Recent Advances of the Theory of Holomorphic Curves and Related Topics

Junjiro Noguchi
Hanoi 2005 September

Abstarct: Beginning with the motivation of the value distribution theory, I will discuss the recent results of the theory of holomorphic curves, applications and some related reuslts.

§0. Introduction

Basic motivation.

- To study a holomorphic map $f : M \to X$ between complex (algebraic) spaces: for example, the shape of the image, the Zariski closure of $f(M)$?

- To study the intersections of $f(M)$ and analytic (algebraic) cycles Z of X.

Fundamental cases

(i) Need to assume X to be compact.

(ii) $\dim M = 1$, $M \equiv \mathbf{C}$.

(iii) $\dim M = \dim X$, $f : M \to X$, differentiably nondegenerate.

(iv) codim $Z = 1$.

(v) codim $Z \geq 2$.

Nevanlina-Cartan Theory.
Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ and D a divisor on $\mathbf{P}^n(\mathbf{C})$.

First Main Theorem (FMT). Assume that $f(\mathbf{C}) \not\subset D$. Then

$$T_f(r, L(D)) = N(r, fD) + m_f(r, D) + O(1).$$

Second Main Theorem (SMT). Assume that f is linearly nondegenerate, and that $D = \sum_{i=1}^q H_i$ is a sum of q hyperplanes in general position.

$$T_f(r, L(D)) + T_f(r, K_{\mathbf{P}^n(\mathbf{C})})$$
$$leq \sum_{i=1}^q N(r, f^* H_i) + S_f(r).$$

Here $S_f(r) = O(\log r) + O(\log T_f(r))||$ is a small term.

As typical applications we have the following.

<u>Picard's Theorem</u>. (i) If $f(\mathbf{C}) \cap \left(\sum_{i=1}^{n+2} H_i\right) = \emptyset$, then f is linearly degenerate.

(ii) If $f(\mathbf{C}) \cap \left(\sum_{i=1}^{2n+1} H_i\right) = \emptyset$, then f is constant.

<u>Unicity Theorem</u> (Fujimoto). Let $g : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve. Assume that $f^*H_i = g^*H_i$ for $i = 1, \ldots, 3n+2$. Then $f \equiv g$.

§1. FMT for $\mathbf{Z}$ of $\mathbf{codim\ Z \geq 1}$

Let X be a compact complex space.
Let Z be an effective analytic (algebraic) cycle on X. Let $\mathcal{I}_Z \subset \mathcal{O}_X$ be the coherent ideal sheaf associated to Z.

Let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf.
Let $X = \cup U_\alpha$ be a finite open covering such that there are local generators
$\sigma_{\alpha i} \in \Gamma(U_\alpha, \mathcal{I}), 1 \leq i \leq \nu_\alpha$.

Set $\rho_\alpha(x) = \sum_i |\sigma_{\alpha i}|^2 \in C^\infty(U_\alpha)$.

Take a particiton of unity $\{c_\alpha(x)\}$ associated with $\{U_\alpha\}$. Set

$$\rho(x) = \sum_\alpha c_\alpha(x)\rho_\alpha(x).$$

For a holomorphic curve $f : \mathbf{C} \to X$, define

$$\omega_{\mathcal{I},f}(z) = -\frac{i}{2\pi}\partial\bar{\partial}\log\rho(f(z)),$$
$$T(r, \omega_{\mathcal{I},f}) = \int_1^r \frac{dt}{t}\int_{|z|<t}\omega_{\mathcal{I},f}.$$

Other choices of $\rho_{\alpha i}$ and c_α yields $\tilde{T}(r, \omega_{\mathcal{I},f})$ and

$$\tilde{T}(r, \omega_{\mathcal{I},f}) - T(r, \omega_{\mathcal{I},f}) = O(1).$$

Lang's Conjecture. If there is an embedding $k \hookrightarrow \mathbf{C}$ such that the obtained complex space $X_{\mathbf{C}}$ is **Kobayashi hyperbolic**, then the cardinality

$$|X(k)| < \infty.$$

Definition (S. Kobayashi 1967). For a connected complex manifold or a space X and its points x, y we take a chain of holomorphic curves

$$f_i : \Delta = \{\zeta \in \mathbf{C}; |\zeta| < 1\} \to X, \ 1 \le i \le l,$$
$$\zeta_i \in \Delta,$$
$$f_1(0) = x, f_i(\zeta_i) = f_{i+1}(0), f_l(\zeta_l) = y.$$

Denoting the Poincaré distance of Δ by $d_\Delta(\cdot,\cdot)$, we set

$$d_X(x, y) = \inf \sum_{i=1}^{l} d_\Delta(0, \zeta_i).$$

Then $d_X(x, y)$ is a pseudo-distance.

X is <u>Kobayashi hyperbolic</u> if $d_X(x, y)$ is a real distance function.

An analogue over function fields was proposed by S. Lang and dealt with by Nog-85, Nog-92, and some others; the following finiteness theorem was a result in a special case:

Theorem 1 (Nog-85, 92). (1) *Let $\mathcal{X} \to R$ be a family of compact hyperbolic spaces over R. Here R may be an open variety such that $\mathcal{X} \to R$ a hyperboliccally embedded compactifications $\bar{\mathcal{X}} \to \bar{R}$ relative over $\bar{R}$.*

Then, if there are infintely many meromrophic cross-sections, there is a constant subfamily in $\mathcal{X} \to R$.

(2) (Specializing to constant family). *Let X be a Kobayashi hyperbolic compact complex space. Let Y be another compact complex space. Then there are only a finite number of surjective meromorphic mappings from Y onto X.*

So far Nevanlinna theory offers a most effective tool to the Kobayashi hyperbolicity problem for complex algebraic varieties, as Diophantine approximation theroy provides a powerful method to the finiteness problem or distributions of rational points.

These relations are described by the following diagram:

Fig. 1

We recall

Kobayashi Conjecture. A "generic" hypersurface $X \subset \mathbf{P}^n(\mathbf{C})$ of high degree ($\geq 2n+1$) is Kobayashi hyperbolic.

Therefore such X defined over k should satisfy

$$|X(k)| < \infty$$

according to Lang's Conjecture.

For the existence of such hypersurfaces we have

Theorem 2 (Masuda-Nog-96). *For every $n \in \mathbf{N}$ there is a number $d(n)$ such that for an arbitrary $d \geq d(n)$ there is a Kobayashi hyperbolic projective hypersurface $X \subset \mathbf{P}^n(\mathbf{C})$ of degree d.*

Examples: In $\mathbf{P}^3(\mathbf{C})$ we set

$$X_d^{(2)} = \{x_0^{4d} + \cdots + x_3^{4d} + t(x_0 \cdots x_3)^d = 0\}, \quad t \neq 0. \tag{3}$$

Then $X_d^{(2)}$ with $d \geq 7$ is Kobayashi hyperbolic.

In $\mathbf{P}^4(\mathbf{C})$, $X_d^{(3)}$ defined by

$$\begin{aligned} z_1^d + \cdots + z_5^d + t_1(z_1^2 z_2)^{d/3} + t_2(z_2^2 z_3)^{d/3} \\ + t_3(z_3^2 z_4)^{d/3} + t_4(z_4^2 z_1)^{d/3} \\ = 0, \quad t_j \in \mathbf{C}^*, \\ d = 3e \geq 192. \end{aligned} \tag{4}$$

is Kobayashi hyperbolic for generic (t_j); in fact, it is so for $(t_j) = (-1, -1, 1, 1)$.

Note that $abc\cdots$-Conjecture would imply

$$|X_d^{(i)}(k)| < \infty, \qquad t \in k^*, i = 2, 3.$$

It is also noted that $X_1^{(2)}$ is a Kummer K3 surface and there is a natural ramified covering $X_d^{(2)} \to X_1^{(2)}$.

Definition. Let X be an algebraic variety defined over k. We say that X satisfies the arithmetic finiteness property if $|X(k')| < \infty$ for all finite extensions k' of k.

Let $S \subset M_k$ be an arbitrarily fixed finite subset of places of k containing all infinite places. Let U_S denote the set of S-units.

Let $X_d^{(i)}(U_S)$ denote the subset of all points of $X_d^{(i)}(k)$ whose coordinates in (3) or in (4) are S-units.

Then by making use of

Schmidt's Subspace Theorem

we deduce the following.

Proposition 5 (Nog-97). *Let X_d be as above. Then $|X_d^{(i)}(U_S)| < \infty$.*

By Masuda-Nog-96 there exist such examples in $\mathbf{P}^n(\mathbf{C})$ of arbitrary dimension.

Notice that $abc\cdots$-Conjecture implies the arithmetic finiteness property of all such projective hypersurfaces.

Therefore it is natural and interesting to ask if there is a projective hypersurface satisfying the arithmetic finiteness property.

In fact we have

Theorem 6 (Nog-03). *There exists a hypersurface $X \subset \mathbf{P}^n_{\mathbf{Q}}$ satisfying the arithmetic finiteness property.*

We follow Shirosaki's construction of a Kobayashi hyperbolic projective hypersurface (Shirosaki-98).

Let $d, e \in \mathbf{N}$ be co-prime, and assume $d \geq 2e + 8$. Set

$$P(w_0, w_1) = w_0^d + w_1^d + w_0^e w_1^{d-e}.$$

We define inductively

$$
\begin{aligned}
&P_1(w_0, w_1) = P(w_1, w_1) \\
&P_n(w_0, \ldots, w_n) \\
&= P_{n-1}(P(w_0, w_1), \ldots, P(w_{n-1}, w_n)) \\
&\qquad\qquad n = 2, 3, \ldots
\end{aligned}
$$

We set $X_{e,d} = \{P_n = 0\} \subset \mathbf{P}^n(\mathbf{C})$.

Theorem 7 (Shirosaki-98). *If $e \geq 2$, then $X_{e,d}$ is Kobayashi hyperbolic.*

The proofs of Theorems 6 and 7 are quite analogous by virtue of Nevanlinna's <u>Second Main Theorem</u> for meromorphic functions and <u>Faltings' Theorem</u> for curves of higher genus (Mordell's Conjecture).

Key Lemma (Yi-95, Shirosaki-98, Nog-03).
(i) *Let* $\alpha, \beta \in \mathbf{C}$ *and* $\alpha \neq 0$. *Then the curve*

$$C_{\alpha\beta} = \{[w_0, w_1, w_2] \in \mathbf{P}^2;$$
$$P(w_0, w_1) = \alpha P(\beta w_1, w_2)\}$$

is hyperbolic for $e \geq 2$, *so that if* $\alpha, \beta \in k$, *then* $C_{\alpha\beta}$ *satisfies the arithmetic finiteness property.*

(ii) *Let* $f_j = [f_{j0}, f_{j1}] : \mathbf{C} \to \mathbf{P}^1$ *be two meromorphic functions satisfying*

$$P(f_{10}, f_{11}) = \exp(g) P(f_{20}, f_{21})$$

with an entire function g. *Then* $f_0 \equiv f_1$.

Then the proof of Theorem 6 is done by the induction on n (Nog-03 for the details).

(a) Analogue over algebraic function fields.

It is interesting to consider the problem over algebraic function fields. The case of algberaic function fields is situated in the middle of the Nevnalinna theory and the number theory.

$$\boxed{\text{Nevanlinna Theory}} \quad \text{———} \quad \boxed{\text{Number Theory}}$$

$$\boxed{\text{Theory/F.F.}}$$

There are a number of works on this subject for $\mathbf{P}^n$ $(n \geq 1)$ over algebraic function fields (Voloch, Mason, Brownawell-Masser, J. T.-Y. Wang, myself,... The problem for abelian varieties was first dealt with by A. Buium.

Theorem 8 (Buium-98). *Let*
$A =$ *an abelian variety;*
$D =$ *a reduced divisor on A which is Kobayashi hyperbolic;*
$C =$ *a smooth compact curve.*

Then $\exists N \in \mathbf{N}$ depending on C, A and D such that for every morphism $f : C \to A$, either

$$f(C) \subset D \quad \textit{or} \quad \mathrm{mult}_x f^* D \le N \quad (\forall x \in C).$$

Corollary 9. *Let the notation be as in Theorem 8. If $f(C) \not\subset D$, then*

$$\text{“ height}\,(f)\text{”} = \deg(f) \le N|f^{-1}(D)|.$$

This is an estimate of type of abc-Conjecture. His proof based on Kolchin's theory of differential algebra and he posed two problems:

- Find a proof by complex geometry.
- The Kobayashi hyperbolicity assumption for D is too strong, and the ampleness should suffice.

Theorem 10 (Nog-Winkelmann-04). *Let*
A = a semi-abelian variety with a smooth equivariant algebraic compactification $\bar{A}$;
$\bar{D}$ = an effective reduced ample divisor on $\bar{A}$, and $D = \bar{D} \cap A$;
C = a smooth algebraic curve with smooth compactification $C \hookrightarrow \bar{C}$.

Then $\exists N \in \mathbf{N}$ such that for every morphism $f : C \to A$ either

$$f(C) \subset D \quad \text{or} \quad \mathrm{mult}_x f^* D \leq N \quad (\forall x \in C).$$

Furthermore, the number N depends only on the numerical data involved as follows:

(i) *The genus of $\bar{C}$ and the number $\#(\bar{C} \setminus C)$ of the boundary (puncture) points of C,*

(ii) *the dimension of* A*,*

(iii) *the toric variety (or, equivalently, the associated "fan") which occurs as closure of the orbit in* $\bar{A}$ *of the maximal connected linear algebraic subgroup* $T \cong (\mathbf{C}^*)^t$ *of* A*,*

(iv) *all intersection numbers of the form* $\bar{D}^h \cdot B_{i_1} \cdots B_{i_k}$*, where the* B_{i_j} *are closures of* A*-orbits in* $\bar{A}$ *of dimension* n_j *and* $h + \sum_j n_j =$ dim A*.*

Corollary 11. *If* $f(C) \not\subset$ Supp D*, then*

$$\deg f^*D \text{ (height)} \leq N \cdot |\mathrm{Supp}\, f^*D|.$$

In particular, if we let A, $\bar{A}$, C and D vary within a flat connected family, then we can find a <u>uniform bound</u> for N. For abelian varieties this specializes to the following result:

Theorem 12 (Noguchi-Winkelmann-04). *There is a function $N : \mathbf{N} \times \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ such that the following statement holds. Let*
$C =$ a smooth compact curve of genus g,
$A =$ an abelian variety of dimension n, and
$D =$ an ample effective divisor on A with intersection number $D^n = d$.

Then for an aribitrary morphism $f : C \to A$, either

$$f(C) \subset D \text{ or}$$
$$\mathrm{mult}_x f^* D \leq N(g, n, d) \quad (\forall x \in C).$$

As an application a *finiteness theorem* was proved for morphisms from a non-compact curve into an abelian variety omitting an ample divisor.

(b) Nevanlinna Theory. Now we see what is happening in Nevanlinna theory for a holomorphic curve $f : \mathbf{C} \to A$ into a semi-abelian variety A. We lately proved the next result.

Theorem 13 (Nog-Winkelmann-Yamanoi-04). *Let D be a reduced divisor on a semi-abelian variety A.*

Then there is an equivariant compactification $\bar{A} \supset A$ of A such that for an arbitrary algebraically non-degenerate holomorphic curve $f : \mathbf{C} \to A$

$$(1-\epsilon)T_f(r; L(\bar{D})) \leq N_1(r; f^*D)||_\epsilon, \quad \forall \epsilon > 0, \tag{14}$$

where $\bar{D}$ is the closure of D in $\bar{A}$.

Here the order (height) function $T_f(r; L(\bar{D}))$ with respect to $L(\bar{D})$ is deined by

$$T_f(r; L(\bar{D})) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* c_1(L(\bar{D})) + O(1).$$

The counting function $N_k(r; f^*D)$ with truncation level $k \leq \infty$ is defined by

$$N_k(r; f^*D) = \int_1^r \frac{dt}{t} \sum_{\zeta \in \Delta(t)} \min\{\mathrm{mult}_\zeta f^*D, k\}.$$

Remark. In Noguchi-Winkelmann-Yamanoi-02 we proved (14) with a higher level truncated counting function $N_k(r; f^*D)$ for some special compactification of A. In the case of abelian A (14) with truncation level one was obtained by Yamanoi-04.

(c) Analogue in Diophantine approximation. Recall

abc-Conjecture. Let $a, b, c \in \mathbf{Z}$ be co-prime numbers satisfying

$$a + b = c. \tag{15}$$

Then for an arbitrary $\epsilon > 0$ there is a number $C_\epsilon > 0$ such that

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \prod_{\text{prime } p|(abc)} p^{1+\epsilon}.$$

Notice that the order of abc at every prime p is counted only by "$1 + \epsilon$" (truncation) when it is positive.

As in §1 we put $x = [a, b] \in \mathbf{P}^1(\mathbf{Q})$. After Vojta's notational dictionary (1987), this is equivalent to

$$(1 - \epsilon)h(x) \leq N_1(x; 0) + N_1(x; \infty) + N_1(x; 1) + C_\epsilon \tag{16}$$

for $x \in \mathbf{P}^1(\mathbf{Q})$ (cf. Nog-96, Vojta-98).

This is quite analogous to (14). Here we follow the notation in Vojta-87 for number theory and Nog-84 for the Nevanlinna theory in particular,

$$\begin{aligned} h(x) &= \text{the height of } x. \\ N_1(x; *) &= \text{the counting function at } * \\ &\quad\ \text{truncated to level 1} \\ &\quad\ \text{(see below).} \end{aligned}$$

Motivated by the results in (a) and (b), we formulate an analogue of abc-Conjecture for semi-abelian varieties. Let k be an algerbaic number field and let $S \subset M_k$ be an arbitrarily fixed finite subset of places of k containing

all infinite places. Let
$A =$ a semi-abelian variety over k;
$D =$ a reduced divisor on A;
$\bar{A} =$ an equivariant compactification of A such that $\bar{D}(\subset \bar{A})$ contains no A-orbit;
$\sigma_{\bar{D}} =$ a regular section of the line bundle $L(\bar{D})$ defining the divisor $\bar{D}$.

abc-Conjecture for semi-abelian variety.
For $\forall\epsilon > 0$, $\exists C_\epsilon >$ such that $\forall x \in A(k) \setminus D$

$$(1-\epsilon)h_{L(\bar{D})}(x) \leq N_1(x; S, \bar{D}) + C_\epsilon. \qquad (17)$$

Here $h_{L(\bar{D})}(x)$ dentoes the height function with respect to $L(\bar{D})$ and $N_1(x, \bar{D}; S)$ denotes the S-counting function truncated to level one:

$$N_1(x; S, \bar{D}) = \frac{1}{[k:\mathbf{Q}]} \sum_{\substack{v \in M_k \setminus S \\ \mathrm{ord}_{\mathfrak{p}_v} \sigma_{\bar{D}}(x) \geq 1}} \log N_{k/\mathbf{Q}}(\mathfrak{p}_v).$$

It may be interesting to specialize the above conjecture in two forms.

abc-Conjecture for S-units. We assume that a and b are S-units in (15); that is, x in (16) is an S-unit.

Then for $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ such that

$$(1-\epsilon)h(x) \leq N_1(x; S, 1) + C_\epsilon. \tag{18}$$

abc-Conjecture for elliptic curve. Let C be an elliptic curve defined as a closure of an affine curve,

$$y^2 = x^3 + c_1 x + c_0, \quad c_i \in k^*.$$

In a neighborhood of $\infty \in C$ $\sigma_\infty = \frac{x}{y}$ gives an affine parameter with $\sigma_\infty(\infty) = 0$.

For $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ such that for $w \in C(k)$

$$
\begin{aligned}
(1-\epsilon)h(w) &\leq N_1(w; S, \infty) + C_\epsilon \\
&= \frac{1}{[k:\mathbf{Q}]} \sum_{\substack{v \in M_k \setminus S \\ \mathrm{ord}_{\mathfrak{p}_v} \sigma\infty(w) \geq 1}} \log N_{k/\mathbf{Q}}(\mathfrak{p}_v) \\
&\quad + C_\epsilon .
\end{aligned}
$$