Inverse of Abelian Integrals and Ramified Riemann Domains*

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Abstract

We deal with the Levi problem (Hartogs’ inverse problem) for ramified Riemann
domains by introducing a positive scalar function $\rho(a, X)$ for a complex manifold
$X$ with a global frame of the holomorphic cotangent bundle by closed Abelian
differentials, which is an analogue of Hartogs’ radius. We obtain some geometric
conditions in terms of $\rho(a, X)$ which imply the validity of the Levi problem for
finitely sheeted ramified Riemann domains over $\mathbb{C}^n$. On the course, we give a new
proof of the Behnke–Stein Theorem.

1 Introduction and main results

1.1 Introduction

In 1943 K. Oka wrote a manuscript in Japanese, solving affirmatively the Levi problem
(Hartogs’ inverse problem) for unramified Riemann domains over complex number space
$\mathbb{C}^n$ of arbitrary dimension $n \geq 2$,¹) and in 1953 he published Oka IX [26] to solve it by
making use of his First Coherence Theorem proved in Oka VII [24]²); there, he put a
special emphasis on the difficulties of the ramified case (see [26], Introduction 2 and §23,
[25], Introduction). H. Grauert also emphasized the problem to generalize Oka’s Theo-
rem (IX) to the case of ramified Riemann domains in his lecture at OKA 100 Conference
Kyoto/Nara 2001. Oka’s Theorem (IX) was generalized for unramified Riemann domains

¹) This fact was written twice in the introductions of his two papers, [25] and [26]: The manuscript
was written as a research report dated 12 Dec. 1943, sent to Teiji Takagi, then Professor at the Imperial
University of Tokyo, and now one can find it in [29].

²) It is noted that Oka VII [24] is different to his original, Oka VII in [27]; therefore, there are two versions
of Oka VII. The English translation of Oka VII in [28] was taken from the latter, but unfortunately in
[28] all the records of the received dates of the papers were deleted.
over complex projective $n$-space $\mathbf{P}^n(\mathbf{C})$ by R. Fujita [10] and A. Takeuchi [32]. On the other hand, H. Grauert [18] gave a counter-example to the problem for ramified Riemann domains over $\mathbf{P}^n(\mathbf{C})$, and J.E. Fornæss [7] gave a counter-example to it over $\mathbf{C}^n$. Therefore, it is natural to look for geometric conditions which imply the validity of the Levi problem for ramified Riemann domains.

Under a geometric condition (Cond A, 1.1) on a complex manifold $X$, we introduce a new scalar function $\rho(a, \Omega)(> 0)$ for a subdomain $\Omega \subset X$, which is an analogue of the boundary distance function in the unramified case (cf. Remark 2.1 (i)). We prove an estimate of Cartan-Thullen type ([4]) for the holomorphically convex hull $\hat{K}_\Omega$ of a compact subset $K \subset \Omega$ with $\rho(a, \Omega)$ (see Theorem 1.3).

In the one-dimensional case, by making use of $\rho(a, \Omega)$ we give a new proof of Behnke–Stein’s Theorem: Every open Riemann surface is Stein. In the known methods one uses a generalization of the Cauchy kernel or some functional analytic method (cf. Behnke–Stein [2], Kusunoki [16], Forster [8], etc.). Here we use Oka’s Jōkū-Ikō combined with Grauert’s Finiteness Theorem, which is now a rather easy result by a simplification of the proof, particularly in the one-dimensional case (see §1.2.2): Oka’s Jōkū-Ikō (transform to a higher space) is a principal method of K. Oka to reduce a difficult problem over a certain general space to the one over a simpler space such as a polydisk, but of higher dimension, and to solve it (cf. K. Oka [27], e.g., [20]). We see here how the scalar $\rho(a, \Omega)$ works well in this case.

Now, let $\pi: X \to \mathbf{C}^n$ be a Riemann domain, possibly ramified, such that $X$ satisfies Cond A. Then, we prove that a domain $\Omega \subset X$ is a domain of holomorphy if and only if $\Omega$ is holomorphically convex (see Theorem 1.12). Moreover, if $X$ is exhausted by a continuous family of relatively compact domains of holomorphy, then $X$ is Stein (see Theorem 1.17; see §3 (a) for a counter-example which does not satisfy Cond A).

We next consider a boundary condition (Cond B, 1.18) with $\rho(a, X)$. We assume that $X$ satisfies Cond A and that $X \xrightarrow{\pi} \mathbf{C}^n$ satisfies Cond B and is finitely sheeted. We prove that if $X$ is locally Stein over $\mathbf{C}^n$, then $X$ is Stein (see Theorem 1.19; see §3 (a) for a counter-example, not satisfying the conditions).

We give the proofs in §2. In §3 we will discuss some examples and properties of $\rho(a, X)$.

Acknowledgment. The author is very grateful to Professor J.E. Fornæss for the clarification that his example ([7]) does not satisfy Cond A (§3 (a)), and to Professor Makoto Abe for interesting discussions on the present theme.

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3) Cf. Definition 1.2
1.2 Main results

1.2.1 Scalar $\rho(a, \Omega)$

Let $X$ be a connected complex manifold of dimension $n$ with holomorphic cotangent bundle $T(X)^*$. We assume:

Condition 1.1 (Cond A). There exists a global frame $\omega = (\omega^1, \ldots, \omega^n)$ of $T(X)^*$ over $X$ such that $dw^j = 0$, $1 \leq j \leq n$.

Let $\Omega \subset X$ be a subdomain. With Cond A we consider an Abelian integral (a path integral) of $\omega$ in $\Omega$ from $a \in \Omega$:

$$\alpha : x \in \Omega \rightarrow \zeta = (\zeta^j) = \left( \int_a^x \omega^1, \ldots, \int_a^x \omega^n \right) \in \mathbb{C}^n.$$  \hfill (1.1)

We denote by $P\Delta = \prod_{j=1}^n \{ |\zeta^j| < 1 \}$ the unit polydisk of $\mathbb{C}^n$ with center at 0 and and set

$$\rho P\Delta = \prod_{j=1}^n \{ |\zeta^j| < \rho \}$$

for $\rho > 0$. Then, $\alpha(x) = \zeta$ has the inverse $\phi_{a,\rho_0}(\zeta) = x$ on a small polydisk $\rho_0 P\Delta$:

$$\phi_{a,\rho_0} : \rho_0 P\Delta \rightarrow U_0 = \phi_{a,\rho_0}(\rho_0 P\Delta) \subset \Omega.$$  \hfill (1.2)

Then we extend analytically $\phi_{a,\rho_0}$ to $\phi_{a,\rho} : \rho P\Delta \rightarrow X$, $\rho \geq \rho_0$, as much as possible, and set

$$\rho(a, \Omega) = \sup \{ \rho > 0 : \exists \phi_{a,\rho} : \rho P\Delta \rightarrow X, \phi_{a,\rho}(\rho P\Delta) \subset \Omega \} \leq \infty.$$  \hfill (1.3)

Then we have the inverse of the Abelian integral $\alpha$ on the polydisk of the maximal radius

$$\phi_a : \rho(a, \Omega) P\Delta \rightarrow \Omega.$$  \hfill (1.4)

To be precise, we should write

$$\rho(a, \Omega) = \rho(a, \omega, \Omega) = \rho(a, P\Delta, \omega, \Omega),$$  \hfill (1.5)

but unless confusion occurs, we use $\rho(a, \Omega)$ for notational simplicity.

We immediately see that (cf. §2.1)

(i) $\rho(a, \Omega)$ is finitely valued and continuous, unless $\rho(a, \Omega) \equiv \infty$;

(ii) $\rho(a, \Omega) \leq \inf \{ |v|_\omega : v \in T(X)_a, F_\Omega(v) = 1 \}$, where $F_\Omega$ denotes the Kobayashi hyperbolic infinitesimal form of $\Omega$, and $|v|_\omega = \max_j |\omega^j(v)|$, the maximum norm of $v$ with respect to $\omega = (\omega^j)$.
For a subset $A \subset \Omega$ we write
\[
\rho(A, \Omega) = \inf \{ \rho(a, \Omega) : a \in A \}.
\]

For a compact subset $K \Subset \Omega$ we denote by $\hat{K}_\Omega$ the holomorphically convex hull of $K$ defined by
\[
\hat{K}_\Omega = \left\{ x \in \Omega : |f(x)| \leq \max_K |f|, \forall f \in \mathcal{O}(\Omega) \right\},
\]
where $\mathcal{O}(\Omega)$ is the set of all holomorphic functions on $\Omega$. If $\hat{K}_\Omega \Subset \Omega$ for every $K \Subset \Omega$, $\Omega$ is called a holomorphically convex domain.

**Definition 1.2.** For a relatively compact subdomain $\Omega \Subset X$ of a complex manifold $X$ we may naturally define the notion of domain of holomorphy: i.e., there is no point $b \in \partial \Omega$ such that there are a connected neighborhood $U$ of $b$ in $X$ and a non-empty open subset $V \subseteq U \cap \Omega$ satisfying that for every $f \in \mathcal{O}(\Omega)$ there exists $g \in \mathcal{O}(U)$ with $f|_V = g|_V$.

The following theorem of the Cartan–Thullen type (cf. [4]) is our first main result.

**Theorem 1.3.** Let $X$ be a complex manifold satisfying Cond A. Let $\Omega \Subset X$ be a relatively compact domain of holomorphy, let $K \Subset \Omega$ be a compact subset, and let $f \in \mathcal{O}(\Omega)$. Assume that
\[
|f(a)| \leq \rho(a, \Omega), \quad \forall a \in K.
\]
Then we have
\[
|f(a)| \leq \rho(a, \Omega), \quad \forall a \in \hat{K}_\Omega.
\]
In particular, we have
\[
\rho(K, \Omega) = \rho(\hat{K}_\Omega, \Omega).
\]

**Corollary 1.4.** Let $\Omega \Subset X$ be a domain of a complex manifold $X$, satisfying Cond A. Then, $\Omega$ is a domain of holomorphy if and only if $\Omega$ is holomorphically convex.

### 1.2.2 The Behnke–Stein Theorem for open Riemann surfaces

We apply the scalar $\rho(a, \Omega)$ introduced above to give a new proof of the Behnke–Stein Theorem for the Steinness of open Riemann surfaces, which is one of the most basic facts in the theory of Riemann surfaces: Here, we do not use the Cauchy kernel generalized on a Riemann surface (cf. [2], [16]), nor a functional analytic method (cf., e.g., [8]), but use Oka’s Jōku-Ikō together with Grauert’s Finiteness Theorem. This is the very difference of our new proof to the known ones.

To be precise, we recall the definition of a Stein manifold:

**Definition 1.5.** A complex manifold $M$ of pure dimension $n$ is called a Stein manifold if the following Stein conditions are satisfied:
M satisfies the second countability axiom.

(ii) For distinct points \( p, q \in M \) there is an \( f \in \mathcal{O}(M) \) with \( f(p) \neq f(q) \).

(iii) For every \( p \in M \) there are \( f_j \in \mathcal{O}(M) \), \( 1 \leq j \leq n \), such that \( df_1(p) \wedge \cdots \wedge df_n(p) \neq 0 \).

(iv) \( M \) is holomorphically convex.

We will rely on the following H. Grauert’s Finiteness Theorem in the one-dimensional case, which is now a rather easy consequence of the Oka–Cartan Fundamental Theorem, thanks to a very simplified proof of L. Schwartz’s Finiteness Theorem based on the idea of Demailly’s Lecture Notes [5], Chap. IX (cf. [20], §7.3 for the present form):

**L. Schwartz’s Finiteness Theorem.** Let \( E \) be a Fréchet space and let \( F \) be a Baire vector space. Let \( A : E \to F \) be a continuous linear surjection, and let \( B : E \to F \) be a completely continuous linear map. Then, \( (A + B)(E) \) is closed and the cokernel \( \text{Coker}(A + B) \) is finite dimensional.

Here, a Baire space is a topological space such that Baire’s category theorem holds. The statement above is slightly generalized than the original one, in which \( F \) is also assumed to be Fréchet (cf. L. Schwartz [30], Serre [31], Bers [3], Grauert-Remmert [13], Demailly [5]).

**Grauert’s Theorem in dimension 1.** Let \( X \) be a Riemann surface, and let \( \Omega \subset X \) be a relatively compact subdomain. Then,

\[
\dim H^1(\Omega, \mathcal{O}_\Omega) < \infty. \tag{1.8}
\]

Here, \( \mathcal{O}_\Omega \) denotes the sheaf of germs of holomorphic functions over \( \Omega \). In case \( \Omega(=X) \) itself is compact, this theorem reduces to the Cartan–Serre Theorem in dimension 1.

**N.B.** It is the very idea of Grauert to claim only the finite dimensionality, weaker than a posteriori statement, \( H^1(\Omega, \mathcal{O}_\Omega) = 0 \): It makes the proof considerably easy.

By making use of this theorem we prove an intermediate result:

**Lemma 1.6.** Every relatively compact domain \( \Omega \) of \( X \) is Stein.

Let \( \Omega \subset \hat{\Omega} \subset X \) be subdomains of an open Riemann surface \( X \). Since \( \hat{\Omega} \) is Stein by Lemma 1.6 and \( H^2(\hat{\Omega}, \mathbb{Z}) = 0 \), we see by the Oka Principle that the line bundle of holomorphic 1-forms over \( \hat{\Omega} \) is trivial, and so we have:

**Corollary 1.7.** There exists a holomorphic 1-form \( \omega \) on \( \hat{\Omega} \) without zeros.

By making use of \( \omega \) above we define \( \rho(a, \Omega) \) as in (1.3) with \( X = \hat{\Omega} \).

Applying Oka’s Jōku-Ikō combined with \( \rho(a, \Omega) \), we give the proofs of the following approximations of the Runge type:
Lemma 1.8. Let $\Omega'$ be a domain such that $\Omega \Subset \Omega' \Subset \tilde{\Omega}$, and let $K \Subset \Omega$ be a compact subset. Assume that
\[
\max_{b \in \partial \Omega} \rho(b, \Omega') < \rho(K, \Omega).
\]
Then, every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on $K$ by elements of $\mathcal{O}(\Omega')$.

Theorem 1.9. Assume that no component of $\tilde{\Omega} \setminus \bar{\Omega}$ is relatively compact in $\tilde{\Omega}$. Then, every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compact subsets of $\Omega$ by elements of $\mathcal{O}(\tilde{\Omega})$.

Finally we give another proof of

Theorem 1.10 (Behnke–Stein [2]). Every open Riemann surface $X$ is Stein.

1.2.3 Riemann domains

Let $X$ be a complex manifold, and let $\pi : X \to \mathbb{C}^n$ (resp. $\mathbb{P}^n(\mathbb{C})$) be a holomorphic map.

Definition 1.11. We call $\pi : X \to \mathbb{C}^n$ (resp. $\mathbb{P}^n(\mathbb{C})$) a Riemann domain (over $\mathbb{C}^n$ (resp. $\mathbb{P}^n(\mathbb{C})$)) if every fiber $\pi^{-1}z$ with $z \in \mathbb{C}^n$ (resp. $\mathbb{P}^n(\mathbb{C})$) is discrete; if $d\pi$ has the maximal rank everywhere, it is called an unramified Riemann domain (over $\mathbb{C}^n$ (resp. $\mathbb{P}^n(\mathbb{C})$)). A Riemann domain which is not unramified, is called a ramified Riemann domain. If the cardinality of $\pi^{-1}z$ is bounded in $z \in \mathbb{C}^n$ (resp. $\mathbb{P}^n(\mathbb{C})$), then we say that $\pi : X \to \mathbb{C}^n$ (resp. $\mathbb{P}^n(\mathbb{C})$) is finitely sheeted or $k$-sheeted with the maximum $k$ of the cardinalities of $\pi^{-1}z$ ($z \in \mathbb{C}^n$ (resp. $\mathbb{P}^n(\mathbb{C})$)).

If $\pi : X \to \mathbb{C}^n$ (resp. $\mathbb{P}^n(\mathbb{C})$) is a Riemann domain, then the pull-back of the Euclidean metric (resp. the Fubini–Study metric) by $\pi$ is a degenerate (pseudo-)hermitian metric on $X$, which leads a distance function on $X$; hence, $X$ satisfies the second countability axiom.

Note that unramified Riemann domains over $\mathbb{C}^n$ naturally satisfy Cond A.

We have:

Theorem 1.12. Let $\pi : X \to \mathbb{C}^n$ be a Riemann domain possibly ramified such that $X$ satisfies Cond A.

(i) Let $\Omega \Subset X$ be a subdomain. Then, $\Omega$ is a domain of holomorphy if and only if $\Omega$ is Stein.

(ii) If $X$ is Stein, then $-\log \rho(a, X)$ is either identically $-\infty$, or continuous plurisubharmonic.

Definition 1.13 (Locally Stein). (i) Let $X$ be a complex manifold. We say that a subdomain $\Omega \Subset X$ is locally Stein if for every $a \in \bar{\Omega}$ (the topological closure) there is a neighborhood $U$ of $a$ in $X$ such that $\Omega \cap U$ is Stein.
Let $\pi : X \to \mathbb{C}^n$ be a Riemann domain, possibly ramified. If for every point $z \in \mathbb{C}^n$ there is a neighborhood $V$ of $z$ such that $\pi^{-1}V$ is Stein or empty, $X$ is said to be \textit{locally Stein over} $\mathbb{C}^n$ (cf. [7]).

In general, the Levi problem is the one to asks if a locally Stein domain (over $\mathbb{C}^n$) is Stein.

\textit{Remark} 1.14. The following statement is a direct consequence of Elencwajg [6], Théorème II combined with Andreotti–Narasimhan [1], Lemma 5:

\textbf{Theorem 1.15.} Let $\pi : X \to \mathbb{C}^n$ be a ramified Riemann domain, and let $\Omega \Subset X$ be a subdomain. If $\Omega$ is locally Stein, then $\Omega$ is a Stein manifold.

Therefore the Levi problem for a ramified Riemann domain $X \xrightarrow{\pi} \mathbb{C}^n$ is essentially at the “infinity” of $X$.

\textit{Definition} 1.16. Let $X$ be a complex manifold in general. A family $\{\Omega_t\}_{0 \leq t \leq 1}$ of subdomains $\Omega_t$ of $X$ is called a \textit{continuous exhaustion family of subdomains of} $X$ if the following conditions are satisfied:

1. $\Omega_t \Subset \Omega_s \Subset \Omega_1 = X$ for $0 \leq t < s < 1$,
2. $\bigcup_{t<s} \Omega_t = \Omega_s$ for $0 < s \leq 1$,
3. $\partial \Omega_t = \bigcap_{s>t} \overline{\Omega_s \setminus \Omega_t}$ for $0 \leq t < 1$.

\textbf{Theorem 1.17.} Let $\pi : X \to \mathbb{C}^n$ be a Riemann domain, possibly ramified. Assume that there is a continuous exhaustion family $\{\Omega_t\}_{0 \leq t \leq 1}$ of subdomains of $X$ such that for $0 \leq t < 1$,

1. $\Omega_t$ satisfies Cond A,
2. $\Omega_t$ is a domain of holomorphy (or equivalently, Stein).

Then, $X$ is Stein, and for any fixed $0 \leq t < 1$ a holomorphic function $f \in \mathcal{O}(\Omega_t)$ can be approximated uniformly on compact subsets by elements of $\mathcal{O}(X)$.

Let $\pi : X \to \mathbb{C}^n$ be a Riemann domain such that $X$ satisfies Cond A and let $\partial X$ denote the ideal boundary of $X$ over $\mathbb{C}^n$ (called the accessible boundary in Fritzsche–Grauert [9], Chap. II §9). We set

$$\Gamma = \overline{\pi(\partial X)}$$ (the topological closure).

To deal with the total space $X$ we consider the following condition which is a sort of localization principle:
Condition 1.18 (Cond B). (i) For any sequence \( \{a_\nu\}_{\nu=1}^\infty \) of points of \( X \) such that it has no accumulation point in \( X \) and \( \{\pi(a_\nu)\}_{\nu=1}^\infty \) is convergent, \( \lim_{\nu \to \infty} \rho(a_\nu, X) = 0 \).

(ii) For every point \( z \in \Gamma \) there are arbitrarily small neighborhoods \( V \subset W \) of \( z \) in \( \mathbb{C}^n \) such that
\[
\rho(a, X) = \rho(a, \tilde{W}), \quad \forall a \in \tilde{V},
\]
where \( \tilde{V} \) (resp. \( \tilde{W} \)) is an arbitrary connected component of \( \pi^{-1}V \) (resp. \( \pi^{-1}W \)) with \( \tilde{V} \subset \tilde{W} \).

For the Levi problem we prove:

**Theorem 1.19.** Let \( \pi : X \to \mathbb{C}^n \) be a finitely sheeted ramified Riemann domain. Assume that Cond A and Cond B are satisfied. If \( X \) is locally Stein over \( \mathbb{C}^n \), \( X \) is a Stein manifold.

**Remark 1.20.** Fornæss’ counter-example ([7]) for the Levi problem in the ramified case is a 2-sheeted Riemann domain over \( \mathbb{C}^n \), but it does not satisfy Cond A (see §3 (a)).

## 2 Proofs

### 2.1 Scalar \( \rho(a, \Omega) \)

Let \( X \) be a complex manifold satisfying Cond A. We here deal with some elementary properties of \( \rho(a, \Omega) \) defined by (1.3) for a subdomain \( \Omega \subset X \). We use the same notation as in §1.2.1.

First, we suppose that \( \rho(a_0, \Omega) = \infty \) at a point \( a_0 \in \Omega \). Then, \( \phi_{a_0} : \mathbb{C}^n \to \Omega \) is surjective, and \( \rho(a, \Omega) \equiv \infty \) for \( a \in \Omega \). In fact, for any \( a \in \Omega \) we take a path \( C_a \) from \( a_0 \) to \( a \) in \( \Omega \) and set \( \zeta = \alpha(a) \). By the definition, \( \phi_{a_0}(\zeta) = a \), and it follows that \( \rho(a, \Omega) = \infty \).

Thus, we have:
\[
\text{either} \quad \rho(a, \Omega) \equiv \infty, \quad \text{or} \quad \rho(a, \Omega) < \infty, \quad \forall a \in \Omega.
\]

(2.1)

Suppose that the latter case above holds. We identify \( \rho_0P\Delta_0 \) and \( U_0 \) in (1.2). For \( b, c \in \rho_0P\Delta \) we have
\[
\rho(b, \Omega) \geq \rho(c, \Omega) - |b - c|,
\]
where \( |b - c| \) denotes the maximum norm with respect to the coordinate system \( (\zeta') \in \rho_0P\Delta \). Thus,
\[
\rho(c, \Omega) - \rho(b, \Omega) \leq |b - c|.
\]

Changing \( b \) and \( c \), we have the converse inequality, so that
\[
|\rho(b, \Omega) - \rho(c, \Omega)| \leq |b - c|, \quad b, c \in \rho_0P\Delta \cong U_0.
\]

(2.2)

Therefore, \( \rho(a, \Omega) \) is a continuous function in \( a \in \Omega \).
Let \( v = \sum_{j=1}^{n} v^j \left( \frac{\partial}{\partial \xi_j} \right)_a \in T(\Omega)_a \) be a holomorphic tangent vector at \( a \in \Omega \). Then, we set
\[
|v|_\omega = \max_{1 \leq j \leq n} |v^j|.
\]
With \( |v|_\omega = 1 \) we have by the definition of the Kobayashi hyperbolic infinitesimal metric \( F_\Omega \) (cf. [15], [21])
\[
F_\Omega(v) \leq 1 / \rho(a, \Omega).
\]
Therefore we have
\[
\rho(a, \Omega) \leq \inf_{v : F_\Omega(v) = 1} |v|_\omega. \tag{2.3}
\]
Provided that \( \partial \Omega \neq \emptyset \), it immediately follows that
\[
\lim_{a \to \partial \Omega} \rho(a, \Omega) = 0. \tag{2.4}
\]

**Remark 2.1.** (i) We consider an unramified Riemann domain \( \pi : X \to \mathbb{C}^n \). Let \((z^1, \ldots, z^n)\) be the natural coordinate system of \( \mathbb{C}^n \) and put \( \omega = (\pi^* dz^j) \) (Cond A). Then the boundary distance function \( \delta_{P\Delta}(a, \partial X) \) to the ideal boundary \( \partial X \) with respect to the unit polydisk \( P\Delta \) is defined as the supremum of such \( r > 0 \) that \( X \) is univalent onto \( \pi(a) + r P\Delta \) in a neighborhood of \( a \) (cf., e.g., [14], [20]). Therefore, in this case we have that
\[
\rho(a, X) = \delta_{P\Delta}(a, \partial X), \tag{2.5}
\]
and Cond B is naturally satisfied. As for the difficulty to deal with the Levi problem for ramified Riemann domains, K. Oka wrote in IX [26], \S 23:

“Pour le deuxième cas les rayons de Hartogs cessent de jouir du rôle; ceci présente une difficulté qui m’apparaît vraiment grande.”

The above “le deuxième cas” is the ramified case.

(ii) For \( X \) satisfying Cond A one can define Hartogs’ radius \( \rho_n(a, X) \) as follows. Consider \( \phi_{a,(r_j)} : P\Delta(r_j) \to X \) for a polydisk \( P\Delta(r_j) \) about 0 with a poly-radius \((r_1, \ldots, r_n) \) \((r_j > 0)\), which is an inverse of \( \alpha \) given by (1.1). Then, one defines \( \rho_n(a, X) \) as the supremum of such \( r_n > 0 \); for other \( j \), it is similarly defined. Hartogs’ radius \( \rho_n(a, \Omega) \) is not necessarily continuous, but lower semi-continuous. In the present paper, the scalar \( \rho(a, X) \) defined under Cond A plays the role of “Hartogs’ radius”.

**Remark 2.2.** Even if \( \rho(a, \omega, X) = \infty \) (cf. (1.5)), “\( \rho(a, \omega', X) < \infty \)” may happen for another choice of \( \omega' \) (cf. \S 3).
2.2 Proof of Theorem 1.3

For \( a \in \Omega \) we let

\[
\phi_a : \rho(a, \Omega)P\Delta \rightarrow \Omega
\]

be as in (1.4). We take an arbitrary element \( u \in \mathcal{O}(\Omega) \). With a fixed positive number \( s < 1 \) we set

\[
L = \bigcup_{a \in K} \phi_a(s|f(a)|P\Delta).
\]

Then it follows from the assumption that \( L \) is a compact subset of \( \Omega \). Therefore there is an \( M > 0 \) such that

\[
|u| < M \text{ on } L.
\]

Let \( \partial_j \) be the dual vector fields of \( \omega_j \), \( 1 \leq j \leq n \), on \( X \). For a multi-index \( \nu = (\nu_1, \ldots, \nu_n) \) with non-negative integers \( \nu_j \in \mathbb{Z}^+ \) we put

\[
\partial^\nu = \partial_1^{\nu_1} \cdots \partial_n^{\nu_n},
\]

\[
|\nu| = \nu_1 + \cdots + \nu_n,
\]

\[
\nu! = \nu_1! \cdots \nu_n!.
\]

By Cauchy’s inequalities for \( u \circ \phi_a \) on \( s|f(a)|P\Delta \) with \( a \in K \) we have

\[
\frac{1}{\nu!} |\partial^\nu u(a)| \cdot |s f(a)|^{||\nu||} \leq M, \quad \forall a \in K, \forall \nu \in (\mathbb{Z}^+)^n.
\]

Note that \( (\partial^\nu u) \cdot f^{||\nu||} \in \mathcal{O}(\Omega) \). By the definition of \( \hat{K}_\Omega \),

\[
\frac{1}{\nu!} |\partial^\nu u(a)| \cdot |s f(a)|^{||\nu||} \leq M, \quad \forall a \in \hat{K}_\Omega, \forall \nu \in (\mathbb{Z}^+)^n. \tag{2.6}
\]

For \( a \in \hat{K}_\Omega \) we consider the Taylor expansion of \( u \circ \phi_a(\zeta) \) at \( a \):

\[
u \in (\mathbb{Z}^+)^n.
\]

\[
\frac{1}{\nu!} |\partial^\nu u(a)| \cdot |s f(a)|^{||\nu||} \leq M, \quad \forall a \in \hat{K}_\Omega, \forall \nu \in (\mathbb{Z}^+)^n. \tag{2.7}
\]

We infer from (2.6) that (2.7) converges at least on \( s|f(a)|P\Delta \). Since \( \Omega \) is a domain of holomorphy, we have that \( \rho(a, \Omega) \geq s|f(a)| \). Letting \( s \nearrow 1 \), we deduce (1.6).

By definition, \( \rho(K, \Omega) \geq \rho(\hat{K}_\Omega, \Omega) \). The converse is deduced by applying the result obtained above for a constant function \( f \equiv \rho(K, \Omega) \); thus (1.7) follows.

Proof of Corollary 1.4: Assume that \( \Omega \Subset X \) is a domain of holomorphy. Let \( K \Subset \Omega \). It follows from (1.7) that \( \hat{K}_\Omega \Subset \Omega \), and hence \( \Omega \) is holomorphically convex. The converse is clear.

Remark 2.3. (i) Replacing \( P\Delta \) by the unit ball \( B \) with center at 0, one may define similarly \( \rho(a, \Omega) \). Then Theorem 1.3 remains to hold. Note that the union of all unitary rotations of \( \frac{1}{\sqrt{n}} P\Delta \) is \( B \).

(ii) Note that \( P\Delta \) may be an arbitrary polydisk with center at 0; still, Theorem 1.3 remains valid. We use the unit polydisk just for simplicity.
2.3 Proof of the Behnke–Stein Theorem

2.3.1 Proof of Lemma 1.6

(a) We take a subdomain $\tilde{\Omega}$ of $X$ such that $\Omega \subset \tilde{\Omega} \subset X$. Let $c \in \partial \Omega$ be any point, and take a local coordinate neighborhood system $(W_0, w)$ in $\tilde{\Omega}$ with holomorphic coordinate $w$ such that $w = 0$ at $c$. We consider Cousin I distributions for $k = 1, 2, \ldots$:

$$\frac{1}{w^k} \text{ on } W_0, \quad 0 \text{ on } W_1 = \tilde{\Omega} \setminus \{c\}.$$

These induce cohomology classes

$$\left[ \frac{1}{w^k} \right] \in H^1(\{W_0, W_1\}, \mathcal{O}_{\tilde{\Omega}}) \hookrightarrow H^1(\tilde{\Omega}, \mathcal{O}_{\tilde{\Omega}}), \quad k = 1, 2, \ldots.$$

Since $\dim H^1(\tilde{\Omega}, \mathcal{O}_{\tilde{\Omega}}) < \infty$ by (1.8) (Grauert’s Theorem), there is a non-trivial linear relation over $\mathbb{C}$

$$\sum_{k=1}^\nu \gamma_k \left[ \frac{1}{w^k} \right] = 0 \in H^1(\tilde{\Omega}, \mathcal{O}_{\tilde{\Omega}}), \quad \gamma_k \in \mathbb{C}, \; \gamma_\nu \neq 0.$$

Hence there is a meromorphic function $F$ on $\tilde{\Omega}$ with a pole only at $c$ such that about $c$

$$F(w) = \frac{\gamma_\nu}{w^\nu} + \cdots + \frac{\gamma_1}{w} + \text{holomorphic term.} \quad (2.8)$$

Therefore the restriction $F|_\Omega$ of $F$ to $\Omega$ is holomorphic and $\lim_{x \to c} |F(x)| = \infty$. Thus we see that $\Omega$ is holomorphically convex.

(b) We show the holomorphic separation property of $\Omega$ (Definition 1.5 (ii)). Let $a, b \in \Omega$ be any distinct points. Let $F$ be the one obtained in (a) above. If $F(a) \neq F(b)$, then it is done. Suppose that $F(a) = F(b)$. We may assume that $F(a) = F(b) = 0$. Let $(U_0, z)$ be a local holomorphic coordinate system about $a$ with $z(a) = 0$. Then we have

$$F(z) = a_{k_0} z^{k_0} + \text{higher order terms}, \quad a_{k_0} \neq 0, \; k_0 \in \mathbb{N}, \quad (2.9)$$

where $\mathbb{N}$ denotes the set of positive integers. We define Cousin I distributions by

$$\frac{1}{z^{k_0}} \text{ on } U_0, \; k \in \mathbb{N}, \quad 0 \text{ on } U_1 = \Omega \setminus \{a\},$$

which lead cohomology classes

$$\left[ \frac{1}{z^{k_0}} \right] \in H^1(\{U_0, U_1\}, \mathcal{O}_{\Omega}) \hookrightarrow H^1(\Omega, \mathcal{O}_{\Omega}), \quad k = 1, 2, \ldots. \quad (2.10)$$
It follows from (1.8) that there is a non-trivial linear relation
\[ \sum_{k=1}^{\mu} \alpha_k \left[ \frac{1}{z^{kk_0}} \right] = 0, \quad \alpha_k \in \mathbb{C}, \quad \alpha_\mu \neq 0. \]

It follows that there is a meromorphic function \( G \) on \( \Omega \) with a pole only at \( a \), where \( G \) is written as
\[ G(z) = \frac{\alpha_\mu}{z^{kk_0}} + \frac{\alpha_{\mu-1}}{z^{(\mu-1)k_0}} + \cdots + \frac{\alpha_1}{z^{k_0}} + \text{holomorphic term}. \]  

With \( g = G \cdot F^\mu \) we have \( g \in \mathcal{O}(\Omega) \) and by (2.9) and (2.11) we see that
\[ g(a) = \alpha_\mu a^{k_0} \neq 0, \quad g(b) = 0. \]

(c) Let \( a \in \Omega \) be any point. We show the existence of an element \( h \in \mathcal{O}(\Omega) \) with non-vanishing differential \( dh(a) \neq 0 \) (Definition 1.5 (iii)). Let \( (U_0, z) \) be a holomorphic local coordinate system about \( a \) with \( z(a) = 0 \). As in (2.10) we consider
\[ \left[ \frac{1}{z^{kk_0-1}} \right] \in H^1(\{U_0, U_1\}, \mathcal{O}_\Omega) \hookrightarrow H^1(\Omega, \mathcal{O}_\Omega), \quad k = 1, 2, \ldots. \]  

In the same as above we deduce that there is a meromorphic function \( H \) on \( \Omega \) with a pole only at \( a \), where \( H \) is written as
\[ H(z) = \frac{\beta_\lambda}{z^{kk_0-1}} + \cdots + \frac{\beta_1}{z^{k_0-1}} \text{ + holomorphic term}, \quad \beta_k \in \mathbb{C}, \quad \beta_\lambda \neq 0, \quad \lambda \in \mathbb{N}. \]  

With \( h = H \cdot F^\lambda \) we have \( h \in \mathcal{O}(\Omega) \) and by (2.9) and (2.13) we get
\[ \frac{dh}{dz}(a) = \beta_\lambda a^{k_0} \neq 0. \]  

Thus, \( \Omega \) is Stein. \( \square \)

2.3.2 Proof of Lemma 1.8

We take a domain \( \tilde{\Omega} \subset X \) with \( \tilde{\Omega} \supset \Omega \). By Lemma 1.6, \( \tilde{\Omega} \) is Stein, and hence there is a holomorphic 1-form on \( \tilde{\Omega} \) without zeros. Then we define \( \rho(a, \tilde{\Omega}) \) as in (1.3) with \( X = \tilde{\Omega} \). With this \( \rho(a, \tilde{\Omega}) \) we have by (1.7):

Lemma 2.4. For a compact subset \( K \Subset \Omega \) we get
\[ \rho(K, \Omega) = \rho(\tilde{K}_{\Omega}, \Omega). \]

Lemma 2.5. Let \( \Omega' \) be a domain such that \( \Omega \Subset \Omega' \Subset \tilde{\Omega} \). Assume that
\[ \max_{b \in \partial \Omega} \rho(b, \Omega') < \rho(K, \Omega). \]  

Then,
\[ \tilde{K}_{\Omega'} \cap \Omega \Subset \Omega. \]
Proof. Since $\hat{K}_{\Omega'}$ is compact in $\Omega'$ by Lemma 1.6, it suffices to show that

$$\hat{K}_{\Omega'} \cap \partial \Omega = \emptyset.$$  

Suppose that there is a point $b \in \hat{K}_{\Omega'} \cap \partial \Omega$. It follows from Lemma 2.4 that

$$\rho(b, \Omega') \geq \rho(\hat{K}_{\Omega'}, \Omega') = \rho(K, \Omega') \geq \rho(K, \Omega).$$

By the assumption, $\rho(b, \Omega') < \rho(K, \Omega)$; this is absurd. \[\square\]

Proof of Lemma 1.8: Here we use Oka’s Jôku-Ikô. By Lemma 1.8 there are holomorphic functions $\psi_j \in \mathcal{O}(\Omega')$ such that a finite union $P$, called an analytic polyhedron, of relatively compact connected components of

$$\{x \in \Omega' : |\psi_j(x)| < 1\}$$

satisfies “$\hat{K}_{\Omega'} \cap \Omega \subset P \subset \Omega'$” and the Oka map

$$\Psi : x \in P \rightarrow (\psi_1(x), \ldots, \psi_N(x)) \in \mathbb{P}\Delta_N$$

is a closed embedding into the $N$-dimensional unit polydisk $\mathbb{P}\Delta_N$.

Let $f \in \mathcal{O}(\Omega)$. We identify $P$ with the image $\Psi(P) \subset \mathbb{P}\Delta_N$ and regard $f|_P$ as a holomorphic function on $\Psi(P)$. Let $\mathcal{I}$ denote the geometric ideal sheaf of the analytic subset $\Psi(P) \subset \mathbb{P}\Delta_N$. Then we have a short exact sequence of coherent sheaves:

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\mathbb{P}\Delta_N} \rightarrow \mathcal{O}_{\mathbb{P}\Delta_N}/\mathcal{I} \rightarrow 0.$$

By Oka’s Fundamental Lemma, $H^1(\mathbb{P}\Delta_N, \mathcal{I}) = 0$ (cf., e.g., [20], §4.3), which implies the surjection

$$H^0(\mathbb{P}\Delta_N, \mathcal{O}_{\mathbb{P}\Delta_N}) \rightarrow H^0(\mathbb{P}\Delta_N, \mathcal{O}_{\mathbb{P}\Delta_N}/\mathcal{I}) \cong \mathcal{O}(P) \rightarrow 0. \quad (2.15)$$

Since $f|_P \in \mathcal{O}(P)$, there is an element $F \in \mathcal{O}(\mathbb{P}\Delta_N)$ with $F|_P = f|_P$. We then expand $F$ to a power series

$$F(w_1, \ldots, w_N) = \sum_{\nu} c_\nu w^\nu, \quad w \in \mathbb{P}\Delta_N,$$

where $\nu$ denote multi-indices in $\{1, \ldots, N\}$. For every $\epsilon > 0$ there is a number $l \in \mathbb{N}$ such that

$$\left| F(w) - \sum_{|\nu| \leq l} c_\nu w^\nu \right| < \epsilon, \quad w \in \Psi(K).$$

Substituting $w_j = \psi_j$, we have that

$$g(x) = \sum_{|\nu| \leq l} c_\nu \Psi^\nu(x) \in \mathcal{O}(\Omega'),$$

$$|f(x) - g(x)| < \epsilon, \quad \forall x \in K. \quad \square$$
2.3.3 Proof of Theorem 1.9

We take a continuous exhaustion family \( \{ \Omega_t \}_{0 \leq t \leq 1} \) of subdomains of \( \tilde{\Omega} \) (cf. Definition 1.16) with \( \Omega_0 = \Omega \). Let \( K \subset \subset \Omega \) be a compact subset and let \( f \in \mathcal{O}(\Omega) \). We set

\[
T = \{ t : 0 < t \leq 1, \mathcal{O}(\Omega_t)|K \text{ is dense in } \mathcal{O}(\Omega)|K \},
\]

where “dense” is taken in the sense of the maximum norm on \( K \). Note that

(i) \( \rho(a, \Omega_t) \) is continuous in \( t \);
(ii) \( \rho(K, \Omega) \leq \rho(K, \Omega_s) < \rho(K, \Omega_t) \) for \( s < t \);
(iii) \( \lim_{t \searrow s} \max_{b \in \partial \Omega_s} \rho(b, \Omega_t) = 0 \).

It follows from Lemma 1.8 that \( T \) is non-empty, open and closed. Therefore, \( T \ni 1 \), so that \( \mathcal{O}(\tilde{\Omega})|K \text{ is dense in } \mathcal{O}(\Omega)|K \).

2.3.4 Proof of Theorem 1.10

We owe the second countability axiom for the Riemann surface \( X \) to T. Radó. We take an increasing sequence of relatively compact domains \( \Omega_j \subset \subset \Omega_{j+1} \subset X, j \in \mathbb{N} \), such that \( X = \bigcup_{j=1}^{\infty} \Omega_j \) and no connected component of \( \Omega_{j+1} \setminus \Omega_j \) is relatively compact in \( \Omega_{j+1} \). Then, \( (\Omega_j, \Omega_{j+1}) \) forms a so-called Rung pair (Theorem 1.9). Since every \( \Omega_j \) is Stein (Lemma 1.6), the Steinness of \( X \) is deduced.

2.4 Proofs for Riemann domains

2.4.1 Proof of Theorem 1.12

(i) Suppose that \( \Omega(\subset X) \) is a domain of holomorphy. It follows from the assumption and Corollary 1.4 that \( \Omega \) is \( K \)-complete in the sense of Grauert and holomorphically convex. Thus, by Grauert’s Theorem ([11]), \( \Omega \) is Stein.

(ii) Let \( Z = \{ \det d\pi = 0 \} \). Then, \( Z \) is a thin analytic subset of \( X \). We first take a Stein subdomain \( \Omega \subset X \) and show the plurisubharmonicity of \( -\log \rho(a, \Omega) \). By Grauert-Remmert [12] it suffices to show that \( -\log \rho(a, \Omega) \) is plurisubharmonic in \( \Omega \setminus Z \). Take an arbitrary point \( a \in \Omega \setminus Z \), and a complex affine line \( \Lambda \subset \mathbb{C}^n \) passing through \( \pi(a) \). Let \( \tilde{\Lambda} \) be the irreducible component of \( \pi^{-1}\Lambda \cap \Omega \) containing \( a \). Let \( \Delta \) be a small disk about \( \pi(a) \) such that \( \tilde{\Delta} := \pi^{-1}\Delta \subset \tilde{\Lambda} \setminus Z \).

Claim. The restriction \( -\log \rho(x, \Omega)|_{\tilde{\Delta}\setminus Z} \) is subharmonic.

By a standard argument (cf., e.g., [14], Proof of Theorem 2.6.7) it suffices to prove that if a holomorphic function \( g \in \mathcal{O}(\tilde{\Lambda}) \) satisfies

\[
-\log \rho(x, \Omega) \leq \Re g(x), \quad x \in \partial \tilde{\Delta},
\]

then

\[
-\log \rho(x, \Omega) \leq \Re g(x), \quad x \in \tilde{\Delta}, \quad \text{(2.16)}
\]
where $\Re$ denotes the real part. Now, we have that
\[ \rho(x, \Omega) \geq |e^{g(x)}|, \quad x \in \partial \tilde{\Delta}. \]

Since $\Omega$ is Stein, there is a holomorphic function $f \in O(\Omega)$ with $f|_{\tilde{\Delta}} = g$ (cf. the arguments for (2.15)). Then,
\[ \rho(x, \Omega) \geq |e^{f(x)}|, \quad x \in \partial \tilde{\Delta}. \]

Since $\tilde{\Delta}_\Omega = \tilde{\Delta}$, it follows from (1.6) that
\[ \rho(x, \Omega) = |e^{f(x)}| = |e^{g(x)}|, \quad x \in \tilde{\Delta}, \]
so that (2.16) follows.

Let $\{\Omega_\nu\}_{n=1}^\infty$ be a sequence of Stein domains of $X$ such that $\Omega_\nu \subset \Omega_{\nu+1}$ for all $\nu$ and $X = \bigcup_\nu \Omega_\nu$. Then, $-\log \rho(a, \Omega_\nu)$, $\nu = 1, 2, \ldots$, are plurisubharmonic and monotone decreasingly converges to $-\log \rho(a, X)$. Therefore, $-\log \rho(a, X)$ is either identically $-\infty$, or plurisubharmonic ($\neq -\infty$). If $-\log \rho(a, X) \neq -\infty$, it is everywhere finitely valued and continuous by (2.1).

**Corollary 2.6.** Let $X$ be a Stein manifold satisfying Cond A. Then, $-\log \rho(a, X)$ is either identically $-\infty$ or continuous plurisubharmonic.

**Proof.** Since $X$ is Stein, there is a holomorphic map $\pi : X \to C^n$ which forms a Riemann domain. The assertion is immediate from (ii) above. \[ \square \]

**Remark 2.7.** As a consequence, one sees with the notation in Corollary 2.6 that if $\Omega \subset X$ is a domain of holomorphy, then Hartogs’ radius $\rho_n(a, \Omega)$ (cf. Remark 2.1 (ii)) is plurisubharmonic. This is, however, opposite to the history: The plurisubharmonicity or the pseudoconvexity of Hartogs’ radius $\rho_n(a, \Omega)$ was found first through the study of the maximal convergence domain of a power series (Hartogs’ series) in several complex variables (cf. Oka [22], VI [23], IX [26], Nishino [19], Chap. I, Fritzsche–Grauert [9], Chap. II).

**Remark 2.8.** We here give a proof of Theorem 1.15 under Cond A by making use of $\rho(a, \Omega)$. Since $\omega$ is defined in a neighborhood of $\bar{\Omega}$, Cond B is satisfied at every point of the boundary $\partial \Omega$; that is, for every $b \in \partial \Omega$ there are neighborhoods $U' \Subset U \Subset X$ of $b$ such that
\[ \rho(a, \Omega) = \rho(a, U \cap \Omega), \quad a \in U'. \]
If $U \cap \Omega$ is Stein, then $-\log \rho(a, \Omega)$ is plurisubharmonic in $a \in U'$ by Theorem 1.12 (iii). Therefore there is a neighborhood $V$ of $\partial \Omega$ in $X$ such that $-\log \rho(a, \Omega)$ is plurisubharmonic in $a \in V \cap \Omega$. Take a real constant $C$ such that
\[ -\log \rho(a, \Omega) < C, \quad a \in \Omega \setminus V. \]
Set
\[ \psi(a) = \max\{-\log \rho(a, \Omega), C\}, \quad a \in \Omega. \]

Then, \( \psi \) is a continuous plurisubharmonic exhaustion function on \( \Omega \). By Theorem 2.10 of Andreotti–Narasimhan below, \( \Omega \) is Stein.

\[
\blacksquare
\]

2.4.2 Proof of Theorem 1.17

In the same way as Lemma 1.8 and its proof we have

**Lemma 2.9.** Let \( \pi : \tilde{\Omega} \to \mathbb{C}^n \) be a Riemann domain such that \( \tilde{\Omega} \) satisfies Cond A. Let \( \Omega \subseteq \Omega' \) be relatively compact subdomains of \( \tilde{\Omega} \) satisfying (1.9): Then, every \( f \in \mathcal{O}(\Omega) \) can be approximated uniformly on \( K \) by elements of \( \mathcal{O}(\Omega') \).

For the proof of the theorem it suffices to show that \( (\Omega_t, \Omega_s) \) is a Runge pair for \( 0 \leq t < s < 1 \). Since any fixed \( \Omega_{s'} \) \( (s < s' < 1) \) satisfies Cond A, we have the scalar \( \rho(a, \Omega_s) \). Take a compact subset \( K \subseteq \Omega_t \). Then, for \( s > t \) sufficiently close to \( t \) we have
\[
\max_{b \in \partial \Omega_t} \rho(b, \Omega_s) < \rho(K, \Omega_t).
\]

It follows from Lemma 2.9 that \( \mathcal{O}(\Omega_s)|_K \) is dense if \( \mathcal{O}(\Omega_t)|_K \). Then, the rest of the proof is the same as in §2.3.3.

\[
\blacksquare
\]

2.4.3 Proof of Theorem 1.19

Here we will use the following result:

**Theorem 2.10** (Andreotti–Narasimhan [1]). Let \( \pi : X \to \mathbb{C}^n \) be a Riemann domain. If \( X \) admits a continuous plurisubharmonic exhaustion function, then \( X \) is Stein.

Let \( z \in \Gamma, (z \in V \subseteq W \) and \( \tilde{V} \subseteq \tilde{W} \) be as in Cond B. Then,
\[
\rho(a, X) = \rho(a, \tilde{W}), \quad a \in \tilde{V}. \quad (2.17)
\]

By the assumption, \( \tilde{W} \) can be chosen to be Stein. By Theorem 1.12 (ii), \( -\log \rho(a, \tilde{W}) \) is plurisubharmonic in \( a \in \tilde{V} \), and hence so is \( -\log \rho(a, X) \) in \( \tilde{V} \). By covering \( \Gamma \) by those \( V \subseteq W \) and making use of Cond B (i), there is a closed subset \( F \subseteq X \) such that

(i) \( F \cap \{x \in X : \|\pi(x)\| \leq R\} \) is compact for every \( R > 0 \),

(ii) \( -\log \rho(a, X) \) is plurisubharmonic in \( a \in X \setminus F \),

(iii) \( \lim_{\nu \to \infty} -\log \rho(a_{\nu}, X) = \infty \) for every sequence \( \{a_{\nu}\} \) of points of \( X \) with no accumulation point in \( X \) such that \( \{\pi(a_{\nu})\} \) is convergent in \( \mathbb{C}^n \).
From this we may construct a continuous plurisubharmonic exhaustion function on $X$ as follows:

We fix a point $a_0 \in F$, and may assume that $\pi(a_0) = 0$. Let $X_\nu$ be a connected component of $\{\|\pi\| < \nu\}$ containing $a_0$. Then, $\bigcup_\nu X_\nu = X$. Put

$$\Omega_\nu = X_\nu \setminus F \in X.$$ 

Take a real constant $C_1$ such that

$$-\log \rho(a, X) < C_1, \quad a \in \bar{\Omega}_1.$$ 

Then we set

$$\psi_1(a) = \max\{-\log \rho(a, X), C_1\}, \quad a \in X.$$ 

Then, $\psi_1$ is plurisubharmonic in $X_1$. We take a positive constant $C_2$ such that

$$-\log \rho(a, X) < C_1 + C_2(\|\pi(a)\|^2 - 1)^+, \quad a \in \bar{\Omega}_2,$$

where $(\cdot)^+ = \max\{\cdot, 0\}$. Put

$$p_2(a) = C_1 + C_2(\|\pi(a)\|^2 - 1)^+,$$

$$\psi_2(a) = \max\{-\log \rho(a, X), p_2(a)\}, \quad a \in X.$$ 

Then, we have:

(i) $p_2(a) \geq C_1 + 2C_2$ in $\{\|\pi\| \geq 2\}$;

(ii) $\psi_1(a) = \psi_2(a)$ in $a \in X_1$;

(iii) $\psi_2(a)$ is plurisubharmonic in $X_2$.

Similarly, we take $C_3 > C_2$ so that

$$-\log \rho(a, X) < p_2(a) + C_3(\|\pi(a)\|^2 - 2^2)^+, \quad a \in \bar{\Omega}_3.$$ 

Put

$$p_3(a) = p_2(a) + C_3(\|\pi(a)\|^2 - 2^2)^+,$$

$$\psi_3(a) = \max\{-\log \rho(a, X), p_3(a)\}, \quad a \in X.$$ 

We then obtain:

(i) $p_3(a) \geq C_1 + 3C_2 + 5C_3$ in $\{\|\pi\| \geq 3\}$;

(ii) $\psi_3(a) = \psi_2(a)$ in $a \in X_2$;

(iii) $\psi_3(a)$ is plurisubharmonic in $X_3$.

Inductively, we may take a continuous function $\psi_\nu(a)$, $\nu = 1, 2, \ldots$, such that $\psi_\nu$ is plurisubharmonic in $X_\nu$ and $\psi_{\nu+1}|_{X_\nu} = \psi_\nu|_{X_\nu}$. It is clear from the construction that

$$\psi(a) = \lim_{\nu \to \infty} \psi_\nu(a), \quad a \in X,$$

is a continuous plurisubharmonic exhaustion function of $X$.

Finally, by Theorem 2.10 of Andreotti–Narasimhan we see that $X$ is Stein.
3 Examples and some more on $\rho(a, X)$

(a) (Fornæss’ example). Fornæss [7] constructed a 2-sheeted ramified Riemann domain $\pi : M \to \mathbb{C}^2$ such that it is locally Stein, $M$ is exhausted by an increasing sequence of relatively compact Stein subdomains, but $M$ is not Stein. We here show that the holomorphic cotangent bundle $T(M)^*$ does not carry a global frame, so that $M$ does not satisfy Cond $A$.

For convenience, we use the same notation as in [7]. Assume that there exists a global frame $\{\lambda_1, \lambda_2\}$. With the coordinates $(z, w)$ we write in a neighborhood $U = \{(z, w) : |z| < \delta, 1 - \delta < |w| < 1 + \delta\} (\delta > 0$, sufficiently small) of $z = 0, |w| = 1$:

$$
\lambda_1 = f(z, w)dz + g(z, w)dw,
\lambda_2 = h(z, w)dz + k(z, w)dw.
$$

Then, we have

$$\lambda_1 \wedge \lambda_2 = (fk - gh)dz \wedge dw.$$

By the assumption, $fk - gh$ has no zero. Put

$$\nu_0 = \frac{1}{2\pi i} \int_{|w|=1} d\log \left( f(z, w)k(z, w) - g(z, w)h(z, w) \right) \in \mathbb{Z}, \quad |z| < \delta.$$

Then we may write

$$f(z, w)k(z, w) - g(z, w)h(z, w) = e^{A(z, w)w^{\nu_0}},$$

where $A(z, w)$ is a holomorphic function of $z$ and $w$. We consider the analytic continuations of the above holomorphic functions as far as possible. In a neighborhood of $(1/n, w)$ with every sufficiently large natural number $n$ and sufficiently small $|w|$ we have another chart,

$$z = \frac{1}{n} + C_n \eta w^{m_n} + \epsilon_n \eta^2, \quad w = w.$$

Then it follows that

$$\lambda_1 \wedge \lambda_2 = (fk - gh)dz \wedge dw
= e^{A(z, w)w^{\nu_0}} \left( \frac{1}{n} + C_n \eta w^{m_n} + \epsilon_n \eta^2 \right) \wedge dw
= e^{A(z, w)w^{\nu_0}} (C_n w^{m_n} + 2\epsilon_n \eta) d\eta \wedge dw. \quad (3.1)$$

Since at $z = 1/n$ and $\eta = 0$ the coefficient function of (3.1) should be holomorphic and should have no zero, we deduce that $m_n = -\nu_0$. But, $m_n \to \infty$ as $n \to \infty$: This is a contradiction.

(b) (Grauert’s example). Grauert [18] gave a counter-example to the Levi problem for ramified Riemann domains over $\mathbb{P}^n(\mathbb{C})$: There is a locally Stein domain $\Omega$ in a complex
torus $M$ such that $\Omega = C$. Then, $M$ satisfies Cond A. One may assume that $M$ is projective algebraic, so that there is a holomorphic finite map $\tilde{\pi} : M \to \mathbb{P}^n(C)$, which is a Riemann domain over $\mathbb{P}^n(C)$. Then, the restriction $\pi = \tilde{\pi}|_\Omega : \Omega \to \mathbb{P}^n(C)$ is a Riemann domain over $\mathbb{P}^n(C)$, which satisfies Cond A and Cond B. Therefore, Theorem 1.19 cannot be extended to a Riemann domain over $\mathbb{P}^n(C)$.

Remark 3.1. Let $\pi : \Omega \to \mathbb{P}^n(C)$ be Grauert’s example as above. Let $C^n$ be an affine open subset of $\mathbb{P}^n(C)$, and let $\pi' : \Omega' \to C^n$ be the restriction of $\pi : \Omega \to \mathbb{P}^n(C)$ to $C^n$. Then, $\Omega'$ is Stein by Theorem 1.19.

The Steinness of $\Omega'$ may be not inferred by a formal combination of the known results on pseudoconvexity, since it is an unbounded domain (cf., e.g., [18], [17]).

(c) Domains in the products of open Riemann surfaces and complex semi-tori (cf. [21], Chap. 5) serve for examples satisfying Cond A.

(d) An open Riemann surface $X$ is not Kobayashi hyperbolic if and only if $X$ is biholomorphic to $C$ or $C^* = C \setminus \{0\}$ (For the Kobayashi hyperbolicity in general, cf. [15], [21]).

(d1) Let $X = C$. If $\omega = dz$, then $\rho(a, dz, C) \equiv \infty$ for every $a \in C$. If $\omega = e^z dz$, then a simple calculation implies that

$$\rho(a, e^z dz, C) = |e^a|.$$

(d2) Let $X = C^*$. If $\omega = z^k dz$ with $k \in \mathbb{Z} \setminus \{-1\}$, then

$$\rho(a, z^k dz, C^*) = \left| \frac{1}{k+1} a^{k+1} \right|.$$

Therefore, $\lim_{a \to 0} \rho(a, z^k dz, C^*) = 0$ for $k \geq 0$, and $\lim_{a \to \infty} \rho(a, z^k dz, C^*) = 0$ for $k \leq -2$. If $\omega = \frac{dz}{z}$, then $\rho(a, \frac{dz}{z}, C^*) \equiv \infty$. It follows that

$$\psi(a) := \max \{ -\log \rho(a, dz, C^*), -\log \rho(a, z^{-2} dz, C^*) \}$$

is continuous subharmonic in $C^*$, and $\lim_{a \to 0, \infty} \psi(a) = \infty$.

Thus, the finiteness or the infiniteness of $\rho(a, \omega, X)$ depends on the choice of $\omega$.

(e) For a Kobayashi hyperbolic open Riemann surface $X$ we take a holomorphic 1-form $\omega$ without zeros, and write

$$\|\omega(a)\|_X = |\omega|, \quad v \in T(X)_a, \quad F_X(v) = 1.$$

Then it follows from (2.3) that $\rho(a, \omega, X) \leq \|\omega(a)\|_X$. We set

$$\rho^+(a, X) = \sup \{ \rho(a, \omega, X) : \omega \text{ hol. 1-form without zeros, } \|\omega(a)\|_X = 1 \},$$

$$\rho^-(a, X) = \inf \{ \rho(a, \omega, X) : \omega \text{ hol. 1-form without zeros, } \|\omega(a)\|_X = 1 \}.$$

Clearly, $\rho^+(a, X)(\leq 1)$ are biholomorphic invariants of $X$, but we do not know the behavior of them.
References


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