A New Unicity Theorem and Erdös' Problem for Polarized Semi-Abelian Varieties

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Analytic Number Theory 2009 Kyoto

14∼16 October 2009
14 October 2009

§1 Introduction

1.1 Nevanlinna's unicity theorem.

Theorem 1.1

(Unicity Theorem) *Let* $f, g : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ *be two non-constant meromorphic functions.*
If $\exists a_i \in \mathbf{P}^1(\mathbf{C}), 1 \leq i \leq 5$, *distinct such that* $\mathrm{Supp}\ f^*a_i = \mathrm{Supp}\ g^*a_i, 1 \leq i \leq 5$, *then* $f \equiv g$.

This follows from Nevanlinna's Second Main Theorem (SMT):

Theorem 1.2

(SMT) *Let* $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ *be a meromorphic function, and* $a_i \in \mathbf{P}^1(\mathbf{C}), 1 \leq i \leq q$, *be distinct* q *points. Then*

$$(q-2)T_f(r) \leq \sum_{i=1}^{q} N(r, \mathrm{Supp}\ f^*a_i) + \textit{small-term}.$$

Proof of Theorem 1.1.

By Nevanlinna's SMT 1.2 we have

$$(5-2=3)\,T_{f(\text{ or } g)}(r) \leq \sum_{i=1}^{5} N(r, \mathrm{Supp}\ f^*(\text{ or } g^*)a_i) + \text{small-term}.$$

Suppose $f \not\equiv g$. Then the assumption implies that

$$\sum_{i=1}^{5} N(r, \mathrm{Supp}\ f^* a_i) \leq N(r, (f-g)_0) \leq T_{f-g}(r) + O(1)$$

$$\leq T_f(r) + T_g(r) + O(1) \leq \frac{2}{3} \sum_{i=1}^{5} N(r, \mathrm{Supp}\ f^* a_i) + \text{small-term}.$$

Thus, $1 \leq \frac{2}{3}$; a contradiction. □

Remark.

The number 5 in the above unicity theorem is optimal for the following trivial reason: Set
$f(z) = e^z$, $g(z) = e^{-z}$; $a_1 = 0, a_2 = \infty, a_3 = 1, a_4 = -1$.
Then $f^* a_i = g^* a_i, 1 \le i \le 4$.
Note that by setting $\sigma(w) = w^{-1}$ and $D = \sum_1^4 a_i$ we have

$$\sigma^* D = D, \quad \sigma \circ f = g; \quad f(z), g(z) \in \mathbf{C}^*.$$

Theorem 1.3

(E.M. Schmid 1971) *Let E be an elliptic curve,*
$a_i \in E, 1 \le i \le 5$, distinct points.
Let $f, g : \mathbf{C} \to E$ be holomorphic maps.
If $\mathrm{Supp}\, f^* a_i = \mathrm{Supp}\, g^* a_i, 1 \le i \le 5$, then $f \equiv g$.

Theorem 1.4

(H. Fujimoto (1975)) *Let* $f, g : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *be holomorphic curves such that at least one of them is linearly non-degenerate;*
$\{H_j\}_{j=1}^{3n+2}$ *be hyperplanes in general position.*
If $f^* H_j = g^* H_j, 1 \leq j \leq 3n+2$ *(as divisors, counting multipicities), then* $f \equiv g$.

Schmid's and Fujimoto's theorems are deduced from some SMT's in the corresponding cases.

The following is a kind of unicity problem in arithmetic theory, which is sometimes called a "support problem":

Erdös' Problem (1988). *Let* x, y *be positive integers. Is it true that*

$$\{p; prime, p|(x^n - 1)\} = \{p; prime, p|(y^n - 1)\}, \forall n \in \mathbf{N}$$

$$\iff x = y \quad ?$$

The answer is Yes:

Theorem 1.5

(Corrales-Rodorigáñez and R. Schoof, JNT 1997)

1. *Suppose that except for finitely many prime* $p \in \mathbf{Z}$

$$y^n \equiv 1 \pmod{p} \text{ whenever } x^n \equiv 1 \pmod{p}, \forall n \in \mathbf{N}.$$

   Then, $y = x^h$ *with* $\exists h \in \mathbf{N}$.

2. *Let* E *be an elliptic curve defined over a number field* k*, and let* $P, Q \in E(k)$*. Suppose that except for finitely many prime* $p \in O(k)$

$$nQ = 0 \text{ whenever } nP = 0 \text{ in } E(k_p).$$

   Then either $Q = \sigma(P)$ *with* $\exists \sigma \in \mathrm{End}(E)$*, or both* P, Q *are torsion points.*

Yamanoi's unicity theorem

Yamanoi proved in Forum Math. 2004 the following striking unicity theorem:

Theorem 1.6

Let
$A_i, i = 1, 2$, *be abelian varieties;*
$D_i \subset A_i$ *be irreducible divisors such that*

$$\mathrm{St}(D_i) = \{a \in A_i; a + D_i = D_i\} = \{0\};$$

$f_i : \mathbf{C} \to A_i$ *be (algebraically) nondegenerate entire holomorphic curves.*
Assume that $f_1^{-1} D_1 = f_2^{-1} D_2$ *as sets.*
Then $\exists$ *isomorphism* $\phi : A_1 \to A_2$ *such that*

$$f_2 = \phi \circ f_1, \quad D_1 = \phi^* D_2.$$

N.B.

1. The new point is that we can determine not only f, but the moduli point of a polarized abelian vareity (A, D) through the distribution of $f^{-1}D$ by a nondegenerate $f : \mathbf{C} \to A$.
2. The assumptions for D_i to be irreducible and the triviality of $\mathrm{St}(D_i)$ are not restrictive. There is a way of reduction.
3. For simplicity we assume them here.

§2 Main Results

We want to uniformize the results in the previous section.

Therefore we deal with semi-abelian varieties.

Let $A_i, i = 1, 2$ be semi-abelian varieties:

$$0 \to (\mathbf{C}^*)^{t_i} \to A_i \to A_{0i} \to 0.$$

Let $D_i \subset A_i, i = 1, 2,$ be irreducible divisors such that

$$\mathrm{St}(D_i) = \{0\} \qquad \text{(for simplicity)}.$$

For real-valued functions $\phi(r)$ and $\psi(r)$ $(r > 1)$, we write $\phi(r) \leq \psi(r)||_E$ if $E \subset [1, \infty)$, Borel, $m(E) < \infty$, and $\phi(r) \leq \psi(r), r \notin E$.

$$\phi(r) \sim \psi(r)|| \iff \exists E, \exists C > 0,\ C^{-1}\phi(r) \leq \psi(r) \leq C\phi(r)||_E.$$

Main Theorem

Main Theorem 2.1

Let $f_i : \mathbf{C} \to A_i$ $(i = 1, 2)$ *be non-degenerate holomorphic curves. Assume that*

$$\text{(2.2)} \qquad \underline{\mathrm{Supp}\ f_1^* D_1}_{\infty} \subset \underline{\mathrm{Supp}\ f_2^* D_2}_{\infty} \ (\textit{germs at } \infty),$$

$$\text{(2.3)} \qquad N_1(r, f_1^* D_1) \sim N_1(r, f_2^* D_2)||.$$

Here $N_1(r, f_1^* D_1) = N(r, \mathrm{Supp}\ f_1^* D_1))$.
Then there is a finite étale morphism $\phi : A_1 \to A_2$ *such that*

$$\phi \circ f_1 = f_2, \quad D_1 \subset \phi^* D_2.$$

If equality holds in (2.2), *then* ϕ *is an isomorphism and* $D_1 = \phi^* D_2$.

N.B. Assumption (2.3) is necessary by example.

The following corollary follows immediately from the Main Theorem 2.1.

Corollary 2.4

1. *Let* $f : \mathbf{C} \to \mathbf{C}^*$ *and* $g : \mathbf{C} \to E$ *with an elliptic curve* E *be holomorphic and non-constant. Then*

$$\underline{f^{-1}\{1\}}_{\infty} \neq \underline{g^{-1}\{0\}}_{\infty}.$$

2. *If* $\dim A_1 \neq \dim A_2$ *in the Main Theorem 2.1, then*

$$\underline{f_1^{-1} D_1}_{\infty} \neq \underline{f_2^{-1} D_2}_{\infty}.$$

N.B.

1. The first statement means that the difference of the value distribution property caused by the quotient $\mathbf{C}^* \to \mathbf{C}^*/\langle\tau\rangle = E$ cannot be recovered by any later choice of f and g, even though they are allowed to be *arbitrarily transcendental*.

$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{f} & \mathbf{C}^* \\
 & \searrow_{g} & \downarrow /\langle\tau\rangle \\
 & & E
\end{array}
$$

2. The second statement implies that the distribution of $f_i^{-1}D_i$ about ∞ contains the topological informations such as $\dim A_i$ and the compactness or non-compactness of A_i.

Example.

Set $A_1 = \mathbf{C}/\mathbf{Z}\ (\cong \mathbf{G}_{\mathrm{m}})$ and let $D_1 = 1$ be the unit element of A_1. Let $f_1 : \mathbf{C} \to A_1$ be the covering map.
Take a number $\tau \in \mathbf{C}$ with $\Im \tau \neq 0$.
Set $A_2 = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$, which is an elliptic curve.
Let $D_2 = 0 \in A_2$ and $f_2 : \mathbf{C} \to A_2$ be the covering map.

Then $f_1^{-1} D_1 = \mathbf{Z} \subset \mathbf{Z} + \tau\mathbf{Z} = f_2^{-1} D_2$: assumption (2.2) of the Main Theorem 2.1 is satisfied.
There is, however, no non-constant morphism $\phi : A_1 \to A_2$. Note that

$$N_1(r, f_1^* D_1) \sim r, \quad N_1(r, f_2^* D_2) \sim r^2.$$

Thus, $N_1(r, f_1^* D_1) \not\sim N_1(r, f_2^* D_2)||$: assumption (2.3) fails.

§3 SMT for semi-abelian varieties.

For a closed subscheme $Z \subset X$ (*compact* complex space) and holomorphic $f : \mathbf{C} \to X$, $f(\mathbf{C}) \not\subset \mathrm{Supp}\, Z$, we write

$$T_f(r, \omega_Z) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* \omega_Z,$$

$$\underline{f^* Z}_{k,a} = \min\{\deg_a f^* Z, k\} \ (k \leq \infty),$$

$$N_k(r, f^* Z) = \int_1^r \frac{dt}{t} \left(\sum_{a \in \Delta(t)} \underline{f^* Z}_{k,a} \right),$$

$$N(r, f^* Z) = N_\infty(r, f^* Z) < T_f(r, \omega_Z) + O(1) \quad (FMT).$$

Let

A be a semi-abelian variety,

$f : \mathbf{C} \to A$ be a holomorphic curve.

Set

- $J_k(A) \cong A \times \mathbf{C}^{nk}$: the k-jet bundle over A;
- $J_k(f) : \mathbf{C} \to J_k(A)$: the k-jet lift of f;
- $X_k(f)$: the Zariski closure of the image $J_k(f)(\mathbf{C})$ in $J_k(A)$.

Theorem 3.1

(N.-Winkelmann-Yamanoi, Acta 2002 & Forum Math. 2008, Yamanoi Forum Math. 2004)

Let $f : \mathbf{C} \to A$ be algebraically non-degenerate.
(i) *Let Z be an algebraic reduced subvariety of $X_k(f)$ $(k \geqq 0)$. Then $\exists \bar{X}_k(f)$, compactification of $X_k(f)$ such that*

$$(3.2) \qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) = N_1(r; J_k(f)^* Z) + o(T_f(r))||.$$

(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geqq 2$, *then*

$$(3.3) \qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) = o(T_f(r))||.$$

(iii) *If $k = 0$ and Z is an effective divisor D on A, then $\bar{A}$ is smooth, equivariant, and independent of f; furthermore, (3.2) takes the form*

$$(3.4) \qquad T_f(r; L(\bar{D})) = N_1(r; f^* D) + o(T_f(r; L(\bar{D})))||.$$

§4 Proof of the Main Theorem 2.1.

Let me first recall

Theorem 4.1

(**Log Bloch-Ochiai**, Nog. 1977 Hiroshima Math.J./81 Nagoya Math.J.)
Let $f : \mathbf{C} \to A$ be a holomoprhic curve into a semi-abelian variety A. Then $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a translate of a subgroup.

Proof of Main Theorem 2.1. With the given $f_i : \mathbf{C} \to A_i$ $(i = 1, 2)$ we set
$g = (f_1, f_2) : \mathbf{C} \to A_1 \times A_2$;
$A_0 = \overline{g(\mathbf{C})}^{\mathrm{Zar}}$ (semi-abelian variety by **Log Bloch-Ochiai**);
$p_i : A_0 \to A_i$ be the projections;
$E_i = p_i^* D_i$.

It follows that

$$T_{f_1}(r) \sim T_{f_2}(r) \sim T_g(r) = T(r).$$

By N. Math. Z. (1998) and a translation we may assume $g(0) = 0 \in E_1$. Let $E_i = \sum_\nu (F_i + a_{i\nu})$ be the irred. decomp. and $F_i \ni 0$.

If $F_1 \neq F_2$, then $\mathrm{codim}_{A_0} F_1 \cap F_2 \geq 2$. It follows from SMT Theorem 3.1 that

$$T(r) \sim N_1(r, f_1^* D_1) \sim N_1(r, g^*(F_1 \cap F_2)) = o(T(r))||.$$

This is a contradiction. Therefore we see that $F_1 = F_2$. Moreover, we deduce that

1. $E_1 \subset E_2$,
2. $\mathrm{St}(E_1) \subset \mathrm{St}(E_2)$, and are finite,
3. p_i are isogenies,
4. $A_1 \cong A_0/\mathrm{St}(E_1) \xrightarrow{\phi} A_0/\mathrm{St}(E_2) \cong A_2$. □

§5 Arithmetic Recurrence.

Due to the well-known correspondence between Number Theory and Nevanlinna Theory, it is tempting to give a number-theoretic analogue of Theorem 2.1 as Pál Erdös Problem–Corrales-Rodorigáñez&Schoof Theorem.

A related problem asks to classify the cases where $x^n - 1$ divides $y^n - 1$ for infinitely many positive integers n.

We would like to deal with the case of a semi-abelian variety with a given divisor, i.e., a polarized semi-abelian variety.

In the present situation, We can prove an analogue of the Main Theorem 2.1 only in the linear toric case, but not in the general case of semi-abelian varieties, that is left to be a *Conjecture*.

Here is our result in the arithmetic case.

Theorem 5.1

Let
$\mathcal{O}_S$ be a ring of S-integers in a number field k;
$\mathbf{G}_1$, $\mathbf{G}_2$ be linear tori;
$g_i \in \mathbf{G}_i(\mathcal{O}_S)$ be elements generating Zariski-dense subgroups.
D_i be reduced divisors defined over k, with defining ideals $\mathcal{I}(D_i)$, such that each irreducible component has a finite stabilizer and $\mathrm{St}(D_2) = \{0\}$.
Suppose that for infinitely many $n \in \mathbf{N}$,

$$(g_1^n)^*\mathcal{I}(D_1) \supset (g_2^n)^*\mathcal{I}(D_2). \tag{5.2}$$

Then $\exists$ étale morphism $\phi : \mathbf{G}_1 \to \mathbf{G}_2$, defined over k, and $\exists h \in \mathbf{N}$ such that $\phi(g_1^h) = g_2^h$ and $D_1 \subset \phi^*(D_2)$.

N.B.

1. Theorem 5.1 is deduced from the main results of Corvaja-Zannier, Invent. Math. 2002.
2. By an example we cannot take $h = 1$ in general.
3. By an example, the condition on the stabilzers of D_1 and D_2 cannot be omitted.
4. Note that inequality (inclusion) (5.2) of ideals is assumed only for an infinite sequence of n, not necessarily for all large n. On the contrary, we need the inequality of ideals, not only of their *supports*, i.e. of the primes containing the corresponding ideals.
5. One might ask for a similar conclusion assuming only the inequality of supports. There is some answer for it, but it is of a weaker form.

§6 1-parameter analytic subgroups

In S. Lang's "Introduction to Transcendental Numbers", Addison-Wesley, 1966, he wrote at the last paragraph of Chap. 3

"Independently of transcendental problem one can raise an interesting question of algebraic-analytic nature, namely given a 1-parameter subgroup of an abelian variety (say Zariski dense), is its intersection with a hyperplane section necessarily non-empty, and infinite unless this subgroup is algebraic?"

In 6 years later, J. Ax (Amer. J. Math. (1972)) took this problem:

Theorem 6.1

Let θ be a reduced theta function on $\mathbf{C}^m$. Let L be a 1-dimensional affine subspace of $\mathbf{C}^m$. Then either $(\theta|L)$ is constant or has an infinite number of zeros; $|\{(\theta|L) = 0\} \cap \Delta(r)| \sim r^2$. .

N.B. It seems to be still open that $|\{(\theta|L) = 0\}/\Gamma| = \infty$ unless $f(\mathbf{C})$ is algebraic.

Theorem 6.2

Let $f : \mathbf{C} \to A$ be a 1-parameter analytic subgroup in a semi-abelian variety A with $v = f'(0)$.
Let D be a reduced divisor on A.

1. *If A is abelian and $H(\cdot, \cdot)$ denotes the Riemann form associated with D, then we have*

$$\begin{aligned} N(r; f^*D) &= H(v, v)\pi r^2 + O(\log r), \\ &= (1 + o(1))N_1(r; f^*D). \end{aligned}$$

2. *Assume that $\dim A \geq 2$. If f is algebraically non-degenerate and if $\mathrm{St}(D)$ is finite, there is an irreducible component D' of D such that then $f(\mathbf{C}) \cap D'$ is Zariski dense in D'; in particular, $|f(\mathbf{C}) \cap D| = \infty$.*

N.B. In fact, the second statement holds for an arbitrary algebraically non-degenerate holomorphic curve $f : \mathbf{C} \to A$.

Proof.

(i) Note that the first Chern class $c_1(L(D))$ is represented by $i\partial\bar{\partial}H(w,w)$. It follows from our SMT Theorem 3.1 that

$$\begin{aligned} N(r; f^*D) &= T_f(r; L(D)) + O(\log r) \\ &= \int_0^r \frac{dt}{t} \int_{\Delta(t)} iH(v,v) dz \wedge d\bar{z} + O(\log r) \\ &= H(v,v)\pi r^2 + O(\log r) \\ &= (1+o(1)) N_1(r, f^*D). \end{aligned}$$

(ii) If the claim does not hold, $\exists$an algebraic subset E such that $f(\mathbf{C}) \cap D \subset E \subsetneq D$ and $\mathrm{codim}_A E \geq 2$. Then our SMT Theorem 3.1 yields that

$$N(r, f^*E) = o(r^2) = N(r, f^*D) \sim r^2 || \ (\textit{contradiction}).$$

□

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