Recent Progress in the Theory of Holomorphic Curves

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We will discuss some new results in the Nevanlinna theory of holomorphic curves into algebraic varieties. The central problem is the following conjecture, strengthened from the original of Griffiths (1972):

Griffiths Conjecture 1 (1972). Let $f : \mathbf{C} \to M$ be an algebraically non-degenerate holomorphic curve into a complex projective manifold M. Let D be an effective reduced divisor of simple normal crossings. Then we have

$$T_f(r; L(D)) + T_f(r; K_M) \leq N_1(r; f^*D) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0. \tag{0.1}$$

Here $N_1(r; f^*D)$ stands for the counting function truncated to level one.

Vojta formulated an analogue of this conjecture in Diophantine approximation theory with the non-truncated counting function $N(r; f^*D)$ and proposed Vojta's dictionary, which has brought interesting observations and motivations in the both theories.

Griffiths Conjecture 1 implies

Griffiths Conjecture 2 (1972). Let X be a (complex) algebraic variety of log general type. Then every holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerate.

1 Order function.

We need to define the order function of f in a more general form than those already known (cf., e.g., Stoll [21], Noguchi-Ochiai [8]).

In what follows X is a compact complex reduced space and a subspace is a closed one. Let $\mathcal{O}_X$ denote the structure sheaf of local holomorphic functions over X. Let Y be a subspace of X, not necessarily reduced, and let $\mathcal{I} \subset \mathcal{O}_X$ be the defining coherent ideal sheaf of Y. Here one may begin with taking a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ and take a subspace Y defined by $\mathcal{I}$. In any case, there are a finite open covering $X = \bigcup U_\lambda$ of X and holomorphic functions $\sigma_{\lambda 1}, \ldots, \sigma_{\lambda l_\lambda}$ on U_λ such that at every point $x \in U_\lambda$ their germs $\underline{\sigma_{\lambda 1}}_x, \ldots, \underline{\sigma_{\lambda l_\lambda}}_x$ generate the fiber $\mathcal{I}_x$ of $\mathcal{I}$ at x. Take relatively compact open covering $V_\lambda \Subset U_\lambda$, $X = \bigcup V_\lambda$. We take $\rho_\lambda \in C_0^\infty(U_\lambda)$ with $\rho_\lambda|_{V_\lambda} \equiv 1$ and set

$$d_Y(x) = d_{\mathcal{I}}(x) = \sum_\lambda \rho_\lambda(x) \left(\sum_{j=1}^{l_\lambda} |\sigma_{\lambda j(x)}|^2 \right)^{1/2}, \qquad x \in N \tag{1.1}$$

(cf. [11] Chap. 2 §3, [25] §2, [16]). Another finite open covering and another local generators of $\mathcal{I}_Y$ yield a function d'_Y by the same construction as above. Then there is a constant $C > 0$ such that

$$|\log d_Y(x) - \log d'_Y(x)| \leq C, \qquad x \in X. \tag{1.2}$$

The function $d_Y(x)$ stands for "a sort of the distance" between x and the subspace Y. We call

$$\phi_Y(x) = \phi_{\mathcal{I}}(x) = -\log d_Y(x), \quad x \in X$$

the *Weil function* or the *proximity (approximation) potential* of Y.

For a holomorphic curve $f : \mathbf{C} \to X$ with $f(\mathbf{C}) \not\subset \operatorname{Supp} Y$ we define

$$\begin{aligned} \omega_{Y,f} = \omega_{\mathcal{I},f} &= -dd^c \phi_Y(z) = -\frac{i}{2\pi}\partial\bar{\partial}\phi_Y(z) \\ &= dd^c \log \frac{1}{d_Y \circ f(z)}, \end{aligned} \tag{1.3}$$

which is a smooth (1,1)-form on $\mathbf{C}$. The order function of f for Y or $\mathcal{I}$ is defined by

$$(1.4)\qquad T(r;\omega_{Y,f})=T(r;\omega_{\mathcal{I},f})=\int_1^r\frac{dt}{t}\int_{|z|<t}\omega_{Y,f}.$$

When $\mathcal{I}$ defines a Cartier divisor D on M, we see that

$$T(r;\omega_{\mathcal{I},f})=T_f(r;L(D))+O(1),$$

where $T_f(r;L(D))$ is the order function defined by the Chern class of L (cf. [8]).

Similarly taking a hermitian metric form ω on X_{red}, we define an order function of f with respect to ω by

$$T_f(r)=T(r;f^*\omega)=\int_1^r\frac{dt}{t}\int_{|z|<t}f^*\omega.$$

Then in general we have

$$T(r;\omega_{\mathcal{I},f})=O(T_f(r)).$$

The *proximity function* (or approximation function) of f for Y is defined by

$$(1.5)\qquad m_f(r,Y)=m_f(r,\mathcal{I})=\int_{|z|=r}\phi_Y\circ f(z)\frac{d\theta}{2\pi}.$$

It follows from (1.1) that the integral is finite, and from (1.2) that $m_f(r,Y)$ is well-defined up to $O(1)$-term.

Let Y, $X=\bigcup U_\lambda$ and $\sigma_{\lambda 1},\dots,\sigma_{\lambda l_\lambda}$ be as above. Suppose that $f(\zeta)\in U_\lambda$. Then $\sigma_{\lambda j}\circ f(z)$ are local holomorphic functions in a neighborhood of ζ vanishing at ζ with multiplicity $\mathrm{mult}_\zeta\,\sigma_{\lambda j}\circ f$. We define the intersection multiplicity of f with Y by

$$\mathrm{mult}_\zeta f^*Y=\min\{\mathrm{mult}_\zeta\sigma_{\lambda j}\circ f;1\le j\le l_\lambda\},$$

which is independent of the choice of local generators $\sigma_{\lambda j}$. The counting function with truncation level $k\le\infty$ is defined by

$$N_k(r;f^*Y)=N_k(r;f^*\mathcal{I})=\int_1^r\frac{dt}{t}\sum_{|\zeta|<t}\min\{\mathrm{mult}_\zeta f^*Y,k\}.$$

We set $N(r;f^*Y)=N(r;f^*\mathcal{I})=N_\infty(r;f^*Y)$.

Theorem 1.6 ([25], [17]) *Let $f:\mathbf{C}\to X$ and $\mathcal{I}$ be as above. Then we have the following:*

(i) (First Main Theorem) $T(r;\omega_{\mathcal{I},f})=N(r;f^*\mathcal{I})+m_f(r;\mathcal{I})-m_f(1;\mathcal{I})$.

(ii) *Let $\mathcal{I}_i$ $(i=1,2)$ be coherent ideal sheaves of $\mathcal{O}_X$ and let Y_i be the subspace defined by $\mathcal{I}_i$. If $\mathcal{I}_1\subset\mathcal{I}_2$ or equivalently $Y_1\supset Y_2$, then*

$$m_f(r;Y_2)\le m_f(r;Y_1)+O(1).$$

(iii) *Let $\phi:X_1\to X_2$ be a holomorphic mappings between compact complex manifolds. Let $\mathcal{I}_2\subset\mathcal{O}_{X_2}$ be a coherent ideal sheaf and let $\mathcal{I}_1\subset\mathcal{O}_{X_1}$ be the coherent ideal sheaf generated by $\phi^*\mathcal{I}_2$. Then*

$$m_f(r;\mathcal{I}_1)=m_{\phi\circ f}(r;\mathcal{I}_2)+O(1).$$

(iv) *Let $\mathcal{I}_i$, $i=1,2$, be two coherent ideal sheaves of $\mathcal{O}_X$. Suppose that $f(\mathbf{C})\not\subset$ Supp $(\mathcal{O}_X/\mathcal{I}_1\otimes\mathcal{I}_2)$. Then we have*

$$T(r;\omega_{\mathcal{I}_1\otimes\mathcal{I}_2,f})=T(r;\omega_{\mathcal{I}_1,f})+T(r;\omega_{\mathcal{I}_2,f})+O(1).$$

(v) *A holomorphic curve $f:\mathbf{C}\to X$ is a rational curve if and only if $T_f(r)=O(\log r)$, provided that X is algebraic.*

Here we recall the classical result for a holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ into the complex projective space of dimension n. We set $T_f(r) = T(r; \Omega)$ with Fubini-Study metric form Ω.

Theorem 1.7 (Nevanlinna-Cartan) *Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate holomorphic curve, i.e., $f(\mathbf{C})$ is not contained in a hyperplane. Let $\{H_j\}_{j=1}^q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. Then*

$$(1.8) \qquad (q-n-1)T_f(r) \leq \sum_{j=1}^{q} N_n(r, f^* H_j) + O(\log r) + O(\log T_f(r))||,$$

where the symbol "$||$" stands for the estimate to hold for $r > 0$ outside a Borel subset of finite total Lebesgue measure.

2 Min Ru's result.

In the Diophantine approximation theory, P. Corvaja and U. Zannier [1] generalized Schmidt's Subspace Theorem to the case of hypersurfaces in the projective space $\mathbf{P}^n$, and then J.-H. Evertse and R.G. Feretti [3], [4] generalized it to the case of subspace $M \subset \mathbf{P}^n$.

Min Ru [18], [19] found their analogue to be valid in the theory of holomorphic curves and proved the following:

Theorem 2.1 *Let $M \subset \mathbf{P}^N(\mathbf{C})$ be a smooth subvariety of dimension n. Let $D_i, 1 \leq i \leq q$ be hypersurfaces of degree d_i in $\mathbf{P}^N(\mathbf{C})$ which are in general position in M; i.e.,*

$$M \cap D_{i_1} \cap \cdots \cap D_{i_{n+1}} = \emptyset$$

for all $1 \leq i_1 < \cdots < i_{n+1} \leq q$. Let $f : \mathbf{C} \to M$ be an algebraically non-degenerate holomorphic curve. Then

$$(q-n-1-\epsilon)T_f(r; O(1)) \leq \sum_{i=1}^{q} \frac{1}{d_i} N(r; f^* D_i)||_\epsilon, \quad \forall \epsilon > 0.$$

In the proof the following approximation theorem due to H. Cartan is one key:

Theorem 2.2 *Let $L_j, j \in Q = \{1, \ldots, q\}$ be linear forms on $\mathbf{P}^n(\mathbf{C})$ in general position. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate holomorphic curve. Then*

$$\int_{|\zeta|=r} \max_K \sum_{j \in K} \log \frac{\|f(\zeta)\| \|L_j\|}{|L_j(f(\zeta))|} \frac{d\theta}{2\pi} \leq (n+1+\epsilon) T_f(r; O(1))||_\epsilon,$$

where $K \subset Q$ runs with $|K| = n+1$.

This is showing the limit how much $f(\zeta)$ can approximate the divisor $\prod_{j \in Q} L_j = 0$ on $\mathbf{P}^n(\mathbf{C})$. They apply a very elaborate combinatorial argument for Veronese embeddings of degree m as $m \to \infty$ ($\epsilon \to 0$).

3 Dethloff-Lu's result.

Theorem 3.1 (Log Bloch-Ochiai (N. '77–'81, N.-Winkelmann [12])) *Let X be a Zariski open subset of a compact Kähler manifold $\bar{X}$ such that the log irregularity $\bar{q}(X) > \dim_{\mathbf{C}} X$. Then no holomorphic curve $f : \mathbf{C} \to X$ has a Zariski dense image in $\bar{X}$.*

Problem. What happens in the case of $\bar{q}(X) = \dim_{\mathbf{C}} X$?

A holomorphic curve $f : \mathbf{C} \to M$ into a compact hermitian manifold M is called a Brody curve if the norm $\|f'(z)\|$ of the differential of f is bounded on $\mathbf{C}$.

As for Griffiths Conjecture 2 G. Dethloff and S. Lu [2] dealt with Brody curves into algebraic surfaces.

Theorem 3.2 *Let X be a smooth algebraic surface of log general type with log irregularity $\bar{q}(X) = 2$, and let $\bar{X}$ be a smooth compactification with s.n.c. $\partial X = \bar{X} \setminus X$. Then every Brody curve $f : \mathbf{C} \to X \subset \bar{X}$ is algebraically degenerate.*

Proposition 3.3 *Let X be an algebraic surface with $\bar{\kappa}(X) = 1$ and $\bar{q}(X) = 2$. Assume that the quasi-Albanese map $\alpha_X : X \to A_X$ is proper (a bit more general assumption works). Then every holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerate.*

By Kawamata's theorem this is easily reduced to the case of $\dim X = \bar{q}(X) = \bar{\kappa}(X) = 1$, and then little Picard's theorem is applied.

They gave an interesting example.

Remark 3.4 *There is an algebraic surface X with $\bar{\kappa}(X) = 1$ and $\bar{q}(X) = 2$ which admits an algebraically non-degenerate $f : \mathbf{C} \to X$.*

On the other hand, J. Winkelmann gave another interesting example:

Remark 3.5 *There is a compact projective threefold X such that*

(i) *$\kappa(X) = 0$ and $q(X) = 3$,*

(ii) *the Kobayashi hyperbolic pseudodistance $d_X \equiv 0$,*

(iii) *there is a holomorphic curve $f : \mathbf{C} \to X$ with the dense image in the sense of the differential topology,*

(iv) *there is a proper subvariety $Z \subset X$ satisfying that for every Brody $g : \mathbf{C} \to X$, $g(\mathbf{C}) \subset Z$.*

4 Semi-abelian varieties.

Let $f : \mathbf{C} \to A$ be a holomorphic curve and let $J_k(f) : \mathbf{C} \to J_k(A)$ denote the k-jet lift of f into the k-jet space $J_k(A)$ over A. Let $X_k(f)$ denote the Zariski closure of the image of $J_k(f)$.

Theorem 4.1 (N.-Winkelmann-Yamanoi [16]) *Let A be a semi-abelian variety. Let $f : \mathbf{C} \to A$ be a holomorphic curve with Zariski dense image.*

(i) *Let Z be an algebraic reduced subvariety of $X_k(f)$ ($k \geq 0$). Then there exists a compactification $\bar{X}_k(f)$ of $X_k(f)$ such that*

$$T(r; \omega_{\bar{Z},J_k(f)}) \leq N_1(r; J_k(f)^* Z) + \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0, \tag{4.2}$$

where $\bar{Z}$ is the closure of Z in $\bar{X}_k(f)$.

(ii) *Moreover, if $\operatorname{codim}_{X_k(f)} Z \geq 2$, then*

$$T(r; \omega_{\bar{Z},J_k(f)}) \leq \epsilon T_f(r)||_\epsilon, \quad \forall \epsilon > 0. \tag{4.3}$$

(iii) *In the case when $k = 0$ and Z is an effective divisor D on A, the compactification $\bar{A}$ of A can be chosen as smooth, equivariant with respect to the A-action, and independent of f; furthermore, (4.2) takes the form*

$$T_f(r; L(\bar{D})) \leq N_1(r; f^* D) + \epsilon T_f(r; L(\bar{D}))||_\epsilon, \quad \forall \epsilon > 0. \tag{4.4}$$

Note that in the above estimate (4.2), (4.3) or (4.4) the error term "$\epsilon T_f(r)$" cannot be replaced by "$O(\log r) + O(\log T_f(r))$" (see [15] Example (5.36)).

Remark 4.5 (i) In N.-Winkelmann-Yamanoi [15] we proved (4.4) with a higher level truncated counting function $N_l(r; f^* D)$. In the case of abelian A (4.4) with truncation level one was obtained by Yamanoi [26].

(ii) Theorem 4.1 is considered as the analogue of abc-Conjecture over semi-abelian varieties. Cf. Vojta [24] for a result without order truncation.

5 Application and conjecture.

As applications for Griffiths Conjecture 2 we have the following (see [17]).

Theorem 5.1 *Let X be a complex algebraic variety and let $\pi : X \to A$ be a finite morphism onto a semi-abelian variety A. Let $f : \mathbf{C} \to X$ be an arbitrary entire holomorphic curve. If $\bar{\kappa}(X) > 0$, then f is algebraically degenerate.*

Moreover, the normalization of the Zariski closure of $f(\mathbf{C})$ is a semi-abelian variety which is a finite étale cover of a translate of a proper semi-abelian subvariety of A.

Corollary 5.2 *Let X be a complex algebraic variety whose quasi-Albanese map is a proper map. Assume that $\bar{\kappa}(X) > 0$ and $\bar{q}(X) \geq \dim X$. Then every entire holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerate.*

Theorem 5.3 *Let $E_i, 1 \leq i \leq q$, be smooth hypersurfaces of the complex projective space $\mathbf{P}^n(\mathbf{C})$ of dimension n such that $E = \sum E_i$ is a divisor of simple normal crossings. Assume that*

(i) $q \geq n+1$.

(ii) $\deg E \geq n+2$.

Then every holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C}) \setminus E$ is algebraically degenerate.

Remark 5.4 In Theorem 5.3 the case when $n = 2$, $E_i, i = 1, 2$, are lines and E_3 is a quadric was a conjecture of M. Green [5].

Let A be a semi-abelian variety and let D be an effective reduced divisor on A. Assume that the stabilizer $\{a \in Aa + D = D\}^0 = \{0\}$. Then there is an equivariant compactification $\bar{A}$ of A such that the closure $\bar{D}$ of D in $\bar{A}$ contains no A-orbit ([24], [16]). Let $\partial A = \bar{A} \setminus A$ denote the boundary divisor, which has only simple normal crossings.

Conjecture. Let $f : \mathbf{C} \to \bar{A}$ be an algebraically non-degenerate holomorphic curve. Then we have

$$m_f(r; \bar{D}) + m_f(r; \partial A) \leq T_f(r; L(\partial D)) + O(\log r) + O(\log T_f(r))||. \tag{5.5}$$

When $f(\mathbf{C}) \cap \partial A = \emptyset$, (5.5) was proved in [15].

6 Analogue in Diophantine approximation.

We first recall

Abc-Conjecture. Let $a, b, c \in \mathbf{Z}$ be co-prime numbers satisfying

$$a + b = c. \tag{6.1}$$

Then for an arbitrary $\epsilon > 0$ there is a number $C_\epsilon > 0$ such that

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \prod_{\text{prime } p|(abc)} p^{1+\epsilon}.$$

Notice that the order of abc at every prime p is counted only by "$1+\epsilon$" (truncation) when it is positive.

As in §1 we put $x = [a, b] \in \mathbf{P}^1(\mathbf{Q})$. After Vojta's notational dictionary ([22]), this is equivalent to

$$(1-\epsilon)h(x) \leq N_1(x; 0) + N_1(x; \infty) + N_1(x; 1) + C_\epsilon \tag{6.2}$$

for $x \in \mathbf{P}^1(\mathbf{Q})$ (cf. [7], [23] for $\mathbf{P}^n$). This is quite analogous to (1.8). Here we follow the notation in Vojta [22] for number theory and Noguchi-Ochiai [8] for the Nevanlinna theory in particular,

$$\begin{aligned} h(x) &= \text{the height of } x. \\ N_1(x; *) &= \text{the counting function at } * \text{ truncated to level 1} \\ &\quad \text{(see below).} \end{aligned}$$

Motivated by the results in sections 3 and 4, we formulate an analogue of abc-Conjecture over semi-abelian varieties. Let k be an algebraic number field and let $S \subset M_k$ be an arbitrarily fixed finite subset of places of k containing all infinite places. Let A be a semi-abelian variety over k, let D be a reduced divisor on A, let $\bar{A}$ be an equivariant compactification of A such that $\bar{D}(\subset \bar{A})$ contains no A-orbit, and let $\sigma_{\bar{D}}$ be a regular section of the line bundle $L(\bar{D})$ defining the divisor $\bar{D}$.

Abc-Conjecture over semi-abelian variety. For an arbitrary $\epsilon > 0$, there exits a constant $C_\epsilon > 0$ such that for all $x \in A(k) \setminus D$

$$(1-\epsilon)h_{L(\bar{D})}(x) \leq N_1(x; S, \bar{D}) + C_\epsilon. \tag{6.3}$$

Here $h_{L(\bar{D})}(x)$ denotes the height function with respect to $L(\bar{D})$ and $N_1(x, \bar{D}; S)$ denotes the S-counting function truncated to level one:

$$N_1(x; S, \bar{D}) = \frac{1}{[k : \mathbf{Q}]} \sum_{\substack{v \in M_k \setminus S \\ \mathrm{ord}_{\mathfrak{p}_v} \sigma_{\bar{D}}(x) \geq 1}} \log N_{k/\mathbf{Q}}(\mathfrak{p}_v).$$

Remark. Cf. [14] for the analogue over algebraic function fields.

It may be interesting to specialize the above conjecture in dimension one.

Abc-Conjecture for S-units. We assume that a and b are S-units in (6.1); that is, x in (6.2) is an S-unit. Then for arbitrary $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$(1-\epsilon)h(x) \leq N_1(x; S, 1) + C_\epsilon. \tag{6.4}$$

Abc-Conjecture for elliptic curves. Let C be an elliptic curve defined as a closure of an affine curve,

$$y^2 = x^3 + c_1 x + c_0, \quad c_i \in k^*.$$

In a neighborhood of $\infty \in C$, $\sigma_\infty = x/y$ gives an affine parameter with $\sigma_\infty(\infty) = 0$. Then for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that for $w \in C(k)$

$$\begin{aligned}(1-\epsilon)h(w) &\leq N_1(w; S, \infty) + C_\epsilon \\ &= \frac{1}{[k : \mathbf{Q}]} \sum_{\substack{v \in M_k \setminus S \\ \mathrm{ord}_{\mathfrak{p}_v} \sigma_\infty(w) \geq 1}} \log N_{k/\mathbf{Q}}(\mathfrak{p}_v) \quad + C_\epsilon.\end{aligned}$$

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