Some open problems in the value distribution theory and Kobayashi hyperbolic manifolds

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§1 Kobayashi hyperbolic manifold

Theorem 1.1

Let X (resp. Y) be a Zariski open of a compact complex space $\bar{X}$ (resp. $\bar{Y}$).
Assume that X is complete hyperbolic and hyperbolically imbedded into $\bar{X}$.
(i) (N. 1988) *If $\bar{Y}$ is smooth and ∂Y is a s.n.c. divisor, then $\mathrm{Hol}(Y, X)$ is relatively compact in $\mathrm{Hol}(\bar{Y}, \bar{X})$.*
(ii) (Makoto Suzuki 1994)
$\mathrm{Mer}_{\mathrm{dom}}(Y, X) = \{f : Y \to X;$ *meromorphic and dominant*$\}$ *is finite.*

S. Lang Conjecture (for compact X) 1974;
Kobayashi-Ochiai 1975 for compact X of general type;
N. 1985, smooth compact Kähler X with $c_1(X) \leq 0$;
C. Horst 1991, smooth compact Kähler X;
N. 1992, general compact X;

N.B. The result (i) has applications to Lang's conjecture and Parshin-Arakelov type theorems over function fields.

Note that $\mathrm{Hol}(Y, X) = \{f \in \mathrm{Hol}(\bar{Y}, \bar{X}); f^*\partial X \subset \partial Y\}$ as sets (after the extension). But, be careful that "$f^*\partial X \subset \partial Y$" is not a closed condition. In a number of papers, it was mistreated as "$f(\partial Y) \subset \partial X$" which is a closed condition. After all, the results hold because of the complete Kobayashi hyperbolicity of X.

For simplicity here, assume that X and Y are compact. The above finiteness theorem is only a part of de Franchis' theorem. There should be a part of Severi's theorem.

Conjecture 1.2

The set of pairs

$$\begin{aligned}\mathrm{Mer}_{\mathrm{dom}}(Y) = &\{(f, X) : X, \textit{Kobayahi hyperbolic},\\ &f : Y \to X; \textit{meromorphic and dominant}\}\end{aligned}$$

is finite.

For the non-equidimensional case we have

Theorem 1.3

(N.-Sunada 1982) *If X is smooth and $\wedge^k T(X)$ is negative in the sense of Grauert, then there are only finitely many meromorphic mappings of rank k or more from Y into X.*

The proof was based on

Lemma 1.4

(N. Hiroshima Math. J. 1977) *For a section of $\omega \in H^0(S^l\Omega^k(X) \otimes [-D])$ with an ample divisor D on X we have "Schwarz' Lemma" for $f : \Delta^k \to X$. If $f : \mathbf{C}^k \to X$, then $f^*\omega \equiv 0$.*

N.B.

A big Picard type theorem follows, and hence it implies an algebraic degeneracy of $f : \mathbf{C} \to X$ for such X under some more condition.

This lemma (idea) was generalized to jet differentials by Green-Griffiths, S. Lu, Y.-T. Siu, ...; and also to log orbifold varieties by Erwan Rousseua in his talk this time.

There are related finiteness results due to T. Urata, Kalka-Shiffman-Wong. Thus it is a problem to investigate the moduli of maps in non-equidimensional case.

Specilaize X to be a locally symmetric space (quotient of a bounded symmetric domain).

The rigidity structure:

Theorem 1.5

(N. 1988; Miyano-N. 1991) *Let* $X = D/\Gamma$ *be a torsion-free co-compact or arithmetic quotient of a bounded symmetric doamin* D.
Let $\mathrm{Hol}(Y, X) = \{f : Y \to X, \ holomorphic\}$ *(harmonic). Then*

1. $\mathrm{Hol}(Y, X)$ *is quasi-projective smooth, and for every connected component*

$$f \in \mathrm{Hol}(Y, X) \to f(x_0) \in X(\forall x_0 \in Y, \mathrm{fixed})$$

is a totally geodesic immersion; hence, $\mathrm{Hol}(Y, X) \cong D_1/\Gamma_1$.

2. *There is "*$Y \to D_2/\Gamma_2$*" such that the natural map*

$$D_1/\Gamma_1 \times D_2/\Gamma_2 \to X$$

is a totally geodesic immersion.

N.B. If Γ has fixed points (resp. co-comppact), then so does Γ_i.

If Γ is co-compact, and $f : Y \to D/\Gamma$ is holomorphic, then the lifting $\tilde{f} : \tilde{Y} \to D$ is proper.

Question 1 (inspired by N.-M. Mok's recent talks). What can we say for the moduli of proper holomorphic mappings $\tilde{f} : \tilde{Y} \to D$?

Moreover, What for a bounded convex domain $D \Subset \mathbf{C}^n$?

Question 2. Let $D \Subset \mathbf{C}^n$ be a convex domain. If $D \cong M_1 \times M_2$ as complex manifolds, then is $M_i \cong D_i \Subset \mathbf{C}^{n_i}$, convex?

This is used in S. Frankel, Acta 1989, in which the main result is

"*If $D \Subset \mathbf{C}^n$ is convex and admits a co-compact discrete $\Gamma \subset \mathrm{Aut}(D)$, then D is isomorphic to a bounded symmetric domain.*"

For such M_i we have

$$\text{(Carathéodory)} C_{M_i}(x, y) = d_{M_i}(x, y) \text{ (Kobayashi)}$$

by Lempert's Theorem.

§2 Extension Problem.

Theorem 2.1

(Kwack 1969) *Let X be a compact Kobayashi hyperbolic manifold, and $\Delta^* \subset \mathbf{C}$ be a punctured disk. Then every holomorphic $f : \Delta^* \to X$ has a holomorphic extension $\bar{f} : \Delta \to X$.*

Theorem 2.2

(Nishino 1979) *Let X be a compact hyperbolic Riemann surface, and $E \subset \Delta$ be a closed subset of capacity 0. Then every holomorphic $f : \Delta \setminus E \to X$ has a holomrophic extension $\bar{f} : \Delta \to X$.*

Masakazu Suzuki gave a simpler proof and generaized it to higher diemsnion.

Theorem 2.3

(Masakazu Suzuki 1987/1989) *Let X be a compact complex manifold whose universal covering is biholomorphic to a bounded polynomially convex domain of $\mathbf{C}^n$, and $E \subset \Delta$ be a closed subset of capacity 0. Then every holomorphic $f : \Delta \setminus E \to X$ has a holomrophic extension $\bar{f} : \Delta \to X$.*

Conjecture 2.4

Let X be a compact Kobayashi hyperbolic manifold, and $E \subset \Delta$ a closed subset of capacity 0. Then every holomorphic $f : \Delta \setminus E \to X$ has a holomrophic extension $\bar{f} : \Delta \to X$.

§3 Nevanlinna theory in higher dimension

(a) Order function and F.M.T.

Let X be a compact complex space, and $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf.

Taking a finite covering $\{U_\mu\}$ of X, and $\sigma_{\mu\nu}$, the generators of $\mathcal{I}$ on U_μ, and $\{c_\mu\}$, a partition of unity, we set

$$\phi_{\mathcal{I}}(x) = -\log \sum_{\mu} c_\mu(x) \sum_{\nu} |\sigma_{\mu\nu}(x)|^2.$$

Then $\phi_{\mathcal{I}}(x)$ gives rise to a Weil function for a subscheme $\mathcal{O}_X/\mathcal{I}$; may be assumed $\phi_{\mathcal{I}}(x) \geqq 0$.

For $z = (z_i) \in \mathbf{C}^m$ we set

$$\alpha = \left(dd^c\|z\|^2\right)^{m-1} = \left(\frac{i}{2\pi}\partial\bar{\partial}\|z\|^2\right)^{m-1},$$
$$\beta = d^c \log\|z\|^2 \wedge \left(dd^c \log\|z\|^2\right)^{m-1}.$$

Let $f : \mathbf{C}^m \to X$ be a meromorphic mapping with $\phi_{\mathcal{I}}(f(z)) \not\equiv 0$. Set

$$\omega_{\mathcal{I},f} = dd^c \phi_{\mathcal{I}}(f),$$

and

$$T(r;\omega_{\mathcal{I},f}) = \int_1^r \frac{dt}{t^{2m-1}} \int_{\|z\|\le t} \omega_{\mathcal{I},f} \wedge \alpha,$$
$$m_f(r;;\mathcal{I}) = \frac{1}{2}\int_{\|z\|=t} -\log \phi_{\mathcal{I}}(f)\beta,$$
$$N_k(r;\mathcal{I}) = \int_1^r \frac{dt}{t^{2m-1}} \int_{(\mathrm{Supp}\, f^*\mathcal{O}/\mathcal{I})\cap\{\|z\|\le t\}} \min\{k, \mathrm{ord} f^*\mathcal{I}\}\alpha,$$
$$1 \le k \le \infty.$$

Theorem 3.1

(F.M.T.) $T(r;\omega_{\mathcal{I},f}) = N_\infty + m_f(r;\mathcal{I}) - m_f(1;\mathcal{I})$.

If $O_X/\mathcal{I}$ is a divisor D, then $\omega_{\mathcal{I},f} = f^* c_1(L(D))$ (the Chern form), and write

$$T_f(r;L(D)) = T(r;\omega_{\mathcal{I},f}).$$

(b) S.M.T.

- "$m = 1$ or $\dim X$ is fundamental.

Two model theorems besides Nevanlinna's:

Theorem 3.2

(Cartan 1933; Ahlfors by Weyls' method 1941)
Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be holomorphic and linearly nondegenerate,
$D = \sum_{i=1}^q H_i$ a sum of hyperplanes in general position.
Then

$$T_r(r;L(D)) + T_f(r;K_{\mathbf{P}^n(\mathbf{C})}) \leqq \sum_{i=1}^q N_n(r;f^*H_i) + S_f(r).$$

Here $S_f(r) = O(\log r + \log T_f(r; O(1)))||_E$, $E \subset [0, \infty)$ and $meas(E) < \infty$.

Theorem 3.3

(Griffiths et al. 1972/3, Shiffman, N.)
Let X be smooth projective, and $\dim X = n$.
Let $D = \sum_i D_i$ be a s.n.c. divisor on X.
Let $f : \mathbf{C}^n \to X$ be meromorphic and $\det df \not\equiv 0$.
Then

$$T_f(r; L(D)) + T_f(r; K_X) \leqq \sum_{i=1}^{q} N_1(r; f^* D_i) + S_f(r).$$

Conjecture 3.4

(Fundamental Conjecture for holomorphic curves)
Let X be smooth projective, and $D = \sum_i^q D_i$ be a s.n.c. divisor on X.
Let $f : \mathbf{C} \to X$ be holomorphic and algebraically nondegenerate.
Then

$$T_f(r; L(D)) + T_f(r; K_X) \leqq \sum_{i=1}^{q} N_1(r; f^* D_i) + \epsilon T_f(r)||_{E(\epsilon)} \quad (\forall \epsilon > 0).$$

N.B. Open for $D = \sum H_i \subset \mathbf{P}^n(\mathbf{C})$ (sum of hyperplanes).

Fund. Conj. for hol. curves implies *Green-Griffiths' Conjecture* and *Kobayashi Conjecture*.

Conjecture 3.5

(Green-Griffiths, log-versin) *Let X be a variety of log general type. Then every holomorphic $f : \mathbf{C} \to X$ is algebraically degenerate.*

Theorem 3.6

(Log Bloch-Ochiai; N.1997/81) *If the log irregularity $\bar{q}(X) > \dim X$, then every holomorphic $f : \mathbf{C} \to X$ is algebraically degenerate.*

This is a consequnce of the following S.M.T.-type inequality.

Theorem 3.7

(N. 1977)

1. (Lemma on log derivative) *Let ω be a logarithmic 1-form on X with poles on ∂X, and set $f^*\omega = \xi(z)dz$ for $f : \mathbf{C} \to \bar{X}$ $(f(\mathbf{C}) \not\subset \partial X)$, holomorphic. Then*
$$m_\xi(r;\infty) = S_f(r).$$

2. (S.M.T.-type) *Let $\alpha_X : X \to A_X$ be the quasi-Albanese map, and Z be the Zariski closure of $\alpha_X(X)$.*
Assume that $\dim Z = \dim X$, *and*
$\mathrm{St}(Z) = \{a \in A_X; a + Z = Z\}^0 = \{0\}$.
Then there is a constant $\lambda > 0$ such that for an algebracally non-degenerate $f : \mathbf{C} \to \bar{X}$
$$\lambda T_f(r) \leqq N_1(r; f^*\partial X) + S_f(r).$$

Question. What is $\lambda > 0$?

Let A be a semi-abelian variety:

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \text{ (abelian variety)} \to 0.$$

The universal covering $\tilde{A} \cong \mathbf{C}^n$, $n = \dim A$.

Let $f : \mathbf{C} \to A$ be a holomorphic curve. Set

- $J_k(A)$: the k-jet bundle over A; $J_k(A) \cong A \times \mathbf{C}^{nk}$;
- $J_k(f) : \mathbf{C} \to J_k(A)$: the k-jet lift of f;
- $X_k(f)$: the Zariski closure of the image $J_k(f)(\mathbf{C})$.

Theorem 3.8

(N.-Winkelmann-Yamanoi, Forum Math. 23(2008))

Let $f : \mathbf{C} \to A$ *be algebraically non-degenerate.*
(i) *Let* Z *be an algebraic reduced subvariety of* $X_k(f)$ $(k \geqq 0)$*. Then* $\exists \bar{X}_k(f)$*, compactification of* $X_k(f)$ *such that*

$$(3.9) \qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) \leqq N_1(r; J_k(f)^* Z) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

(ii) *Moreover, if* $\mathrm{codim}\,_{X_k(f)} Z \geqq 2$*, then*

$$(3.10) \qquad T_{J_k(f)}(r; \omega_{\bar{Z}}) \leqq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

(iii) *If* $k = 0$ *and* Z *is an effective divisor* D *on* A*, then* $\bar{A}$ *is smooth, equivariant, and independent of* f*; furthermore,* (3.9) *takes the form*

$$(3.11) \qquad T_f(r; L(\bar{D})) \leqq N_1(r; f^* D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon, \quad \forall \epsilon > 0.$$

As an application for degeneracy problem of holomorphic curves we have

Theorem 3.12

(N.-Winkelmann-Yamanoi, J. Math. Pure. Appl. 2007)
Let X be an algebraic variety such that

(i) $\bar{q}(X) \geqq \dim X$ *(log. irregularity);*
(ii) $\bar{\kappa}(X) > 0$ *(log. Kodaira dimension);*
(iii) *the Albanese map $X \to A$ is proper.*

Then $\forall f : \mathbf{C} \to X$ is algebraically degenerate.
Moreover, the normalization of $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ is a semi-abelian variety which is finite étale over a translate of a proper semi-abelian subvariety of A.

N.B. There is a related result due to Dethloff-Lu for the surface and Brody curves.

Example.

Let $X = \mathbf{P}^n(\mathbf{C}) \setminus \sum_{i=1}^{q} D_i$ with distinct irreducible hypersurfaces D_i.

Even restricting to this elementary example case, you can see steady advances in the value distribution theroy. The algebraic degeneracy of $f : \mathbf{C} \to X$ has been proved as follows:

1. $q \geqq n+2$; $q = n+2$, critical, and $\forall \deg D_i = 1$ $\cdots$ E. Borel (Acta 1897).
2. $q = n+2$ and $\deg D_i \geqq 1$ $\cdots$ M. Green (1975), ..., T. Nishino (1989?).
   From the viewpoint of Log Bloch-Ochiai (N. (1977/81)), $\bar{q}(X) = q - 1 = n + 1 > n$.
3. $\bar{q}(X) = q - 1 = n$, and $\bar{\kappa}(X) > 0$ or $X \not\cong (\mathbf{C}^*)^n$ ($\Longrightarrow \deg \sum D_i \geqq n+2$) $\cdots$ Theorem 3.12.
   If $\sum D_i$ has only s.n.c. and $\deg \sum D_i \geqq n+2$, then $\bar{\kappa}(X) = n$.

Corollary 3.13

Assume $q \geqq n+1$ and that $\sum_{i=1}^{q} D_i$ has only s.n.c. Then every $f : \mathbf{C} \to X$ is algebracally degenerate.

Question. Allow singularities for D_i, but assume $\sum_{i=1}^{q} D_i$ is in general position; i.e.,

every intersection of k number of D_i's has pure codimension k.

Then is $\bar{\kappa}(X) > 0$?

In Theorem 3.8 (iii) we take an equivariant compactification $\bar{A}$ such that the closure $\bar{D}$ contains no A-orbit; this is a general position condition for the divisor $\bar{D} + \partial A$ in $\bar{A}$.

Conjecture 3.14

Let $\bar{A}$ and $\bar{D} + \partial A$ be as above. Let $f : \mathbf{C} \to \bar{A}$ be algebraically nondegenerate.
Then

$$T_f(r; L(\bar{D} + \partial A)) + T_f(r; K_{\bar{A}}) = T_f(r; L(\bar{D}))$$
$$\leqq N_1(r; f^*\bar{D}) + N_1(r; f^*\partial A) + \epsilon T_f(r)||_{E(\epsilon)}.$$

Specializing the conjecure, we have

Conjecture 3.15

Let $D \subset (\mathbf{C}^*)^2 = A$ be irreducible such that $\mathrm{St}(D) = \{0\}$ and $\bar{D} \subset \bar{A}$ be in general position. Let $f : \mathbf{C} \to \bar{A}$ be algebraically nondegenerate. Then it is conjectured that

$$T_f(r; L(\bar{D} + \partial A)) + T_f(r; K_{\bar{A}}) = T_f(r; L(\bar{D}))$$
$$\leqq N_1(r; f^*\bar{D}) + N_1(r; f^*\partial A) + \epsilon T_f(r)||_{E(\epsilon)}.$$

Example. With $(z, w) \in (\mathbf{C}^*)^2$ we set

$$D = \left\{ z + \frac{1}{z} + w + \frac{1}{w} - \frac{1}{zw} = 0 \right\}$$

Then $\bar{D} \subset (\mathbf{P}^1)^2 = \bar{A}$ is in general position. Let $f = (f_1, f_2) : \mathbf{C} \to (\mathbf{P}^1)^2$ be algebraically nondegenerate. Then

$$(3.16) \quad 2T_f(r) + 2T_f(r) \leqq N_1(r; f^*\bar{D}) + N_1(r; (f_1)_0) + N_1(r; (f_1)_\infty)$$
$$+ N_1(r; (f_2)_0) + N_1(r; (f_2)_\infty) + \epsilon T_f(r)||_{E(\epsilon)}.$$

There is some evidence:

Proposition 3.17

In the above Example, assume that one of f_i, say f_2 satisfies $T_{f_2}(r) \leqq \epsilon T_f(r)||_{E(\epsilon)}$. Then (3.16) *holds.*

This follows from Yamanoi's abc-Theorem, which we recall.

Let $p : X \to S$ be a surjective morphism between smooth projective algebraic varieties with relative canonical bundle $K_{X/S}$.

Theorem 3.18

(Yamanoi's abc, 2004, 2006) *Assume that*

- $\dim X/S = 1$;
- $D \subset X$ *is a reduced divisor ;*
- $f : \mathbf{C} \to X$ *is algebraically nondegenerate ;*
- $g = p \circ f : \mathbf{C} \to S$.

Then for $\forall \epsilon > 0$, $\exists C(\epsilon) > 0$ *such that*

$$(3.19) \quad T_f(r; L(D)) + T_f(r; K_{X/S}) \leq N_1(r; f^* D) + \epsilon T_f(r) + C(\epsilon) T_g(r) ||_{E(\epsilon)}.$$

Thank You Very Much!!