Value Distribution and Distribution of Rational Points at Mittag-Leffler

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2008 March 27

§1 Introduction; a basic observation

- Analogues between value distribution theory and Diophantine approximaion theory.
- Some (not all) results motivated by the analogues.

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$$a + b = c. \tag{1.1}$$

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We recall the unit equation with variables a, b, c:

$$a + b = c. \tag{1.1}$$

Why is this equation interesting?

There might be several answers, but one should be that

(1.1) gives a *hyperbolic space*.

In fact, equation (1.1) defines a subvariety of the projective 2-space,

$$X \subset \mathbf{P}^2$$

with homogeneous coordinates $[a, b, c]$.

Since the variables are assumed to be units, X is isomorphic to $\mathbf{P}^1$ minus three distinct points, to say, 0,1, and ∞:

$$X \cong \mathbf{P}^1 \setminus \{0, 1, \infty\}.$$

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In complex function theory, (1.1) was studied by E. Picard for *units of entire functions*.

Picard's Theorem (1879). *A meromorphic function f on $\mathbf{C}$ omitting three distinct values of $\mathbf{P}^1$ must be constant.*

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These are certain estimates of order (height) functions by the counting functions (the functions counting orders at finite places); explicit formulae will be given later, soon.

It is of importance and interest to study a unit equation in several variables,

$$a + b + c + \cdots + f = 0 \quad (n \text{ variables}). \tag{1.2}$$

Equation (1.2) defines a variety isomorphic to

$$\mathbf{P}^{n-2} \setminus \{n \text{ hyperplanes in general position}\}.$$

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In complex function theory (1.2) was studied by E. Borel for *units of entire functions*:

E. Borel (1897): *Subsum Theorem for units of entire functions holds*; i.e., a proper shorter subsum of $a, b, c, \ldots, f$ vanishes constantly.

W. Schmidt (1971): *Subsum Theorem for S-units of an algebraic number field holds.*

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Following it, we can formulate the n variable version of abc Conjecture, named "*abc ⋯ Conjecture*":

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$$\text{Second Main Theorem for hol. curves} \iff abc\cdots \text{ Conjecture.}$$

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N.B.

- First Main Theorem $\iff$ Product Formula.
- Kobayashi hyperbolicity?

§2 Lang's Conjectures & Kobayashi Hyperbolicity

Set

- k: an algebraic number field (finite over $\mathbf{Q}$) ;
- X: an algebraic variety defined over k;
- $X(k)$: the set of k-rational points of X.

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Lang's Conjecture ('74). If there is an embedding $k \hookrightarrow \mathbf{C}$ such that the obtained complex space $X_{\mathbf{C}}$ is *Kobayashi hyperbolic*, then the cardinality $|X(k)| < \infty$.

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The *analogue of the conjecture over algebraic function fields* was proposed in the same paper; in paticular for split case, he conjectured

- *There exist only finitely many surjective rational maps from an algebraic variety to a Kobayashi hyperbolic compact algebraic variety.*

Theorem 2.1

(N. '85∼'92) (i)('85) *Let $\mathcal{X} \to R$ be a family of compact hyperbolic spaces over R. Here R may be an open variety such that*

(*) *$\mathcal{X} \to R$ carries a hyperbolically embedded compactification $\bar{\mathcal{X}} \to \bar{R}$ relative over $\bar{R}$.*

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N.B. For (ii) (Curve case): Y. Manin ('63), H. Grauert ('65), N. ('85), Imayoshi-Shiga ('88), R.F. Coleman ('90), C.-L. Chai ('91).

The Mandala of the analogues:

"Value Dist."		"Dist. of Rational Points"
$\boxed{f : \mathbf{C} \to X}$	$\Longleftrightarrow$	Infin. Family of Rat'l Pts
Kobay. Hyperbolic.	$\Longleftrightarrow$ Lang's Conj.	Finiteness of Rat'l Pts
$\Uparrow$		$\Uparrow$
Nevan. Theory	$\Longleftrightarrow$ Vojta's Dict.	Dioph. Approx.

§3 abc Conjecture and Nevanlinna's S.M.T.

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abc Conjecture. Let $a, b, c \in \mathbf{Z}$ be co-prime integers satisfying

$$a + b = c. \tag{3.1}$$

Then for ${}^{\forall}\epsilon > 0$, ${}^{\exists}C_\epsilon > 0$ such that

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \prod_{\text{prime } p|(abc)} p^{1+\epsilon}. \tag{3.2}$$

N.B. The order of *abc* at every prime p is counted only by "$1 + \epsilon$" (truncation), when it is positive.

As in §1 we put $x = [a, -b] \in \mathbf{P}^1(\mathbf{Q})$, and set

$$\mathrm{h}(x) = \log \max\{|a|, |b|\} \geq 0 \quad \text{(height)}, \tag{3.3}$$

$$N_1(x; \infty) = \sum_{p|a} \log p \quad \text{(counting function truncated to level 1)}, \tag{3.4}$$

$$N_1(x; 0) = \sum_{p|b} \log p \qquad (\quad '' \quad),$$

$$N_1(x; 1) = \sum_{p|c} \log p \qquad (\quad '' \quad).$$

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Then abc Conjecture (3.2) is rewritten as

$$\text{(3.5)} \qquad (1-\epsilon)\mathrm{h}(x) \leq N_1(x; 0) + N_1(x; \infty) + N_1(x; 1) + C_\epsilon, \ x \in \mathbf{P}^1(\mathbf{Q}).$$

For q distinct points $a_i \in \mathbf{P}^1(\mathbf{Q}), 1 \leq i \leq q$,

$$(3.6) \qquad (q-2-\epsilon)\mathrm{h}(x) \leq \sum_{i=1}^{q} N_1(x; a_i) + C_\epsilon$$

(N '96, Vojta '98).

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Theorem 3.7

(Nevanlinna's S.M.T.) *Let f be a meromorphic function in $\mathbf{C}$. For q distinct points $a_i \in \mathbf{P}^1(\mathbf{C}), 1 \leq i \leq q$,*

$$(q-2)T_f(r) \leq \sum_{i=1}^{q} N_1(r, f^* a_i) + O(\log^+(rT_f(r)))||.$$

§4 abc· · · Conjecture

abc· · · Conjecture 1. Let $a, b, c, \ldots, e, f \in \mathbf{Z}$ be n integers without common factor satisfying

$$a + b + c + \cdots + e + f = 0.$$

Then for ${}^{\forall}\epsilon > 0$, ${}^{\exists}C_\epsilon$ and a proper algebraic subset ${}^{\exists}E_\epsilon \subset \mathbf{P}^{n-2}_{\mathbf{Z}}$ such that for $[a, b, \ldots, e] \notin E_\epsilon$

(4.1) $(1 - \epsilon) \log \max\{|a|, \ldots, |f|\} \leq \sum_{p|a} \log p + \cdots + \sum_{p|f} \log p + C_\epsilon$.

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For the sake of notational convenience, we set

- $a = x_0, b = x_1, \dots, e = x_n$ ($n+1$ variables).
- $x = [x_0, \dots, x_n] \in \mathbf{P}^n(\mathbf{Q})$.
- $\mathrm{h}(x) = \log \max_{0 \leq j \leq n}\{|x_j|\}$: the height of x.
- $H_j = x_j, 0 \leq j \leq n$, $H_{n+1} = -\sum_{j=0}^n x_j$: $n+2$ linear forms in general position.
- $N_1(x; H_j)$: the counting function truncated to level 1.

Then (4.1) is equivalent to

$$(1-\epsilon)\mathrm{h}(x) \leq \sum_{j=0}^{n+1} N_1(x; H_j) + C_\epsilon. \tag{4.2}$$

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We consider a bit more general case.
Let S be a finte set of primes and let $l \leq \infty$.
We define an S-counting function truncated to level l by

$$N_l(x; S, H_j) = \sum_{p \notin S,\, p | H_j(x)} \min\{\deg_p H_j(x)), l\} \cdot \log p. \tag{4.3}$$

abc$\cdots$ Conjecture 2. Let $H_j, 1 \le j \le q$ be q ($\ge n+2$) linear forms on $\mathbf{P}^n_{\mathbf{Q}}$ in general position. Then for ${}^{\forall}\epsilon > 0$, ${}^{\exists}C_\epsilon$ and a proper algebraic subset ${}^{\exists}E_\epsilon \subset \mathbf{P}^n_{\mathbf{Q}}$ such that

$$(4.4) \quad (q-n-1-\epsilon)\mathrm{h}(x) \le \sum_{j=1}^{q} N_1(x; S, H_j) + C_\epsilon, \quad x \in \mathbf{P}^n(\mathbf{Q}) \setminus E_\epsilon$$

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Schmidt's Subspace Theorem is stated as follows.

Theorem 4.5

Let the notaion be as above.
For $^\forall \epsilon > 0$, $^\exists C_\epsilon$ and a finite union $^\exists E_\epsilon$ of proper linear subspaces of $\mathbf{P}^n_{\mathbf{Q}}$ such that

$$(q-n-1-\epsilon)\mathrm{h}(x) \le \sum_{j=1}^q N_\infty(x; S, H_j) + C_\epsilon, \quad x \in \mathbf{P}^n(\mathbf{Q}) \setminus E_\epsilon.$$

N.B. When $n = 1$, this is Roth's Theorem.

Theorem 4.6

(H. Cartan's S.M.T., '33) *Let* $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *be a linearly non-degenerate holomorphic curve.*
Let H_j *be* q *hyperplanes of* $\mathbf{P}^n(\mathbf{C})$ *in general position.*
Then

$$(q-n-1)T_f(r) \le \sum_{i=1}^{q} N_n(r, f^*H_j) + O(\log^+(rT_f(r)))||$$

§5 Analogue over algebraic function fields

It is interesting to consider the problem over algebraic function fields. The case of algebraic function fields is situated in the middle of the Nevanlinna theory and the number theory.

$$\boxed{\text{Nevanlinna Theory}} \quad \text{———} \quad \boxed{\text{Number Theory}}$$

$$\diagdown \qquad \boxed{\text{Theory/ F.F.}} \qquad \diagup$$

We skip this today.

§6 abc··· Theorem for holomorphic curves into semi-abelian varieties

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Let A be a semi-abelian variety:

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \text{ (abelian variety)} \to 0.$$

The universal covering $\tilde{A} \cong \mathbf{C}^n$, $n = \dim A$.

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The universal covering $\tilde{A} \cong \mathbf{C}^n$, $n = \dim A$.

Let $f : \mathbf{C} \to A$ be a holomorphic curve. Set

- $J_k(A)$: the k-jet bundle over A; $J_k(A) \cong A \times \mathbf{C}^{nk}$;
- $J_k(f) : \mathbf{C} \to J_k(A)$: the k-jet lift of f;
- $X_k(f)$: the Zariski closure of the image $J_k(f)(\mathbf{C})$.

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Lemma 6.1

(N. '77)
(i) *For* $f : \mathbf{C} \to A$,

$$T_{I_k \circ J_k(f)}(r) = O(\log^+(rT_f(r))) \,||.$$

(ii) *For* $f : \mathbf{C} \to \bar{A}$ *(compactification)*,

$$m(r; I_k \circ J_k(f)) \stackrel{\text{def}}{=} \int_{|z|=r} \log^+ \|I_k \circ J_k(f)(z)\| \frac{d\theta}{2\pi} = O(\log^+(rT_f(r))) \,||.$$

N.B. This is Lemma on logarithmic derivatives in higher dimension.

Theorem 6.2

(N.-Winkelmann-Yamanoi, to appear in Forum Math. '08)

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Let $f : \mathbf{C} \to A$ be algebraically non-degenerate.
(i) *Let Z be an algebraic reduced subvariety of $X_k(f)$ ($k \geqq 0$). Then ${}^{\exists}\bar{X}_k(f)$, compactification of $X_k(f)$ such that*

$$(6.3) \qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) \leqq N_1(r; J_k(f)^* Z) + \epsilon T_f(r)||_\epsilon, \quad {}^{\forall}\epsilon > 0.$$

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(ii) *Moreover, if $\mathrm{codim}\,_{X_k(f)} Z \geqq 2$, then*

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(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geqq 2$, *then*

$$(6.4) \qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) \leqq \epsilon T_f(r)||_\epsilon, \quad {}^\forall \epsilon > 0.$$

(iii) *If $k = 0$ and Z is an effective divisor D on A, then $\bar{A}$ is smooth, equivariant, and independent of f; furthermore, (6.3) takes the form*

$$(6.5) \qquad T_f(r; L(\bar{D})) \leqq N_1(r; f^* D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon, \quad {}^\forall \epsilon > 0.$$

N.B. (1) In N.-W.-Y. Acta '02, we proved (6.5) with a higher level truncated counting function $N_k(r; f^*D)$ for some special compactification of A and with a better error term "$O(\log^+(rT_f(r)))$".

(2) For the truncation of level 1, the error term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log^+(rT_f(r)))$".

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Because of the trunction level 1, we have the following interesting application.

Theorem 6.6

(Conjectured by M. Green, '74) *Assume that $f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C})$ omits two lines $\{x_i = 0\}, i = 1, 2$, and the conic $\{x_0^2 + x_1^2 + x_2^2 = 0\}$. Then f is algebraically degenerate.*

Lately, Corvaja-Zannier obtained the corresponding result over algebraic function fields (preprint).

§7 Application

Theorem 7.1

(N.-Winkelmann-Yamanoi, J. Math. Pure. Appl. '07)

Let X be an algebraic variety such that

(i) $\bar{q}(X) \geqq \dim X$ *(log. irregularity);*

(ii) $\bar{\kappa}(X) > 0$ *(log. Kodaira dimension);*

(iii) *the Albanese map $X \to A$ is proper.*

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Then $^{\forall} f : \mathbf{C} \to X$ is algebraically degenerate.
Moreover, the normalization of $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a semi-abelian variety which is finite étale over a translate of a proper semi-abelian subvariety of A.

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N.B. The case "$\bar{q}(X) > \dim X$" was known as Log-Bloch-Ochiai's Theorem (N. '77-'81). The proof for the case "$\bar{q}(X) = \dim X$" requires our new Theorem 6.2.

As a special case we have

Theorem 7.2

Let $D = \sum_{i=1}^{q} D_i \subset \mathbf{P}^n(\mathbf{C})$ be an s.n.c. divisor.
Assume that $q > n$ and $\deg D > n + 1$.
Then ${}^\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus D$ is algebraically degenerate.

This is even more than Green's Conjecture.

Question. Let $D = \sum_{i=1}^{q} D_i \subset \mathbf{P}^n(\mathbf{C})$ be a divisor in general position (the codimensions of intersections of D_i's decrease exactly as the number of D_i's), possibly with singularities.

Assume that $q > n$ and $\deg D > n + 1$.

Then, is $\bar{\kappa}(\mathbf{P}^n(\mathbf{C}) \setminus D) > 0$?

§8 Yamanoi's abc Theorem

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His method:

- Ahlfors' covering theory;
- Mumford's theory of the compactification of curve moduli;
- The tree theory for point configurations.

We recall his result in a form suitable to the present talk.

Let $p : X \to S$ be a surjective morphism between smooth projective algebraic varieties with relative canonical bundle $K_{X/S}$.

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Theorem 8.1

(Yamanoi, '04, '06) *Assume that*

- $\dim X/S = 1$;
- $D \subset X$ *is a reduced divisor ;*
- $f : \mathbf{C} \to X$ *is algebraically nondegenerate ;*
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Then for ${}^{\forall}\epsilon > 0$, ${}^{\exists}C(\epsilon) > 0$ *such that*

$$(8.2) \quad T_f(r; [D]) + T_f(r; K_{X/S}) \leq N_1(r; f^*D) + \epsilon T_f(r) + C(\epsilon) T_g(r) ||_\epsilon.$$

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Let $\mathbf{C}(X)$ be the rational function field of X. Then,

$$\text{transc-deg}_{\mathbf{C}} \mathbf{C}(X) = \dim X.$$

Set $\mathcal{S}(f) = \{\phi \in \mathbf{C}(X); \mathrm{Supp}\,(\phi)_\infty \not\supset f(\mathbf{C}),\, T_{\phi\circ f}(r) \leq \epsilon T_f(r)||_\epsilon,\ \forall \epsilon > 0\}.$

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Then $\mathcal{S}(f)$ is a subfield of $\mathbf{C}(X)$ by F.M.T., and we define

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Proposition 8.3

If $\mathrm{s\text{-}dim}(f) = \dim X$, *then* f *is algebraically degenerate.*

Otherwise, $T_f(r) \leq \epsilon T_f(r)||$; this is easily follows from the general theory of F.M.T.

N.B. If $\dim X = 1$ and genus$(X) \geq 2$, then Lemma 6.1 (L.L.D.) & F.M.T. imply $\mathrm{s\text{-}dim}(f) = 1$ for non-constant $f : \mathbf{C} \to X$.

The proofs of many important degeneracy theorems for holomorphic curves rely on "$\mathrm{s\text{-}dim}(f) = \dim X$", at least in part.

As an application of Yamanoi's abc Theorem we have

Theorem 8.4

Assume that $\dim X = 2$, *and that* X *is of general type.*
Let $f : \mathbf{C} \to X$ *be a holomorphic curve such that* $\mathrm{s\text{-}dim}(f) = 1$.
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Proof. Suppose that f is algebraically nondegenerate. The assumption implies that

$^{\exists}S$, a curve, and
$^{\exists}p : X \to S$, a morphism (after some birational change) such that
$g = p \circ f : \mathbf{C} \to S$ is non-constant and satisfies

$$T_g(r) \leq \epsilon T_f(r)||_\epsilon.$$

Therefore S is rational or elliptic. Thus, $-K_S$ is effective.
Since K_X is big and $K_{X/S} = K_X - p^* K_S$, $K_{X/S}$ is big. Hence,

$$T_f(r; K_{X/S}) \sim T_f(r).$$

Yamanoi's abc Theorem 8.2 (with $D = 0$) implies that

$$T_f(r; K_{X/S}) = O(T_g(r))||.$$

Since $T_g(r) \le \epsilon T_f(r)||_\epsilon$ and $K_{X/S}$ is big,

$$\text{s-dim}(f) = \dim X, \text{ or } T_f(r) \le \epsilon T_f(r)|| \quad ({}^\forall \epsilon > 0).$$

This is a contradiction!; Q.E.D..

§9 Fundamental Conjecture for holomrophic curves

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Let X be a smooth algebraic variety, and let $D = \sum_i D_i$ be a reduced s.n.c. divisor on X with irreducible D_i.

Then, for an algebraically non-degenerate $f : \mathbf{C} \to X$ we have

$$T_f(r; L(D)) + T_f(r; K_X) \leq \sum_i N_1(r; D_i) + \epsilon T_f(r)||, \ ^{\forall}\epsilon > 0. \tag{9.1}$$

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This implies Green-Griffiths' Conjecture and Kobayashi's Conjecture.

Even when $X = \mathbf{P}^n(\mathbf{C})$ and D_i are hyperplanes, the Fundamental Conjecture is open; if $N_1(r; D_i)$ are replaced by $N_n(r; D_i)$, this is Cartan's Theorem 4.6, where f suffices to be linearly non-degenerate.

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If $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ omits $n+1$ hyperplanes $H_i, 1 \leq i \leq n+1$ in general position, then $\mathbf{P}^n(\mathbf{C}) \setminus \sum_{i=1}^{n+1} H_i \cong (\mathbf{C}^*)^n$. In this case, the Fundamental Conjecture is true because of Theorem 6.2.