On the Degeneracy of Holomorphic Curves

Junjiro Noguchi
(MSU-Tokyo)
Complex Geometry in Osaka
for Professor Fujiki's KANREKI (60th birthday)
Osaka, 1-5 November 2007

§1 Introduction.

I would like to discuss the degeneracy problem of holomorphic curves into complex projective varieties.

In this talk, algebraic varieties are those defined over $\mathbf{C}$, and $f : \mathbf{C} \to X$ stands for a holomorphic curve into an algebraic variety X, unless otherwise mentioned. We say that $f : \mathbf{C} \to X$ is *degenerate* if it is algebraically degenerate; i.e., the image $f(\mathbf{C})$ is contained by a proper algebraic subset of X.

Green-Griffiths Conjecture (Log version). Let X be an algebraic variety of general type. Then $\forall f : \mathbf{C} \to X$ is degenerate.

Kobayashi Conjecture. Let $V \subset \mathbf{P}^n(\mathbf{C})$ be a "generic" hypersurface of $\deg V \geq 2n-1$. Then V is Kobayashi hyperbolic; equivalently by Brody's Theorem $\forall f : \mathbf{C} \to X$ is constant.

Fundamental Conjecture for holomorphic curves. Let V be a smooth projective variety and let D be an s.n.c. (simple normal crossing) divisor on V. Then for a nondegenerate $f : \mathbf{C} \to V$

$$T_f(r; [D]) + T_f(r; K_V) \leq N_1(r; f^*D) + \epsilon T_f(r)||, \ \forall \epsilon > 0.$$

Here, "$||_\epsilon$" means that the stated estimate holds for $r > 0$ except for those of a Borel subset of $(0, \infty)$ dependent on $\epsilon > 0$ with finite measure.

This conjecture is modeled after a successful generalization of Nevanlinna's theory to the case of differentially nondegenerate holomorphic mappings $f : \mathbf{C}^n \to V$ ($n = \dim V$) (Carlson, Griffiths, King, ...). Note that

Fundamental Conjecture for holomorphic curves

$\Rightarrow$ Green-Griffiths Conjecture (immediate),

and then

$\Rightarrow$ Kobayashi Conjecture (not so immediate).

§2 Notation and First Main Theorem.

Let $E = \sum_{\mu=1}^{\infty} \nu_\mu z_\mu$ be a divisor on $\mathbf{C}$ with distinct $z_\mu \in \mathbf{C}$. Then we set

$$\mathrm{ord}_z E = \begin{cases} \nu_\mu, & z = z_\mu, \\ 0, & z \notin \{z_\mu\}. \end{cases}$$

We define the *counting functions* of E truncated to $l \leqq \infty$ by

$$n_l(t; E) = \sum_{\{|z_\mu|<t\}} \min\{\nu_\mu, l\}, \quad N_l(r; E) = \int_1^r \frac{n_l(t; E)}{t} dt.$$

When $l = \infty$, we write

$$n(t; E) = n_\infty(t; E), \quad N(r; E) = N_\infty(r; E).$$

Let X be a compact reduced complex space with structure sheaf $\mathcal{O}_X$, and let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf. For a holomorphic curve

$$f : \mathbf{C} \to X, \quad f(\mathbf{C}) \not\subset \operatorname{Supp} \mathcal{O}_X/\mathcal{I}$$

we are going to define 3 quantities, $m_f(r; *), N_l(r; *), T(r; *)$ as follows.

Let $\{U_j\}$ be a finite open covering of X such that

(i) there are finitely many sections

$$\sigma_{jk} \in \Gamma(U_j, \mathcal{I}), k = 1, 2, \ldots,$$

generating every fiber $\mathcal{I}_x$ over $x \in U_j$;

(ii) there is a partition of unity $\{c_j\}$ subordinate to $\{U_j\}$.

Setting

$$\rho_{\mathcal{I}}(x) = \left(\sum_j c_j(x) \sum_k |\sigma_{jk}(x)|^2 \right)^{1/2},$$

we take a constant $C > 0$ so that

$$\hat{\rho}_{\mathcal{I}}(x) = C\rho_{\mathcal{I}}(x) \leqq 1, \quad x \in M.$$

Using the compactness of X, one easily verifies that $\log \hat{\rho}_{\mathcal{I}}$ (Weil function in arithmetic) is well-defined up to a bounded function on X.

We define the *approximation (proximity)* function of f for $\mathcal{I}$ or for the subspace $Y = (\operatorname{Supp} \mathcal{O}_X/\mathcal{I}, \mathcal{O}/\mathcal{I})$ (possibly non-reduced) by

$$m_f(r; \mathcal{I}) = m_f(r; Y) = \int_{|z|=r} \log \frac{1}{\hat{\rho}_{\mathcal{I}}(f(z))} \frac{d\theta}{2\pi} \quad (\geqq 0).$$

- $\hat{\rho}_{\mathcal{I}} \circ f(z)$ is C^∞ over $\mathbf{C} \setminus f^{-1}(\operatorname{Supp} Y)$.
- For $z_0 \in f^{-1}(\operatorname{Supp} Y)$, $\exists$ neighborhood $U \ni z_0$ and $\exists \nu \in \mathbf{Z}_{>0}$ such that $(f^*\mathcal{I})|_U = ((z - z_0)^\nu)$.

Then

$$\log \hat{\rho}_{\mathcal{I}} \circ f(z) = \nu \log |z - z_0| + \psi(z), \quad z \in U,$$

where $\psi(z)$ is C^∞ on U. We define the *counting function*

$$N(r; f^*\mathcal{I}), \quad N_l(r; f^*\mathcal{I})$$

by using ν in the same way as $N(r; E)$ and $N_l(r; E)$. Moreover we define

$$\begin{aligned}\omega_{\mathcal{I},f} = \omega_{Y,f} &= -dd^c\psi(z) = -\frac{i}{2\pi}\partial\bar{\partial}\psi(z)\\ &= dd^c \log \frac{1}{\hat{\rho}_{\mathcal{I}} \circ f(z)} \quad (z \in U),\end{aligned}$$

which is well-defined on $\mathbf{C}$ as a smooth (1,1)-form. The *order function* of f for $\mathcal{I}$ or Y is defined by

$$T(r; \omega_{\mathcal{I},f}) = T(r; \omega_{Y,f}) = \int_1^r \frac{dt}{t} \int_{|z|<t} \omega_{\mathcal{I},f}.$$

If $\mathcal{I}$ is the ideal sheaf defined by a Cartier effective divisor D on X, in terms of commonly used notation we have

$$\begin{aligned} m_f(r; \mathcal{I}) &= m_f(r; D) + O(1),\\ T(r; \omega_{\mathcal{I},f}) &= T_f(r; [D]) + O(1).\end{aligned}$$

Fix a hermitian form ω_X on X. We define a *standard* order function by

$$T_f(r) = T(r; \omega_X) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\omega_X.$$

Theorem 1 (First Main Theorem, N. '03). *Let $f : \mathbf{C} \to X$ and let $\mathcal{I}$ be as above. Then*

$$T(r; \omega_{\mathcal{I},f}) = N(r; f^*\mathcal{I}) + m_f(r; \mathcal{I}) - m_f(1; \mathcal{I}).$$

§3 Classical results.

E. Borel's Theorem (1897). *Let $H_i \subset \mathbf{P}^n(\mathbf{C})$, $1 \le i \le l$, be hyperplanes in general position. If $l > n + 1$, then $\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus \cup_{i=1}^l H_i$ is (linearly) degenerate.*

N.B. The log irregularity

$$\bar{q}(\mathbf{P}^n(\mathbf{C}) \setminus \cup_1^q H_i) = l - 1 > n \Longleftrightarrow l > n + 1.$$

Bloch-Ochiai's Theorem ('26-'77). *Let V be an n-dimensional projective algebraic variety. If the irregularity $q(V) > n$, then $\forall f : \mathbf{C} \to V$ is degenerate.*

Log Bloch-Ochiai's Theorem (N. '77-'81). *Let X be an algebraic variety of dimension n. If $\bar{q}(X) > n$, then $\forall f : \mathbf{C} \to X$ is degenerate.*

N.B. Log Bloch-Ochiai's theorem unifies Borel's and Bloch-Ochiai's Theorems in terms of log irregularities.

The proof is reduced as follows: Let V be a smooth n-dimensional projective variety, let D be a reduced divisor on V, and set $X = V \setminus D$. Let $\alpha : X \to A_X$ be the quasi-Albanese map.

Theorem (N. '77-'81). *If the log Kodaira dimension* $\bar{\kappa}(\overline{\alpha(X)}^{\mathrm{Zar}}) = n$, *then* $\exists \lambda > 0$ *such that for* $\forall$ *nondegenerate* $f : \mathbf{C} \to V$

$$\lambda T_f(r) \leq N_1(r; f^*D) + O(\delta \log r + \log T_f(r))||_\delta,\ \forall \delta > 0.$$

Problem 1. What is the best λ?

The proof of Log Bloch-Ochiai's Theorem is reduced to the case where X is of general type. Therefore it is considered as a special case of Green-Griffiths' Conjecture. Thus, we may ask

Problem 2. Assuming $\bar{\kappa}(X) = \dim X$, can we decrease the lower bound "$n < \bar{q}(X)$" in Log Bloch-Ochiai's Theorem?

E.g.: Given 4 lines $L_i \subset \mathbf{P}^2 (1 \leq i \leq 4)$ in general position, we merge 2 lines L_3 and L_4 to a quadric D_3, so that $L_1 + L_2 + D_3$ has only s.n.c., and set

$$X = \mathbf{P}^2 \setminus (L_1 \cup L_2 \cup D_3).$$

Then $\bar{q}(X) = \bar{\kappa}(X) = 2$.

M. Green's Conjecture ('74). Is $\forall f : \mathbf{C} \to X$ degenerate?

M. Green proved this for f of finite order.

N.B. $\bar{\kappa}(X) = \bar{q}(X) = 2$ and the quasi-Albanese $\alpha_X : X \to (\mathbf{C}^*)^2$ is finite.

We proved M. Green's Conjecture in much more general form (cf. §5). Note that in the case of Diophantine approximation, the the analogous problem is open; Corvaja and Zannier lately dealt with the problem over function fields in a preprint.

§4 Semi-abelian varieties.

Let A be a semi-abelian variety; i.e.,

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \to 0,$$

where A_0 is an abelian variety. Let $J_k(A)(k \geq 0)$ be the k-jet space over A, let $f : \mathbf{C} \to A$ be a holomorphic curve, let $J_k(f) : \mathbf{C} \to J_k(X)$ be the k-jet lift, and set

$$X_k(f) = \overline{J_k(f)(\mathbf{C})}^{\mathrm{Zar}} \subset J_k(A).$$

As an answer to Problem 1 we have

Theorem 2 (N.-Winkelmann-Yamanoi [3]). *Assume that* $f : \mathbf{C} \to A$ *is nondegenerate.*

(i) *Let* Z *be an algebraic reduced subvariety of* $X_k(f)$. *Then* $\exists \bar{X}_k(f)$, *a compactification of* $X_k(f)$ *such that*

$$T(r; \omega_{\bar{Z}, J_k(f)}) \leqq N_1(r; J_k(f)^* Z) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0,$$

where $\bar{Z}$ is the closure of Z in $\bar{X}_k(f)$.

(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geqq 2$, *then*

$$T(r; \omega_{\bar{Z}, J_k(f)}) \leqq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

(iii) *When $k = 0$ and Z is an effective reduced divisor D on A, $\bar{A}$ can be chosen as smooth, equivariant with respect to A-action, and independent of f, and we have*

$$T_f(r; L(\bar{D})) \leqq N_1(r; f^*D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon, \, \forall \epsilon > 0.$$

The above (iii) gives yet another proof of Lang's conjecture:

Corollary 3 (Siu-Yeung '96, N. '98, McQuillan '01). *Let D be an effective reduced divisor on A (semi-abelian). Then $\forall f : \mathbf{C} \to A \setminus D$ is degenerate.*

§5 Finite cover of semi-abelian variety.

As an application of Theorem 2 we give an answer to Problem 2.

Let X be a normal variety, not necessarily compact, and let A be a semi-abelian variety.

Theorem 4 (N.-Winkelmann-Yamanoi [1]). *Assume that*

(i) *$\exists$a finite morphism $\pi : X \to A$;*

(ii) $\bar{\kappa}(X) > 0$.

Then $\forall f : \mathbf{C} \to X$ is degenerate.

N.B. (Kawamata '81) Under condition (i),

$$\bar{\kappa}(X) > 0 \Leftrightarrow X \text{ is } \underline{\text{not}} \text{ isomorphic to a semi-abelian variety.}$$

Corollary 5 *Assume that the quasi-Albanese map of X is proper, and moreover that*

$$\bar{\kappa}(X) > 0, \quad \bar{q}(X) \geq \dim X.$$

Then $\forall f : \mathbf{C} \to X$ is degenerate.

For the proof of Theorem 4 we need the following precise resolution and compactification of X.

Theorem 6 *Let $\pi : X \to A$ be a finite morphism from a normal variety X of* $\dim X = n$ *onto a semi-abelian variety A. Let $\bar{A}$ be a smooth equivariant compactification of A. Let D denote the critical locus of π; i.e. the closure of the set of all $\pi(z)$, where $z \in X_{\mathrm{reg}}$ and* $\operatorname{rank} d\pi < n$.

Then there exist

(a) *a desingularization* $\tau : \tilde{X} \to X$ *and a smooth compactification* $j : \tilde{X} \hookrightarrow \hat{X}$ *such that the boundary divisor* $\partial \hat{X} = \hat{X} \setminus j(\tilde{X})$ *has only s.n.c.;*

(b) *a proper holomorphic map* $\psi : \hat{X} \to \bar{A}$ *such that* $\psi \circ j = \pi \circ \tau$ *with* $\psi^{-1}(A) = \tilde{X}$*;*

(c) *an effective divisor* Θ *on* $\hat{X}$*;*

(d) *a subvariety* $\hat{S} \subset \hat{X}$

such that

(i) $\Theta \sim K_{\hat{X}} + \partial \hat{X}$, *the log canonical divisor of* $\tilde{X}$,

(ii) $\psi(\mathrm{Supp}\, \Theta) \subset \bar{D}$,

(iii) $\mathrm{codim}\,_A \psi(\hat{S}) \geq 2$,

(iv) *for* $\forall f : \Delta \to A$, *holomorphic curve from a disk* Δ *in* $\mathbf{C}$ *with lifting* $F : \Delta \to \tilde{X}$ *and for* $z \in F^{-1}(\mathrm{Supp}\, \Theta \setminus \hat{S})$ *we have*

$$\mathrm{mult}_z F^* \Theta \leq \mathrm{mult}_z f^* D - 1. \tag{7}$$

Proof of Theorem 4:

By some theorem of Kawamata the case is easily reduced to $\bar{\kappa}(X) = \dim X$. Then the above Θ is big and hence so is $\bar{D}$. Set

$$F = \pi \circ f : \mathbf{C} \to A.$$

Then $\exists C > 0$ such that

$$C^{-1} T_f(r) < T_F(r) < C T_f(r). \tag{8}$$

In this case we write

$$T_f(r) \sim T_F(r).$$

By Theorem 2 (S.M.T.)

$$m_F(r; \Theta) \leq m_f(r; \bar{D}) \leq \epsilon T_f(r)||_\epsilon, \ \forall \epsilon > 0. \tag{9}$$

Theorem 6 (iv) implies

$$N(r; F^* \Theta) \leq N(r; F^* \hat{S}) + N(r; f^* D) - N_1(r; f^* D). \tag{10}$$

Now $\mathrm{codim}\,_A \psi(\hat{S}) \geq 2$. Therefore we can infer from Theorem 2 (S.M.T.) that

$$N(r; F^* \hat{S}) \leq N(r; f^*(\psi_* \hat{S})) \leq \epsilon T_f(r)||_\epsilon. \tag{11}$$

By virtue of Theorems 1 (F.M.T.) and 2 (S.M.T.) we have

$$\begin{aligned} N(r; f^* D) - N_1(r; f^* D) &\leq T_f(r; L(\bar{D})) - N_1(r; f^* D) \\ &\leq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0. \end{aligned} \tag{12}$$

Now one infers from (10)–(12) that

$$N(r; F^*\Theta) \le \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0. \tag{13}$$

Since Θ is big,

$$T_F(r) \sim T(r; F^*\Theta).$$

Thus

$$T_F(r) \le \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0,$$

and so by (8)

$$T_F(r) \le \epsilon T_F(r)||_\epsilon. \quad \forall \epsilon > 0.$$

This is a contradiction.

Example and Question. Let $D_i, 1 \le i \le q$, be irreducible hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ *in general position*; i.e., for distinct $1 \le i_1 < \cdots < i_k \le q$,

$$\operatorname{codim} D_{i_1} \cap \cdots \cap D_{i_k} = \begin{cases} k, & k \le n \\ \emptyset, & k > n. \end{cases}$$

Assume that $\deg \sum_{i=1}^q D_i > n+1$.

Then, if $q > n+1$, then Log Bloch-Ochiai's Theorem implies the degeneracy of $\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus \sum_{i=1}^q D_i$.

Suppose $q = n+1$. Then $\bar{q}\,(\mathbf{P}^n(\mathbf{C}) \setminus \sum_{i=1}^q D_i) = n$. If every D_i are smooth, then $\bar{\kappa}(\mathbf{P}^n(\mathbf{C}) \setminus \sum_{i=1}^q D_i) = n$, and hence Theorem 4 implies the degeneracy of $\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus \sum_{i=1}^q D_i$; with $n = 2$ and $q = 3$ this resolves M. Green's Conjecture in §3.

Question. Without the smoothness condition for each D_i, can we have

$$\bar{\kappa}\left(\mathbf{P}^n(\mathbf{C}) \setminus \sum_{i=1}^{n+1} D_i\right) > 0?$$

§6 Kobayashi Conjecture.

There is a nice preparation from algebraic side:

Theorem 14 (Ein '88 '91, Xu '94, Voisin '96). *Let $V \subset \mathbf{P}^n(\mathbf{C})$ be a general hypersurface of* $\deg V \ge 2n-1$. *Then all subvarieties of V whatsoever are of general type.*

Therefore, Green-Griffiths' Conjecture implies the Kobayashi Conjecture.

We consider a specialized Fundamental Conjecture: Let V be a projective manifold of dimension n with embedding $X \hookrightarrow \mathbf{P}^N(\mathbf{C})$.

Let $\pi : V \to \mathbf{P}^n(\mathbf{C})$ be a generic projection, let D be the ramification divisor of π on V, and let E be the critical value divisor on $\mathbf{P}^n(\mathbf{C})$.

In the sequel we have this setting in mind. Note that if π moves, then D and E are *deformed*. Thus we may consider the following conjectures only for a generic projection $\pi : X \to \mathbf{P}^n(\mathbf{C})$.

We specialize the Fundamental Conjecture as follows:

Conjecture A. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be nondegenerate, and let E be a reduced divisor E on $\mathbf{P}^n(\mathbf{C})$, allowing singularities. Then we have

$$\{\deg E - n - 1\} T_f(r) \leq N_1(r; f^* E) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

We consider a milder conjecture for general projective V:

Conjecture B. Let $f : \mathbf{C} \to V$ be nondegenerate, let D be a reduced divisor on V, allowing singularities. Then we have

$$N(r; f^* D) - N_1(r; f^* D) \leq \epsilon T_f(r)||_\epsilon, \ \forall \epsilon > 0.$$

Conjectures A and B imply Kobayashi Conjecture:

For we assume that $f : \mathbf{C} \to X$ is nondegenerate. Let $\pi : X \to \mathbf{P}^n(\mathbf{C})$ be a projection with the ramification divisor D on X and the critical value divisor E on $\mathbf{P}^n(\mathbf{C})$, for which Conjectures A and B hold.

Conjecture A implies that

$$(\deg E - n - 1) T_g(r) \leq N_1(r; g^* E) + \epsilon T_g(r)||_\epsilon.$$

Thus, by combining this with Theorem 1 we obtain

$$\begin{aligned} & N(r; g^* E) - N_1(r; g^* E) \\ & \leq (n+1) T_g(r) - m_g(r; E) + \epsilon T_g(r)||_\epsilon. \end{aligned}$$

Because of ramifications we get

$$\begin{aligned} & N_1(r; f^* D) \leq N(r; g^* E) - N_1(r; g^* E) \\ & \leq (n+1) T_g(r) - m_g(r; E) + \epsilon T_g(r)||_\epsilon. \end{aligned}$$

Conjecture B implies that

$$T_f(r; [D]) \leq N_1(r; f^* D) + m_f(r, D) + \epsilon T_f(r)||_\epsilon.$$

Combining the above two with $m_f(r; D) \leq m_g(r; E)$, we have

$$\begin{aligned} T_f(r; [D]) & \leq (n+1) T_g(r) - m_g(r; E) + \epsilon T_g(r) + m_f(r; D) \\ & \leq (n+1) T_g(r) + \epsilon T_g(r)||_\epsilon. \end{aligned}$$

Note that $K_X = \pi^* K_{\mathbf{P}^n(\mathbf{C})} + D$, so that

$$T_f(r; K_X) = -(n+1) T_g(r) + T_f(r; [D]).$$

Therefore,

$$T_f(r; K_X) \leq \epsilon T_g(r)||_\epsilon.$$

Since K_X is big, $T_f(r; K_X) \sim T_g(r)$, so that we have a contradiction:

$$T_g(r) \leq \epsilon T_g(r)||_\epsilon, \quad \forall \epsilon > 0.$$

N.B. Similar Conjectures A and B for a finite cover X over a semi-abelian variety A, being replaced $\mathbf{P}^n(\mathbf{C})$ by A, are true by Theorem 2.

References

1. 野口潤次郎, 多変数ネヴァンリンナ理論とディオファントス近似 (Nevanlinna Theory in Several Variables and Diophantine Approximation), viii+264 pp., 共立出版社 (Kyoritsu Publ.), 2003.

2. Noguchi, J., Winkelmann, J. and Yamanoi, K., Degeneracy of Holomorphic Curves into Algebraic Varieties, J. Math. Pures Appl. **88** Issue 3, (2007), 293–306.

3. Noguchi, J., Winkelmann, J. and Yamanoi, K., The second main theorem for holomorphic curves into semi-abelian varieties II, preprint UTMS 2004-17, in Press, Forum Math.

See bibliographies of the above references for more.