Hayama Symposium on Several Complex Variables 1998

Tensors, Jet Bundles, Holomorphic Curves, and (D, S)-Integral Point Sets*

Junjiro Noguchi

The Graduate School of Mathematical Sciences
The University of Tokyo, Meguro, Tokyo 153-8914

The above title was slightly modified from the original one. The purpose of this talk is to report and discuss some new results on the subjects of the title. Some new results are joint works with J. Winkelmann.

§1. Tensors and holomorphic mappings

We are interested in the following properties:

[Little Picard] *Let $f : \mathbf{C}^k \to M$ be a holomorphic mapping from the k-dimensional complex affine space into a compact complex manifold M. Assume that* $\operatorname{rank} df = k$, *then the image of f is not Zariski dense (algebraically degenerate).*

[Big Picard] *Let $f : \Delta^* \times \Delta^{k-1} \to M$ be a holomorphic mapping. Assume that* $\operatorname{rank} df = k$, *and the image of f is Zariski dense in M. Then, f extends meromorphically over $\Delta \times \Delta^{k-1}$.*

For instance, if $k = 1$, Schwarz' Lemma implies that

if the holomorphic tangent bundle $\mathbf{T}(M)$ carries a hermitian metric with negative curvature, then M is Kobayashi hyperbolic; $f : \mathbf{C} \to M$ is constant.

[Kobayashi, 75] *If the cotangent bundle $\mathbf{T}^*(M)$ is ample, then M is Kobayashi hyperbolic.*

Hence, we have little and big Picard's theorems for such M.

For $k = m = \dim M$, it follows that

[Kobayashi-Ochiai, 71, 75] *if the canonical bundle $\mathbf{K}_M$ is big, little and big Picard's theorems hold.*

For general $1 \leqq k \leqq m$, we have

[Carlson, 72] *If $\bigwedge^k \mathbf{T}^*(M)$ carries a hermitian metric with positive curvature in the sense of Griffiths, then little and big Picard's theorems hold.*

Let L be a line bundle over M, and let E be a vector bundle over M. Let

$$\pi : \mathbf{P}(E^*) = (E^* \setminus \{O\})/\mathbf{C}^* \to M$$

*Partially supported by Grant-in-Aid for Scientific Research (A)(1) 09304014.

be the projective bundle. Let $H \to \mathbf{P}(E^*)$ be the hyperplane bundle such that the sections of H correspond to those of E. Let $\tilde{B}(E, L)$ be the stable base locus for $lH - \pi^{-1}L, l = 1, 2, \ldots$. Set

$$B(E, L) = \pi(\tilde{B}(E, L)).$$

Then it follows that the condition

(C1) $\qquad B(E, L) \neq M$ for big L

is independent of the choice of L, and bimeromorphically invariant.

[Nog., 77] *Assume (C1) for* $E = \bigwedge^k \mathbf{T}^*(M)$. *Then little and big Picard's theorems hold.*

It is better to obtain a condition described only in terms of $\bigwedge^k \mathbf{T}^*(M)$, which is equivalent to (C1). For that purpose we need to look into a bit more detailed structure than the stable base locus.

In general, let $\lambda : F \to M$ be a line bundle, and set

$$\Phi_l : M \to \mathbf{P}(H^0(N, lF)^*).$$

Let B_l be the base locus of lF, and set

$$\begin{aligned} C_l = & B_l \bigcup \{y \in F \setminus \lambda^{-1} B_l; \\ & y \in \Phi_l^{-1}(\Phi(y)) \text{ is not an isolated point}\}. \end{aligned}$$

Then C_l is an analytic subset. We call C_l the *critical locus* of lF. We set

$$C(F) = \bigcap_{l=1}^{\infty} C_l,$$

and call it the *stable critical locus* of F.

Let $\pi : E \to M$ be a vector bundle, and $H \to \mathbf{P}(E^*)$ be the dual of the tautological line bundle.

Definition. We say that E is *fairly big* if $\pi(C(H)) \neq M$.

Lemma 1.1. *E is fairly big if and only if E satisfies (C1).*

Theorem 1.2. *If $\bigwedge^k \mathbf{T}^*(M)$ is fairly big, then little and big Picard's theorems hold.*

In the proof we use the Stein factorization, and the fact that the pull-back of an ample line bundle by a finite holomorphic mapping is again ample.

Remark. Let M be a surface of general type.

(1) [Bogomolov, 77] If $c_1^2(M) > c_2(M)$, then $\mathbf{T}^*(M)$ is big.

(2) [Lu-Yau, 90] If $c_1^2(M) > 2c_2(M)$, then $\mathbf{T}^*(M)$ is fairly big.

Lu and Yau proved (C1) for such M, and deduced little and big Picard's theorems for $f : \mathbf{C}$ or $\Delta^* \to M$. Write the ratio in order:

$$1 < \frac{c_1^2}{c_2} < 2 < \frac{c_1^2}{c_2} \leqq 3.$$

The last is Miyaoka's inequality. These are all about 1-jets.

§2. Jet differentials

In this section we deal with the case of $k = 1$; $f : \mathbf{C} \to M$. Let $\pi_k : J_k(M) \to M$ be the k-jet bundle over M. A holomorphic functional on $J_k(M)$ which is a polynomial on every fiber is called a (global) k-jet differential. Let $\mathcal{JD}_{k,d}$ denote the sheaf of k-jet differentials which are polynomials of weighted degree d on fibers. Note that

[Nog., 86] *if $\mathcal{JD}_{k,d}$ is "fairly big" in a sense, then little and big Picard's theorems hold for holomorphic curves in M.*

There is its logarithmic version. It is expected that $\mathcal{JD}_{k,d}$ carries more detailed information than holomorphic tensors which are of jet-level 1.

[Basic Idea] It is the basic idea originally due to Bloch that if there are enough many jet differentials $\phi_j, 1 \leqq j \leqq N$, so that the transcendental basis of the function field of M (here M is assumed to be algebraic) is reproduced by them, then for a non-degenerate (in a sense) holomorphic curve $f : \mathbf{C} \to M$, we have estimates of $\phi_j \circ J_k f$, where $J_k f : \mathbf{C} \to J_k(M)$ denotes the k-jet lifting, so that the order function $T_f(r)$ is bounded as

$$T_f(r) \leqq O(\log r T_f(r)) \quad ||.$$

This implies the degeneracy of f. The problem is reduced to find enough good jet differentials

$$\Phi = (\phi_1, \dots, \phi_N) : J_k(M) \to \mathbf{C}^N.$$

Definition. A jet differential $\phi : J_k(M) \to \mathbf{C}$ is said to be "*invariant*" or "*conformal*" if for a holomorphic mapping $g : z \in \Delta \to g(z) \in M$ and a change of variable, $z = z(\zeta)$,

$$\phi(J_k(g \circ z)(\zeta)) = \left(\frac{dz(\zeta)}{d\zeta}\right)^d \phi(J_k g(z)).$$

By taking a subspace of the k-jet bundle $J_k(M)$ and and its projectivization $\pi_k : \mathbf{P}_k(M) \to M$, which is called the Semple jet bundle, we have a line bundle $L_k \to \mathbf{P}_k(M)$ such that a global section of L_k is equivalent to a conformal jet differential on $J_k(M)$ ([Demailly, 97]). Let $C_k \subset \mathbf{P}_k(M)$ be the stable critical locus of L_k. Assume that

(C2) $$C_\infty = \bigcap_{k=1}^{\infty} \pi_k(C_k) \neq M.$$

Then we can apply the Basic Idea to a holomorphic curve $f : \mathbf{C} \to M$ to conclude that it has an image included in C_∞; hence it is algebraically degenerate.

The following is an application of this Basic Idea.

[Demailly-Goul, preprint 98] *Let M be a surface of general type. Assume the following:*

1. $\mathrm{Pic}(M) = \mathbf{Z}$;

2. $c_1^2 - \frac{9}{10} c_2 > 0$;

3. $H^0(S^l\mathbf{T}^*(M)) = \{O\}, \forall l$;
4. $H^0(\mathcal{E}_{k,d} \otimes (-tK_M)) = 0$ *for all* $t > 3/4$ *such that* tK_M *is an integral divisor.*

Then every holomorphic $f : \mathbf{C} \to M$ *is algebraically degenerate at the level of 2-jet.* [Demailly-Goul, preprint 98] *Let* M *be a generic hypersurface of* $\mathbf{P}^3$ *of degree* d*. Then we have*

1. $\mathrm{Pic}(M) = \mathbf{Z}$;
2. $10c_1^2 - 9c_2 = d(d^2 - 44d + 104) > 0$ *for* $d \geqq 42$;
3. $H^0(S^l\mathbf{T}^*(M) \otimes \mathcal{O}(k)) = \{O\}, \forall l > 0, k \leqq l$;
4. $H^0(\mathcal{E}_{k,d} \otimes (-tK_M)) = 0$ *for* $d \geqq 11$ *and* $t > 1/2$ *such that* tK_M *is an integral divisor.*

Then every holomorphic $f : \mathbf{C} \to M$ *is algebraically degenerate at the level of 2-jet.*

This combined with McQuillan's work [preprint, 1997] and G. Xu [X94] would imply Theorem [Demailly-Goul, preprint 98]. *A generic hypersurface of* $\mathbf{P}^3$ *of degree* $\geqq 42$ *is Kobayashi hyperbolic.*

Remarks. (1) $c_1^2 = d(d-4)^2 < c_2 = d(d^2 - 4d + 6)$.
(2) Bogomolov's result, the above 2 and 3 imply that

$$\frac{9}{10} < \frac{c_1^2}{c_2} \leqq 1.$$

Thus, the hyperbolicity problem of hypersurfaces of $\mathbf{P}^3$ may be difficult.

Recall another application of the Basic Idea, which is older than the above.

Logarithmic Bloch-Ochiai's Theorem [Nog., 77~81; cf., Dethloff-Lu, 98]. *Let* M *be a complex projective algebraic manifold, and let* D *be a hypersurface. Assume that* $q(M \setminus D) = \dim H^0(\Omega^1(M \log D)) > \dim M$*. Then every entire holomorphic curve* $f : \mathbf{C} \to M \setminus D$ *has a non Zariski dense image.*

In the case where $D = \emptyset$, it is easy to show that the same holds for a compact Kähler manifold M. Thus it is natural to ask the case of Kähler M with $D \neq \emptyset$.

Theorem [Nog.-Winkel., 99]. *Let* M *be a compact Kähler manifold and let* D *be a hypersurface of* M*. If the logarithmic irregularity* $q(M \setminus D) > dim M$*, then the image of an entire holomorphic curve* $f : \mathbf{C} \to M \setminus D$ *is contained in a proper analytic subset of* M*.*

For the proof, we first take the quasi-Albanese mapping $\alpha : M \setminus D \to \mathcal{T}$. Then $\mathcal{T}$ is a quasi-torus:

$$0 \to (\mathbf{C})^t \to \mathcal{T} \to T_0 \to 0,$$

where T_0 is the Albanese torus of M. Let B be the maximal closed subgroup which leaves the Zariski closure of $\alpha(M \setminus D)$ invariant. It is a point to show that

the quotient $\mathcal{T}/B$ is again a quasi-torus.

Then one may reduce it to the algebraic case.

In the Diophantine approximation, Vojta generalized Faltings' theorem to

Theorem [Vojta, 96]. *Let K be a number field and S be a finite set of a proper set of inequivalent places (valuations) of K with product formula such that S contains all archimedean places. Let V be an algebraic smooth variety defined over a number field K, and let D be a hypersurface of V. If $q(V \setminus D) > \dim V$, then any (D, S)-integral point set A is not Zariski dense in V.*

This is an analogue of logarithmic Bloch-Ochiai's theorem. Here the counter objects are

$$\text{a non-constant holomorphic curve } f : \mathbf{C} \to V \setminus D$$
$$\Longleftrightarrow \text{ an infinite } (D, S)\text{-integral point set of } V.$$

To explain what is a (D, S)-integral point set, we take $K = \mathbf{Q}$. Then S consists of the ordinary absolute value $|\bullet|$ and finitely many primes, $p_i, 1 \leqq i \leqq q < \infty$. Then a rational number of type

$$a = \frac{b}{p_1^{e_1} p_2^{e_2} \cdots p_q^{e_q}}, \quad b, e_i \in \mathbf{Z}$$

is called an S-integer. For the sake of simplicity, assume that D is very ample. Taking a basis $\{\sigma_j\}_{j=0}^N$ of $H^0(V, [D])$ with $(\sigma_0) = D$, we have an affine embedding

$$\Psi = \left(\frac{\sigma_1}{\sigma_0}, \ldots, \frac{\sigma_N}{\sigma_0} \right) : V \setminus D \to \mathbf{A}^N.$$

A subset A of the set $V(K)$ of all K-rational points of V is called a (D, S)-integral point set if there is such Ψ that all points of $\Psi(A)$ are S-integral points; that is, its coordinates are S-integers. Cf. S. Lang [L83, L87, L91].

Note that any finite set A is a (D, S)-integral point set, after multiplying large integers to the coordinates; in particular, one point set is always a (D, S)-integral point set. Thus the definition makes sense only for infinite A.

Let M be a compact Kähler manifold of dimension m and let $\{D_i\}_{i=1}^l$ be a family of hypersurfaces of M.

Definition. We say that $\{D_i\}_{i=1}^l$ is *in general position* if for any distinct indices $1 \leqq i_1, \ldots, i_k \leqq l$, the codimension of every irreducible component of the intersection $\bigcap_{j=1}^k D_{i_j}$ is k for $k \leqq m$, and $\bigcap_{j=1}^k D_{i_j} = \emptyset$ for $k > m$.

This notion is defined for singular M as well.

Let $\operatorname{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^l$ denote the $\mathbf{Z}$-rank of the subgroup of $H^2(M, \mathbf{R})$ generated by $\{c_1(D_i)\}_{i=1}^l$. Let $\mathrm{NS}(M)$ denote the Neron-Severi group of M; i.e., $\mathrm{NS}(M) = \mathrm{Pic}(M)/\mathrm{Pic}^0(M)$. We know that

$$\operatorname{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^l \leqq \operatorname{rank}_{\mathbf{Z}} \mathrm{NS}(M).$$

Theorem [Nog.-Winkel., 98]. *Let $\{D_i\}_{i=1}^l$ be a family of hypersurfaces of M in general position. Let $W \subset M$ be a subvariety such that there is a non-constant holomorphic curve $f : \mathbf{C} \to W \setminus \bigcup_{D_i \not\supset W} D_i$ with Zariski dense image. Then we have that*

1. $\#\{W \cap D_i \neq W\} + q(W) \leqq \dim W + \mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^l$*;*
2. *Assume that all D_i are ample. Then we have*

$$(l-m)\dim W \leqq m\left(\mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^l - q(W)\right)^+.$$

Here $(\cdot)^+$ stands for the maximum of 0 and the number. We have the following corollary which provides also examples.

Corollary [Nog.-Winkel., 98]. *Let the notation be as above.*

1. *Assume that all D_i are ample and that $l > m(\mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(M)+1)$. Then $M \setminus \bigcup_{i=1}^l D_i$ is complete hyperbolic and hyperbolically imbedded into M.*
2. *Let $X \subset \mathbf{P}^m(\mathbf{C})$ be an irreducible subvariety, and let $D_i, 1 \leqq i \leqq l$, be distinct hypersurface cuts of X that are in general position as hypersurfaces of X. If $l > 2\dim X$, then $X \setminus \bigcup_{i=1}^l D_i$ is complete hyperbolic and hyperbolically imbedded into X.*
3. *Let $\{D_i\}_{i=1}^l$ be a family of ample hypersurfaces of M in general position. Let $f : \mathbf{C} \to M$ be a holomorphic curve such that for every D_i, either $f(\mathbf{C}) \subset D_i$, or $f(\mathbf{C}) \cap D_i = \emptyset$. Assume that $l > m$. Then $f(\mathbf{C})$ is contained in an algebraic subspace W of M such that*

$$\dim W \leqq \frac{m}{l-m}\mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(M).$$

In special, if $M = \mathbf{P}^m(\mathbf{C})$, then we have

$$\dim W \leqq \frac{m}{l-m}.$$

Remark. The above Corollary, (ii) for $X = \mathbf{P}^m(\mathbf{C})$ was given by Babets [B84], but his proof seems to carry some incompleteness and confusion. In the case of $\mathbf{P}^m(\mathbf{C})$ and hyperplanes D_i, the above (ii) with $X = \mathbf{P}^m(\mathbf{C})$ and (iii) for $f : \mathbf{C} \to \mathbf{P}^m(\mathbf{C}) \setminus \bigcup_{i=1}^l D_i$ were first proved by Fujimoto [F72] and Green [G72], where the linearity of W was also proved, and by their examples the dimension estimate is best possible in general.

In the Diophantine approximation we have the following analogues.

Theorem [Nog.-Winkel., 98]. *Assume that everything is defined over a number field K, and S is a finite subset of a proper set $M(K)$ of inequivalent places of K with product formula such that S contains all infinite places. Let V be a projective smooth variety of dimension m. Let $\{D_i\}_{i=1}^l$ be a family of ample hypersurfaces of V in general position. Let $W \subset V$ be a subvariety of V. Assume that there exists a Zariski dense $(\sum_{D_i \not\supset W} D_i \cap W, S)$-integral point set of $W(K)$. Then we have*

$$(l-m)\dim W \leqq m\left(\mathrm{rank}_{\mathbf{Z}}\{c_1(D_i)\}_{i=1}^l - q(W)\right)^+.$$

Corollary [Nog.-Winkel., 98]. *Let the notation be as above.*

1. *Assume that all D_i are ample and that $l > m(\mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(V)+1)$. Then any $(\sum_{i=1}^{l} D_i, S)$-integral point set of $V(K)$ is finite.*
2. *Let $X \subset \mathbf{P}_K^m$ be an irreducible subvariety, and let $D_i, 1 \leqq i \leqq l$, be distinct hypersurface cuts of X that are in general position as hypersurfaces of X. If $l > 2\dim X$, then any $(\sum_{i=1}^{l} D_i, S)$-integral point set of $X(K)$ is finite.*
3. *Let $D_i, 1 \leqq i \leqq l$, be ample divisors of V in general position. Let A be a subset of $V(K)$ such that for every D_i, either $A \subset D_i$, or A is a $(\sum_{D_i \not\supset A} D_i, S)$-integral point set. Assume that $l > m$. Then A is contained in an algebraic subvariety W of V such that*

$$\dim W \leqq \frac{m}{l-m} \mathrm{rank}_{\mathbf{Z}}\,\mathrm{NS}(V).$$

In special, if $V = \mathbf{P}_K^m$, then we have

$$\dim W \leqq \frac{m}{l-m}.$$

The dimension estimates obtained above are optimal. These generalize and improve the result of M. Ru and P.-M. Wong [RW91], where they dealt with the case of $V = \mathbf{P}_K^m$ and hyperplanes D_i. In fact, they proved that if A is a $(\sum_{i=1}^{l} D_i, S)$-integral point set, then A is contained in a finite union W of linear subspaces such that

$$\dim W \leqq (2m+1-l)^{+}.$$

Cf. $\dim W \leqq m/(l-m)$ of Corollary, 3.

References

[B84] V.A. Babets, Picard-type theorems for holomorphic mappings, Siberian Math. J. **25** (1984), 195-200.

[Bl26] A. Bloch, Sur les systèmes de fonctions uniformes satisfaisant à l'équation d'une variété algébrique dont l'irrégularité dépasse la dimension, J. Math. Pures Appl. **5** (1926), 9-66.

[D97] Demailly, J.-P.: Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, Algebraic geometry–Santa Cruz 1995, p.p. 285-360, Proc. Sympos. Pure Math., **62**, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[DL98] G.-E Dethloff and S. S.-Y. Lu, Logarithmic jet bundles and applications, preprint 1998.

[I76] S. Iitaka, Logarithmic forms of algebraic varieties, J. Fac. Sci. Univ. Tokyo, Sect. IA **23** (1976), 525-544.

[K80] Y. Kawamata, On Bloch's conjecture, Invent. Math. **57** (1980), 97-100.

[Ko70] S. Kobayashi, Hyperbolic Manifolds and Holomorphic Mappings, Marcel Dekker, New York, 1970.

[Ko98] S. Kobayashi, Hyperbolic Complex Spaces, Grundlehren der mathematischen Wissenschaften **318**, Springer-Verlag, Berlin-Heidelberg, 1998.

[L83] S. Lang, Fundamentals of Diophantine Geometry, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1983

[L87] S. Lang, Introduction to Complex Hyperbolic Spaces, Springer-Verlag, New York-Berlin-Heidelberg, 1987.

[L91] S. Lang, Number Theory III, Encycl. Math. Sci. vol. **60**, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong-Barcelona, 1991.

[MN96] K. Masuda and J. Noguchi, A construction of hyperbolic hypersurfaces of $\mathbf{P}^n(\mathbf{C})$, Math. Ann. **304** (1996), 339-362.

[N77] J. Noguchi, Holomorphic curves in algebraic varieties, Hiroshima Math. J. **7** (1977), 833-853.

[N80] J. Noguchi, Supplement to "Holomorphic curves in algebraic varieties", Hiroshima Math. J. **10** (1980), 229-231.

[N81] J. Noguchi, Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties, Nagoya Math. J. **83** (1981), 213-233.

[N91] J. Noguchi, Hyperbolic Manifolds and Diophantine Geometry, Sugaku Exposition Vol. **4** pp. 63-81, Amer. Math. Soc., Rhode Island, 1991.

[N92] J. Noguchi, Meromorphic mappings into compact hyperbolic complex spaces and geometric Diophantine problems, International J. Math. **3** (1992), 277-289.

[N95] J. Noguchi, A short analytic proof of closedness of logarithmic forms, Kodai Math. J. **18** (1995), 295-299.

[N96] J. Noguchi, On Nevanlinna's second main theorem, Geometric Complex Analysis, Proc. the Third International Research Institute, Math. Soc. Japan, Hayama, 1995, pp. 489-503, World Scientific, Singapore, 1996.

[N97] J. Noguchi, Nevanlinna-Cartan theory over function fields and a Diophantine equation, J. reine angew. Math. **487** (1997), 61-83.

[N98] J. Noguchi, On holomorphic curves in semi-Abelian varieties, Math. Z. **228** (1998), 713-721.

[NO$\frac{84}{90}$] J. Noguchi and T. Ochiai, Geometric Function Theory in Several Complex Variables, Japanese edition, Iwanami, Tokyo, 1984; English Translation, Transl. Math. Mono. **80**, Amer. Math. Soc., Providence, Rhode Island, 1990.

[NW98] J. Noguchi and J. Winkelmann, Holomorphic Curves and Integral Points off Divisors, preprint 1998.

[O77] T. Ochiai, On holomorphic curves in algebraic varieties with ample irregularity, Invent. Math. **43** (1977), 83-96.

[RW91] M. Ru and P.-M. Wong, Integral points of $\mathbf{P}^n - \{2n+1$ hyperplanes in general position$\}$, Invent. Math. **106** (1991), 195-216.

[S-Y96] Y.-T. Siu and S.-K. Yeung, A generalized Bloch's theorem and the hyperbolicity of the complement of an ample divisor in an Abelian variety, Math. Ann. **306** (1996), 743-758.

[V96] P. Vojta, Integral points on subvarieties of semiabelian varieties, I, Invent. Math. **126** (1996), 133-181.

[X94] G. Xu, Subvarieties of general hypersurfaces in projective space, J. Diff. Geom. **39** (1994), 139-172.