正則曲線と有理点分布について
Holomorphic curves and distribution of rational points

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§1 Introduction; a basic observation

- Analogues between value distribution theory and Diophantine approximaion theory.
- Some (not all) results motivated by the analogues.

We recall the unit equation with variables a, b, c:

$$a + b = c. \tag{1.1}$$

Why is this equation interesting?

There might be several answers, but one should be that

(1.1) gives a *hyperbolic space*.

In fact, equation (1.1) defines a subvariety of the projective 2-space,

$$X \subset \mathbf{P}^2$$

with homogeneous coordinates $[a, b, c]$.

Since the variables are assumed to be units, X is isomorphic to $\mathbf{P}^1$ minus three distinct points, to say, 0,1, and ∞:

$$X \cong \mathbf{P}^1 \setminus \{0, 1, \infty\}.$$

In complex function theory, (1.1) was studied by E. Picard for *units of entire functions*.

Picard's Theorem (1879). *A meromorphic function f on $\mathbf{C}$ omitting three distinct values of $\mathbf{P}^1$ must be constant.*

How are they related?

If f omits $0, 1, \infty$, then $f, (1 - f)$ and 1 are units in the ring of entire functions, and satisfy

$$f + (1 - f) = 1.$$

R. Nevanlinna (Acta '25): Quantitative theory to measure the frequencies for non-constant f to take those three values.

Second Main Theorem $\Longleftrightarrow$ abc Conjecture

(Masser-Oesterlé ('85)).

These are certain estimates of order (height) functions by the counting functions (the functions counting orders at finite places); explicit formulae will be given later, soon.

It is of importance and interest to study a unit equation in several variables,

$$(1.2) \qquad a + b + c + \cdots + f = 0 \quad (n \text{ variables}).$$

Equation (1.2) defines a variety isomorphic to

$$\mathbf{P}^{n-2} \setminus \{n \text{ hyperplanes in general position}\}.$$

In complex function theory (1.2) was studied by E. Borel for *units of entire functions*:

E. Borel (1897): *Subsum Theorem for units of entire functions holds*; i.e., a proper shorter subsum of $a, b, c, \ldots, f$ vanishes constantly.

W. Schmidt (1971): *Subsum Theorem for S-units of an algebraic number field holds.*

In complex function theory the corresponding quantitative theory was established by H. Cartan ('33), also by Weyls and Ahlfors ('41), which generalized Nevanlinna's theory.

Following it, we can formulate the n variable version of abc Conjecture, named "*abc* $\cdots$ *Conjecture*":

Second Main Theorem for hol. curves $\Longleftrightarrow$ *abc* $\cdots$ Conjecture.

These are the topics we are going to discuss.

N.B.

- First Main Theorem $\Longleftrightarrow$ Product Formula.
- Related topics: Kobayashi hyperbolicity.

§2 Log Bloch-Ochiai & Faltings-Vojta

Here I show a fine analogue in the distribution of holomorphic curves and the distribution of rational points.

Theorem 2.1

(Log Bloch-Ochiai (26-77), N. (77-81)). *Let X be a complex algebraic variety with logarithmic irregularity $\bar{q}(X) > \dim X$.*
Then every holomorphic curve $f : \mathbf{C} \to X$ is degenerate.

Here we say that f is degenerate if $f(\mathbf{C})$ is not Zariski dense in X.

Theorem 2.2

(Faltings (91)-Vojta (96)). *Let k be a number field and let S be a finite subset of inequivalent places of k containing all infinite places.*
Let X be defined over k with a compactification $\bar{X}$ and $D = \bar{X} \setminus X$.
Assume $\bar{q}(X) > \dim X$.
Then every (D, S)-integral point set is contained in a proper algebraic subset of X.

Theorem 2.3

Let M = a projective manifold of dimension m (or defined over k; the same in below);
$\{D_i\}_{i=1}^{l}$ = *a family of ample hypersurfaces of M in general position;*
$W(\subset M)$ = *a subvariety such that*

(CA) $\exists$ *non-degenerate holomorphic curve* $f : \mathbf{C} \to W \setminus \bigcup_{D_i \not\supset W} D_i$.

or

(Ar) $\exists$ *Zariski dense* $(\sum_{D_i \not\supset W} D_i \cap W, S)$*-integral point set in W.*

Then we have

(i) $(l - m) \dim W \leqq m \left(\mathrm{rank}_{\mathbf{Z}} \{c_1(D_i)\}_{i=1}^{l} - q(W)\right)^{+}$.

(ii) **(CA)** *Let $f : \mathbf{C} \to M$ be a holomorphic curve such that for every D_i, either $f(\mathbf{C}) \subset D_i$, or $f(\mathbf{C}) \cap D_i = \emptyset$:*

or

(Ar) *Let Z be a subset of $V(k)$ such that for every D_i, either $Z \subset D_i$, or $Z \cap D_i = \emptyset$ and Z is a $(\sum_{D_i \not\supset Z} D_i, S)$-integral point set.*

Assume that $l > m$.

Then $f(\mathbf{C})$ or Z is contained in an algebraic subspace W of M such that

$$\dim W \leqq \frac{m}{l-m} \mathrm{rank}_{\mathbf{Z}} \mathrm{NS}(M).$$

In particular, if $\mathrm{rank}_{\mathbf{Z}} \mathrm{NS}(M) = 1$ (e.g., $M = \mathbf{P}^m(\mathbf{C})$), then we have

$$\dim W \leqq \frac{m}{l-m}; \quad W \text{ is finite for } l > 2m.$$

The Mandala of the analogues:

"Value Dist."		"Dist. of Rational Points"
$f : \mathbf{C} \to X$	$\Longleftrightarrow$	Infin. Family of Rat'l Pts
Kobay. Hyperbolic.	$\Longleftrightarrow$ Lang's Conj.	Finiteness of Rat'l Pts
$\Uparrow$		$\Uparrow$
Nevan. Theory	$\Longleftrightarrow$ Vojta's Dict.	Dioph. Approx.

§3 abc Conjecture and Nevanlinna's S.M.T.

What is abc Conjecture?

abc Conjecture. For $^\forall \epsilon > 0$, $^\exists C_\epsilon > 0$ such that if co-prime integers $a, b, c \in \mathbf{Z}$ satisfyies

$$a + b = c, \tag{3.1}$$

then

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \prod_{\text{prime } p|(abc)} p^{1+\epsilon}. \tag{3.2}$$

N.B. The order of *abc* at every prime *p* is counted only by "$1 + \epsilon$" (truncation), when it is positive.

As in §1 we put $x = [a, -b] \in \mathbf{P}^1(\mathbf{Q})$, and set

$$(3.3) \qquad \mathrm{h}(x) = \log \max\{|a|, |b|\} \geq 0 \quad \text{(height)},$$

$$(3.4) \qquad N_1(x; \infty) = \sum_{p|a} \log p \quad \text{(counting function truncated to level 1)},$$

$$N_1(x; 0) = \sum_{p|b} \log p \qquad (\quad \text{〃} \quad),$$

$$N_1(x; 1) = \sum_{p|c} \log p \qquad (\quad \text{〃} \quad).$$

Then abc Conjecture (3.2) is rewritten as

$$(3.5) \quad (1-\epsilon)\mathrm{h}(x) \leq N_1(x; 0) + N_1(x; \infty) + N_1(x; 1) + C_\epsilon, \ x \in \mathbf{P}^1(\mathbf{Q}).$$

For q distinct points $a_i \in \mathbf{P}^1(\mathbf{Q}), 1 \leq i \leq q$,

$$(3.6) \qquad (q-2-\epsilon)\mathrm{h}(x) \leq \sum_{i=1}^{q} N_1(x; a_i) + C_\epsilon$$

(formulated by N '96, Vojta '98).

Theorem 3.7

(Nevanlinna's S.M.T.) *Let f be a meromorphic function in $\mathbf{C}$. For q distinct points $a_i \in \mathbf{P}^1(\mathbf{C}), 1 \leq i \leq q$,*

$$(q-2)T_f(r) \leq \sum_{i=1}^{q} N_1(r, f^* a_i) + O(\log^+(rT_f(r)))||.$$

$$T_f(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^*(\text{F.-S. metric}) \text{ (due to Shimizu).}$$

If f is entire, $T_f(r) \sim \log \max_{|z| \leq r} |f(z)|$.

§4 abc· · · Conjecture

abc· · · Conjecture 1. Let $a, b, c, \dots, e, f \in \mathbf{Z}$ be n integers without common factor satisfying

$$a + b + c + \cdots + e + f = 0.$$

Then for $^{\forall}\epsilon > 0$, $^{\exists}C_\epsilon$ and a proper algebraic subset $^{\exists}E_\epsilon \subset \mathbf{P}^{n-2}_{\mathbf{Z}}$ such that for $[a, b, \dots, e] \notin E_\epsilon$

(4.1) $(1-\epsilon)\log\max\{|a|, \dots, |f|\} \le \sum_{p|a} \log p + \cdots + \sum_{p|f} \log p + C_\epsilon$.

For the sake of notational convenience, we set

- $a = x_0, b = x_1, \dots, e = x_n$ ($n+1$ variables).
- $x = [x_0, \dots, x_n] \in \mathbf{P}^n(\mathbf{Q})$.
- $\mathrm{h}(x) = \log\max_{0\le j\le n}\{|x_j|\}$: the height of x.
- $H_j = x_j, 0 \le j \le n$, $H_{n+1} = -\sum_{j=0}^{n} x_j$: $n+2$ linear forms in general position.
- $N_1(x; H_j)$: the counting function truncated to level 1.

Then (4.1) is equivalent to

$$(1-\epsilon)\mathrm{h}(x) \leq \sum_{j=0}^{n+1} N_1(x; H_j) + C_\epsilon. \tag{4.2}$$

We consider a bit more general case.
Let S be a finte set of primes and let $l \leq \infty$.
We define an S-counting function truncated to level l by

$$N_l(x; S, H_j) = \sum_{p \notin S,\, p | H_j(x)} \min\{\deg_p H_j(x)), l\} \cdot \log p. \tag{4.3}$$

abc· · · Conjecture 2. Let $H_j, 1 \le j \le q$ be q ($\ge n+2$) linear forms on $\mathbf{P}^n_{\mathbf{Q}}$ in general position.
Then for $^{\forall}\epsilon > 0$, $^{\exists}C_\epsilon$ and a proper algebraic subset $^{\exists}E_\epsilon \subset \mathbf{P}^n_{\mathbf{Q}}$ such that

$$(4.4)\quad (q-n-1-\epsilon)\mathrm{h}(x) \le \sum_{j=1}^{q} N_1(x; S, H_j) + C_\epsilon, \quad x \in \mathbf{P}^n(\mathbf{Q}) \setminus E_\epsilon$$

(N '96, Vojta '98).

Schmidt's Subspace Theorem is stated as follows.

Theorem 4.5

Let the notaion be as above.
For $^{\forall}\epsilon > 0$, $^{\exists}C_\epsilon$ and a finite union $^{\exists}E_\epsilon$ of proper linear subspaces of $\mathbf{P}^n_{\mathbf{Q}}$ such that

$$(q-n-1-\epsilon)\mathrm{h}(x) \le \sum_{j=1}^{q} N_\infty(x; S, H_j) + C_\epsilon, \quad x \in \mathbf{P}^n(\mathbf{Q}) \setminus E_\epsilon.$$

N.B. When $n = 1$, this is Roth's Theorem.

Theorem 4.6

(H. Cartan's S.M.T., '33) *Let* $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *be a linearly non-degenerate holomorphic curve.*
Let H_j *be* q *hyperplanes of* $\mathbf{P}^n(\mathbf{C})$ *in general position.*
Then

$$(q-n-1)T_f(r) \le \sum_{i=1}^{q} N_n(r, f^*H_j) + O(\log^+(rT_f(r)))||$$

§6 abc··· Theorem for hppolomorphic curves into semi-abelian varieties

Let A be a semi-abelian variety.
The universal covering $\tilde{A} \cong \mathbf{C}^n$, $n = \dim A$. Let

- $f : \mathbf{C} \to A$, be a holomorphic curve ;
- $J_k(A) =$ k-jet bundle over A; $J_k(A) \cong A \times \mathbf{C}^{nk}$;
- $J_k(f) : \mathbf{C} \to J_k(A)$, k-jet lift of f;
- $X_k(f) =$ Zariski closure of the image $J_k(f)(\mathbf{C})$.
- $I_k : J_k(A) \cong A \times \mathbf{C}^{nk} \to \mathbf{C}^{nk}$, jet projection.

Lemma 5.1

(N. '77)
(i) *For* $f : \mathbf{C} \to A$,

$$T_{I_k \circ J_k(f)}(r) = O(\log^+(rT_f(r)))\,||.$$

(ii) *For* $f : \mathbf{C} \to \bar{A}$ *(compactification)*,

$$m(r; I_k \circ J_k(f)) \stackrel{\mathrm{def}}{=} \int_{|z|=r} \log^+ \|I_k \circ J_k(f)(z)\| \frac{d\theta}{2\pi} = O(\log^+(rT_f(r)))\,||.$$

N.B. This is Lemma on logarithmic derivatives in higher dimension.

Theorem 5.2

(N.-Winkelmann-Yamanoi, Acta '02, Forum Math. '08)

Let $f : \mathbf{C} \to A$ *be non-degenerate.*

(i) *Let* Z *be an algebraic reduced subvariety of* $X_k(f)$ $(k \geqq 0)$*.*
Then ${}^{\exists}\bar{X}_k(f)$*, compactification of* $X_k(f)$ *such that*

$$(5.3)\qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) \leqq N_1(r; J_k(f)^* Z) + \epsilon T_f(r)||_\epsilon, \quad {}^{\forall}\epsilon > 0.$$

(ii) *Moreover, if* $\mathrm{codim}\,_{X_k(f)} Z \geqq 2$*, then*

$$(5.4)\qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) \leqq \epsilon T_f(r)||_\epsilon, \quad {}^{\forall}\epsilon > 0.$$

(iii) *If* $k = 0$ *and* Z *is an effective divisor* D *on* A*, then* $\bar{A}$ *is smooth, equivariant, and independent of* f*; furthermore,* (5.3) *takes the form*

$$(5.5)\qquad T_f(r; L(\bar{D})) \leqq N_1(r; f^* D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon, \quad {}^{\forall}\epsilon > 0.$$

N.B. (1) In N.-W.-Y. Acta '02, we proved (5.5) with a higher level truncated counting function $N_k(r; f^*D)$ for some special compactification of A and with a better error term "$O(\log^+(rT_f(r)))$".

(2) For the truncation of level 1, the error term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log^+(rT_f(r)))$".

(3) When A is an abelian vareity, (iii) was obtained by Yamanoi, '04.

Because of the trunction level 1, we have the following:

Theorem 5.6

(Conjectured by M. Green, '74) *Assume that $f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C})$ omits two lines $\{x_i = 0\}, i = 1, 2$, and the conic $\{x_0^2 + x_1^2 + x_2^2 = 0\}$. Then f is degenerate.*

There is a historical reason in this case of 2 lines and 1 conic.

Lately, Corvaja-Zannier obtained some corresponding result over algebraic function fields (J.A.G. 2008).

§6 Application

Theorem 6.1

(N.-Winkelmann-Yamanoi, J. Math. Pure. Appl. '07)
Let X be an algebraic variety such that

(i) $\bar{q}(X) \geqq \dim X$ *(log. irregularity);*

(ii) $\bar{\kappa}(X) > 0$ *(log. Kodaira dimension);*

(iii) *the quasi- Albanese map $X \to A$ is proper.*

Then $^{\forall} f : \mathbf{C} \to X$ is degenerate.
Moreover, the normalization of $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a semi-abelian variety which is finite étale over a translate of a proper semi-abelian subvariety of A.

N.B. The case "$\bar{q}(X) > \dim X$" was known as Log-Bloch-Ochiai's Theorem (N. '77-'81). The proof for the case "$\bar{q}(X) = \dim X$" requires our new Theorem 5.2.

As a special case we have

Theorem 6.2

Let $D = \sum_{i=1}^{q} D_i \subset \mathbf{P}^n(\mathbf{C})$ be an s.n.c. divisor.
Assume that $q > n$ and $\deg D > n+1$.
Then $^{\forall} f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus D$ is degenerate.

Here are more applications:

Theorem 6.3

Let A be a semi-abelian variety and D a reduced divisor on A.
Let $f : \mathbf{C} \to A$ be a holomorphic curve such that

$$\deg_{\zeta} f^* D \geqq 2, \quad {}^{\forall}\zeta \in f^{-1} D.$$

Then f is degenerate.

Theorem 6.4

Let $D = \sum_{i=1}^{n+1} D_i$ be an s.n.c. divisor on $\mathbf{P}^n(\mathbf{C})$ and D_{n+2} a reduced divisor not contained in D.
Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus D$ be a holomorphic curve such that

$$\deg_\zeta f^* D_{n+2} \geqq 2, \quad {}^\forall \zeta \in f^{-1} D_{n+2}.$$

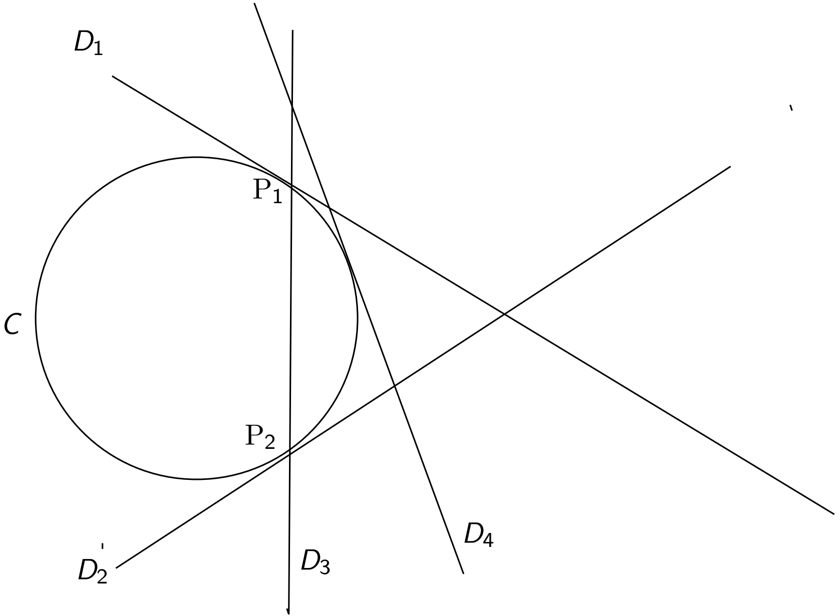
Then f is degenerate.

Example. Let $D = \sum_{i=1}^{3} D_i$, be a sum of lines of $\mathbf{P}^2$ in general position, and set

$$\mathrm{P}_1 = D_1 \cap D_2, \quad \mathrm{P}_2 = D_1 \cap D_3.$$

Let C be a smooth conic intersecting D at $\mathrm{P}_1, \mathrm{P}_2$, tangent to D_2 and D_3, respectively. Let D_4 be a line tangent to C such that $\sum_{i=1}^{4} D_i$ is in general position. Let $f : \mathbf{C} \to C \setminus \{\mathrm{P}_1, \mathrm{P}_2\}$ be a holomorphic curve.d:/Sagyo/PAPER/d:/Sagyo/PAPER/ Then

$$\deg_\zeta f^* D_4 \geqq 2, \quad {}^\forall \zeta \in f^{-1} D_4.$$


D_1
P_1
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P_2
D_2
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D_4

§8 Yamanoi's abc Theorem

In Acta '04, K. Yamanoi proved a striking S.M.T. for meromorphic functions with respect to moving targets, where the counting functions are truncated to level 1; it gives the best answer to Nevanlinna's Conjecture for moving targets, and more.

It is considered to be "*abc Theorem*" for fields of meromorphic functions, which are transcendental in general.

His method:

- Ahlfors' covering theory;
- Mumford's theory of the compactification of curve moduli;
- The tree theory for point configurations.

We recall his result in a form suitable to the present talk.

Let $p : X \to S$ be a surjective morphism between smooth projective algebraic varieties with relative canonical bundle $K_{X/S}$.

Theorem 7.1

(Yamanoi, '04, '06) *Assume that*

- $\dim X/S = 1$;
- $D \subset X$ *is a reduced divisor ;*
- $f : \mathbf{C} \to X$ *is nondegenerate ;*
- $g = p \circ f : \mathbf{C} \to S$.

Then for ${}^{\forall}\epsilon > 0$, ${}^{\exists}C(\epsilon) > 0$ *such that*

$$(7.2) \quad T_f(r; [D]) + T_f(r; K_{X/S}) \leq N_1(r; f^*D) + \epsilon T_f(r) + C(\epsilon) T_g(r) ||_{\epsilon}.$$

§9 Fundamental Conjecture for holomrophic curves

The titled conjecture is as follows:

Fund. Conj. for hol. curves.

Let X be a smooth algebraic variety, and let $D = \sum_i D_i$ be a reduced s.n.c. divisor on X with irreducible D_i.

Then, for a non-degenerate $f : \mathbf{C} \to X$ we have

$$T_f(r; L(D)) + T_f(r; K_X) \le \sum_i N_1(r; f^* D_i) + \epsilon T_f(r)||, \ {}^\forall \epsilon > 0. \tag{8.1}$$

Even in the case when $X = \mathbf{P}^n(\mathbf{C})$ and D is allowed to have some singularities, the fundamental conjecture implies Green-Griffiths' Conjecture and Kobayashi's Conjecture.

Green-Griffiths' Conjecture. Let X be a variety of general type. Then $^{\forall}f : \mathbf{C} \to X$ is degenerate.

Kobayashi's Conjecture. A generic hypersurface $X \subset \mathbf{P}^n(\mathbf{C})$ of high degree ($\geq 2n-1$) is Kobayashi hyperbolic.

Even when $X = \mathbf{P}^n(\mathbf{C})$ and D_i are hyperplanes, the Fundamental Conjecture is open; if $N_1(r; D_i)$ are replaced by $N_n(r; D_i)$, this is Cartan's Theorem 4.6, where f suffices to be linearly non-degenerate.

If $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ omits $n+1$ hyperplanes $H_i, 1 \leq i \leq n+1$ in general position, then $\mathbf{P}^n(\mathbf{C}) \setminus \sum_{i=1}^{n+1} H_i \cong (\mathbf{C}^*)^n$. In this case, the Fundamental Conjecture is true because of Theorem 5.2.

After establishing the case of semi-abelian varieties, it is interesting to deal with K3 surfaces.

Problem 1. Let X be a K3 surface. Does there exist a non-degenerate holomorphic curve $f : \mathbf{C} \to X$?

Problem 2. Let X be a K3 surface and let D be a reduced non-zero divisor on X. Is every $f : \mathbf{C} \to X \setminus D$ degenerate?

After Green's conjecture the following is interesting:

Problem 3. Let $D = \sum_{i=1}^{q} D_i$ be an s.n.c. divisor on $\mathbf{P}^2$ such that

$$\deg D = 4 \text{ and } q = 1, 2.$$

Is every $f : \mathbf{C} \to \mathbf{P}^2 \setminus D$ degenerate?