Topics on Holomorphic Curves and Some Open Problems

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We will discuss some new results in Nevanlinna theory of holomorphic curves into algebraic varieties. The central problem is the following conjecture, strengthened from the original of Griffiths (1972):

Griffiths Conjecture 1 (1972) Let
$M :=$ a complex projective manifold,
$D :=$ an s.n.c. divisor on M,
$f : \mathbf{C} \to M$ be an algebraically non-degenerate holomorphic curve.
Then we have

$$(1) \qquad T_f(r; L(D)) + T_f(r; K_M) \leq N_1(r; f^*D) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

Here $N_1(r; f^*D)$ stands for the counting function truncated to level one.

Griffiths Conjecture 1 implies

Griffiths Conjecture 2 (1972). Let
$X :=$ an algebraic variety of log general type. Then $\forall f : \mathbf{C} \to X$ is algebraically degenerate.

§1 Order function.

We need to define the order function of f in a more general form than those already known (e.g., Stoll, Griffiths, Noguchi-Ochiai).

$X :=$ a compact complex reduced space.
$\mathcal{O}_X :=$ the structure sheaf of X.
$Y :=$ a (closed) subspace of X,
not necessarily reduced,
$\mathcal{I} \subset \mathcal{O}_X :=$ the defining coherent ideal sheaf of Y.

We take
$X = \bigcup U_\lambda$, open covering,
$\sigma_{\lambda j}, 1 \leq j \leq l_\lambda$, generators of $\mathcal{I}$ over U_λ,
$V_\lambda \Subset U_\lambda, X = \bigcup V_\lambda$,
$\rho_\lambda \in C_0^\infty(U_\lambda)$, $\rho_\lambda|_{V_\lambda} \equiv 1$.

Set

$$(2) \qquad d_Y(x) = d_{\mathcal{I}}(x) = \sum_{\lambda} \rho_\lambda(x) \left(\sum_{j=1}^{l_\lambda} |\sigma_{\lambda j(x)}|^2 \right)^{1/2}.$$

Other choices yield another d'_Y; $\exists C > 0$ s.t.

$$(3) \qquad |\log d_Y(x) - \log d'_Y(x)| \leq C, \; x \in X.$$

We call

$$\phi_Y(x) = \phi_{\mathcal{I}}(x) = -\log d_Y(x), \quad x \in X$$

the *Weil function* or the *proximity potential* of Y.

For $f : \mathbf{C} \to X$, $f(\mathbf{C}) \not\subset \mathrm{Supp} Y$, the *proximity function* of Y or $\mathcal{I}$ is defined by

$$m_f(r, Y) = m_f(r, \mathcal{I}) = \int_{|z|=r} \phi_Y \circ f(z) \frac{d\theta}{2\pi}.$$

Cf. J.H. Silberman Math. Ann. (1987); J. Noguchi, Book (2003); K. Yamanoi Nagoya J. (2004); N.-W.-Y. preprint (2005).

We further define

$$\text{(4)} \quad \omega_{Y,f} = \omega_{\mathcal{I},f} = -dd^c\phi_Y(z)$$
$$= -\frac{i}{2\pi}\partial\bar{\partial}\phi_Y(z) = dd^c \log \frac{1}{d_Y \circ f(z)},$$

which is a smooth (1,1)-form on $\mathbf{C}$.

The order function of f for Y or $\mathcal{I}$ is defined by

$$\text{(5)} \qquad T(r; \omega_{Y,f}) = T(r; \omega_{\mathcal{I},f})$$
$$= \int_1^r \frac{dt}{t} \int_{|z|<t} \omega_{Y,f}.$$

If $\mathcal{I}$ defines a Cartier divisor D on M,

$$T(r; \omega_{\mathcal{I},f}) = T_f(r; L(D)) + O(1)$$
$$= \int_1^r \frac{dt}{t} \int_{|z|<t} f^* c_1(L(D)) + O(1).$$

Take

$\omega :=$ a hermitian metric form on X.

An order function of f w.r.t. ω is defined by

$$T_f(r) = T(r; f^*\omega) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\omega.$$

In general, $T(r; \omega_{\mathcal{I},f}) = O(T_f(r))$.

Suppose that $f(\zeta) \in U_\lambda \cap \mathrm{Supp}\, Y$.
$\mathrm{mult}_\zeta\, \sigma_{\lambda j} \circ f$:= the mutiplicity of zero at ζ.
$\mathrm{mult}_\zeta f^* Y := \min\{\mathrm{mult}_\zeta \sigma_{\lambda j} \circ f; 1 \leq j \leq l_\lambda\}$.
The counting function with truncation level $k \leq \infty$ is defined by

$$N_k(r; f^*Y) = \int_1^r \frac{dt}{t} \sum_{|\zeta|<t} \min\{\mathrm{mult}_\zeta f^* Y, k\},$$

$$N(r; f^*Y) = N(r; f^*\mathcal{I}) = N_\infty(r; f^*Y).$$

Theorem 6 (Yamanoi, N.-Winkelmann-Y.)

(i) (First Main Theorem)

$$T(r;\omega_{\mathcal{I},f}) = N(r; f^*\mathcal{I}) + m_f(r;\mathcal{I}) - m_f(1;\mathcal{I}).$$

(ii) *If* $Y_1 \supset Y_2$ *(*$\mathcal{I}_1 \subset \mathcal{I}_2$*),*

$$m_f(r;Y_2) \le m_f(r;Y_1) + O(1).$$

(iii) $\phi : X_1 \to X_2$ *be a holomorphic map between compact complex manifolds.*
For $Y_2 \subset X_2$*,*

$$m_f(r;\phi^* Y_2) = m_{\phi\circ f}(r;Y_2) + O(1).$$

(iv) *Let* $Y_i \subset X$*,* $i = 1, 2$*.*

$$T(r;\omega_{Y_1\cup Y_2,f}) \\ = T(r;\omega_{Y_1,f}) + T(r;\omega_{Y_2,f}) + O(1).$$

(v) *Assume that* X *is algebraic.*
$f : \mathbf{C} \to X$ *is rational* $\Leftrightarrow T_f(r) = O(\log r)$*.*

Recall the classical result.

Theorem 7 (Nevanlinna-Cartan) *Let* $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *be a linearly non-degenerate holomoprhic curve,* $\{H_j\}_{j=1}^q$ *be hyperplanes in general position. Then*

$$(q-n-1)T_f(r; O(1)) \leq \sum_{j=1}^{q} N_n(r, f^*H_j) + O(\log r) + O(\log T_f(r))||,$$

where the symbol "||" stands for the estimate to hold for $r > 0$ *outside a Borel subset of finite total Lebesgue measure.*

§2 Min Ru's result.

In the Diophantine approximation theory, P. Corvaja and U. Zannier (2004) generalized Schmidt's Subsapce Theorem to the case of hypersurfaces in the projective space $\mathbf{P}^n$, and then J.-H. Evertse and R.G. Feretti generalized it to the case of a submanifold $M \subset \mathbf{P}^n$.

Min Ru (2004–05) found their analogue to be valid in the theory of holomorphic curves and proved the follwoing:

Theorem 8 *Let*
$M \subset \mathbf{P}^N(\mathbf{C})$ *be an* n*-dim. submanifold,*
$D_i, 1 \le i \le q$*, be hypersurfaces of degree* d_i *in general position in* M*; i.e.,*

$$M \cap D_{i_1} \cap \cdots \cap D_{i_{n+1}} = \emptyset$$

for all $1 \le i_1 < \cdots < i_{n+1} \le q$*,*
$f : \mathbf{C} \to M$ *be algebraically non-degenerate. Then*

$$\begin{aligned}&(q-n-1-\epsilon)T_f(r; O(1))\\ &\le \sum_{i=1}^{q} \frac{1}{d_i} N(r; f^* D_i)||_\epsilon, \quad \forall \epsilon > 0.\end{aligned}$$

In the proof the following approximation theorem due to H. Cartan is one key:

Theorem 9 *Let*
$L_j, j \in Q = \{1, \dots, q\}$, *be linear forms on* $\mathbf{P}^n(\mathbf{C})$ *in general position,*
$f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ *be linearly non-degenerate.*
Then

$$\int_{|\zeta|=r} \max_{K \subset Q, |K|=n+1} \sum_{j \in K} \log \frac{\|f(\zeta)\| \|L_j\|}{|L_j(f(\zeta))|} \frac{d\theta}{2\pi}$$
$$\leq (n+1+\epsilon) T_f(r; O(1))||_\epsilon.$$

This is showing the limit how close $f(\zeta)$ can approximate the divisor $\prod_{j \in Q} L_j = 0$.

The second key is a very elaborate combinatorial argument for Veronese embeddings of degree m as $m \to \infty$ ($\epsilon \to 0$).

§3 Dethloff-Lu's result.

Theorem 10 (Log Bloch-Ochiai (N. '77–'81), N.-Winkelmann (2002)) *Let*
X := a Zariski open subset of a compact Kähler manifold $\bar{X}$.
Assume log.-irreg. $\bar{q}(X) > \dim_{\mathbf{C}} X$.
Then no $f : \mathbf{C} \to X$ has a Zariski dense image in $\bar{X}$.

Problem. What happens if $\bar{q}(X) = \dim_{\mathbf{C}} X$?

$f : \mathbf{C} \to M$ (compact hermitian) is said to be Brody if $\sup \|f'(z)\| < \infty$.

As for Griffiths Conjecture 2, Dethloff and Lu (2005) deal with Brody curves into algebraic surfaces.

Theorem 11 *Let*
$X :=$ *a smooth algebraic surface of log general type with* $\bar{q}(X) = 2$,
$\bar{X} :=$ *a smooth compactification with s.n.c.* ∂X.
Then every Brody curve $f : \mathbf{C} \to X \subset \bar{X}$ *is algebraically degenerate.*

Proposition 12 *Let* X *be an algebraic surface with* $\bar{\kappa}(X) = 1$ *and* $\bar{q}(X) = 2$.
Assume that the quasi-Albanese map
$\alpha_X : X \to A_X$ *is proper*
(a bit more general condition works, but in general $\exists$*counter-example as below).*
Then $\forall f : \mathbf{C} \to X$ *is algebraically degenerate.*

By Kawamata's theorem this is easily reduced to the case of

$$\dim X' = \bar{q}(X') = \bar{\kappa}(X') = 1,$$

$\exists \phi : X \to X'$, dominant and then little Picard's theorem is applied.

They give an interesting example.

Example **1** $\exists$algebraic surface X with $\bar{\kappa}(X) = 1$ and $\bar{q}(X) = 2$, which admits an algebraically non-degenerate $f : \mathbf{C} \to X$.

On the other hand, J. Winkelmann gives another interesting example (2005):

Example **2** $\exists$compact projective 3-fold X s.t.

(i) $\kappa(X) = 0$, $q(X) = 3$,

(ii) Kobayashi $d_X \equiv 0$,

(iii) $\exists f : \mathbf{C} \to X$ with the dense image in the sense of the differential topology,

(iv) $\exists$proper subvareity $Z \subset X$ s.t. for $\forall$ Brody $g : \mathbf{C} \to X$, $g(\mathbf{C}) \subset Z$.

§4 Semi-abelian varieties.

A semi-abelian variety A is an algebraic group carring the representation

$$0 \to (\mathbf{C}^*)^t \to A \to A_0 \to 0,$$

where A_0 is an abelian variety.

Let

$f : \mathbf{C} \to A$ be holomorphic,

$J_k(A) :=$ the k-jet space over A,

$J_k(f) : \mathbf{C} \to J_k(A)$ be the k-jet lift,

$X_k(f) := \overline{J_k(f)(\mathbf{C})}^{\mathrm{Zar}}$.

The following S.M.T. is due to N.-Winkelmann-Yamanoi (2004):

Theorem 13 *Let*

$A :=$ *a semi-abelian variety,*

$f : \mathbf{C} \to A$ *be algebraically nondegenerate.*

(i) *Let* $Z :=$ *a reduced subvariety of* $X_k(f)$ $(k \geq 0)$.

Then $\exists$ *a compactification* $\bar{X}_k(f)$ *of* $X_k(f)$ *s.t.*

$$T(r; \omega_{\bar{Z}, J_k(f)}) \leq N_1(r; J_k(f)^* Z) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0.$$

(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geq 2$, *then*

$$T(r; \omega_{\bar{Z}, J_k(f)}) \leq \epsilon T_f(r)||_\epsilon, \ \forall \epsilon > 0.$$

(iii) *When* $k = 0$ *and* Z *is an effective reduced divisor* D *on* A, *the compactification* $\bar{A}$ *of* A *can be chosen as smooth, equivariant with respect to the* A*-action, and independent of* f*; furthermore,* (i) *takes* *the form*

$$T_f(r; L(\bar{D})) \leq N_1(r; f^*D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon,$$
$$\forall \epsilon > 0.$$

Remark-Example. Note that in the above estimate, the error term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log r) + O(\log T_f(r))$" (N.-W.-Y. (2002) Example (5.36)). Set

$E = \mathbf{C}/(\mathbf{Z} + i\mathbf{Z})$,

$D =$ an irreducible divisor on $E \times E$ with cusp of order N at $0 \in E^2$.

Let $f : z \in \mathbf{C} \to (z, z^2) \in E^2$.

Then $f(\mathbf{C})$ is Zariski dense in E^2, and

$$T_f(r, L(D)) \sim r^4(1 + o(1)).$$

Note that $f^{-1}(0) = \mathbf{Z} + i\mathbf{Z}$ and

$$f^*D \geqq N(\mathbf{Z} + i\mathbf{Z}).$$

For an arbitrary fixed k_0, we take $N > k_0$, and then have

$$N(r, f^*D) - N_{k_0}(r, f^*D) \geqq (N - k_0) r^2 (1 + o(1)).$$

The above left-hand side cannot be bounded by $S_f(r, c_1(D)) = O(\log r)$.

Remark. (i) In N.-W.-Y. (2002) we proved the above (iii) with a higher level truncated counting function $N_l(r; f^*D)$.
For abelian A, (iii) is due to Yamanoi (2004).

(ii) Theorem 13 is considered as the analogue of abc-Conjecture over semi-abelian varieties. Cf. Vojta (1999) for a result without order truncation.

§5 Applications.

As applications for Griffiths Conjecture 2 we have the following due to N.-W.-Y. (2005):

Theorem 14 *Assume that*

$X :=$ *a complex algebraic variety;*

$\exists \pi : X \to A$, *a finite morphism onto a semi-abelian variety* A*;*

$\bar{\kappa}(X) > 0$.

Then $\forall f : \mathbf{C} \to X$ *is algebraically degenerate.*

Moreover, the normalization of the Zariski closure of $f(\mathbf{C})$ *is a semi-abelian variety which is a finite étale cover of a translate of a proper semi-abelian subvariety of* A.

Corollary 15 *Let* X *be a complex algebraic variety whose quasi-Albanese map is proper. Assume that*

$$\bar{\kappa}(X) > 0, \quad \bar{q}(X) \geq \dim X.$$

Then $\forall f : \mathbf{C} \to X$ *is algebraically degenerate.*

Theorem 16 *Let*
$E_i \subset \mathbf{P}^n(\mathbf{C}), 1 \leq i \leq q$, *be smooth hypersurfaces s.t.* $E = \sum E_i$ *is s.n.c.*
Assume that

$$q \geq n+1, \quad \deg E \geq n+2.$$

Then $\forall f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus E$ *is algebraically degenerate.*

Remark. In Theorem 16 the case when

$n = 2$,
$E_i, i = 1, 2$, are lines,
E_3 is a quadric

was a conjecture of M. Green (1974).

The case when

$n = 2$,
$E_i, 1 \leq i \leq 3$, are 3 quadrics

was due to H. Grauert (1989) and Dethloff-Schumacher-P.M. Wong (1995). They proved the hyperbolcity of the complement $\mathbf{P}^2(\mathbf{C}) \setminus E$ for E_i in generic position.

§6 Open problem.

By the results discussed in §§4,5 it is interesting to ask

<u>*Conjecture.*</u> Let $A :=$ a semi-abelian variety,
$D :=$ an effective reduced divsior on A s.t.
$\{a \in A; a + D = D\}$,
$\bar{A} :=$ an equivariant compactification s.t.

$\bar{D} \not\supset$ no A-orbit in $\bar{A}$,

$f : \mathbf{C} \to \bar{A}$ algebraically nondegenerate.
Then

$$m_f(r; \bar{D}) + m_f(r; \partial A) \leq T_f(r; L(\partial A)) + O(\log r + \log T_f(r))||.$$

This implies a weak S.M.T. of Griffiths Conjecture 1:

$$T_f(r; L(\bar{D})) + T_f(r; K_{\bar{A}}) \leq N(r; f^*\bar{D}) + N(r; f^*\partial A) + \epsilon T_f(r)||_\epsilon, \ \forall \epsilon > 0.$$