

A Unicity Theorem and Erdős' Problem for Polarized Semi-Abelian Varieties

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§1 Introduction

The following is a kind of the unicity problem in arithmetic recurrence:
 Erdős' Problem (1988). *Let x, y be positive integers. Is it true that*

$$\{p; \text{prime}, p|(x^n - 1)\} = \{p; \text{prime}, p|(y^n - 1)\}, \forall n \in \mathbf{N}$$

$$\iff x = y \quad ?$$

The answer is Yes:

Theorem 1.1

(Corrales-Rodrigáñez and R. Schoof, JNT 1997)

$$\textcircled{1} \quad \{p; \text{prime}, p|(x^n - 1)\} \subset \{p; \text{prime}, p|(y^n - 1)\}, \forall n \in \mathbf{N}$$

$$\implies y = x^h \quad (\exists h \in \mathbf{N}).$$

$$\textcircled{2} \quad \textit{The elliptic version holds, too.}$$

In complex analysis, Yamanoi proved in Forum Math. 2004 the following striking unicity theorem:

Theorem 1.2

Let

$A_i, i = 1, 2$, be abelian varieties;

$D_i \subset A_i$ be irreducible divisors such that

$$\text{St}(D_i) = \{a \in A_i; a + D_i = D_i\} = \{0\};$$

$f_i : \mathbb{C} \rightarrow A_i$ be (algebraically) nondegenerate entire holomorphic curves.

Assume that $f_1^{-1}D_1 = f_2^{-1}D_2$ as sets.

Then \exists isomorphism $\phi : A_1 \rightarrow A_2$ such that

$$f_2 = \phi \circ f_1, \quad D_1 = \phi^* D_2.$$

N.B.

- ① The new point is that we can determine not only f , but the moduli point of a polarized abelian variety (A, D) through the distribution of $f^{-1}D$ by a nondegenerate $f : \mathbf{C} \rightarrow A$.
- ② The assumptions for D_i to be irreducible and the triviality of $\text{St}(D_i)$ are not restrictive. There is a way of reduction.
- ③ For simplicity we assume them here.

§2 Main Results

We want to uniformize the results in the previous section.

Therefore we have to deal with semi-abelian varieties.

Let $A_i, i = 1, 2$ be semi-abelian varieties:

$$0 \rightarrow (\mathbf{C}^*)^{t_i} \rightarrow A_i \rightarrow A_{0i} \rightarrow 0.$$

Let D_i be an irreducible divisor on A_i such that

$$\text{St}(D_i) = \{0\}$$

for simplicity.

Main Theorem

Main Theorem 2.1

Let $f_i : \mathbf{C} \rightarrow A_i$ ($i = 1, 2$) be non-degenerate holomorphic curves.
Assume that

$$(2.2) \quad \underline{\text{Supp} f_1^* D_1}_\infty \subset \underline{\text{Supp} f_2^* D_2}_\infty \text{ (germs at } \infty),$$

$$(2.3) \quad N_1(r, f_1^* D_1) \sim N_1(r, f_2^* D_2)||;$$

$$\text{i.e., } C^{-1} N_1(r, f_1^* D_1) \leq N_1(r, f_2^* D_2) \leq C N_1(r, f_1^* D_1)||, \exists C > 0.$$

Then there is a finite étale morphism $\phi : A_1 \rightarrow A_2$ such that

$$\phi \circ f_1 = f_2, \quad D_1 \subset \phi^* D_2.$$

If equality holds in (2.2), then ϕ is an isomorphism and $D_1 = \phi^* D_2$.

N.B. Assumption (2.3) is necessary by example.

The following corollary follows immediately from the Main Theorem 2.1.

Corollary 2.4

- ① *Let $f : \mathbf{C} \rightarrow \mathbf{C}^*$ and $g : \mathbf{C} \rightarrow E$ with an elliptic curve E be holomorphic and non-constant. Then*

$$\underline{f^{-1}\{1\}}_{\infty} \neq \underline{g^{-1}\{0\}}_{\infty}.$$

- ② *If $\dim A_1 \neq \dim A_2$ in the Main Theorem 2.1, then*

$$\underline{f_1^{-1}D_1}_{\infty} \neq \underline{f_2^{-1}D_2}_{\infty}.$$

N.B.

- 1 The first statement means that the difference of the value distribution property caused by the quotient $\mathbf{C}^* \rightarrow \mathbf{C}^*/\langle \tau \rangle = E$ cannot be recovered by the choice of f and g , even though they are allowed to be *arbitrarily transcendental*.

$$\begin{array}{ccc}
 \mathbf{C} & \xrightarrow{f} & \mathbf{C}^* \\
 & \searrow g & \downarrow / \langle \tau \rangle \\
 & & E
 \end{array}$$

- 2 The second statement implies that the distribution of $f_i^{-1}D_i$ about ∞ contains the *topological informations* of $\dim A_i$ and the compact or non-compactness of A_i ;
It is interesting to observe that this works even for one parameter subgroups with Zariski dense image.

Due to the well-known correspondence between Number Theory and Nevanlinna Theory, it is tempting to give a number-theoretic analogue of Theorem 2.1 as Pál Erdős Problem–Corrales-Rodorigáñez&Schoof Theorem.

A related problem asks to classify the cases where $x^n - 1$ divides $y^n - 1$ for infinitely many positive integers n .

We would like to deal with the case of a semi-abelian variety with a given divisor, i.e., a polarized semi-abelian variety.

We can prove an analogue of the Main Theorem 2.1 in the [linear toric case](#), but not in the general case of [semi-abelian varieties](#), that is left to be a [Conjecture](#).

§3 Arithmetic Recurrence

Here is our result in the arithmetic case.

Theorem 3.1

Let

\mathcal{O}_S be a ring of S -integers in a number field k ;

$\mathbf{G}_1, \mathbf{G}_2$ be linear tori;

$g_i \in \mathbf{G}_i(\mathcal{O}_S)$ be elements generating Zariski-dense subgroups.

D_i be reduced divisors defined over k , with defining ideals $\mathcal{I}(D_i)$, such that each irreducible component has a finite stabilizer and $\text{St}(D_2) = \{0\}$.

Suppose that for infinitely many $n \in \mathbf{N}$,

$$(3.2) \quad (g_1^n)^* \mathcal{I}(D_1) \supset (g_2^n)^* \mathcal{I}(D_2).$$

Then \exists étale morphism $\phi : \mathbf{G}_1 \rightarrow \mathbf{G}_2$, defined over k , and $\exists h \in \mathbf{N}$ such that $\phi(g_1^h) = g_2^h$ and $D_1 \subset \phi^(D_2)$.*

N.B.

- ① Theorem 3.1 is deduced from the main results of Corvaja-Zannier, Invent. Math. 2002.
- ② By an example we cannot take $h = 1$ in general.
- ③ By an example, the condition on the stabilizers of D_1 and D_2 cannot be omitted.
- ④ Note that inequality (inclusion) (3.2) of ideals is assumed only for an infinite sequence of n , not necessarily for all large n . On the contrary, we need the inequality of ideals, not only of their *supports*, i.e. of the primes containing the corresponding ideals.
- ⑤ One might ask for a similar conclusion assuming only the inequality of supports. There is some answer for it, but it is of a weaker form.

Proof of the Main Theorem

The next two theorems are crucial in the proof.

Theorem 4.1

(Log Bloch-Ochiai, Nog. 1977 Hiroshima Math.J./81 Nagoya Math.J.)
Let $f : \mathbf{C} \rightarrow A$ be a holomorphic curve into a semi-abelian variety A . Then $\overline{f(\mathbf{C})}^{\text{Zar}}$ is a translate of a subgroup.

Theorem 4.2

(NWY-II (N.-Winkelmann-Yamanoi) 2008 Forum Math.)

Let $f : \mathbf{C} \rightarrow A$ be nondegenerate;

$Z \subset A$ be an algebraic cycle on A .

- ① *If $\text{codim } Z = 1$, $\exists \bar{A} \supset \bar{Z}$ s.t. $T_f(r, L(\bar{Z})) = N_1(r, f^*Z) + o(T_f(r))$.*
- ② *If $\text{codim } Z \geq 2$, $T_{f, \bar{Z}}(r) = N(r, f^*Z) + O(\log r T_f(r)) = o(T_f(r))$.*

Proof.

With the given f_i ($i = 1, 2$) in the Main Theorem we set

$$g = (f_1, f_2) : \mathbf{C} \rightarrow A_1 \times A_2 ;$$

$$A_0 = \overline{g(\mathbf{C})}^{\text{Zar}} \text{ (semi-abelian variety by \textbf{Log Bloch-Ochiai})};$$

$p_i : A_0 \rightarrow A_i$ be the projections;

$$E_i = p_i^* D_i.$$

We apply **NWY-II**, Theorem 4.2 for g and E_i .

We deduce that

- ① $E_1 \subset E_2$,
- ② $\text{St}(E_1) \subset \text{St}(E_2)$, and are finite,
- ③ p_i are isogenies,
- ④ $A_1 \cong A_0/\text{St}(E_1) \xrightarrow{\phi} A_0/\text{St}(E_2) \cong A_2$.

Problem

Characterize a polarized algebraic variety (V, D)
in terms of $f^{-1}D$ or f^*D
by a nondegenerate holomorphic curve $f : \mathbf{C} \rightarrow V$.

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