{array} & \boxed{\text{Dioph. Approx.}}
\end{array}
$$

§4 Higher dimensional generalizations

The higher dimensional generalization of Nevanlinna's theory was started almost the same time as Nevanlinna's work.

1. A. Bloch (1926∼), H. Cartan (1928∼), H.&J. Weyl (1938∼), A.L. Ahlfors (1941): the target $\mathbf{P}^1(\mathbf{C}) \Rightarrow \mathbf{P}^n(\mathbf{C})$.
2. W. Stoll (1953∼): the domain $\mathbf{C} \Rightarrow \mathbf{C}^m$, $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$.
3. A. Bloch (1926∼): $f : \mathbf{C} \to V$, algebraic variety.
4. Griffiths ét al. (1972∼): $f : \mathbf{C}^m \to V^n$ dominant ($f(\mathbf{C}^m) \supset$ open subset $\neq \emptyset$) into algebraic variety V.

The generalization mainly consists of two parts: the first main theorem (FMT) and the second main theorem (SMT).

FMT for coherent ideal sheaves

Let $f : \mathbf{C}^m \to X$ be a meromorphic map into a compact complex space (reduced).

Let $\mathcal{I}$ be a coherent ideal sheaf of the structure sheaf $\mathcal{O}_X$.

Let $\{U_j\}$ be a finite open covering of X such that

1. ${}^{\exists}$partition of unity $\{c_j\}$ associated with $\{U_j\}$,
2. ${}^{\exists}$finitely many sections $\sigma_{jk} \in \Gamma(U_j, \mathcal{I})$, $k = 1, 2, \ldots$, generating every fiber $\mathcal{I}_x$ over ${}^{\forall}x \in U_j$.

Setting $\rho_{\mathcal{I}}(x) = C \left(\sum_j c_j(x) \sum_k |\sigma_{jk}(x)|^2 \right)$ with constant $C > 0$, we may have

$$\rho_{\mathcal{I}}(x) \leqq 1, \qquad x \in X.$$

Set $Y = \mathcal{O}_X / \mathcal{I}$ (possibly non-reduced subspace), and

$$\omega_Y = \omega_{\mathcal{I}} = -dd^c \log \rho_{\mathcal{I}}, \qquad d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$$

Suppose that $f(\mathbf{C}^m) \not\subset \operatorname{Supp} Y$. Set

$$T_f(r,\omega_{\mathcal{I}}) = \int_1^r \frac{dt}{t} \int_{\|z\|<t} f^*\omega_{\mathcal{I}} \wedge (dd^c\|z\|^2)^{m-1},$$

$$N(r,f^*Y) = N(r,f^*\mathcal{I}) = \int_1^r \frac{dt}{t} \int_{\{\|z\|<t\}\cap f^*Y} (dd^c\|z\|^2)^{m-1},$$

$$N_k(r,f^*Y) = N_k(r,f^*\mathcal{I}) = \int_1^r \frac{dt}{t} \int_{\{\|z\|<t\}\cap \underline{f^*Y}_k} (dd^c\|z\|^2)^{m-1},$$

$$m_f(r,Y) = m_f(r,\mathcal{I}) = \frac{1}{2}\int_{\|z\|=r} \log \frac{1}{\rho_{\mathcal{I}}\circ f}\eta.$$

Theorem 4.1

(FMT, N. 2003)

$$T_f(r,\omega_{\mathcal{I}}) = N(r,f^*\mathcal{I}) + m_f(r,\mathcal{I}) - m_f(1,\mathcal{I}).$$
$$N(r,f^*\mathcal{I}) < T_f(r,\omega_{\mathcal{I}}) + O(1).$$

N.B. If Y is a Cartier divisor D, $\omega_D \in c_1(L(D))$ and

$$T_f(r,\omega_{\mathcal{I}}) = T_f(r,\omega_D) = T_f(r,L(D)).$$

SMT.

The following SMT is a typical case due to H. Cartan, A.L. Ahlfors, W. Stoll, ...

Theorem 4.2

Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate meromorphic map. Let $H_j, 1 \leq j \leq q$ be q hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. Then

$$(q-n-1)T_f(r) \leq \sum_{j=1}^{q} N_n(r, f^*H_j) + O(\log^+(rT_f(r)))||$$

The follwoing is due to Carlson, Griffiths, King, Shiffman, N. ...

Theorem 4.3

Let $f : \mathbf{C}^m \to V$ be a dominant meromorphic map into a projective algebraic manifold V, and
$D = \sum_j D_j$ an snc divisor on V.

$$T_f(r, L(D)) + T_f(r, K_V) < \sum_j N_1(r, f^* D_j) + O(\log r T_f(r))||.$$

Corollary 4.4

If V is of general type, then $f : \mathbf{C}^m \to V$ is necessarily non-dominant.

N.B. The above SMT in the dominant case is very satisfactory, and the case of holomorphic curves remains open.

Further generalizations:

- E.I. Nochka (1982): Linearly degenerate $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ (Cartan Conjecture): Let l be the dimension of the linear span of $f(\mathbf{C}^m)$.

$$(q - 2n + l - 1)T_f(r) \leq \sum_{j=1}^{q} N_l(r, f^* H_j) + O(\log^+(rT_f(r)))||.$$

- A.E. Eremenko and M.L. Sodin (1992): For $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ non-constant and $D_j \subset \mathbf{P}^n(\mathbf{C})$ hypersurfaces in general position,

$$(q - 2n)T_f(r) \leq \sum_{j=1}^{q} \frac{1}{\deg D_j} N(r, f^* D_j) + \epsilon T_f(r) + O(\log r)||.$$

- Corvaja-Zannier-Ru (2004): Hypersurfaces $D_j \subset \mathbf{P}^n(\mathbf{C})$ in general position and algebraically non-degenerate $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ ($f(\mathbf{C})$ Zariski dense in $\mathbf{P}^n(\mathbf{C})$).

$$(q - n - 1)T_f(r) \leq \sum_{j=1}^{q} \frac{1}{\deg D_j} N(r, f^* D_j) + \epsilon T_f(r) + O(\log r)||.$$

- **Inequality of SMT type** (N. (1977∼'81)): Let D be a reduced divisor on V such that the quasi-Albanese mapping $\alpha : V \setminus D \to A_{V\setminus D}$ satisfies the following conditions:
  1. α is dominant;
  2. $\mathrm{St}(W) = \{a \in A_{V\setminus D}; a + W = W\}$ is finite, where $W = \overline{\alpha(V \setminus D)}^{\mathrm{Zar}}$.

  Then ${}^{\exists}\kappa > 0$ such that for algebraically non-degenerate $f : \mathbf{C} \to V$,

$$\kappa T_f(r) \leq N_1(r, f^*D) + O(\log rT_f(r))||.$$

- **Log Bloch-Ochiai Theorem** (N. 1981): If the logarithmic irregularity $\bar{q}(V \setminus D) > \dim V$, then ${}^{\forall}f : \mathbf{C} \to V \setminus D$ is algebraically degenerate.

  N.B. (i) This unifies E. Borel's Theorem (1897) and Bloch-Ochiai's Theorem (1926/77) for projective V.
  (ii) Log Bloch-Ochiai's Theorem holds for compact Kähler V (N.-Winkelmann (2002)).

§5 Conjectures for holomrophic curves

Referring to the above mentioned results, one may pose the following:

Fund. Conj. for hol. curves.

Let X be a smooth compact algebraic variety, and let $D = \sum_j D_j$ be a reduced s.n.c. divisor on X with smooth D_j.

Then, for an algebraically non-degenerate $f : \mathbf{C} \to X$ we have

$$T_f(r; L(D)) + T_f(r; K_X) \leq \sum_j N_1(r; f^* D_j) + \epsilon T_f(r) ||, \ {}^\forall \epsilon > 0. \tag{5.1}$$

This implies

Green-Griffiths' Conjecture (1980). If X is a variety of (log) general type, then ${}^\forall f : \mathbf{C} \to X$ is algebraically degenerate.

$T_f(r) \leq \epsilon T_f(r)||$.

This implies

Kobayashi Conjecture (1970). If $X \subset \mathbf{P}^n(\mathbf{C})$ is a general hypersurface of $\deg X \geq 2n-1$, then X is Kobayashi hyperbolic.

The implication is supported by

Theorem 5.2

(C. Voisin (1996/98))

Let $X \subset \mathbf{P}^n(\mathbf{C})$ be a general hypersurface of deg $\geq 2n-1$ for $n \geq 3$. Then every subvariety of X is of general type.

§6 Yamanoi's *abc* Theorem (SMT)

R. Nevanlinna dealt with the distibution of the roots of $f(z) - a = 0$ for a meromorhic function f and constant values a.

He conjectured the same to hold for moving targets $a(z)$ of small order functions (called Nevanlinna's Conjecture).

The term "moving target" is due to W. Stoll, but such a study can go back to E. Borel, Acta 1897.

Nevanlinna's Conjecture was proved with non-truncated counting functions (Osgood (1985), Steinmetz (1986)), and as well for moving hyperplanes of $\mathbf{P}^n(\mathbf{C})$ (M. Ru-W. Stoll (1991)).

In Acta 2004, K. Yamanoi proved the best SMT for meromorphic functions with respect to moving targets, where the counting functions are truncated to level 1.

It is considered to be "*abc Theorem*" for the field of meromorphic functions.

Let $p : X \to S$ be a surjective morphism between smooth projective algebraic varieties with relative canonical bundle $K_{X/S}$.

Theorem 6.1

(Yamanoi, '04, '06) *Assume that*

- $\dim X/S = 1$;
- $D \subset X$ *is a reduced divisor ;*
- $f : \mathbf{C} \to X$ *is algebraically nondegenerate ;*
- $g = p \circ f : \mathbf{C} \to S$.

Then for $^\forall \epsilon > 0$, $^\exists C(\epsilon) > 0$ *such that*

$$(6.2) \quad T_f(r; L(D)) + T_f(r; K_{X/S}) \leq N_1(r; f^* D) + \epsilon T_f(r) + C(\epsilon) T_g(r)||_\epsilon.$$

His method:

- Ahlfors' covering theory;
- Mumford's theory of the compactification of curve moduli;
- The tree theory for point configurations.

§7 SMT (*abc*) for semi-abelian varieties

Let A be a semi-abelian variety:

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \text{ (abelian variety)} \to 0.$$

The universal covering $\tilde{A} \cong \mathbf{C}^n$, $n = \dim A$.

N.B. E. Borel's case: $\mathbf{P}^n(\mathbf{C}) \setminus n+1$ hyperplanes in gen. pos. $= (\mathbf{C}^*)^n$.

Let $f : \mathbf{C} \to A$ be a holomorphic curve. Set

- $J_k(A)$: the k-jet bundle over A; $J_k(A) \cong A \times \mathbf{C}^{nk}$;
- $J_k(f) : \mathbf{C} \to J_k(A)$: the k-jet lift of f;
- $X_k(f)$: the Zariski closure of the image $J_k(f)(\mathbf{C})$.

Set the jet projection

$$I_k : J_k(A) \cong A \times \mathbf{C}^{nk} \to \mathbf{C}^{nk}.$$

Lemma 7.1

(N. '77)
(i) *For* $f : \mathbf{C} \to A$,

$$T_{I_k \circ J_k(f)}(r) = O(\log^+(rT_f(r))) \,||.$$

(ii) *For* $f : \mathbf{C} \to \bar{A}$ *(compactification),*

$$m(r; I_k \circ J_k(f)) \stackrel{\mathrm{def}}{=} \int_{|z|=r} \log^+ \|I_k \circ J_k(f)(z)\| \frac{d\theta}{2\pi} = O(\log^+(rT_f(r))) \,||.$$

N.B. This is Lemma on logarithmic derivatives in higher dimension.

Theorem 7.2

(N.-Winkelmann-Yamanoi, Acta 2002 & Forum Math. 2008, Yamanoi Forum Math. 2004)

Let $f : \mathbf{C} \to A$ be algebraically non-degenerate. (i) *Let Z be an algebraic reduced subvariety of $X_k(f)$ $(k \geqq 0)$. Then* $^{\exists}\bar{X}_k(f)$, *compactification of $X_k(f)$ such that*

$$(7.3) \qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) \leqq N_1(r; J_k(f)^* Z) + \epsilon T_f(r)||_{\epsilon}, \quad {}^{\forall}\epsilon > 0.$$

(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geqq 2$, *then*

$$(7.4) \qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) \leqq \epsilon T_f(r)||_{\epsilon}, \quad {}^{\forall}\epsilon > 0.$$

(iii) *If $k = 0$ and Z is an effective divisor D on A, then $\bar{A}$ is smooth, equivariant, and independent of f; furthermore, (7.3) takes the form*

$$(7.5) \qquad T_f(r; L(\bar{D})) \leqq N_1(r; f^* D) + \epsilon T_f(r; L(\bar{D}))||_{\epsilon}, \quad {}^{\forall}\epsilon > 0.$$

N.B. (1) In N.-W.-Y. Acta 2002, we proved (7.5) with a higher level truncated counting function $N_k(r; f^*D)$ for some special compactification of A and with a better error term "$O(\log^+(rT_f(r)))$".

(2) For the truncation of level 1, the error term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log^+(rT_f(r)))$".

The trunction level 1 implies a number of applications. Here we mention two instant ones.

Theorem 7.6

(Conjectured by M. Green, 1974) *Assume that $f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C})$ omits two lines $\{x_i = 0\}, i = 1, 2$, and the conic $\{x_0^2 + x_1^2 + x_2^2 = 0\}$. Then f is algebraically degenerate.*

N.B. This holds in an even more general setting as seen in the next section.

N.B.

Theorem 7.6 is optimal in two senses:

1. the number of irreducible components is 3;
2. the total degree is 4.
3. The case of 4 lines is due to E. Borel (1897).
4. $^{\exists}$Dominant $f : \mathbf{C}^2 \to \mathbf{P}^2(\mathbf{C}) \setminus D$ for $\deg D \leq 3$ (Buzzard-Lu (1990)).

Theorem 7.7

1. *Let $f : \mathbf{C} \to A$ be alg.-nondegen., and $D \subset A$ a reduced divisor. Then $f(\mathbf{C}) \cap D$ is Zariski dense in D.*
2. *Let $f : \mathbf{C} \to A$ (abelian) be a 1-par. analy. subgroup, $a = f'(0)$ (speed vector), $D \subset A$ a reduced divisor with Riemann form $H(\cdot,\cdot)$, $D \not\supset f(\mathbf{C})$. Then*

$$N(r; f^*D) = H(a,a)\pi r^2 + O(\log r) = (1 + o(1))N_1(r; f^*D).$$

§8 Application I: Degeneracy theorems

Theorem 8.1

(N.-Winkelmann-Yamanoi, J. Math. Pure. Appl. '07)
Let X be an algebraic variety such that

(i) $\bar{q}(X) \geqq \dim X$ *(log irregularity);*

(ii) $\bar{\kappa}(X) > 0$ *(log Kodaira dimension);*

(iii) *the (quasi-)Albanese map $X \to A$ is proper.*

Then ${}^{\forall} f : \mathbf{C} \to X$ is algebraically degenerate.
Moreover, the normalization of $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a semi-abelian variety which is finite étale over a translate of a proper semi-abelian subvariety of A.

N.B. The case "$\bar{q}(X) > \dim X$" was "Log Bloch-Ochiai's Theorem" (N. '77-'81). The proof for the case "$\bar{q}(X) = \dim X$" requires the new S. M. Theorem 7.2.

Specializing $X = \mathbf{P}^n(\mathbf{C})$, we have

Theorem 8.2

Let $D = \sum_{j=1}^{q} D_j \subset \mathbf{P}^n(\mathbf{C})$ be an s.n.c. divisor.
Assume that $q > n$ and $\deg D > n+1$.
Then ${}^\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus D$ is algebraically degenerate.

M. Green's Conjecture: $n = 2$, $q = 3$.

Question 1. Let $D = \sum_{i=1}^{q} D_i \subset \mathbf{P}^n(\mathbf{C})$ be a divisor in general position (the codimensions of intersections of D_i's decrease exactly as the number of D_i's), possibly with singularities.

Assume that $q > n$ and $\deg D > n+1$. Then, is $\bar{\kappa}(\mathbf{P}^n(\mathbf{C}) \setminus D) > 0$?

Question 2. Let $D = D_1 + D_2 \subset \mathbf{P}^2(\mathbf{C})$ be an s.n.c. divisor with two conics D_j. Then, is ${}^\forall f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C}) \setminus D$ algebraically degenerate?

§9 Application II: A new unicity theorem

The following is a famous application of Nevnlinna-Cartan's SMT.

Theorem 9.1

(Unicity Theorem, Nevanlinna $n = 1$ (1926); Fujimoto $n \geq 2$ (1975))
Let $f, g : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be holomorphic curves such that at least one of two is linearly non-degenerate;
$\{H_j\}_{j=1}^{3n+2}$ be hyperplanes in general position.
If $\mathrm{Supp}\, f^* H_j = \mathrm{Supp}\, g^* H_j, 1 \leq j \leq 3n + 2$, then $f \equiv g$.

Erdös' Problem (1988) (unicity problem for arithmetic recurrences). *Let x, y be positive integers. Is it true that*

$$\{p; prime, p|(x^n - 1)\} = \{p; prime, p|(y^n - 1)\}, \forall n \in \mathbf{N}$$

$$\iff x = y \quad ?$$

The answer is Yes by Corrales-Rodorigáñez and R. Schoof, JNT 1997, with the elliptic analogue.

In complex analysis, K. Yamanoi proved the following striking unicity theorem in Forum Math. 2004:

Theorem 9.2

(Yamanoi's Unicity Theorem) *Let*
$A_i, i = 1, 2$, *be abelian varieties;*
$D_i \subset A_i$ *be irreducible divisors such that* $\mathrm{St}(D_i) = \{0\}$;
$f_i : \mathbf{C} \to A_i$ *be algebraically nondegenerate.*
Assume that $f_1^{-1}D_1 = f_2^{-1}D_2$ *as sets.*
Then $\exists$ *isomorphism* $\phi : A_1 \to A_2$ *such that*

$$f_2 = \phi \circ f_1, \quad D_1 = \phi^* D_2.$$

N.B. The new point is that we can determine not only f, but the moduli point of a polarized abelian vareity (A, D) through the distribution of $f^{-1}D$ by an algebraically nondegenerate $f : \mathbf{C} \to A$.

We want to generalize this to semi-abelian varieties to have a uniformized theory.

Let $A_i, i = 1, 2$ be semi-abelian varieties:

$$0 \to (\mathbf{C}^*)^{t_i} \to A_i \to A_{0i} \to 0.$$

Let D_i be an irreducible divisor on A_i such that

$$\mathrm{St}(D_i) = \{0\}$$

just for simplicity; this is not restrictive.

Theorem 9.3

(Corvaja-N., preprint 2009) *Let* $f_i : \mathbf{C} \to A_i$ $(i = 1, 2)$ *be non-degenerate holomorphic curves.*
Assume that

$$\text{(9.4)} \qquad \underline{\mathrm{Supp}\, f_1^* D_1}_\infty \subset \underline{\mathrm{Supp}\, f_2^* D_2}_\infty \ (\textit{germs at } \infty),$$

$$\text{(9.5)} \qquad {}^\exists C > 0 \text{ such that }, \ C^{-1} N_1(r, f_1^* D_1) \leq N_1(r, f_2^* D_2)||.$$

Then ${}^\exists$*a finite étale morphism* $\phi : A_1 \to A_2$ *such that*

$$\phi \circ f_1 = f_2, \quad D_1 \subset \phi^* D_2.$$

If equality holds in (9.4), *then* ϕ *is an isomorphism and* $D_1 = \phi^* D_2$.

N.B. Assumption (9.5) is necessary by example.

The following is immediate from Theorem 9.3.

Corollary 9.6

1. *Let* $f : \mathbf{C} \to \mathbf{C}^*$ *and* $g : \mathbf{C} \to E$ *with an elliptic curve* E *be holomorphic and non-constant. Then*

$$\underline{f^{-1}\{1\}}_{\infty} \neq \underline{g^{-1}\{0\}}_{\infty}.$$

2. *If* $\dim A_1 \neq \dim A_2$ *in Theorem 9.3, then*

$$\underline{f_1^{-1}D_1}_{\infty} \neq \underline{f_2^{-1}D_2}_{\infty}.$$

N.B.

$$\begin{tikzcd} \mathbf{C} \arrow[r, "f"] \arrow[dr, "g"'] & \mathbf{C}^* \arrow[d, "/\langle\tau\rangle"] \\ & E \end{tikzcd}$$

§10 Arithmetic Recurrence

It is natural to expect an analogue in arithmetic as in the new unicity Theorem 9.3.

For the linear tori we can prove such a result,
but the general semi-abeian case is left to be an open conjecture.

Let

- $\mathcal{O}_S$ be a ring of S-integers in a number field k;
- $\mathbf{G}_1$, $\mathbf{G}_2$ be linear tori;
- D_i be reduced divisors on $\mathbf{G}_i$ defined over k;
- $\mathcal{I}(D_i)$ be the defining ideals;
- $^\forall$irreducible component of D_i has a finite stabilizer; $\mathrm{St}(D_2) = \{0\}$.

Theorem 10.1

(Corvaja-N., preprint 2009) *Let $g_i \in \mathbf{G}_i(\mathcal{O}_S)$ be elements generating Zariski-dense subgroups.*
Suppose that for infinitely many $n \in \mathbf{N}$,

$$(g_1^n)^* \mathcal{I}(D_1) \supset (g_2^n)^* \mathcal{I}(D_2). \tag{10.2}$$

Then $^\exists$finite étale morphism $\phi : \mathbf{G}_1 \to \mathbf{G}_2$, defined over k, and $^\exists h \in \mathbf{N}$ such that $\phi(g_1^h) = g_2^h$ and $D_1 \subset \phi^*(D_2)$.

N.B.

1. Theorem 10.1 is deduced from Corvaja-Zannier, Invent. Math. 2002.
2. By an example we cannot take $h = 1$ in general.
3. By an example, the condition on the stabilzers of D_1 and D_2 cannot be omitted.
4. Note that inequality (inclusion) (10.2) of ideals is assumed only for an infinite sequence of n, not necessarily for all large n. On the contrary, we need the inequality of ideals, not only of their *supports*, i.e. of the primes containing the corresponding ideals.
5. One might ask for a similar conclusion assuming only the inequality of supports. There is some answer for it, but it is of a weaker form.

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