A Survey of Estimates of SMT-type for Holomorphic Curves

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§1 Introduction.

I would like to survey and discuss an estimate of SMT-type for holomorphic curves into (complex) projective algebraic varieties in relation with the following conjecture due to P.A. Griffiths in 1972:

Griffiths' Conjecture 1. Every holomorphic curve $f : \mathbf{C} \to X$ into a variety X of general type is algebraically degenerate; i.e., $f(\mathbf{C})$ is not Zariski dense in X.

Griffiths' Conjecture 2. For every algebraically nondegenerate holomorphic curve $f : \mathbf{C} \to X$ into a smooth projective algebraic variety X and an s.n.c. divisor D on X we have

$$T_f(r; [D]) + T_f(r; K_X)$$
$$\leq N(r; f^*D) + S_f(r).$$

This conjecture was modeled after a successful generalization of Nevanlinna's theory to the case of differentiably nondegenerate holomorphic mappings $f : \mathbf{C}^n \to X^n$ (Griffiths, Carlson, King,...).

Introducing a notion of Jacobian sections to use Griffiths' "Ricci curvature method", W. Stoll obtained a formal SMT for holomorphic curves $f : \mathbf{C} \to X$ that involves the counting function of the Jacobian section: it is not canonical. It is totally non-trivial to get an estimate for the Jacobian section.

N.B. Conjecture 2 implies Conjecture 1.

§2 Notation and FMT.

Let $E = \sum_{\mu=1}^{\infty} \nu_\mu z_\mu$ be a divisor on $\mathbf{C}$ with distinct $z_\mu \in \mathbf{C}$. Then we set

$$\mathrm{ord}_z E = \begin{cases} \nu_\mu, & z = z_\mu, \\ 0, & z \notin \{z_\mu\}. \end{cases}$$

We define the counting functions of E truncated to $l \leqq \infty$ by

$$n_l(t; E) = \sum_{\{|z_\mu|<t\}} \min\{\nu_\mu, l\},$$
$$N_l(r; E) = \int_1^r \frac{n_l(t; E)}{t} dt.$$

We define the counting functions of E by

$$n(t; E) = n_\infty(t; E),$$
$$N(r; E) = N_\infty(r; E).$$

Let X be a compact reduced complex space and $\mathcal{O}_X$ the structure sheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf. For a holomorphic curve

$$f : \mathbf{C} \to X, \quad f(\mathbf{C}) \not\subset \mathrm{Supp}\, \mathcal{O}_X / \mathcal{I}$$

we are going to define several quantities as follows.

Let $\{U_j\}$ be a finite open covering of X such that

(i) there are finitely many sections

$$\sigma_{jk} \in \Gamma(U_j, \mathcal{I}), k = 1, 2, \dots ,$$

generating every fiber $\mathcal{I}_x$ over $x \in U_j$.

(ii) there is a partition of unity $\{c_j\}$ associated with $\{U_j\}$,

Setting $\rho_{\mathcal{I}}(x) = \left(\sum_j c_j(x) \sum_k |\sigma_{jk}(x)|^2\right)^{1/2}$, we take a positive constant C so that

$$C\rho_{\mathcal{I}}(x) \leqq 1, \qquad x \in M.$$

Using the compactness of X, one easily verifies that $\log \rho_{\mathcal{I}}$ (Weil's function in arithmetic) is well-defined up to a bounded continuous function on X.

We define the proximity function of f for $\mathcal{I}$ or for the subspace $Y = (\mathrm{Supp}\, \mathcal{O}_X/\mathcal{I}, \mathcal{O}/\mathcal{I})$ (possibly non-reduced) by

$$\begin{aligned} m_f(r;\mathcal{I}) &= m_f(r;Y) \\ &= \int_{|z|=r} \log \frac{1}{C\rho_{\mathcal{I}}(f(z))} \frac{d\theta}{2\pi} \quad (\geqq 0). \end{aligned}$$

The function $\rho_{\mathcal{I}} \circ f(z)$ is smooth over $\mathbf{C} \backslash f^{-1}(\mathrm{Supp} Y)$. For $z_0 \in f^{-1}(\mathrm{Supp} Y)$ choose an open neighborhood U of z_0 and a positive integer ν such that $f^*\mathcal{I} = ((z - z_0)^{\nu})$. Then

$$\log \rho_{\mathcal{I}} \circ f(z) = \nu \log |z - z_0| + \psi(z), \qquad z \in U$$

for some C^{∞}-function $\psi(z)$ defined on U. We define the counting function $N(r; f^*\mathcal{I})$ and $N_l(r; f^*\mathcal{I})$ by using ν in the same way as using $\mathrm{ord}_{z_0}(E)$ in the definition of $N(r;E)$ and

$N_l(r; E)$. Moreover we define

$$\omega_{\mathcal{I},f} = \omega_{\mathrm{Y},f} = -dd^c\psi(z) = -\frac{i}{2\pi}\partial\bar{\partial}\psi(z)$$
$$= dd^c \log\frac{1}{\rho_{\mathcal{I}} \circ f(z)} \quad (z \in U),$$

which is well-defined on $\mathbf{C}$ as a smooth (1,1)-form. The order function of f for $\mathcal{I}$ or Y is defined by

$$T(r; \omega_{\mathcal{I},f}) = T(r; \omega_{\mathrm{Y},f}) = \int_1^r \frac{dt}{t}\int_{|z|<t}\omega_{\mathcal{I},f}.$$

If $\mathcal{I}$ is the ideal sheaf defined by a Cartier effective divisor D on X, in terms of commonly used notation we have

$$m_f(r; \mathcal{I}) = m_f(r; D) + O(1),$$
$$T(r; \omega_{\mathcal{I},f}) = T_f(r; [D]) + O(1).$$

Let $\mathcal{I}_i$ $(i = 1, 2)$ be coherent ideal sheaves of $\mathcal{O}_M$ and let Y_i be the subspace defined by $\mathcal{I}_i$. We write $Y_1 \supset Y_2$ if $\mathcal{I}_1 \subset \mathcal{I}_2$.

Fix a hermitian form ω_X on X. We define an order function

$$T_f(r) = T(r; \omega_X) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^* \omega_X. \qquad (1)$$

Theorem 2 *Let $f : \mathbf{C} \to X$ and $\mathcal{I}$ be as above. Then we have the following:*

(i) (FMT)

$$T(r; \omega_{\mathcal{I},f}) = N(r; f^*\mathcal{I}) + m_f(r; \mathcal{I}) - m_f(1; \mathcal{I}).$$

(ii) *If X is projective, $T(r; \omega_{\mathcal{I},f}) = O(T_f(r))$.*

(iii) *Let $\mathcal{I}_i$ $(i = 1, 2)$ be coherent ideal sheaves of $\mathcal{O}_M$ and let Y_i be the subspace defined by $\mathcal{I}_i$. If $\mathcal{I}_1 \subset \mathcal{I}_2$ $(Y_1 \supset Y_2)$, then*

$$m_f(r; \mathcal{I}_2) \leqq m_f(r; \mathcal{I}_1) + O(1),$$

or equivalently,

$$m_f(r; Y_2) \leqq m_f(r; Y_1) + O(1).$$

(iv) *Let* $\phi : X_1 \to X_2$ *be a holomorphic mappings between compact complex spaces. Let* $Y_i \subset X_i$ $(i = 1, 2)$ *be subschemes such that* $\phi(Y_1) \subset Y_2$*. Then*

$$m_f(r; Y_1) \le m_{\phi \circ f}(r; Y_2) + O(1).$$

(v) *Let* $\mathcal{I}_i$*,* $i = 1, 2$ *be two coherent ideal sheaves of* $\mathcal{O}_M$*. Suppose that* $f(\mathbf{C}) \not\subset \mathrm{Supp}\,(\mathcal{O}_M/\mathcal{I}_1 \otimes \mathcal{I}_2)$*. Then we have*

$$T(r; \omega_{\mathcal{I}_1 \otimes \mathcal{I}_2, f}) = T(r; \omega_{\mathcal{I}_1, f}) + T(r; \omega_{\mathcal{I}_2, f}) + O(1).$$

§3 Cartan's and Weyls-Ahlfors' Theories.

Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve, and $\tilde{f} = (f_0 : \cdots : f_n)$ be a reduced representation. Let

$$T_f(r) = T(r; f^*(\text{F.-S.}))$$

be the order function with respect to the Fubini-Study metric form.

Theorem 3 *Assume that f is linearly nondegenerate. Let $H_j, 1 \leq j \leq q$ be hyperplanes in general position. Then*

$$(q-n-1)T_f(r) \leq \sum_{j=1}^{q} N_n(r; f^*H_i) + S_f(r),$$

where $S_f(r) = O(\log^+ r + \log^+ T_f(r))||$.

N.B. Since $-(n+1)T_f(r) = T_f(r; K_{\mathbf{P}^n(\mathbf{C})})$, Griffiths' Conjecture 2 holds in this case.

The reason of "$n+1$" in Cartan's Theorem comes from the order (size) of the Wronskian $W(f_0, \cdots, f_n)$ (non-trivial invariant jet differential in Demailly's terminology) and the identity (almost Riesz' decomposition)

$$\log \frac{\prod_{j\in Q} |\hat{H}_j \circ \tilde{f}|}{|W(\tilde{f})|} = \log \frac{\prod_{j\in R} |\hat{H}_j \circ \tilde{f}|}{|W(\tilde{f})|} + \log \prod_{j \in Q \setminus R} |\hat{H}_j \circ \tilde{f}|,$$

for $\forall R \subset Q$ with $|R| = n+1$, where $Q = \{1, \ldots, q\}$.

Let $\tilde{f}^k = (f_0^{(k)} : \cdots : f_n^{(k)})$ be the derivatives of $\tilde{f}$ and consider the derived (associated) curves into the Grassmannians

$$\begin{aligned} f^{(k)} : \mathbf{C} \to & [\tilde{f} \wedge \tilde{f}^{(1)} \wedge \cdots \wedge \tilde{f}^{(k)}] \\ & \in \mathrm{Gr}(k+1, n+1) \\ & \subset \mathbf{P}^{\binom{n+1}{k+1}-1}(\mathbf{C}). \end{aligned}$$

We have

$$T_f(r) \sim T_{f^{(k)}}; \text{ i.e.,}$$
$$C^{-1} T_f(r) \leq T_{f^{(k)}}(r) \leq C T_f(r), \ \exists C > 0.$$

Theorem 4 (Weyls-Ahlfors-Fujimoto) *Let f be linearly nondegenerate and let $A_j, 1 \leq j \leq q$, be decomposable hyperplanes in* $\mathrm{Gr}(k+1, n+1) \subset \mathbf{P}^{\binom{n+1}{k+1}-1}(\mathbf{C})$ *in general position. Then*

$$\begin{aligned} & \left(q - \binom{n+1}{k+1}\right) T_{f^{(k)}} \\ & \leq \sum_{j=1}^{q} N_{(k+1)(n-k)}(r; {f^{(k)}}^* A_j) \\ & \quad + S_{f^{(k)}}(r). \end{aligned}$$

N.B. (i) W. Stoll generalized Weyls-Ahlfors' theory to the case of $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ (1953-54), defining the Wronksian by some polynomial vector field on $\mathbf{C}^m$.

(ii) Fujimoto (1982) obtained the above explicit trunction in the counting function. It is the point of the present talk to enphasize the importance of intoducing truncations in counting functions.

Cartan-Nochka's SMT. (1982) *Let* $H_j, 1 \leq j \leq q$, *be hyperplanes of* $\mathbf{P}^n(\mathbf{C})$ *in general position. For a holomorphic curve* $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *whose image has an* l*-dimensional linear span,*

$$\begin{aligned}&(q-2n+l-1)T_f(r)\\&\quad\leq \sum_{j=1}^{q} N_l(r; f^*H_j) + S_f(r).\end{aligned}$$

N.B. (i) W. Chen obtained a similar SMT for the derived curves $f^{(k)}$ (1990).

(ii) There are other proofs of Cartan's SMT due to B. Shiffman (1977) by applying Griffiths' equidimensional SMT, and that of Weyls-Ahlfors' SMT due to Cowen-Griffiths (1976) which put an emphasis on the metric-curvature method.

§4 Hypersurfaces of $\mathbf{P}^n(\mathbf{C})$.

Eremenko and Sodin dealt with the case of hypersurfaces $D_j, 1 \le j \le q$, of $\mathbf{P}^n(\mathbf{C})$ (1992).

Definition. $D_j, 1 \le j \le q$, are said to be in general position if for an arbitrary $R \subset \{1, \ldots, q\}$, codim $\cap_{j \in R} D_j = |R|$ for $|R| \le n$, and $\cap_{j \in R} D_j = \emptyset$ for $|R| > n$.

Theorem 5 *Let $D_j, 1 \le j \le q$, be hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ in general position. Then for $\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ we have*

$$(q-2n)T_f(r) \le \sum_{j=1}^{q} \frac{1}{\deg D_j} N(r; f^* D_j) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

The proof is very different to those of Cartan's and Weyls-Ahlfors' SMT, and purely depends on the potential theory and the plane topology of $\mathbf{C}$.

N.B. It is an interesting open problem if the counting functions $N(r; f^*D_j)$ can be replaced by truncated ones.

As an analogue to Corvaja-Zannier's generalization of Schmidt's Subspace Theorem to the case of hypersurfaces of $\mathbf{P}^n$ in the Diophantine approximation theory, Min Ru lately proved the following, which was conjectured by B. Shiffman (1979).

Theorem 6 (M. Ru, 2004) Let $D_j, 1 \leq j \leq q$, be hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ in general position. Then for an algebraically nondegenerate $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$

$$
\begin{aligned}
&(q-n-1)T_f(r) \\
&\leq \sum_{j=1}^{q} \frac{1}{\deg D_j} N(r; f^*D_j) + \epsilon T_f(r) ||_\epsilon .
\end{aligned}
$$

It is a problem to introduce a truncation in the counting functions $N(r; f^*D_j)$.

§5 Siu's meromorphic connection.

Let X be an n-dimensional smooth projective algebraic variety with a line bundle $L > 0$. Let $D_j \in |L|, 1 \le j \le q$, be smooth reduced divisor given by $s_j \in H^0(X, L)$.

Let $\Gamma^{\gamma}_{\alpha\beta}$ be a meromorphic connection in the holomorphic tangent bundle $\mathbf{T}(X)$.
Let F be the line bundle defined by the poles of $\Gamma^{\gamma}_{\alpha\beta}$ and let $t \in H^0(X, F)$ be a section such that $t\Gamma^{\gamma}_{\alpha\beta}$ are holomorphic.

Let $f : \mathbf{C} \to X$ be a transcendental holomorphic curve. Assume that

(i) for any fixed indices α and β, $t\mathbf{D}_\alpha\partial_\beta s_j$ is a linear combination of $\partial_\alpha s_j$,$\partial_\beta s_j$ and s_j with smooth coefficients, where ∂_α is the $(1,0)$ partial derivative and $\mathbf{D}_\alpha$ is the $(1,0)$ covariant differentiation with respect to $\Gamma^\gamma_{\alpha\beta}$.

(ii) $f_z \wedge t\mathbf{D}_z f_z \wedge \cdots \wedge (t\mathbf{D}_z)^{(n-1)} f_z \in K_X^{-1} \otimes F^{n(n-1)/2}$ does not vanish identically.

Theorem 7 (Y. Siu, 1987) *Let the notation and the assumption be as above. Let k be a positive integer such that no more than k of D_j has a non-empty intersection (if $\sum D_j$ is s.n.c., then $k = 1$). Then*

$$\begin{aligned} & qT_f(r; L) + kT_f(r; K_X) \\ & - k\frac{n(n-1)}{2} T_f(r; F) \\ & \leq \sum_{j=1}^{q} N(r; f^* D_j) + \epsilon T_f(r) ||_\epsilon . \end{aligned}$$

N.B. It is in general very difficult to deduce the algebraic degeneracy of f from the failure of the above assumption (ii).

§6 Compactification of bounded symmetric domain.

Let Ω be a bounded symmetric domain in $\mathbf{C}^n$ equipped with the Bergman metric g normalized as Ricci$(g) = -g$.
Let κ_Ω be a positive number such that every holomorphic sectional curvature

$$H(v, v) \leq -\kappa_\Omega.$$

One has $1/n \leq \kappa_\Omega \leq 1$, and if Ω is a ball, $\kappa_\Omega = 2/(n+1)$.

Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free arithmetic discrete subgroup and let $X = \overline{\Omega/\Gamma}$ be a smooth toroidal compactification. Set

$$D = X \setminus (\Omega/\Gamma).$$

Theorem 8 (A. Nadel, 1989; Y. Aihara-N., 1991) *Let $f : \mathbf{C} \to X$ be an arbitrary holomorphic curve such that $f(\mathbf{C}) \not\subset D$. Then*

$$\begin{aligned} &\kappa_\Omega \{T_f(r; [D]) + T_f(r; K_X)\} \\ &\leq N_1(r; f^*D) + S_f(r). \end{aligned}$$

N.B. A similar estimate holds for $f : \{0 < |z| < 1\} \to X$, which gives big Picard's theorem for $f : \{0 < |z| < 1\} \to \Omega/\Gamma$.

§7 Deligne's logarithmic 1-forms.

Let X be an n-dimensional smooth projective algebraic variety.
Let D be a reduced divisor on X and let $\Omega^1_X(\log D)$ be the sheaf of germs of Deligne's logarithmic 1-forms along D.

Let $\alpha : X \setminus D \to A_X$ be the quasi-Albanese mapping and let Y be the Zariski closure of $\alpha(X \setminus D)$ in A_X.

Theorem 9 (N., 1977/81) *Assume that*
(i) $\dim Y = n$,
(ii) Y *is of (log) general type.*

Then there is a constant $\kappa > 0$ such that for every algebraically non-degenerate holomorphic curve $f : \mathbf{C} \to X$

$$\kappa T_f(r) \leq N_1(r, f^*D) + S_f(r).$$

Set the log irregularity of $X \setminus D$ by

$$\bar{q}(X \setminus D) = h^0(X, \Omega^1_X(\log D));$$

when $D = \emptyset$, write $q(X) = \bar{q}(X)$.

Corollary 10 (Log Bloch-Ochiai's Theorem)
Assume that $\bar{q}(X \setminus D) > n$. *Then*
$\forall f : \mathbf{C} \to X \setminus D$ *is algebraically degenerate.*

Corollary 11 (Bloch-Ochiai's Theorem, 1926/77; ...)
Assume that $q(X) > n$. *Then*
$\forall f : \mathbf{C} \to X$ *is algebraically degenerate.*

A. Bloch (1926) could not prove a lemma on (elliptic or abelian) logarithmic derivative, and T. Ochiai (1977) proved it with much clarification in general case.

Corollary 12 (Borel's Theorem, 1897) *Let $H_j, 1 \leq j \leq n+2$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. Then*
$\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus \sum_{j=1}^{n+2} H_j$ *is algebraically (linearly) degenerate.*

N.B. $\bar{q}(\mathbf{P}^n(\mathbf{C}) \setminus \sum_{j=1}^{n+2} H_j) = n+1 > n$. Thus, Log Bloch-Ochiai's Theorem unifies Bloch-Ochiai's and Borel's Theorems.

§7 Semi-abelian varieties.

Let A be a semi-abelian variety; i.e.,

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \to 0,$$

where A_0 is an abelian variety. Let
$J_k(A)$ be the k-jet space over A;
$f : \mathbf{C} \to A$ be a holomorphic curve;
$J_k(f) : \mathbf{C} \to J_k(X)$ be the k-jet lift;
$X_k(f)$ be the Zariski closure of $J_k(f)(\mathbf{C})$ in $J_k(A)$.

Theorem 13 (N.-Winkelmann-Yamanoi, 2007) *Assume that $f : \mathbf{C} \to A$ is algebraically non-degenerate.*

(i) *Let Z be an algebraic reduced subvariety of $X_k(f)$ ($k \geqq 0$). Then $\exists \bar{X}_k(f)$, a compactification of $X_k(f)$ such that*

$$T(r; \omega_{\bar{Z}, J_k(f)}) \leqq N_1(r; J_k(f)^* Z) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0,$$

where $\bar{Z}$ is the closure of Z in $\bar{X}_k(f)$.

(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geqq 2$, *then*

$$T(r; \omega_{\bar{Z}, J_k(f)}) \leqq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

(iii) *When $k = 0$ and Z is an effective divisor D on A, the compactification $\bar{A}$ of A can be chosen as smooth, equivariant with respect to the A-action, and independent of f, and we have*

$$T_f(r; L(\bar{D})) \leqq N_1(r; f^* D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon, \quad \forall \epsilon > 0.$$

The above (iii) gives yet another proof of Lang's conjecture:

Corollary 14 (Y.T. Siu-Z.K. Yeung, 1996; N., 1998; M. McQuillan, 2001) *Let*
A be a semi-abelian variety;
D be an effective reduced divisor on A.

Then, $\forall f : \mathbf{C} \to A \setminus D$ is algebraically degenerate.

§8 Finite cover of semi-abelian variety.

As an application one can improve Log Bloch Ochiai's Theorem. Let
X be an algebraic normal variety, which is not necessarily compact;
$\bar{\kappa}(X)$ denote the log Kodaira dimension of X;
A be a semi-abelian variety.

Theorem 15 (N.-Winkel.-Yam., 2007)
Assume that
(i) $\exists \pi : X \to A$,*a finite morphism;*
(ii) $\bar{\kappa}(X) > 0$.
Then $\forall f : \mathbf{C} \to X$ *is algebraically degenerate.*

N.B. The condition "$\bar{\kappa}(X) > 0$" prohibits X to be a semi-abelian variety.

Corollary 16 *Let X be an algebraic variety whose quasi-Albanese map is proper.*
Assume that $\bar{\kappa}(X) > 0$ and $\bar{q}(X) \geq \dim X$.
Then $\forall f : \mathbf{C} \to X$ *is algebraically degenerate.*

§9 Yamanoi's relative SMT.

In Acta 2004, K. Yamanoi proved a striking SMT for meromorphic functions with respect to moving targets, where the counting functions are truncated to level 1; it gives the best answer to Nevanlinna's conjecture for moving targets. We recall his result in a form suitable to the present purpose.

Let $p : X \to S$ be a surjective morphism between smooth projective algebraic varieties with relative canonical bundle $K_{X/S}$.

Theorem 17 (Yamanoi, 2004)
Assume that $\dim X/S = 1$*. Let*
$D \subset X$ *be a reduced divisor.*
$f : \mathbf{C} \to X$ *be algebraically nondegenerate;*
$g = p \circ f : \mathbf{C} \to S$*.*

Then for $\forall \epsilon > 0$, $\exists C(\epsilon) > 0$ *such that*

$$\begin{aligned}&T_f(r; [D]) + T_f(r; K_{X/S}) \\ &\quad \leq N_1(r; f^*D) + \epsilon T_f(r) + C(\epsilon) T_g(r)||_\epsilon.\end{aligned}$$

Here we introduce a new notion of the *small-dimension* s-dim(f) for $f : \mathbf{C} \to X$ as follows.

Let $\mathbf{C}(X)$ be the rational function field of X. Then, transc-deg$_{\mathbf{C}}\, \mathbf{C}(X) = \dim X$. Set

$$\begin{aligned}\mathcal{S}(f) = \{\phi \in \mathbf{C}(X); &\operatorname{Supp}(\phi)_\infty \not\supset f(\mathbf{C}), \\ &T(r; f^*\phi) \leq \epsilon T_f(r)||_\epsilon,\ \forall \epsilon > 0\}.\end{aligned}$$

Then $\mathcal{S}(f)$ is a subfield of $\mathbf{C}(X)$, and we set

$$\text{s-dim}(f) = \text{transc-deg}_{\mathbf{C}}\, \mathcal{S}(f).$$

We have

Proposition 18 *If* $\text{s-dim}(f) = \dim X$, *then* f *is algebraically degenerate.*

In the proofs of Corollary 10 (Log Bloch-Ochiai) and Corollary 14 (Lang's conjecture), assuming the algebraic nondegeneracy of f, we actually prove that

$$\text{s-dim}(J_k(f)) = \dim \overline{J_k(f)(\mathbf{C})}^{\text{Zar}}$$

for some high order jet $J_k(f)(k >> 1)$, which gives a contradiction.

As an application of Yamanoi's relative SMT we have

Theorem 19 *For $f : \mathbf{C} \to X$, assume that*

$$\dim X = 2, \quad \text{s-dim}(f) = 1.$$

Then f is algebraically degenerate.

Proof. Suppose that f is algebraically nondegenerate. The assumption implies that
$\exists S$, a curve, and
$\exists p : X \to S$, a morphism (after some birational change)
such that $g = p \circ f : \mathbf{C} \to S$ is algebraically nondegenerate and satisfies

$$T_g(r) \leq \epsilon T_f(r)||_\epsilon.$$

Therefore S is rational or elliptic. In general, the Kodaira dimensions satisfy

$$\kappa(X) \leq \kappa(p^{-1}(t)) + \dim S.$$

Thus, $\kappa(p^{-1}(t)) = 1$. Hence $K_{X/S}$ is big and

$$T_f(r; K_{X/S}) \sim T_f(r).$$

Combining this with Yamanoi's relative SMT, we get a contradiction

$$T_f(r) \leq \epsilon T_f(r)||_\epsilon. \quad \text{(q.e.d)}$$

Here we are not using the full force of Yamanoi's relative SMT. It is interesting to find a path (induction?) to reach to Griffiths' Conjecture 1 by making use of Yamanoi's relative SMT.

§10 Some easier conjectures.

We consider Conjecture 1:
Let X be an n-dimensinal projective manifold with embedding $X \hookrightarrow \mathbf{P}^N(\mathbf{C})$.

Let
$\pi : X \to \mathbf{P}^n(\mathbf{C})$ be a genraic projection;
D be the ramification divisor on X;
E be the critical value divisor on $\mathbf{P}^n(\mathbf{C})$.

In the sequel we have this setting in mind. Note that if π moves, then D and E are deformed.

From the above observations it is plausible to conjecture

Conjecture 2'. Let
$\forall f : \mathbf{C} \to X$ be algebraically nondegenerate;
D be an s.n.c. divisor on X.

Then

$$T_f(r; [D]) + T_f(r; K_X) \\ \leq N_1(r; f^* D) + \epsilon T_f(r)||_\epsilon, \\ \forall \epsilon > 0.$$

But this is more difficult than the original. So we specialize it:

Conjecture A. Let
$f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be alg. nondegenerate;
E be a reduced divisor E on $\mathbf{P}^n(\mathbf{C})$, allowing some singularities.

Then we have

$$\{\deg E - n - 1\} T_f(r) \\ \leq N_1(r; f^* E) + \epsilon T_f(r)||_\epsilon, \\ \forall \epsilon > 0.$$

We consider a milder conjecture for general X, which is a variant of FMT:

Conjecture B. Let
$f : \mathbf{C} \to X$ be algebraically nondegenerate;
D be a reduced divisor on X, allowing some singularities.

Then we have

$$\begin{aligned} T_f(r; [D]) \leq & N_1(r; f^* D) + m_f(r, D) \\ & + \epsilon T_f(r) ||_\epsilon, \quad \forall \epsilon > 0. \end{aligned}$$

Conjectural Proof of Conjecture 1.

Assume that $f : \mathbf{C} \to X$ is algebraically nondegenerate. Let $\pi : X \to \mathbf{P}^n(\mathbf{C})$ be a projection with the ramification divisor D on X and the critical value divisor E on $\mathbf{P}^n(\mathbf{C})$, for which Conjectures A and B hold.

Conjecture A implies that

$$\begin{aligned} & (\deg E - n - 1) T_g(r) \\ & \leq N_1(r; g^* E) + \epsilon T_g(r) ||_\epsilon. \end{aligned}$$

Thus,

$$N(r; g^*E) - N_1(r; g^*E)$$
$$\leq (n+1)T_g(r) - m_g(r; E) + \epsilon T_g(r)||_\epsilon.$$

Because of ramifications we get

$$N_1(r; f^*D) \leq N(r; g^*E) - N_1(r; g^*E)$$
$$\leq (n+1)T_g(r) - m_g(r; E) + \epsilon T_g(r)||_\epsilon.$$

Conjecture B implies that

$$T_f(r; [D]) \leq N_1(r; f^*D) + m_f(r, D) + \epsilon T_f(r)||_\epsilon$$

Combining the above two with $m_f(r; D) \leq m_g(r; E)$, we have

$$T_f(r; [D]) \leq (n+1)T_g(r) - m_g(r; D) + \epsilon T_g(r) + m_f(r; D)$$
$$\leq (n+1)T_g(r) + \epsilon T_g(r)||_\epsilon.$$

Note that $K_X = \pi^* K_{\mathbf{P}^n(\mathbf{C})} + D$, so that

$$T_f(r; K_X) = -(n+1)T_g(r) + T_f(r; [D]).$$

Therefore,

$$T_f(r; K_X) \leq \epsilon T_g(r)||_\epsilon.$$

Since K_X is big, $T_f(r; K_X) \sim T_g(r)$, so that we have a contradiction:

$$T_g(r) \leq \epsilon T_g(r)||_\epsilon, \quad \forall \epsilon > 0. \quad \text{q.e.d.?}$$

N.B. Conjectures A and B for a finite cover $X \to A$ over a semi-abelian variety A, being replaced $\mathbf{P}^n(\mathbf{C})$ by A, are true.