

A Scalar Associated with the Inverse of Some Abelian Integrals on Open Riemann Surfaces and a Ramified Riemann Domain

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March 17, 2015

Abstract

We introduce a positive scalar function $\rho(a, \Omega)$ for a domain Ω of a complex manifold X with a global holomorphic frame of the cotangent bundle by closed Abelian differentials, which is an analogue of Hartogs' radius. We prove an *estimate of Cartan–Thullen type with $\rho(a, \Omega)$* for holomorphically convex hulls of compact subsets. In one dimensional case, we apply the obtained estimate of $\rho(a, \Omega)$ to give a new proof of Behnke–Stein's Theorem for the Steiness of open Riemann surfaces. We then extend the idea to deal with the Levi problem (Hartogs' inverse problem) for ramified Riemann domains over $\mathbf{C}^n$. We obtain some geometric conditions in terms of $\rho(a, X)$ which imply the validity of the Levi problem (Hartogs' inverse problem) for a finitely sheeted Riemann domain over $\mathbf{C}^n$.

1 Introduction and main results

1.1 Introduction

In 1943 K. Oka wrote a manuscript in Japanese, solving affirmatively the Levi problem (Hartogs' inverse problem) for unramified Riemann domains over complex Euclidean space $\mathbf{C}^n$ of arbitrary dimension $n \geq 2$,¹⁾ and in 1953 he published Oka IX [25] to solve it by making use of his Coherence Theorem proved in Oka VII [23]²⁾; there, he put a special emphasis on the difficulties of the ramified case (see [25], Introduction 2 and

*Research supported in part by Grant-in-Aid for Scientific Research (B) 23340029.

Keywords: Abelian integral; open Riemann surface; Levi problem; Riemann domain; Stein manifold.

¹⁾This fact was written twice in the introductions of his two papers, [24] and [25]: The manuscript was written as a research report *dated 12 Dec. 1943*, sent to Teiji Takagi, then Professor at the Imperial University of Tokyo, and now one can find it in [28].

²⁾It is noted that Oka VII [23] is different to his original, Oka VII in [26]; therefore, there are two versions of Oka VII. The English translation of Oka VII in [27] was taken from the latter, but unfortunately in [27] all the records of the received dates of the papers were deleted.

§23). H. Grauert also emphasized the Levi problem for ramified Riemann domains in his lecture at OKA 100 Conference Kyoto/Nara 2001. Oka's Theorem was generalized for unramified Riemann domains over complex projective n -space $\mathbf{P}^n(\mathbf{C})$ by R. Fujita [10] and A. Takeuchi [31]. Later, H. Grauert [18] gave a counter-example to the problem for ramified Riemann domains over $\mathbf{P}^n(\mathbf{C})$, and J.E. Fornæss [7] gave a counter-example to it over $\mathbf{C}^n$. Therefore, it is natural to look for geometric conditions which imply the validity of the Levi problem for ramified Riemann domains.

Under a geometric condition (Cond A, 1.1) on a complex manifold X , we introduce a new scalar function $\rho(a, \Omega) (> 0)$ for a subdomain $\Omega \subset X$, which is an analogue of the boundary distance function in the unramified case (cf. Remark 2.4 (i)). We prove an *estimate of Cartan-Thullen type* ([4]) for the holomorphically convex hull $\hat{K}_\Omega$ of a compact subset $K \Subset \Omega$ with $\rho(a, \Omega)$ (see Theorem 1.7).

In one dimensional case, by making use of $\rho(a, \Omega)$ we give a *new proof of Behnke-Stein's Theorem*: Every Riemann surface is Stein. In the known methods one uses a generalization of the Cauchy kernel or some functional analytic method (cf. Behnke-Stein [2], Kusunoki [16], Forster [8], etc.). Here we will use Oka's Jôku-Ikô combined with Grauert's finiteness theorem, which is now a rather easy result by a simplification of the proof, particularly in 1-dimensional case (see §1.2.2). We see here how the scalar $\rho(a, \Omega)$ works well in this case.

Now, let $\pi : X \rightarrow \mathbf{C}^n$ be a Riemann domain, possibly ramified, such that X satisfies Cond A. Then, we prove that a domain $\Omega \Subset X$ is a *domain of holomorphy if and only if Ω is holomorphically convex* (see Theorem 1.20). Moreover, if X is exhausted by a continuous family of domains of holomorphy, then X is Stein (see Theorem 1.25).

We next consider a boundary condition (Cond B, 1.26) with $\rho(a, X)$. We assume that X satisfies Cond A and that $X \xrightarrow{\pi} \mathbf{C}^n$ satisfies Cond B and is finitely sheeted. We prove that *if X is locally Stein over $\mathbf{C}^n$, then X is Stein* (see Theorem 1.28).

We give the proofs in §2. In §3 we will discuss some examples and properties of $\rho(a, X)$.

Acknowledgment. The author is very grateful to Professor Makoto Abe for interesting discussions on the present theme.

1.2 Main results

1.2.1 Scalar $\rho(a, \Omega)$

Let X be a connected complex manifold of dimension n with holomorphic cotangent bundle $\mathbf{T}(X)^*$. We assume:

Condition 1.1 (Cond A). There exists a global frame $\omega = (\omega^1, \dots, \omega^n)$ of $\mathbf{T}(X)^*$ over X such that $d\omega^j = 0$, $1 \leq j \leq n$.

Let $\Omega \subset X$ be a subdomain. With Cond A we consider an Abelian integral (a path

integral) of ω in Ω from $a \in \Omega$:

$$(1.2) \quad \alpha : x \in \Omega \longrightarrow \zeta = (\zeta^j) = \left(\int_a^x \omega^1, \dots, \int_a^x \omega^n \right) \in \mathbf{C}^n.$$

We denote by $P\Delta = \prod_{j=1}^n \{|\zeta^j| < 1\}$ the unit polydisk of $\mathbf{C}^n$ with center at 0 and set

$$\rho P\Delta = \prod_{j=1}^n \{|\zeta^j| < \rho\}$$

for $\rho > 0$. Then, $\alpha(x) = \zeta$ has the inverse $\phi_{a,\rho_0}(\zeta) = x$ on a small polydisk $\rho_0 P\Delta$:

$$(1.3) \quad \phi_{a,\rho_0} : \rho_0 P\Delta \longrightarrow U_0 = \phi_{a,\rho_0}(\rho_0 P\Delta) \subset \Omega.$$

Then we extend analytically ϕ_{a,ρ_0} to $\phi_{a,\rho} : \rho P\Delta \rightarrow X$, $\rho \geq \rho_0$, as much as possible, and set

$$(1.4) \quad \rho(a, \Omega) = \sup\{\rho > 0 : \exists \phi_{a,\rho} : \rho P\Delta \rightarrow X, \phi_{a,\rho}(\rho P\Delta) \subset \Omega\}.$$

Then we have the inverse of the Abelian integral α on the polydisk of the maximal radius

$$(1.5) \quad \phi_a : \rho(a, \Omega) P\Delta \longrightarrow \Omega.$$

To be precise, we should write

$$(1.6) \quad \rho(a, \Omega) = \rho(a, \omega, \Omega) = \rho(a, P\Delta, \omega, \Omega),$$

but unless confusion occurs, we use $\rho(a, \Omega)$ for notational simplicity.

We immediately see that (cf. §2.1)

- (i) $\rho(a, \Omega)$ is continuous;
- (ii) $\rho(a, \Omega) \leq \inf\{|v|_\omega : v \in \mathbf{T}(X)_a, F_\Omega(v) = 1\}$, where F_Ω denotes the Kobayashi hyperbolic infinitesimal form of Ω , and $|v|_\omega = \max_j |\omega^j(v)|$, the maximum norm of v with respect to $\omega = (\omega^j)$.

For a subset $A \subset \Omega$ we write

$$\rho(A, \Omega) = \inf\{\rho(a, \Omega) : a \in A\}.$$

For a compact subset $K \Subset \Omega$ we denote by $\hat{K}_\Omega$ the holomorphically convex hull of K defined by

$$\hat{K}_\Omega = \left\{ x \in \Omega : |f(x)| \leq \max_K |f|, \forall f \in \mathcal{O}(\Omega) \right\},$$

where $\mathcal{O}(\Omega)$ is the set of all holomorphic functions on Ω . If $\hat{K}_\Omega \Subset \Omega$ for every $K \Subset \Omega$, Ω is called a holomorphically convex domain.

The following theorem of Cartan-Thullen type (cf. [4]) is our first main result.

Theorem 1.7. *Let X be a complex manifold satisfying Cond A. Let $\Omega \Subset X$ be a relatively compact subdomain, let $K \Subset \Omega$ be a compact subset, and let $f \in \mathcal{O}(\Omega)$. Assume that*

$$|f(a)| \leq \rho(a, \Omega), \quad \forall a \in K.$$

Then we have

$$(1.8) \quad |f(a)| \leq \rho(a, \Omega), \quad \forall a \in \hat{K}_\Omega.$$

In particular, we have

$$(1.9) \quad \rho(K, \Omega) = \rho(\hat{K}_\Omega, \Omega).$$

For a relatively compact subdomain $\Omega \Subset X$ we may naturally define the notion of domain of holomorphy: i.e., there is no point $b \in \partial\Omega$ such that there are a connected neighborhood U of b in X and a non-empty open subset $V \subset U \cap \Omega$ satisfying that for every $f \in \mathcal{O}(\Omega)$ there exists $g \in \mathcal{O}(U)$ with $f|_V = g|_V$.

Corollary 1.10. *Let $\Omega \Subset X$ be as above. Then, Ω is a domain of holomorphy if and only if Ω is holomorphically convex.*

1.2.2 Behnke-Stein's Theorem for open Riemann surfaces

We apply the scalar $\rho(a, \Omega)$ introduced above to give a new proof of Behnke-Stein's Theorem for the Steiness of open Riemann surfaces, which is one of the most basic facts in the theory of Riemann surfaces: Here, we do not use the Cauchy kernel generalized on a Riemann surface (cf. [2], [16]), nor a functional analytic method (cf., e.g., [8]), but use Oka's Jôku-Ikô together with Grauert's finiteness theorem, which is now a rather easy result, particularly in 1-dimensional case. This is the very difference of our new proof to the known ones.

To be precise, we recall the definition of Stein manifold:

Definition 1.11. A complex manifold M of pure dimension n is called a Stein manifold if the following Stein conditions are satisfied:

- (i) M satisfies the second countability axiom.
- (ii) For distinct points $p, q \in M$ there is an $f \in \mathcal{O}(M)$ with $f(p) \neq f(q)$.
- (iii) For every $p \in M$ there are $f_j \in \mathcal{O}(M)$, $1 \leq j \leq n$, such that $df_1(p) \wedge \cdots \wedge df_n(p) \neq 0$.
- (iv) M is holomorphically convex.

We will rely on the following H. Grauert's Finiteness Theorem in 1-dimensional case, which is now a rather easy consequence of Oka–Cartan's Fundamental Theorem, particularly in 1-dimensional case, thanks to a very simplified proof of L. Schwartz's Finiteness

Theorem based on the idea of Demailly's Lecture Notes [5], Chap. IX (cf. [20], §7.3 for the present form):

L. Schwartz' Finiteness Theorem. *Let E be a Fréchet space and let F be a Baire vector space. Let $A : E \rightarrow F$ be a continuous linear surjection, and let $B : E \rightarrow F$ be a completely continuous linear map. Then, $(A + B)(E)$ is closed and the cokernel $\text{Coker}(A + B)$ is finite dimensional.*

Here, a Baire space is a topological space such that Baire's category theorem holds. The statement above is slightly generalized than the one usually stated, in which F is also assumed to be Fréchet (cf. L. Schwartz [29], Serre [30], Bers [3], Grauert-Remmert [13], Demailly [5]).

Grauert's Theorem in dimension 1. *Let X be an open Riemann surface, and let $\Omega \Subset X$ be a relatively compact subdomain. Then,*

$$(1.12) \quad \dim H^1(\Omega, \mathcal{O}_\Omega) < \infty.$$

Here, $\mathcal{O}_\Omega$ denotes the sheaf of germs of holomorphic functions over Ω .

N.B. It is the very idea of Grauert to claim only the finite dimensionality, weaker than a posteriori statement, $H^1(\Omega, \mathcal{O}_\Omega) = 0$: It makes the proof considerably easy.

By making use of this theorem we prove an intermediate result:

Lemma 1.13. *Every relatively compact domain Ω of X is Stein.*

Let $\Omega \Subset \tilde{\Omega} \Subset X$ be subdomains of an open Riemann surface X . Since $\tilde{\Omega}$ is Stein by Lemma 1.13 and $H^2(\tilde{\Omega}, \mathbf{Z}) = 0$, we see by the Oka Principle that the line bundle of holomorphic 1-forms over $\tilde{\Omega}$ is trivial, and so we have:

Corollary 1.14. *There exists a holomorphic 1-form ω on $\tilde{\Omega}$ without zeros.*

By making use of ω above we define $\rho(a, \Omega)$ as in (1.4) with $X = \tilde{\Omega}$.

Applying Oka's Jôku-Ikô combined with $\rho(a, \Omega)$, we give the proofs of the following approximations of Runge type:

Lemma 1.15. *Let Ω' be a domain such that $\Omega \Subset \Omega' \Subset \tilde{\Omega}$. Assume that*

$$(1.16) \quad \max_{b \in \partial\Omega} \rho(b, \Omega') < \rho(K, \Omega).$$

Then, every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on K by elements of $\mathcal{O}(\Omega')$.

Theorem 1.17. *Assume that no component of $\tilde{\Omega} \setminus \bar{\Omega}$ is relatively compact in $\tilde{\Omega}$. Then, every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compact subsets of Ω by elements of $\mathcal{O}(\tilde{\Omega})$.*

Finally we give a proof of

Theorem 1.18 (Behnke-Stein [2]). *Every open Riemann surface X is Stein.*

1.2.3 Riemann domains

Let X be a complex manifold, and let $\pi : X \rightarrow \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) be a holomorphic map.

Definition 1.19. We call $\pi : X \rightarrow \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) a *Riemann domain* (over $\mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$)) if every fiber $\pi^{-1}z$ with $z \in \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) is discrete; if $d\pi$ has the maximal rank everywhere, it is called an *unramified* Riemann domain (over $\mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$)). A Riemann domain which is not unramified, is called a *ramified Riemann domain*. If the cardinality of $\pi^{-1}z$ is bounded in $z \in \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$), then we say that $\pi : X \rightarrow \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) is *finitely sheeted* or *k-sheeted* with the maximum k of the cardinalities of $\pi^{-1}z$ ($z \in \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$)).

If $\pi : X \rightarrow \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) is a Riemann domain, then the pull-back of Euclidean metric (resp. Fubini-Study metric) by π is a degenerate (pseudo-)hermitian metric on X , which leads a distance function on X ; hence, X satisfies the second countability axiom.

Note that unramified Riemann domains over $\mathbf{C}^n$ naturally satisfy Cond A.

We have:

Theorem 1.20. *Let $\pi : X \rightarrow \mathbf{C}^n$ be a Riemann domain such that X satisfies Cond A.*

- (i) *Let $\Omega \Subset X$ be a subdomain. Then, Ω is a domain of holomorphy if and only if Ω is Stein.*
- (ii) *If X is Stein, then $-\log \rho(a, X)$ is either identically $-\infty$, or continuous plurisubharmonic.*

Definition 1.21 (Locally Stein). (i) Let X be a complex manifold. We say that a subdomain $\Omega \Subset X$ is *locally Stein* if for every $a \in \bar{\Omega}$ (the topological closure) there is a neighborhood U of a in X such that $\Omega \cap U$ is Stein.

- (ii) Let $\pi : X \rightarrow \mathbf{C}^n$ be a Riemann domain. If for every point $z \in \mathbf{C}^n$ there is a neighborhood V of z such that $\pi^{-1}V$ is Stein, X is said to be *locally Stein over $\mathbf{C}^n$* (cf. [7]).

In general, the Levi problem is the one to asks if a locally Stein domain (over $\mathbf{C}^n$) is Stein.

Remark 1.22. The following statement is a direct consequence of Elencwajg [6], Théorème II combined with Andreotti-Narasimhan [1], Lemma 5:

Theorem 1.23. *Let $\pi : X \rightarrow \mathbf{C}^n$ be a Riemann domain, and let $\Omega \Subset X$ be a subdomain. If Ω is locally Stein, then Ω is a Stein manifold.*

Therefore the Levi problem for a ramified Riemann domain $X \xrightarrow{\pi} \mathbf{C}^n$ is essentially at the “infinity” of X .

Definition 1.24. Let X be a complex manifold in general. A family $\{\Omega_t\}_{0 \leq t \leq 1}$ of subdomains Ω_t of X is called a *continuous exhaustion family of subdomains of X* if the following conditions are satisfied:

- (i) $\Omega_t \Subset \Omega_s \Subset \Omega_1 = X$ for $0 \leq t < s < 1$,
- (ii) $\bigcup_{t < s} \Omega_t = \Omega_s$ for $0 < s \leq 1$,
- (iii) $\partial\Omega_t = \bigcap_{s > t} \overline{\Omega_s} \setminus \overline{\Omega_t}$ for $0 \leq t < 1$.

Theorem 1.25. *Let $\pi : X \rightarrow \mathbf{C}^n$ be a Riemann domain. Assume that there is a continuous exhaustion family $\{\Omega_t\}_{0 \leq t \leq 1}$ of subdomains of X such that for $0 \leq t < 1$,*

- (i) Ω_t satisfies Cond A,
- (ii) Ω_t is a domain of holomorphy (or equivalently, Stein).

Then, X is Stein, and for any fixed $0 \leq t < 1$ a holomorphic function $f \in \mathcal{O}(\Omega_t)$ can be approximated uniformly on compact subsets by elements of $\mathcal{O}(X)$.

Let $\pi : X \rightarrow \mathbf{C}^n$ be a Riemann domain such that X satisfies Cond A and let ∂X denote the ideal boundary of X over $\mathbf{C}^n$ (called the accessible boundary in Fritzsche-Grauert [9], Chap. II §9). To deal with the total space X we consider the following condition which is a sort of *localization principle*:

Condition 1.26 (Cond B). (i) $\lim_{a \rightarrow \partial X} \rho(a, X) = 0$,

- (ii) For every ideal boundary point $b \in \partial X$ there are neighborhoods $V \Subset W$ of $\pi(b)$ in $\mathbf{C}^n$ such that for the connected components $\tilde{V} \subset \tilde{W}$ which are elements of the defining filter of b ,

$$(1.27) \quad \rho(a, X) = \rho(a, \tilde{W}), \quad \forall a \in \tilde{V}.$$

For the Levi problem in case (ii) we prove:

Theorem 1.28. *Let $\pi : X \rightarrow \mathbf{C}^n$ be a finitely sheeted Riemann domain. Assume that Cond A and Cond B are satisfied. Then, if X is locally Stein over $\mathbf{C}^n$, X is a Stein manifold.*

Remark 1.29. Fornæss' counter-example ([7]) for the Levi problem in the ramified case is a 2-sheeted Riemann domain over $\mathbf{C}^n$.

2 Proofs

2.1 Scalar $\rho(a, \Omega)$

Let X be a complex manifold satisfying Cond A. We here deal with some elementary properties of $\rho(a, \Omega)$ defined by (1.4) for a subdomain $\Omega \Subset X$. We use the same notion as in §1.2.1. We identify $\rho P\Delta_0$ and U_0 in (1.3). For $b, c \in \rho_0 P\Delta$ we have

$$\rho(b, \Omega) \geq \rho(c, \Omega) - |b - c|,$$

where $|b - c|$ denotes the maximum norm with respect to the coordinate system $(\zeta^j) \in \rho_0 P\Delta$. Thus,

$$\rho(c, \Omega) - \rho(b, \Omega) \leq |b - c|.$$

Changing b and c , we have the converse inequality, so that

$$(2.1) \quad |\rho(b, \Omega) - \rho(c, \Omega)| \leq |b - c|, \quad b, c \in \rho_0 P\Delta \cong U_0.$$

Therefore, $\rho(a, \Omega)$ is a continuous function in $a \in \Omega$.

Let $v = \sum_{j=1}^n v^j \left(\frac{\partial}{\partial \zeta^j} \right)_a \in \mathbf{T}(\Omega)_a$ be a holomorphic tangent at $a \in \Omega$. Then,

$$|v|_\omega = \max_{1 \leq j \leq n} |v^j|.$$

With $|v|_\omega = 1$ we have by the definition of the Kobayashi hyperbolic infinitesimal metric F_Ω (cf. [15], [21])

$$F_\Omega(v) \leq \frac{1}{\rho(a, \Omega)}.$$

Therefore we have

$$(2.2) \quad \rho(a, \Omega) \leq \inf_{v: F_\Omega(v)=1} |v|_\omega.$$

Provided that $\partial\Omega \neq \emptyset$, it immediately follows that

$$(2.3) \quad \lim_{a \rightarrow \partial\Omega} \rho(a, \Omega) = 0.$$

Remark 2.4. (i) We consider an *unramified* Riemann domain $\pi : X \rightarrow \mathbf{C}^n$. Let $(z^1, \dots, z^n)$ be the natural coordinate system of $\mathbf{C}^n$ and put $\omega = (\pi^* dz^j)$. Then the boundary distance function $\delta_{P\Delta}(a, \partial X)$ to the ideal boundary ∂X with respect to the unit polydisk $P\Delta$ is defined as the supremum of such $r > 0$ that X is univalent onto $\pi(a) + rP\Delta$ in a neighborhood of a (cf., e.g., [14], [20]). Therefore, in this case we have that

$$(2.5) \quad \rho(a, X) = \delta_{P\Delta}(a, \partial X).$$

As for the difficulty to deal with the Levi problem for ramified Riemann domains, K. Oka wrote in IX [25], §23:

“ Pour le deuxième cas les rayons de *Hartogs* cessent de jouer du rôle; ceci présente une difficulté qui m'apparaît vraiment grande.”

The above “le deuxième cas” is the ramified case.

(ii) For X satisfying Cond A one can define *Hartogs' radius* $\rho_n(a, X)$ as follows. Consider $\phi_{a, (r_j)} : P\Delta(r_j) \rightarrow X$ for a polydisk $P\Delta(r_j)$ about 0 with a poly-radius $(r_1, \dots, r_n)$ ($r_j > 0$), which is an inverse of α given by (1.2). Then, one defines $\rho_n(a, X)$ as the supremum of such $r_n > 0$; for other j , it is similarly defined. Hartogs' radius $\rho_n(a, \Omega)$ is not necessarily continuous, but lower semi-continuous. In the present paper, the scalar $\rho(a, X)$ defined under Cond A plays the role of “Hartogs' radius”.

Remark 2.6. Let X be a complex manifold satisfying Cond A. We see that if $\rho(a_0, X) = \infty$ at a point $a_0 \in X$, then $\phi_{a_0} : \mathbf{C}^n \rightarrow X$ is surjective, and $\rho(a, X) \equiv \infty$ for $a \in X$. In fact, suppose that $\rho(a_0, X) = \infty$. Then, for any $a \in X$ we take a path C_a from a_0 to a and set $\zeta = \alpha(a)$. By the definition, $\phi_{a_0}(\zeta) = a$, and it follows that $\rho(a, X) = \infty$. Even if $\rho(a, \omega, X) = \infty$ (cf. (1.6)), “ $\rho(a, \omega', X) < \infty$ ” may happen for another choice of ω' (cf. §3).

2.2 Proof of Theorem 1.7

For $a \in \Omega$ we let

$$\phi_a : \rho(a, \Omega)P\Delta \longrightarrow \Omega$$

be as in (1.5). We take an arbitrary element $u \in \mathcal{O}(\Omega)$. With a fixed positive number $s < 1$ we set

$$L = \bigcup_{a \in K} \phi_a (s|f(a)|\overline{P\Delta}).$$

Then it follows from the assumption that L is a compact subset of Ω . Therefore there is an $M > 0$ such that

$$|u| < M \text{ on } L.$$

Let ∂_j be the dual vector fields of ω^j , $1 \leq j \leq n$, on X . For a multi-index $\nu = (\nu_1, \dots, \nu_n)$ with non-negative integers $\nu_j \in \mathbf{Z}^+$ we put

$$\begin{aligned} \partial^\nu &= \partial_1^{\nu_1} \cdots \partial_n^{\nu_n} \\ |\nu| &= \nu_1 + \cdots + \nu_n, \\ \nu! &= \nu_1! \cdots \nu_n!. \end{aligned}$$

By Cauchy's inequalities for $u \circ \phi_a$ on $s|f(a)|\overline{P\Delta}$ with $a \in K$ we have

$$\frac{1}{\nu!} |\partial^\nu u(a)| \cdot |sf(a)|^{|\nu|} \leq M, \quad \forall a \in K, \quad \forall \nu \in (\mathbf{Z}^+)^n.$$

Note that $(\partial^\nu u) \cdot f^{|\nu|} \in \mathcal{O}(\Omega)$. By the definition of $\hat{K}_\Omega$,

$$(2.7) \quad \frac{1}{\nu!} |\partial^\nu u(a)| \cdot |sf(a)|^{|\nu|} \leq M, \quad \forall a \in \hat{K}_\Omega, \quad \forall \nu \in (\mathbf{Z}^+)^n.$$

For $a \in \hat{K}_\Omega$ we consider the Taylor expansion of $u \circ \phi_a(\zeta)$ at a :

$$(2.8) \quad u \circ \phi_a(\zeta) = \sum_{\nu \in (\mathbf{Z}^+)^n} \frac{1}{\nu!} \partial^\nu u(a) \zeta^\nu.$$

We infer from (2.7) that (2.8) converges at least on $s|f(a)|P\Delta$. Since Ω is a domain of holomorphy, we have that $\rho(a, \Omega) \geq s|f(a)|$. Letting $s \nearrow 1$, we deduce (1.8).

By definition, $\rho(K, \Omega) \geq \rho(\hat{K}_\Omega, \Omega)$. The converse is deduced by applying the result obtained above for a constant function $f \equiv \rho(K, \Omega)$; thus (1.9) follows. $\square$

Proof of Corollary 1.10: Assume that $\Omega \Subset X$ is a domain of holomorphy. Let $K \Subset \Omega$. It follows from (1.9) that $\hat{K}_\Omega \Subset \Omega$, and hence Ω is holomorphically convex. The converse is clear. $\square$

Remark 2.9. (i) Replacing $P\Delta$ by the unit ball B with center at 0, one may define similarly $\rho(a, \Omega)$. Then Theorem 1.7 remains to hold. Note that the union of all unitary rotations of $\frac{1}{\sqrt{n}}P\Delta$ is B .

(ii) Note that $P\Delta$ may be an arbitrary polydisk with center at 0; still, Theorem 1.7 remains valid. We use the unit polydisk just for simplicity.

2.3 Proof of Behnke-Stein's Theorem

2.3.1 Proof of Lemma 1.13

(a) We take a subdomain $\tilde{\Omega}$ of X such that $\Omega \Subset \tilde{\Omega} \Subset X$. Let $c \in \partial\Omega$ be any point, and take a local coordinate neighborhood system (W_0, w) in $\tilde{\Omega}$ with holomorphic coordinate w such that $w = 0$ at c . We consider Cousin I distributions for $k = 1, 2, \dots$:

$$\begin{aligned} & \frac{1}{w^k} \quad \text{on} \quad W_0, \\ & 0 \quad \text{on} \quad W_1 = \tilde{\Omega} \setminus \{c\}. \end{aligned}$$

These induce cohomology classes

$$\left[\frac{1}{w^k} \right] \in H^1(\{W_0, W_1\}, \mathcal{O}_{\tilde{\Omega}}) \hookrightarrow H^1(\tilde{\Omega}, \mathcal{O}_{\tilde{\Omega}}), \quad k = 1, 2, \dots$$

Since $\dim H^1(\tilde{\Omega}, \mathcal{O}_{\tilde{\Omega}}) < \infty$ by (1.12) (Grauert's Theorem), there is a non-trivial linear relation over $\mathbf{C}$

$$\sum_{k=1}^{\nu} \gamma_k \left[\frac{1}{w^k} \right] = 0 \in H^1(\tilde{\Omega}, \mathcal{O}_{\tilde{\Omega}}), \quad \gamma_k \in \mathbf{C}, \quad \gamma_\nu \neq 0.$$

Hence there is a meromorphic function F on $\tilde{\Omega}$ with a pole only at c such that about c

$$(2.10) \quad F(w) = \frac{\gamma_\nu}{w^\nu} + \cdots + \frac{\gamma_1}{w} + \text{holomorphic term}.$$

Therefore the restriction $F|_\Omega$ of F to Ω is holomorphic and $\lim_{x \rightarrow c} |F(x)| = \infty$. Thus we see that Ω is holomorphically convex.

(b) We show the holomorphic separation property of Ω (Definition 1.11 (ii)). Let $a, b \in \Omega$ be any distinct points. Let F be the one obtained in (a) above. If $F(a) \neq F(b)$, then it is done. Suppose that $F(a) = F(b)$. We may assume that $F(a) = F(b) = 0$. Let (U_0, z) be a local holomorphic coordinate system about a with $z(a) = 0$. Then we have

$$(2.11) \quad F(z) = a_{k_0} z^{k_0} + \text{higher order terms}, \quad a_{k_0} \neq 0, \quad k_0 \in \mathbf{N},$$

where $\mathbf{N}$ denotes the set of natural numbers (positive integers). We define Cousin I distributions by

$$\begin{aligned} \frac{1}{z^{kk_0}} & \quad \text{on} \quad U_0, \quad k \in \mathbf{N}, \\ 0 & \quad \text{on} \quad U_1 = \Omega \setminus \{a\}, \end{aligned}$$

which lead cohomology classes

$$(2.12) \quad \left[\frac{1}{z^{kk_0}} \right] \in H^1(\{U_0, U_1\}, \mathcal{O}_\Omega) \hookrightarrow H^1(\Omega, \mathcal{O}_\Omega), \quad k = 1, 2, \dots$$

It follows from (1.12) that there is a non-trivial linear relation

$$\sum_{k=1}^{\mu} \alpha_k \left[\frac{1}{z^{kk_0}} \right] = 0, \quad \alpha_k \in \mathbf{C}, \quad \alpha_\mu \neq 0.$$

It follows that there is a meromorphic function G on Ω with a pole only at a , where G is written as

$$(2.13) \quad G(z) = \frac{\alpha_\mu}{z^{\mu k_0}} + \cdots + \frac{\alpha_1}{z^{k_0}} + \text{holomorphic term}.$$

With $g = G \cdot F^\mu$ we have $g \in \mathcal{O}(\Omega)$ and by (2.11) and (2.13) we see that

$$g(a) = \alpha_\mu a_{k_0}^\mu \neq 0, \quad g(b) = 0.$$

(c) Let $a \in \Omega$ be any point. We show the existence of an $h \in \mathcal{O}(\Omega)$ with non vanishing differential $dh(a) \neq 0$ (Definition 1.11 (iii)). Let (U_0, z) be a holomorphic local coordinate system about a with $z(a) = 0$. As in (2.12) we consider

$$(2.14) \quad \left[\frac{1}{z^{kk_0-1}} \right] \in H^1(\{U_0, U_1\}, \mathcal{O}_\Omega) \hookrightarrow H^1(\Omega, \mathcal{O}_\Omega), \quad k = 1, 2, \dots$$

In the same as above we deduce that there is a meromorphic function H on Ω with a pole only at a , where H is written as

$$(2.15) \quad H(z) = \frac{\beta_\lambda}{z^{\lambda k_0 - 1}} + \cdots + \frac{\beta_1}{z^{k_0 - 1}} + \text{holomorphic term}, \quad \beta_k \in \mathbf{C}, \beta_\lambda \neq 0, \lambda \in \mathbf{N}.$$

With $h = H \cdot F^\lambda$ we have $h \in \mathcal{O}(\Omega)$ and by (2.11) and (2.15) we get

$$dh(a) = \beta_\lambda a_{k_0}^\lambda \neq 0.$$

Thus, Ω is Stein. □

2.3.2 Proof of Lemma 1.15

We take a domain $\tilde{\Omega} \Subset X$ with $\tilde{\Omega} \supset \Omega$. By Lemma 1.13, $\tilde{\Omega}$ is Stein, and hence there is a holomorphic 1-form on $\tilde{\Omega}$ without zeros. Then we define $\rho(a, \Omega)$ as in (1.4) with $X = \tilde{\Omega}$. With this $\rho(a, \Omega)$ we have by (1.9):

Lemma 2.16. *For a compact subset $K \Subset \Omega$ we get*

$$\rho(K, \Omega) = \rho(\hat{K}_\Omega, \Omega).$$

Lemma 2.17. *Let Ω' be a domain such that $\Omega \Subset \Omega' \Subset \tilde{\Omega}$. Assume that*

$$(2.18) \quad \max_{b \in \partial\Omega} \rho(b, \Omega') < \rho(K, \Omega).$$

Then,

$$\hat{K}_{\Omega'} \cap \Omega \Subset \Omega.$$

Proof. Since $\hat{K}_{\Omega'}$ is compact in Ω' by Lemma 1.13, it suffices to show that

$$\hat{K}_{\Omega'} \cap \partial\Omega = \emptyset.$$

Suppose that there is a point $b \in \hat{K}_{\Omega'} \cap \partial\Omega$. It follows from Lemma 2.16 that

$$\rho(b, \Omega') \geq \rho(\hat{K}_{\Omega'}, \Omega') = \rho(K, \Omega') \geq \rho(K, \Omega).$$

By assumption, $\rho(b, \Omega') < \rho(K, \Omega)$; this is absurd. □

Proof of Lemma 1.15: Here we use Oka's Jôku-Ikô (transform to a higher space), which is a principal method of K. Oka to reduce a difficult problem to the one over a simpler space such as a polydisk, but of higher dimension, and to solve it (cf. K. Oka [26], e.g., [20]).

By Lemma 1.15 there are holomorphic functions $g_j \in \mathcal{O}(\Omega')$ such that a finite union P , called an analytic polyhedron, of relatively compact components of

$$\{x \in \Omega' : |g_j(x)| < 1\}$$

satisfies “ $\hat{K}_{\Omega'} \cap \Omega \Subset P \Subset \Omega$ ” and the Oka map

$$\Psi : x \in P \longrightarrow (g_1(x), \dots, g_N(x)) \in P\Delta_N$$

is a closed embedding into the N -dimensional unit polydisk $P\Delta_N$.

Let $f \in \mathcal{O}(\Omega)$. We identify P with the image $\Psi(P) \subset P\Delta_N$ and regard $f|_P$ as a holomorphic function on $\Psi(P)$. Let $\mathcal{I}$ denote the geometric ideal sheaf of the analytic subset $\Psi(P) \subset P\Delta_N$. Then we have a short exact sequence of coherent sheaves:

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{P\Delta_N} \rightarrow \mathcal{O}_{P\Delta_N}/\mathcal{I} \rightarrow 0.$$

By Oka’s Fundamental Lemma, $H^1(P\Delta_N, \mathcal{I}) = 0$ (cf., e.g., [20], §4.3), which implies the surjection

$$(2.19) \quad H^0(P\Delta_N, \mathcal{O}_{P\Delta_N}) \rightarrow H^0(P\Delta_N, \mathcal{O}_{P\Delta_N}/\mathcal{I}) \cong \mathcal{O}(P) \rightarrow 0.$$

Since $f|_P \in \mathcal{O}(P)$, there is an element $F \in \mathcal{O}(P\Delta_N)$ with $F|_P = f|_P$. We then expand F to a power series

$$F(w_1, \dots, w_N) = \sum_{\nu} c_{\nu} w^{\nu}, \quad w \in P\Delta_N,$$

where ν denote multi-indices in $\{1, \dots, N\}$. For every $\epsilon > 0$ there is a number $l \in \mathbf{N}$ such that

$$\left| F(w) - \sum_{|\nu| \leq l} c_{\nu} w^{\nu} \right| < \epsilon, \quad w \in \Psi(K).$$

Substituting $w_j = g_j$, we have that

$$\begin{aligned} g(x) &= \sum_{|\nu| \leq l} c_{\nu} g^{\nu} \in \mathcal{O}(\Omega'), \\ |f(x) - g(x)| &< \epsilon, \quad \forall x \in K. \end{aligned}$$

□

2.3.3 Proof of Theorem 1.17

We take a continuous exhaustion family $\{\Omega_t\}_{0 \leq t \leq 1}$ of subdomains of $\tilde{\Omega}$ (cf. Definition 1.24) with $\Omega_0 = \Omega$. Let $K \Subset \Omega$ be a compact subset and let $f \in \mathcal{O}(\Omega)$. We set

$$T = \{t : 0 < t \leq 1, \mathcal{O}(\Omega_t)|K \text{ is dense in } \mathcal{O}(\Omega)|K\},$$

where “dense” is taken in the sense of the maximum norm on K . Note that

- (i) $\rho(a, \Omega_t)$ is continuous in t ;
- (ii) $\rho(K, \Omega) \leq \rho(K, \Omega_s) < \rho(K, \Omega_t)$ for $s < t$;
- (iii) $\lim_{t \searrow s} \max_{b \in \partial\Omega_s} \rho(b, \Omega_t) = 0$.

It follows from Lemma 1.15 that T is non-empty, open and closed. Therefore $T \ni 1$, so that $\mathcal{O}(\tilde{\Omega})|K$ is dense in $\mathcal{O}(\Omega)|K$. □

2.3.4 Proof of Theorem 1.18

We owe the second countability axiom for Riemann surface X to T. Radó. We take an increasing sequence of relatively compact domains $\Omega_j \Subset \Omega_{j+1} \Subset X$, $j \in \mathbf{N}$, such that $X = \bigcup_{j=1}^{\infty} \Omega_j$ and no component of $\Omega_{j+1} \setminus \bar{\Omega}_j$ is relatively compact in Ω_{j+1} . Then, (Ω_j, Ω_{j+1}) forms a so-called Rung pair (Theorem 1.17). Since every Ω_j is Stein (Lemma 1.13), the Steiness of X is deduced. $\square$

2.4 Proofs for Riemann domains

2.4.1 Proof of Theorem 1.20

(i) Suppose that $\Omega(\Subset X)$ is a domain of holomorphy. It follows from the assumption and Corollary 1.10 that Ω is K -complete in the sense of Grauert and holomorphically convex. Thus, by Grauert's Theorem ([11]), Ω is Stein.

(ii) Let $Z = \{\det d\pi = 0\}$. Then, Z is a thin analytic subset of X .

We first take a Stein subdomain $\Omega \Subset X$ and show the plurisubharmonicity of $-\log \rho(a, \Omega)$. By Grauert-Remmert [12] it suffices to show that $-\log \rho(a, \Omega)$ is plurisubharmonic in $\Omega \setminus Z$. Take an arbitrary point $a \in \Omega \setminus Z$, and a complex affine line $\Lambda \subset \mathbf{C}^n$ passing through $\pi(a)$. Let $\tilde{\Lambda}$ be the irreducible component of $\pi^{-1}\Lambda \cap \Omega$ containing a . Let Δ be a small disk about $\pi(a)$ such that $\tilde{\Delta} = \pi^{-1}\Delta \cap \tilde{\Lambda} \Subset \tilde{\Lambda} \setminus Z$.

Claim. The restriction $-\log \rho(x, \Omega)|_{\tilde{\Lambda} \setminus Z}$ is subharmonic.

By a standard argument (cf. e.g., [14], Proof of Theorem 2.6.7) it suffices to prove that if a holomorphic function $g \in \mathcal{O}(\tilde{\Lambda})$ satisfies

$$-\log \rho(x, \Omega) \leq \Re g(x), \quad x \in \partial \tilde{\Delta},$$

then

$$(2.20) \quad -\log \rho(x, \Omega) \leq \Re g(x), \quad x \in \tilde{\Delta},$$

where $\Re$ denotes the real part. Now, we have that

$$\rho(x, \Omega) \geq |e^{g(x)}|, \quad x \in \partial \tilde{\Delta},$$

Since Ω is Stein, there is a holomorphic function $f \in \mathcal{O}(\Omega)$ with $f|_{\tilde{\Lambda}} = g$ (cf. the arguments for (2.19)). Then,

$$\rho(x, \Omega) \geq |e^{f(x)}|, \quad x \in \partial \tilde{\Delta},$$

Since $\widehat{\tilde{\Delta}}_{\Omega} = \tilde{\Delta}$, it follows from (1.8) that

$$\rho(x, \Omega) \geq |e^{f(x)}| = |e^{g(x)}|, \quad x \in \tilde{\Delta},$$

so that (2.20) follows.

Let $\{\Omega_\nu\}_{\nu=1}^\infty$ be a sequence of Stein domains of X such that $\Omega_\nu \Subset \Omega_{\nu+1}$ for all ν and $X = \bigcup_\nu \Omega_\nu$. Then, $-\log \rho(a, \Omega_\nu)$, $\nu = 1, 2, \dots$, are plurisubharmonic and monotone decreasingly converges to $-\log \rho(a, X)$. Therefore if $-\log \rho(a, X)$ is either identically $-\infty$, or plurisubharmonic ($\neq -\infty$).

Suppose that $-\log \rho(a, X) \not\equiv -\infty$. Then, the subset $A := \{a \in X : -\log \rho(a, X) \neq -\infty\}$ is dense in X . Take any point $a \in X$ and $U_0 (\cong P\Delta(\rho_0))$ as in (1.3). Then, there is a point $b \in A$. Since $\rho(b, X) < \infty$, we infer that $\rho(a, X) < \infty$. Therefore, $A = X$, and (2.1) remains valid for $\Omega = X$. Thus, $\rho(a, X)$ is continuous in X . $\square$

Corollary 2.21. *Let X be a Stein manifold satisfying Cond A. Then, $-\log \rho(a, X)$ is either identically $-\infty$ or continuous plurisubharmonic.*

Proof. Since X is Stein, there is a holomorphic map $\pi : X \rightarrow \mathbf{C}^n$ which forms a Riemann domain. The assertion is immediate from (ii) above. $\square$

Remark 2.22. As a consequence, one sees with the notation in Corollary 2.21 that if $\Omega \subset X$ is a domain of holomorphy, then Hartogs' radius $\rho_n(a, \Omega)$ (cf. Remark 2.4 (ii)) is plurisubharmonic. This is, however, opposite to the history: The plurisubharmonicity or the pseudoconvexity of Hartogs' radius $\rho_n(a, \Omega)$ was found first through the study of the maximal convergence domain of a power series (Hartogs' series) in several complex variables (cf. Oka VI [22], IX [25], Nishino [19], Chap. I, Fritzsche-Grauert [9], Chap. II).

Remark 2.23. We here give a proof of Theorem 1.23 under Cond A by making use of $\rho(a, \Omega)$.

Since ω is defined in a neighborhood of $\bar{\Omega}$, Cond B is satisfied at every point of the boundary $\partial\Omega$; that is, for every $b \in \partial\Omega$ there are neighborhoods $U' \Subset U \Subset X$ of b such that

$$\rho(a, \Omega) = \rho(a, U \cap \Omega), \quad a \in U'.$$

If $U \cap \Omega$ is Stein, then $-\log \rho(a, \Omega)$ is plurisubharmonic in $a \in U'$ by Theorem 1.20 (iii). Therefore there is a neighborhood V of $\partial\Omega$ in X such that $-\log \rho(a, \Omega)$ is plurisubharmonic in $a \in V \cap \Omega$. Take a real constant C such that

$$-\log \rho(a, \Omega) < C, \quad a \in \Omega \setminus V.$$

Set

$$\psi(a) = \max\{-\log \rho(a, \Omega), C\}, \quad a \in \Omega.$$

Then, ψ is a continuous plurisubharmonic exhaustion function on Ω . By Andreotti-Narasimhan's Theorem 2.26 in below, Ω is Stein. $\square$

2.4.2 Proof of Theorem 1.25

In the same way as Lemma 1.15 and its proof we have

Lemma 2.24. *Let $\pi : \tilde{\Omega} \rightarrow \mathbf{C}^n$ be a Riemann domain such that $\tilde{\Omega}$ satisfies Cond A. Let $\Omega \Subset \Omega' \Subset \tilde{\Omega}$ be domains such that*

$$(2.25) \quad \max_{b \in \partial\Omega} \rho(b, \Omega') < \rho(K, \Omega).$$

Then, every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on K by elements of $\mathcal{O}(\Omega')$.

For the proof of the theorem it suffices to show that (Ω_t, Ω_s) is a Runge pair for $0 \leq t < s < 1$. Since any fixed $\Omega_{s'}$ ($s < s' < 1$) satisfies Cond A, we have the scalar $\rho(a, \Omega_s)$. Take a compact subset $K \Subset \Omega_t$. Then, for $s > t$ sufficiently close to t we have

$$\max_{b \in \partial\Omega_t} \rho(b, \Omega_s) < \rho(K, \Omega_t).$$

It follows from Lemma 2.24 that $\mathcal{O}(\Omega_s)|_K$ is dense if $\mathcal{O}(\Omega_t)|_K$. Then, the rest of the proof is the same as in §2.3.3. $\square$

2.4.3 Proof of Theorem 1.28

Here we will use the following result:

Theorem 2.26 (Andreotti-Narasimhan [1]). *Let $\pi : X \rightarrow \mathbf{C}^n$ be a Riemann domain. If X admits a continuous plurisubharmonic exhaustion function, then X is Stein.*

For an ideal boundary point $b \in \partial X$ there are connected open subsets $\tilde{V} \subset \tilde{W}$ as in Cond B such that

$$(2.27) \quad \rho(a, X) = \rho(a, \tilde{W}).$$

By the assumption, $\tilde{W}$ can be chosen to be Stein. By Theorem 1.20 (ii), $-\log \rho(a, \tilde{W})$ is plurisubharmonic in $a \in \tilde{V}$, and hence so is $-\log \rho(a, X)$ in $\tilde{V}$. Since $\lim_{a \rightarrow \partial X} \rho(a, X) = 0$ by Cond B, there is a closed subset $F \subset X$ such that

- (i) $F \cap \{x \in X : \|\pi(x)\| \leq R\}$ is compact for every $R > 0$,
- (ii) $-\log \rho(a, X)$ is plurisubharmonic in $a \in X \setminus F$,
- (iii) $-\log \rho(a, X) \rightarrow \infty$ as $a \rightarrow \partial X$.

From this we may construct a continuous plurisubharmonic exhaustion function on X as follows:

We fix a point $a_0 \in F$, and may assume that $\pi(a_0) = 0$. Let X_ν be a connected component of $\{\|\pi\| < \nu\}$ containing a_0 . Then, $\bigcup_\nu X_\nu = X$. Put

$$\Omega_\nu = X_\nu \setminus F \Subset X.$$

Take a real constant C_1 such that

$$-\log \rho(a, X) < C_1, \quad a \in \bar{\Omega}_1.$$

Then we set

$$\psi_1(a) = \max\{-\log \rho(a, X), C_1\}, \quad a \in X.$$

Then, ψ_1 is plurisubharmonic in X_1 . We take a positive constant C_2 such that

$$-\log \rho(a, X) < C_1 + C_2(\|\pi(a)\|^2 - 1)^+, \quad a \in \bar{\Omega}_2,$$

where $(\cdot)^+ = \max\{\cdot, 0\}$. Put

$$\begin{aligned} p_2(a) &= C_1 + C_2(\|\pi(a)\|^2 - 1)^+, \\ \psi_2(a) &= \max\{-\log \rho(a, X), p_2(a)\}. \end{aligned}$$

Then $\psi_1(a) = \psi_2(a)$ in $a \in X_1$ and $\psi_2(a)$ is plurisubharmonic in X_2 . Similarly, we take $C_3 > C_2$ so that

$$-\log \rho(a, X) < p_2(a) + C_3(\|\pi(a)\|^2 - 2^2)^+, \quad a \in \bar{\Omega}_3,$$

Put

$$\begin{aligned} p_3(a) &= p_2(a) + C_3(\|\pi(a)\|^2 - 2^2)^+, \\ \psi_3(a) &= \max\{-\log \rho(a, X), p_3(a)\}. \end{aligned}$$

Then $\psi_3(a) = \psi_2(a)$ in $a \in X_2$ and $\psi_3(a)$ is plurisubharmonic in X_3 . Inductively, we may take a continuous function $\psi_\nu(a)$, $\nu = 1, 2, \dots$, such that ψ_ν is plurisubharmonic in X_ν and $\psi_{\nu+1}|_{X_\nu} = \psi_\nu|_{X_\nu}$. It is clear from the construction that

$$\psi(a) = \lim_{\nu \rightarrow \infty} \psi_\nu(a), \quad a \in X$$

is a continuous plurisubharmonic exhaustion function of X .

Finally by Andreotti-Narasimhan's Theorem 2.26 we see that X is Stein.

3 Examples and some more on $\rho(a, X)$

(a) (Grauert's example). Grauert [18] gave a counter-example to the Levi problem for ramified Riemann domains over $\mathbf{P}^n(\mathbf{C})$: There is a locally Stein domain Ω in a complex

torus M such that $\mathcal{O}(\Omega) = \mathbf{C}$. Then, M satisfies Cond A. One may assume that M is projective algebraic, so that there is a holomorphic finite map $\tilde{\pi} : M \rightarrow \mathbf{P}^n(\mathbf{C})$, which is a Riemann domain over $\mathbf{P}^n(\mathbf{C})$. Then, the restriction $\pi = \tilde{\pi}|_{\Omega} : \Omega \rightarrow \mathbf{P}^n(\mathbf{C})$ is a Riemann domain over $\mathbf{P}^n(\mathbf{C})$, which satisfies Cond A and Cond B. Therefore, Theorem 1.28 cannot be extended to a Riemann domain over $\mathbf{P}^n(\mathbf{C})$.

Remark 3.1. Let $\pi : \Omega \rightarrow \mathbf{P}^n(\mathbf{C})$ be Grauert's example as above. Let $\mathbf{C}^n$ be an affine open subset of $\mathbf{P}^n(\mathbf{C})$, and let $\pi' : \Omega' \rightarrow \mathbf{C}^n$ be the restriction of $\pi : \Omega \rightarrow \mathbf{P}^n(\mathbf{C})$ to $\mathbf{C}^n$. Then, Ω' is Stein by Theorem 1.28.

The Steinness of Ω' may be not inferred by a formal combination of the known results on pseudoconvexity, since it is an unbounded domain (cf., e.g., [18], [17]).

(b) Domains in the products of open Riemann surfaces and complex semi-tori (cf. [21], Chap. 5) serve for examples satisfying Cond A.

(c) An open Riemann surface X is *not* Kobayashi hyperbolic if and only if X is biholomorphic to $\mathbf{C}$ or $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ (For the Kobayashi hyperbolicity in general, cf. [15], [21]).

Let $X = \mathbf{C}$. If $\omega = dz$, then $\rho(a, dz, \mathbf{C}) \equiv \infty$ for every $a \in \mathbf{C}$. If $\omega = e^z dz$, then a simple calculation implies that

$$\rho(a, e^z dz, \mathbf{C}) = |e^a|.$$

Let $X = \mathbf{C}^*$. If $\omega = dz$, then $\rho(a, dz, \mathbf{C}^*) = |a|$. If $\omega = \frac{dz}{z}$, then $\rho(a, \frac{dz}{z}, \mathbf{C}^*) \equiv \infty$.

Therefore, the finiteness or the infiniteness of $\rho(a, \omega, X)$ depends on the choice of ω .

(d) For a Kobayashi hyperbolic open Riemann surface X we take a holomorphic 1-form ω without zeros, and write

$$\|\omega(a)\|_X = |\omega(v)|, \quad v \in \mathbf{T}(X)_a, \quad F_X(v) = 1.$$

Then it follows from (2.2) that $\rho(a, \omega, X) \leq \|\omega(a)\|_X$. We set

$$\begin{aligned} \rho^+(a, X) &= \sup\{\rho(a, \omega, X) : \omega \text{ hol. 1-form without zeros, } \|\omega(a)\|_X = 1\}, \\ \rho^-(a, X) &= \inf\{\rho(a, \omega, X) : \omega \text{ hol. 1-form without zeros, } \|\omega(a)\|_X = 1\}. \end{aligned}$$

Clearly, $\rho^{\pm}(a, X) (\leq 1)$ are biholomorphic invariants of X , but we do not know the behavior of them.

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