On the Degeneracy of Holomorphic Curves
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§1 Introduction.
I would like to discuss the degeneracy problem of holomorphic curves into complex projective varieties.

In this talk, algebraic varieties are those defined over $\mathbb{C}$, and $f : \mathbb{C} \to X$ stands for a holomorphic curve into an algebraic variety $X$, unless otherwise mentioned. We say that $f : \mathbb{C} \to X$ is degenerate if it is algebraically degenerate; i.e., the image $f(\mathbb{C})$ is contained by a proper algebraic subset of $X$.

Green-Griffiths Conjecture (Log version). Let $X$ be an algebraic variety of general type. Then $\forall f : \mathbb{C} \to X$ is degenerate.

Kobayashi Conjecture. Let $V \subset \mathbb{P}^n(\mathbb{C})$ be a “generic” hypersurface of $\deg V \geq 2n - 1$. Then $V$ is Kobayashi hyperbolic; equivalently by Brody’s Theorem $\forall f : \mathbb{C} \to X$ is constant.

Fundamental Conjecture for holomorphic curves. Let $V$ be a smooth projective variety and let $D$ be an s.n.c. (simple normal crossing) divisor on $V$. Then for a nondegenerate $f : \mathbb{C} \to V$

$$T_f(r; [D]) + T_f(r; K_V) \leq N_1(r; f^*D) + \epsilon T_f(r)||, \forall \epsilon > 0.$$ 

Here, “$||\epsilon$” means that the stated estimate holds for $r > 0$ except for those of a Borel subset of $(0, \infty)$ dependent on $\epsilon > 0$ with finite measure.

This conjecture is modeled after a successful generalization of Nevanlinna’s theory to the case of differentially nondegenerate holomorphic mappings $f : \mathbb{C}^n \to V$ ($n = \dim V$) (Carlson, Griffiths, King, ...). Note that

Fundamental Conjecture for holomorphic curves 
$\Rightarrow$ Green-Griffiths Conjecture (immediate),

and then 
$\Rightarrow$ Kobayashi Conjecture (not so immediate).

§2 Notation and First Main Theorem.
Let $E = \sum_{\mu=1}^{\infty} \nu_\mu z_\mu$ be a divisor on $\mathbb{C}$ with distinct $z_\mu \in \mathbb{C}$. Then we set

$$\text{ord}_z E = \begin{cases} 
\nu_\mu, & z = z_\mu, \\
0, & z \notin \{z_\mu\}. 
\end{cases}$$
We define the *counting functions* of $E$ truncated to $l \leq \infty$ by

$$n_l(t; E) = \sum_{\{z_\mu \prec t\}} \min\{\nu_\mu, l\}, \quad N_l(r; E) = \int_1^r \frac{n_l(t; E)}{t} dt.$$ 

When $l = \infty$, we write

$$n(t; E) = n_\infty(t; E), \quad N(r; E) = N_\infty(r; E).$$

Let $X$ be a compact reduced complex space with structure sheaf $\mathcal{O}_X$, and let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf. For a holomorphic curve $f : \mathbb{C} \to X$, $f(\mathbb{C}) \not\subset \text{Supp} \mathcal{O}_X/\mathcal{I}$ we are going to define 3 quantities, $m_f(r; \ast), N_l(r; \ast), T(r; \ast)$ as follows.

1. there are finitely many sections $\sigma_{jk} \in \Gamma(U_j, \mathcal{I}), k = 1, 2, \ldots,$ generating every fiber $\mathcal{I}_x$ over $x \in U_j$;
2. there is a partition of unity $\{c_j\}$ subordinate to $\{U_j\}$.

Setting

$$\rho_{\mathcal{I}}(x) = \left( \sum_j c_j(x) \sum_k |\sigma_{jk}(x)|^2 \right)^{1/2},$$

we take a constant $C > 0$ so that

$$\hat{\rho}_{\mathcal{I}}(x) = C\rho_{\mathcal{I}}(x) \leq 1, \quad x \in M.$$ 

Using the compactness of $X$, one easily verifies that $\log \hat{\rho}_{\mathcal{I}}$ (Weil function in arithmetic) is well-defined up to a bounded function on $X$.

We define the *approximation (proximity)* function of $f$ for $\mathcal{I}$ or for the subspace $Y = (\text{Supp} \mathcal{O}_X/\mathcal{I}, \mathcal{O}/\mathcal{I})$ (possibly non-reduced) by

$$m_f(r; \mathcal{I}) = m_f(r; Y) = \int_{|z|=r} \log \frac{1}{\hat{\rho}_{\mathcal{I}}(f(z))} \frac{d\theta}{2\pi} \quad (\geq 0).$$

- $\hat{\rho}_{\mathcal{I}} \circ f(z)$ is $C^\infty$ over $\mathbb{C} \setminus f^{-1}(\text{Supp} Y)$.
- For $z_0 \in f^{-1}(\text{Supp} Y)$, $\exists$ neighborhood $U \ni z_0$ and $\exists \nu \in \mathbb{Z}_{>0}$ such that $(f^*\mathcal{I})|_U = ((z - z_0)^\nu)$.
Then
\[ \log \hat{\rho}_\mathcal{I} \circ f(z) = \nu \log |z - z_0| + \psi(z), \quad z \in U, \]
where \( \psi(z) \) is \( C^\infty \) on \( U \). We define the counting function
\[ N(r; f^*\mathcal{I}), \quad N_i(r; f^*\mathcal{I}) \]
by using \( \nu \) in the same way as \( N(r; E) \) and \( N_i(r; E) \). Moreover we define
\[ \omega_{\mathcal{I}, f} = \omega_{Y, f} = -dd^c\psi(z) = -\frac{i}{2\pi} \partial \bar{\partial} \psi(z) \]
\[ = dd^c \log \frac{1}{\hat{\rho}_\mathcal{I} \circ f(z)} \quad (z \in U), \]
which is well-defined on \( \mathbb{C} \) as a smooth \((1,1)\)-form. The order function of \( f \) for \( \mathcal{I} \) or \( Y \) is defined by
\[ T(r; \omega_{\mathcal{I}, f}) = \int_1^r \frac{dt}{t} \int_{|z| < t} \omega_{\mathcal{I}, f}. \]

If \( \mathcal{I} \) is the ideal sheaf defined by a Cartier effective divisor \( D \) on \( X \), in terms of commonly used notation we have
\[ m_f(r; \mathcal{I}) = m_f(r; D) + O(1), \]
\[ T(r; \omega_{\mathcal{I}, f}) = T_f(r; [D]) + O(1). \]

Fix a hermitian form \( \omega_X \) on \( X \). We define a standard order function by
\[ T_f(r) = T(r; \omega_X) = \int_1^r \frac{dt}{t} \int_{|z| < t} f^*\omega_X. \]

**Theorem 1** (First Main Theorem, N. ’03). Let \( f : \mathbb{C} \to X \) and let \( \mathcal{I} \) be as above. Then
\[ T(r; \omega_{\mathcal{I}, f}) = N(r; f^*\mathcal{I}) + m_f(r; \mathcal{I}) - m_f(1; \mathcal{I}). \]

§3 Classical results.

**E. Borel’s Theorem** (1897). Let \( H_i \subset \mathbb{P}^n(\mathbb{C}), 1 \leq i \leq l, \) be hyperplanes in general position. If \( l > n + 1 \), then \( \forall f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^l H_i \) is (linearly) degenerate.

**N.B.** The log irregularity
\[ \bar{q}(\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i}^l H_i) = l - 1 > n \iff l > n + 1. \]

**Bloch-Ochiai’s Theorem** (’26-’77). Let \( V \) be an \( n \)-dimensional projective algebraic variety. If the irregularity \( q(V) > n \), then \( \forall f : \mathbb{C} \to V \) is degenerate.

**Log Bloch-Ochiai’s Theorem** (N. ’77-’81). Let \( X \) be an algebraic variety of dimension \( n \). If \( \bar{q}(X) > n \), then \( \forall f : \mathbb{C} \to X \) is degenerate.

**N.B.** Log Bloch-Ochiai’s theorem unifies Borel’s and Bloch-Ochiai’s Theorems in terms of log irregularities.
The proof is reduced as follows: Let $V$ be a smooth $n$-dimensional projective variety, let $D$ be a reduced divisor on $V$, and set $X = V \setminus D$. Let $\alpha : X \to A_X$ be the quasi-Albanese map.

**Theorem** (N. '77-'81). If the log Kodaira dimension $\tilde{\kappa}(\alpha(X)^{\text{Zar}}) = n$, then $\exists \lambda > 0$ such that for $\forall$ nondegenerate $f : C \to V$

$$\lambda T_f(r) \leq N_1(r; f^*D) + O(\delta \log r + \log T_f(r))||\delta, \forall \delta > 0.$$ 

**Problem 1.** What is the best $\lambda$?

The proof of Log Bloch-Ochiai’s Theorem is reduced to the case where $X$ is of general type. Therefore it is considered as a special case of Green-Griffiths’ Conjecture. Thus, we may ask

**Problem 2.** Assuming $\tilde{\kappa}(X) = \dim X$, can we decrease the lower bound “$n < \tilde{q}(X)$” in Log Bloch-Ochiai’s Theorem?

E.g.: Given 4 lines $L_i \subset P^2 (1 \leq i \leq 4)$ in general position, we merge 2 lines $L_3$ and $L_4$ to a quadric $D_3$, so that $L_1 + L_2 + D_3$ has only s.n.c., and set

$$X = P^2 \setminus (L_1 \cup L_2 \cup D_3).$$

Then $\tilde{q}(X) = \tilde{\kappa}(X) = 2$.

**M. Green’s Conjecture** (’74). Is $\forall f : C \to X$ degenerate?

M. Green proved this for $f$ of finite order.

N.B. $\tilde{\kappa}(X) = \tilde{q}(X) = 2$ and the quasi-Albanese $\alpha_X : X \to (C^*)^2$ is finite.

We proved M. Green’s Conjecture in much more general form (cf. §5). Note that in the case of Diophantine approximation, the analogous problem is open; Corvaja and Zannier lately dealt with the problem over function fields in a preprint.

§4 Semi-abelian varieties.

Let $A$ be a semi-abelian variety; i.e.,

$$0 \to (C^*)^t \to A \to A_0 \to 0,$$

where $A_0$ is an abelian variety. Let $J_k(A)(k \geq 0)$ be the $k$-jet space over $A$, let $f : C \to A$ be a holomorphic curve, let $J_k(f) : C \to J_k(X)$ be the $k$-jet lift, and set

$$X_k(f) = \overline{J_k(f)(C)^{\text{Zar}}} \subset J_k(A).$$

As an answer to Problem 1 we have

**Theorem 2** (N.-Winkelmann-Yamanoi [3]). Assume that $f : C \to A$ is nondegenerate.

(i) Let $Z$ be an algebraic reduced subvariety of $X_k(f)$. Then $\exists \tilde{X}_k(f), a compactification of X_k(f) such that

$$T(r; \omega_{Z,J_k(f)}^X) \leq N_1(r; J_k(f)^*Z) + \epsilon T_f(r)||\epsilon, \forall \epsilon > 0,$$

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where $\hat{Z}$ is the closure of $Z$ in $\hat{X}_k(f)$.

(ii) Moreover, if $\text{codim}_{X_k(f)} Z \geq 2$, then

$$T(r; \omega_{\hat{Z}, X_k(f)}) \leq \epsilon T_f(r) \|_{\epsilon}, \quad \forall \epsilon > 0.$$ 

(iii) When $k = 0$ and $Z$ is an effective reduced divisor $D$ on $A$, $\hat{A}$ can be chosen as smooth, equivariant with respect to $A$-action, and independent of $f$, and we have

$$T_f(r; L(\hat{D})) \leq N_1(r; f^* D) + \epsilon T_f(r; L(\hat{D})) \|_{\epsilon}, \quad \forall \epsilon > 0.$$ 

The above (iii) gives yet another proof of Lang’s conjecture:

**Corollary 3** (Siu-Yeung ’96, N. ’98, McQuillan ’01). Let $D$ be an effective reduced divisor on $A$ (semi-abelian). Then $\forall f : C \to A \setminus D$ is degenerate.

§5 Finite cover of semi-abelian variety.

As an application of Theorem 2 we give an answer to Problem 2.

Let $X$ be a normal variety, not necessarily compact, and let $A$ be a semi-abelian variety.

**Theorem 4** (N.-Winkelmann-Yamanoi [1]). Assume that

(i) $\exists a$ finite morphism $\pi : X \to A$;

(ii) $\bar{\kappa}(X) > 0$.

Then $\forall f : C \to X$ is degenerate.

N.B. (Kawamata ’81) Under condition (i),

$$\bar{\kappa}(X) > 0 \Leftrightarrow X \text{ is not isomorphic to a semi-abelian variety.}$$

**Corollary 5** Assume that the quasi-Albanese map of $X$ is proper, and moreover that

$$\bar{\kappa}(X) > 0, \quad \bar{q}(X) \geq \dim X.$$

Then $\forall f : C \to X$ is degenerate.

For the proof of Theorem 4 we need the following precise resolution and compactification of $X$.

**Theorem 6** Let $\pi : X \to A$ be a finite morphism from a normal variety $X$ of $\dim X = n$ onto a semi-abelian variety $A$. Let $\hat{A}$ be a smooth equivariant compactification of $A$. Let $D$ denote the critical locus of $\pi$; i.e. the closure of the set of all $\pi(z)$, where $z \in X_{\text{reg}}$ and $\text{rank } d\pi < n$.

Then there exist
(a) a desingularization \( \tau : \tilde{X} \to X \) and a smooth compactification \( j : \tilde{X} \hookrightarrow \hat{X} \) such that the boundary divisor \( \partial \hat{X} = \hat{X} \setminus j(\tilde{X}) \) has only s.n.c.;

(b) a proper holomorphic map \( \psi : \hat{X} \to \bar{A} \) such that \( \psi \circ j = \pi \circ \tau \) with \( \psi^{-1}(A) = \tilde{X} \);

(c) an effective divisor \( \Theta \) on \( \hat{X} \);

(d) a subvariety \( \hat{S} \subset \hat{X} \)

such that

(i) \( \Theta \sim K_{\tilde{X}} + \partial \hat{X} \), the log canonical divisor of \( \tilde{X} \),

(ii) \( \psi(\text{Supp} \Theta) \subset \bar{D} \),

(iii) \( \text{codim}_A \psi(\hat{S}) \geq 2 \),

(iv) for all \( f : \Delta \to A \), holomorphic curve from a disk \( \Delta \) in \( C \) with lifting \( F : \Delta \to \tilde{X} \) and for \( z \in F^{-1}(\text{Supp} \Theta \setminus \hat{S}) \) we have

\[
\text{mult}_z F^* \Theta \leq \text{mult}_z f^* D - 1. \tag{7}
\]

Proof of Theorem 4:

By some theorem of Kawamata the case is easily reduced to \( \kappa(X) = \dim X \). Then the above \( \Theta \) is big and hence so is \( \bar{D} \). Set

\[ F = \pi \circ f : C \to A. \]

Then \( \exists \epsilon > 0 \) such that

\[
C^{-1} T_f(r) < T_F(r) < CT_f(r). \tag{8}
\]

In this case we write

\[ T_f(r) \sim T_F(r). \]

By Theorem 2 (S.M.T.)

\[
m_F(r; \Theta) \leq m_f(r; \bar{D}) \leq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0. \tag{9}
\]

Theorem 6 (iv) implies

\[
N(r; F^* \Theta) \leq N(r; F^* \hat{S}) + N(r; f^* D) - N_1(r; f^* D). \tag{10}
\]

Now \( \text{codim}_A \psi(\hat{S}) \geq 2 \). Therefore we can infer from Theorem 2 (S.M.T.) that

\[
N(r; F^* \hat{S}) \leq N(r; f^* (\psi_\ast \hat{S})) \leq \epsilon T_f(r)||_\epsilon. \tag{11}
\]

By virtue of Theorems 1 (F.M.T.) and 2 (S.M.T.) we have

\[
N(r; f^* D) - N_1(r; f^* D) \leq T_f(r; L(\bar{D})) - N_1(r; f^* D) \leq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0. \tag{12}
\]
Now one infers from (10)–(12) that
\[ N(r; F^* \Theta) \leq \epsilon T_f(r) ||\epsilon, \quad \forall \epsilon > 0. \]  
(13)

Since \( \Theta \) is big,
\[ T_F(r) \sim T(r; F^* \Theta). \]
Thus
\[ T_F(r) \leq \epsilon T_f(r) ||\epsilon, \quad \forall \epsilon > 0, \]
and so by (8)
\[ T_F(r) \leq \epsilon T_F(r) ||\epsilon, \quad \forall \epsilon > 0. \]
This is a contradiction.

**Example and Question.** Let \( D_i, 1 \leq i \leq q \), be irreducible hypersurfaces of \( \mathbb{P}^n(\mathbb{C}) \) in general position; i.e., for distinct \( 1 \leq i_1 < \cdots < i_k \leq q \),

\[ \text{codim } D_{i_1} \cap \cdots \cap D_{i_k} = \begin{cases} k, & k \leq n \\ \emptyset, & k > n. \end{cases} \]

Assume that \( \text{deg} \sum_{i=1}^q D_i > n + 1 \).

Then, if \( q > n + 1 \), then Log Bloch-Ochiai’s Theorem implies the degeneracy of \( \forall f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \setminus \sum_{i=1}^q D_i \).

Suppose \( q = n + 1 \). Then \( \bar{q}(\mathbb{P}^n(\mathbb{C}) \setminus \sum_{i=1}^q D_i) = n \). If every \( D_i \) are smooth, then \( \bar{\kappa}(\mathbb{P}^n(\mathbb{C}) \setminus \sum_{i=1}^q D_i) = n \), and hence Theorem 4 implies the degeneracy of \( \forall f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \setminus \sum_{i=1}^q D_i \); with \( n = 2 \) and \( q = 3 \) this resolves M. Green’s Conjecture in §3.

**Question.** Without the smoothness condition for each \( D_i \), can we have
\[ \bar{\kappa}(\mathbb{P}^n(\mathbb{C}) \setminus \sum_{i=1}^{n+1} D_i) > 0? \]

§6 Kobayashi Conjecture.

There is a nice preparation from algebraic side:

**Theorem 14** (Ein ’88 ’91, Xu ’94, Voisin ’96). Let \( V \subset \mathbb{P}^n(\mathbb{C}) \) be a general hypersurface of \( \text{deg} V \geq 2n - 1 \). Then all subvarieties of \( V \) whatsoever are of general type.

Therefore, Green-Griffiths’ Conjecture implies the Kobayashi Conjecture.

We consider a specialized Fundamental Conjecture: Let \( V \) be a projective manifold of dimension \( n \) with embedding \( X \hookrightarrow \mathbb{P}^N(\mathbb{C}) \).

Let \( \pi : V \to \mathbb{P}^n(\mathbb{C}) \) be a generic projection, let \( D \) be the ramification divisor of \( \pi \) on \( V \), and let \( E \) be the critical value divisor on \( \mathbb{P}^n(\mathbb{C}) \).

In the sequel we have this setting in mind. Note that if \( \pi \) moves, then \( D \) and \( E \) are deformed. Thus we may consider the following conjectures only for a generic projection \( \pi : X \to \mathbb{P}^n(\mathbb{C}) \).
We specialize the Fundamental Conjecture as follows:

**Conjecture A.** Let $f : C \to P^n(C)$ be nondegenerate, and let $E$ be a reduced divisor $E$ on $P^n(C)$, allowing singularities. Then we have
\[
\{\deg E - n - 1\}T_f(r) \leq N_1(r; f^*E) + \epsilon T_f(r)||\epsilon, \quad \forall \epsilon > 0.
\]

We consider a milder conjecture for general projective $V$:

**Conjecture B.** Let $f : C \to V$ be nondegenerate, let $D$ be a reduced divisor on $V$, allowing singularities. Then we have
\[
N(r; f^*D) - N_1(r; f^*D) \leq \epsilon T_f(r)||\epsilon, \quad \forall \epsilon > 0.
\]

Conjectures A and B imply Kobayashi Conjecture:

For we assume that $f : C \to X$ is nondegenerate. Let $\pi : X \to P^n(C)$ be a projection with the ramification divisor $D$ on $X$ and the critical value divisor $E$ on $P^n(C)$, for which Conjectures A and B hold.

Conjecture A implies that
\[
(\deg E - n - 1)T_g(r) \leq N_1(r; g^*E) + \epsilon T_g(r)||\epsilon.
\]
Thus, by combining this with Theorem 1 we obtain
\[
N(r; g^*E) - N_1(r; g^*E) \\
\leq (n + 1)T_g(r) - m_g(r; E) + \epsilon T_g(r)||\epsilon.
\]

Because of ramifications we get
\[
N_1(r; f^*D) \leq N(r; g^*E) - N_1(r; g^*E) \\
\leq (n + 1)T_g(r) - m_g(r; E) + \epsilon T_g(r)||\epsilon.
\]

Conjecture B implies that
\[
T_f(r; [D]) \leq N_1(r; f^*D) + m_f(r, D) + \epsilon T_f(r)||\epsilon.
\]
Combining the above two with $m_f(r; D) \leq m_g(r; E)$, we have
\[
T_f(r; [D]) \leq (n + 1)T_g(r) - m_g(r; E) + \epsilon T_g(r) + m_f(r; D) \\
\leq (n + 1)T_g(r) + \epsilon T_g(r)||\epsilon.
\]

Note that $K_X = \pi^*K_{P^n(C)} + D$, so that
\[
T_f(r; K_X) = -(n + 1)T_g(r) + T_f(r; [D]).
\]

Therefore,
\[
T_f(r; K_X) \leq \epsilon T_g(r)||\epsilon.
\]
Since $K_X$ is big, $T_f(r; K_X) \sim T_g(r)$, so that we have a contradiction:

$$T_g(r) \leq \epsilon T_g(r) ||_\epsilon, \quad \forall \epsilon > 0.$$ 

_N.B._ Similar Conjectures A and B for a finite cover $X$ over a semi-abelian variety $A$, being replaced $\mathbb{P}^n(\mathbb{C})$ by $A$, are true by Theorem 2.

**References**

1. 野口潤次郎, 多変数ネヴァンリンナ理論とディオファントス近似 (Nevanlinna Theory in Several Variables and Diophantine Approximation), viii+264 pp., 共立出版 (Kyoritsu Publ.), 2003.


See bibliographies of the above references for more.