Holomorphic Curves and Rational Points

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§1 Introduction; a basic observation

- Analogues between value distribution theory and Diophantine approximaion theory.
- Some (not all) results motivated by the analogues.

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$$a + b = c. \tag{1.1}$$

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$$a + b = c. \tag{1.1}$$

Why is this equation interesting?

There might be several answers, but one should be that

(1.1) gives a *hyperbolic space*.

In fact, equation (1.1) defines a subvariety of the projective 2-space,

$$X \subset \mathbf{P}^2$$

with homogeneous coordinates $[a, b, c]$.

Since the variables are assumed to be units, X is isomorphic to $\mathbf{P}^1$ minus three distinct points, to say, 0,1, and ∞:

$$X \cong \mathbf{P}^1 \setminus \{0, 1, \infty\}.$$

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In complex function theory, (1.1) was studied by E. Picard for *units of entire functions*.

Picard's Theorem (1879). *A meromorphic function f on $\mathbf{C}$ omitting three distinct values of $\mathbf{P}^1$ must be constant.*

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If f omits $0, 1, \infty$, then $f, (1 - f)$ and 1 are units in the ring of entire functions, and satisfy

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These are certain estimates of order (height) functions by the counting functions (the functions counting orders at finite places); explicit formulae will be given later, soon.

It is of importance and interest to study a unit equation in several variables,

$$a + b + c + \cdots + f = 0 \quad (n \text{ variables}). \tag{1.2}$$

Equation (1.2) defines a variety isomorphic to

$$\mathbf{P}^{n-2} \setminus \{n \text{ hyperplanes in general position}\}.$$

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E. Borel (1897): *Subsum Theorem for units of entire functions holds*; i.e., a proper shorter subsum of $a, b, c, \ldots, f$ vanishes constantly.

W. Schmidt (1971): *Subsum Theorem for S-units of an algebraic number field holds.*

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Following it, we can formulate the n variable version of abc Conjecture, named "$abc\cdots$ *Conjecture*":

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N.B.

- First Main Theorem $\Longleftrightarrow$ Product Formula.
- Related topics: Kobayashi hyperbolicity.

§2 Log Bloch-Ochiai & Faltings-Vojta

Here I show a fine analogue in the distribution of holomorphic curves and the distribution of rational points.

Theorem 2.1

(Log Bloch-Ochiai (26-77), N. (77-81)). *Let X be a complex algebraic variety with logarithmic irregularity $\bar{q}(X) > \dim X$.*
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Theorem 2.2

(Faltings (91)-Vojta (96)). *Let X be defined over a number field k with $\bar{q}(X) > \dim X$.*
Then $X(k)$ is contained in a proper subvariety of X.

Theorem 2.3

Let
M = *a projective manifold of dimension* m;
$\{D_i\}_{i=1}^{l}$ = *a family of ample hypersurfaces of* M *in general position;*
$W(\subset M)$ = *a subvariety such that* $\exists$ *non-constant holomorphic curve* $f : \mathbf{C} \to W \setminus \bigcup_{D_i \not\supset W} D_i$ *with Zariski dense image.*
Then we have
(i) $(l - m)\dim W \leqq m\left(\mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^{l} - q(W)\right)^{+}.$

(ii) *Let* $f : \mathbf{C} \to M$ *be a holomorphic curve such that for every* D_i*, either* $f(\mathbf{C}) \subset D_i$*, or* $f(\mathbf{C}) \cap D_i = \emptyset$.

Assume that $l > m$.

Then $f(\mathbf{C})$ *is contained in an algebraic subspace* W *of* M *such that*

$$\dim W \leqq \frac{m}{l-m} \mathrm{rank}_{\mathbf{Z}} \mathrm{NS}(M).$$

In particular, if $\mathrm{rank}_{\mathbf{Z}} \mathrm{NS}(M) = 1$ *(e.g.,* $M = \mathbf{P}^m(\mathbf{C})$*), then we have*

$$\dim W \leqq \frac{m}{l-m}; \quad W \text{ is finite for } l > 2m.$$

Theorem 2.4

Assume that everything is defined over a number field k, and Let
S = a finite subset of inequivalent non-trivial places of k containing all infinite places;
V = a projective smooth variety of dimension m;
$\{D_i\}_{i=1}^{l}$ = a family of ample hypersurfaces of V in general position;
$W(\subset V)$ = a subvariety of V.
Assume that there exists a Zariski dense $(\sum_{D_i \not\supset W} D_i \cap W, S)$-integral point set of $W(k)$ in W.
Then we have

1. $(l-m)\dim W \leqq m\left(\mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^{l} - q(W)\right)^{+}$.
2. *Let $D_i, 1 \leqq i \leqq l$, be ample divisors of V in general position.*
   Let Z be a subset of $V(k)$ such that for every D_i, either $Z \subset D_i$, or Z is a $(\sum_{D_i \not\supset A} D_i, S)$-integral point set.
   Assume that $l > m$.
   Then Z is contained in an algebraic subvariety W of V such that

$$\dim W \leqq \frac{m}{l-m}\mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(V).$$

In particular, if $\mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(V) = 1$ $(V = \mathbf{P}^m_K)$, then we have

$$\dim W \leqq \frac{m}{l-m}; \quad W \textit{ is finite for } l > 2m.$$

The Mandala of the analogues:

“Value Dist.”		“Dist. of Rational Points”
$f : \mathbf{C} \to X$	$\Longleftrightarrow$	Infin. Family of Rat’l Pts
Kobay. Hyperbolic.	$\Longleftrightarrow$ Lang’s Conj.	Finiteness of Rat’l Pts
$\Uparrow$		$\Uparrow$
Nevan. Theory	$\Longleftrightarrow$ Vojta’s Dict.	Dioph. Approx.

§3 abc Conjecture and Nevanlinna's S.M.T.

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(3.1) $$a + b = c,$$

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$$a + b = c, \tag{3.1}$$

then

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \prod_{\text{prime } p|(abc)} p^{1+\epsilon}. \tag{3.2}$$

N.B. The order of *abc* at every prime p is counted only by "$1 + \epsilon$" (truncation), when it is positive.

As in §1 we put $x = [a, -b] \in \mathbf{P}^1(\mathbf{Q})$, and set

$$(3.3) \qquad \mathrm{h}(x) = \log \max\{|a|, |b|\} \geq 0 \quad \text{(height)},$$

$$(3.4) \qquad N_1(x; \infty) = \sum_{p|a} \log p \quad \text{(counting function truncated to level 1)},$$

$$N_1(x; 0) = \sum_{p|b} \log p \qquad (\quad '' \quad),$$

$$N_1(x; 1) = \sum_{p|c} \log p \qquad (\quad '' \quad).$$

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$$N_1(x; 1) = \sum_{p|c} \log p \qquad (\quad '' \quad).$$

Then abc Conjecture (3.2) is rewritten as

$$\text{(3.5)} \qquad (1 - \epsilon)\mathrm{h}(x) \leq N_1(x; 0) + N_1(x; \infty) + N_1(x; 1) + C_\epsilon, \ x \in \mathbf{P}^1(\mathbf{Q}).$$

For q distinct points $a_i \in \mathbf{P}^1(\mathbf{Q}), 1 \leq i \leq q$,

$$(3.6) \qquad (q-2-\epsilon)\mathrm{h}(x) \leq \sum_{i=1}^{q} N_1(x; a_i) + C_\epsilon$$

(formulated by N '96, Vojta '98).

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Theorem 3.7

(Nevanlinna's S.M.T.) *Let f be a meromorphic function in $\mathbf{C}$. For q distinct points $a_i \in \mathbf{P}^1(\mathbf{C}), 1 \le i \le q$,*

$$(q-2)T_f(r) \le \sum_{i=1}^{q} N_1(r, f^* a_i) + O(\log^+(rT_f(r))||.$$

$$T_f(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^*(\text{F.-S. metric}) \text{ (due to Shimizu)}.$$

If f is entire, $T_f(r) \sim \log \max_{|z| \le r} |f(z)|$.

§4 abc$\cdots$ Conjecture

abc$\cdots$ Conjecture 1. Let $a, b, c, \ldots, e, f \in \mathbf{Z}$ be n integers without common factor satisfying

$$a + b + c + \cdots + e + f = 0.$$

Then for $^{\forall}\epsilon > 0$, $^{\exists}C_\epsilon$ and a proper algebraic subset $^{\exists}E_\epsilon \subset \mathbf{P}^{n-2}_{\mathbf{Z}}$ such that for $[a, b, \ldots, e] \not\in E_\epsilon$

(4.1) $(1 - \epsilon)\log\max\{|a|, \ldots, |f|\} \leq \sum_{p|a} \log p + \cdots + \sum_{p|f} \log p + C_\epsilon$.

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(4.1) $(1-\epsilon) \log \max\{|a|, \ldots, |f|\} \le \sum_{p|a} \log p + \cdots + \sum_{p|f} \log p + C_\epsilon$.

For the sake of notational convenience, we set

- $a = x_0, b = x_1, \ldots, e = x_n$ ($n+1$ variables).
- $x = [x_0, \ldots, x_n] \in \mathbf{P}^n(\mathbf{Q})$.
- $\mathrm{h}(x) = \log \max_{0 \le j \le n}\{|x_j|\}$: the height of x.
- $H_j = x_j, 0 \le j \le n$, $H_{n+1} = -\sum_{j=0}^n x_j$: $n+2$ linear forms in general position.
- $N_1(x; H_j)$: the counting function truncated to level 1.

Then (4.1) is equivalent to

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We consider a bit more general case.
Let S be a finte set of primes and let $l \leq \infty$.
We define an S-counting function truncated to level l by

$$N_l(x; S, H_j) = \sum_{p \notin S,\, p | H_j(x)} \min\{\deg_p H_j(x)), l\} \cdot \log p. \tag{4.3}$$

abc$\cdots$ Conjecture 2. Let $H_j, 1 \le j \le q$ be q $(\ge n+2)$ linear forms on $\mathbf{P}^n_{\mathbf{Q}}$ in general position.
Then for $^\forall \epsilon > 0$, $^\exists C_\epsilon$ and a proper algebraic subset $^\exists E_\epsilon \subset \mathbf{P}^n_{\mathbf{Q}}$ such that

$$(4.4) \quad (q-n-1-\epsilon)\mathrm{h}(x) \le \sum_{j=1}^{q} N_1(x; S, H_j) + C_\epsilon, \quad x \in \mathbf{P}^n(\mathbf{Q}) \setminus E_\epsilon$$

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abc$\cdots$ Conjecture 2. Let $H_j, 1 \leq j \leq q$ be q $(\geq n+2)$ linear forms on $\mathbf{P}^n_{\mathbf{Q}}$ in general position.
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(N '96, Vojta '98).

Schmidt's Subspace Theorem is stated as follows.

Theorem 4.5

Let the notaion be as above.
For ${}^\forall\epsilon > 0$, ${}^\exists C_\epsilon$ and a finite union ${}^\exists E_\epsilon$ of proper linear subspaces of $\mathbf{P}^n_{\mathbf{Q}}$ such that

$$(q-n-1-\epsilon)\mathrm{h}(x) \leq \sum_{j=1}^{q} N_\infty(x; S, H_j) + C_\epsilon, \quad x \in \mathbf{P}^n(\mathbf{Q}) \setminus E_\epsilon.$$

N.B. When $n = 1$, this is Roth's Theorem.

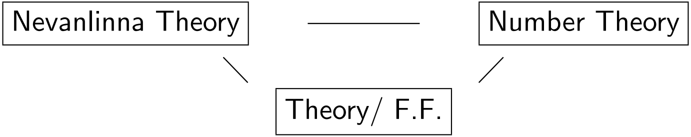
Theorem 4.6

(H. Cartan's S.M.T., '33) *Let* $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *be a linearly non-degenerate holomorphic curve.*
Let H_j *be* q *hyperplanes of* $\mathbf{P}^n(\mathbf{C})$ *in general position.*
Then

$$(q-n-1)T_f(r) \leq \sum_{i=1}^{q} N_n(r, f^* H_j) + O(\log^+(rT_f(r)))||$$

§5 Analogue over algebraic function fields

It is interesting to consider the problem over algebraic function fields. The case of algebraic function fields is situated in the middle of the Nevanlinna theory and the number theory.



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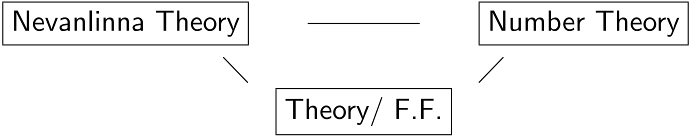

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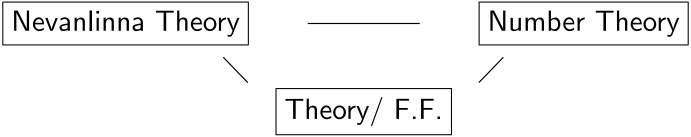

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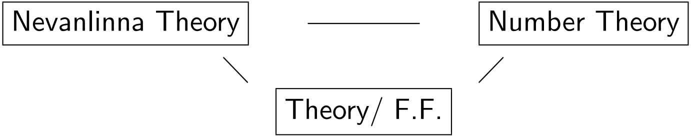

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(b) Deformation of a family of p.p. abelian varieties over function fields (Family of families ..., Kuga-Ihara (77)).

(c) A gap theorem for contact orders in abelian or semi-abelian varieties.

We skip (a) today.

Problem (b) $\rightarrow$ Deformation of a holomorphic map $y : B \rightarrow D/\Gamma$ (Siegel domain/Γ). Here

B denotes a smooth algebraic variety with the given function field,

D a bunded symmetric domain in general, and

Γ is arithmetic or co-compact discrete sugroup of $\mathrm{Aut}(D)$.

Problem (b) $\rightarrow$ Deformation of a holomorphic map $y : B \rightarrow D/\Gamma$ (Siegel domain/Γ). Here

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By making use of the Kobayashi hyperbolic metric and the theory of harmonic maps we have

Theorem 5.1

(N. (88), Miyano-N. (91)). *For the simplicity, assume that* D/Γ *is smooth.*
(i) *The moduli space* $\mathrm{Hol}(B, D/\Gamma)$ *of all holomorphic maps from* B *into* D/Γ *is a smooth quasi-projective algebraic variety.*
For every component Z_1 *of* $\mathrm{Hol}(B, D/\Gamma)$, *the evaluation map at* $x \in B$

$$\Phi_x : y \in Z_1 \to y(x) \in D/\Gamma$$

is a proper holomorphic immersion onto a totally geodesic submanifold of D/Γ, *and hence*

$$Z_1 \cong D_1/\Gamma_1.$$

(ii) *There is a natural holomorphic map*

$$\eta : x \in B \to \Phi_x \in \mathrm{Hol}(D_1/\Gamma_1, D/\Gamma),$$

and the natural map (2nd evaluation map)

$$\Phi_2 : (D_1/\Gamma_1) \times (D_2/\Gamma_2) \to D/\Gamma,$$

is a proper holomorphic embedding onto a totally geodesic submanifold of D/Γ *such that* $y(x) = \Phi_2(y, \eta(x))$ *for* $(y, x) \in (D_1/\Gamma_1) \times B$.

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Corollary 5.2

If D/Γ *admits no non-trivial product structure of totally geodesic submanifolds, then every non-constant* $y : B \to D/\Gamma$ *is rigid, and hence there are only finitely many such* y.

(c) A gap theorem. The problem for abelian varieties was first dealt with by A. Buium.

Theorem 5.3

(Buium-98). *Let*
$A =$ *an abelian variety;*
$D =$ *a reduced divisor on* A *which is Kobayashi hyperbolic;*
$C =$ *a smooth compact curve.*
Then $\exists N \in \mathbf{N}$ *depending on* C, A *and* D *such that for every morphism* $f : C \to A$, *either* $\mathrm{mult}_x f^* D \leq N \quad (\forall x \in C)$, *or* $f(C) \subset D$.

Corollary 5.4

Let the notation be as in Theorem 5.3. If $f(C) \not\subset D$, *then*

$$\text{“height}\,(f)\text{''} = \deg(f) \leq N|f^{-1}(D)|.$$

This is a problem of type of *abc*-Conjecture. His proof based on Kolchin's theory of differential algebra and he posed two problems:

- Find a proof by complex geometry.
- The Kobayashi hyperbolicity assumption for D is too strong, and the ampleness should suffice.

Definition. A complex algebraic group A is semi-abelian if

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \,(= \text{abelian}) \to 0.$$

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Theorem 5.5

(Nog.-Winkelmann ('04)). *Let*
A = a semi-abelian variety with a smooth equivariant algebraic compactification $\bar{A}$;
$\bar{D}$ = an effective reduced ample divisor on $\bar{A}$, and $D = \bar{D} \cap A$;
C = a smooth algebraic curve with smooth compactification $C \hookrightarrow \bar{C}$.
Then $\exists N \in \mathbf{N}$ such that for every morphism $f : C \to A$ either

$$f(C) \subset D \quad \text{or} \quad \mathrm{mult}_x f^* D \leq N \quad (\forall x \in C).$$

Furthermore, the number N depends only on the numerical data involved as follows:

1. *The genus of $\bar{C}$ and the number $\#(\bar{C} \setminus C)$ of the boundary (puncture) points of C (only the genus in compact case),*
2. *the dimension of A,*
3. *the toric variety (or, equivalently, the associated "fan") which occurs as closure of the orbit in $\bar{A}$ of the maximal connected linear algebraic subgroup $T \cong (\mathbf{C}^*)^t$ of A,*
4. *all intersection numbers of the form $\bar{D}^h \cdot B_{i_1} \cdots B_{i_k}$, where the B_{i_j} are closures of A-orbits in $\bar{A}$ of dimension n_j and $h + \sum_j n_j = \dim A$ (only D^n in compact case).*

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Corollary 5.6

If $f(C) \not\subset \mathrm{Supp}\, D$, then

$$\deg f^* D \;(\mathrm{height}) \leq N \cdot |\mathrm{Supp}\, f^* D|.$$

In particular, if we let A, $\bar{A}$, C and D vary within a flat connected family, then we can find a uniform bound for N.

As an application, a *finiteness theorem* was obtained for morphisms from a non-compact curve into an abelian variety omitting an ample divisor.

§6 abc$\cdots$ Theorem for holomorphic curves into semi-abelian varieties

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The universal covering $\tilde{A} \cong \mathbf{C}^n$, $n = \dim A$.

§6 abc··· Theorem for holomorphic curves into semi-abelian varieties

Let A be a semi-abelian variety.
The universal covering $\tilde{A} \cong \mathbf{C}^n$, $n = \dim A$. Let

- $f : \mathbf{C} \to A$, be a holomorphic curve ;
- $J_k(A) =$ k-jet bundle over A; $J_k(A) \cong A \times \mathbf{C}^{nk}$;
- $J_k(f) : \mathbf{C} \to J_k(A)$, k-jet lift of f;
- $X_k(f) =$ Zariski closure of the image $J_k(f)(\mathbf{C})$.
- $I_k : J_k(A) \cong A \times \mathbf{C}^{nk} \to \mathbf{C}^{nk}$, jet projection.

Lemma 6.1

(N. '77)
(i) *For* $f : \mathbf{C} \to A$,

$$T_{I_k \circ J_k(f)}(r) = O(\log^+(rT_f(r)))\,||.$$

(ii) *For* $f : \mathbf{C} \to \bar{A}$ *(compactification),*

$$m(r; I_k \circ J_k(f)) \stackrel{\text{def}}{=} \int_{|z|=r} \log^+ \|I_k \circ J_k(f)(z)\| \frac{d\theta}{2\pi} = O(\log^+(rT_f(r)))\,||.$$

N.B. This is Lemma on logarithmic derivatives in higher dimension.

Theorem 6.2

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Let $f : \mathbf{C} \to A$ be algebraically non-degenerate.
(i) *Let Z be an algebraic reduced subvariety of $X_k(f)$ ($k \geqq 0$). Then $^\exists \bar{X}_k(f)$, compactification of $X_k(f)$ such that*

$$T_{J_k(f)}(r; \omega_{\bar{Z}}) \leqq N_1(r; J_k(f)^* Z) + \epsilon T_f(r) ||_\epsilon, \quad ^\forall \epsilon > 0. \tag{6.3}$$

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(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geqq 2$*, then*

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(iii) *If $k = 0$ and Z is an effective divisor D on A, then $\bar{A}$ is smooth, equivariant, and independent of f; furthermore, (6.3) takes the form*

$$(6.5) \qquad T_f(r; L(\bar{D})) \leqq N_1(r; f^* D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon, \quad {}^{\forall}\epsilon > 0.$$

N.B. (1) In N.-W.-Y. Acta '02, we proved (6.5) with a higher level truncated counting function $N_k(r; f^*D)$ for some special compactification of A and with a better error term "$O(\log^+(rT_f(r)))$".

(2) For the truncation of level 1, the error term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log^+(rT_f(r)))$".

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Because of the trunction level 1, we have the following interesting application.

Theorem 6.6

(Conjectured by M. Green, '74) *Assume that $f : \mathbf{C} \to \mathbf{P}^2(\mathbf{C})$ omits two lines $\{x_i = 0\}, i = 1, 2$, and the conic $\{x_0^2 + x_1^2 + x_2^2 = 0\}$. Then f is algebraically degenerate.*

Lately, Corvaja-Zannier obtained some corresponding result over algebraic function fields (J.A.G. 2008).

§7 Application

Theorem 7.1

(N.-Winkelmann-Yamanoi, J. Math. Pure. Appl. '07)
Let X be an algebraic variety such that

(i) $\bar{q}(X) \geqq \dim X$ *(log. irregularity);*

(ii) $\bar{\kappa}(X) > 0$ *(log. Kodaira dimension);*

(iii) *the quasi- Albanese map $X \to A$ is proper.*

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N.B. The case "$\bar{q}(X) > \dim X$" was known as Log-Bloch-Ochiai's Theorem (N. '77-'81). The proof for the case "$\bar{q}(X) = \dim X$" requires our new Theorem 6.2.

As a special case we have

Theorem 7.2

Let $D = \sum_{i=1}^{q} D_i \subset \mathbf{P}^n(\mathbf{C})$ *be an s.n.c. divisor.*
Assume that $q > n$ *and* $\deg D > n + 1$.
Then ${}^{\forall} f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus D$ *is algebraically degenerate.*

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Question. Let $D = \sum_{i=1}^{q} D_i \subset \mathbf{P}^n(\mathbf{C})$ be a divisor in general position (the codimensions of intersections of D_i's decrease exactly as the number of D_i's), possibly with singularities.

Assume that $q > n$ and $\deg D > n + 1$.

Then, is $\bar{\kappa}(\mathbf{P}^n(\mathbf{C}) \setminus D) > 0$?