A New Introductory Lecture of Several Complex Variables
– the Oka Theory

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§1 Introduction.

The **Big 3 Problems** summarized by Behnke–Thullen in 1934:

1. Levi (Hartogs' Inverse) Problem (Chap. IV).
2. Cousin I(/resp. II) Problem (Chap. V).
3. Approximation Problem (Development of functions) (Chap. VI).

Kiyoshi Oka solved all 3 in the opposite order (1936–'53).

Two comments on their difficulties from

"Kiyoshi Oka, Collected Papers", ed. R. Remmert, translated by R. Narasimhan, Springer 1984:

H. Cartan: .... : il se fixa pour tâche de résoudre ces problèmes difficiles, tâche quasi-surhumaine.

.... : he fixed himself to the task to solve these difficult problems, the task quasi-superhumane.

R. Remmert: ..... Er löste Probleme, die als unangreifbar galten; ...

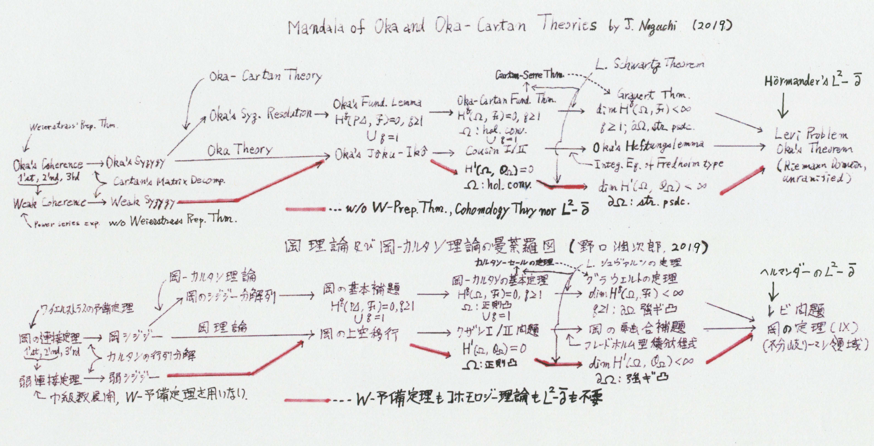
.... He solved problems which were believed to be unsolvable;

....

- For the Big 3 Problems Oka wrote 9 papers, I (1936)–IX ('53).
- They are classified into two groups:
- [1] I('36)–VI('42)+IX('53 ('43)) (essential part of IX was done in '43 for the solutions of the Big 3 Problems for unramified Riemann domains over $\mathbf{C}^n$; in the ramified case, $^\exists$ counter example, but why? $\cdots$ open).
- [2] VII('48)+VIII('51) written *beyond* the 3 Big Problems, intending to solve the Levi (Hartogs' Inverse) Problem for singular ramified Riemann domains over $\mathbf{C}^n$.
- Our "Introductory Lecture" covers the solutions of the Big 3 Problems solved in the first part [1] by a

  "Weak Coherence Theorem".

H. Cartan: .... But we must admit that the technical aspects of his proofs and the mode of presentation of his results make it difficult to read, and that it is possible only at the cost of a real effort to grasp the scope of its results, which is considerable. .....



N.B. The present lectures are mainly concerned with the univalent domain case. In the end I will briefly mention the multivalent case.

Reference:

1. [AFT] N–, Analytic Function Theory of Several Variables—Elements of Oka's Coherence, Springer, 2016.
   Translated from: Analytic Function Theory in Several Variables (in Japanese), Asakurashoten, Tokyo, 2013.
2. N–, A brief chronicle of the Levi (Hartogs' Inverse) Problem, Coherence and an open problem, to appear in Notices Intern. Cong. Chin. Math., 2019, Intern. Press.
3. N–, A weak coherence theorem and remarks to the Oka theory, to appear in Kodai Math. J., 2019.

Point: No use of W-Prep. Thm., cohomology thry., nor L^2-$\bar{\partial}$.

§2. Domain of holomorphy and holomorphic convexity

Let

$\Omega \subset \mathbf{C}^n$ be a domain, and

$\mathcal{O}(\Omega)$ denote the set of all holomorphic functions in Ω.

If $\Omega' \supset \Omega$ and $\mathcal{O}(\Omega') \cong \mathcal{O}(\Omega)$, Ω' is called an *extension of holomorphy* of Ω. If $\Omega' \underset{\neq}{\supset} \Omega$, such simultaneous analytic continuation is called a *Hartogs' phenomenon*: The maximal extension of holomorphy $\hat{\Omega}$ of Ω is called the *envelope of holomorphy* of Ω (not necessarily univalent, even if Ω is).

Definition. If $\hat{\Omega} = \Omega$, then Ω is called a *domain of holomorphy*.

For $K \subset \Omega$ we define the **holomorphic convex hull** of K by

$$\hat{K}_\Omega = \hat{K}_{\mathcal{O}(\Omega)} = \left\{ z \in \Omega : |f(z)| \le \sup_K |f|, \ {}^\forall f \in \mathcal{O}(\Omega) \right\}.$$

Definition. Ω is said to be *holomorphically convex* if for all $K \Subset \Omega$,

$$\hat{K}_{\mathcal{O}(\Omega)} \Subset \Omega.$$

Theorem 2.1 (Cartan–Thullen, '32)

A domain is hol. convex if and only if it is a domain of hol.

Hartogs domains.

Let $n \geq 2$, $a = (a_j) \in \mathbf{C}^n$, $0 < \delta_j < \gamma_j$, $1 \leq j \leq n$, $\gamma = (\gamma_j)$. Set

$$\begin{aligned}
\mathrm{P}\Delta(a,\gamma) &= \{z = (z_j) \in \mathbf{C}^n : |z_j - a_j| < \gamma_j, {}^{\forall} j\}, \\
\Omega_1 &= \{z = (z_j) \in \mathrm{P}\Delta(a,\gamma) : |z_j - a_j| < \delta_j,\ j \geq 2\}, \\
\Omega_2 &= \{z \in \mathrm{P}\Delta(a,\gamma) : \delta_1 < |z_1 - a_1| < \gamma_1\}, \\
\Omega_{\mathrm{H}}(a;\gamma) &= \Omega_1 \cup \Omega_2 \quad \text{(Fig. 1)}.
\end{aligned}$$

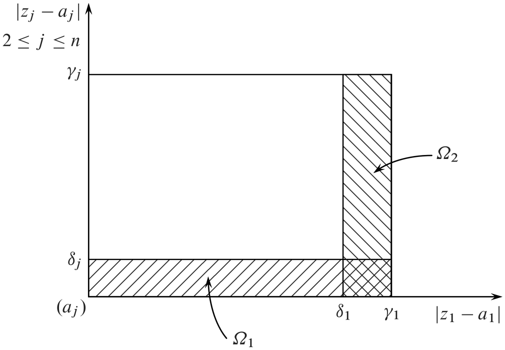


Figure: 1. Hartogs domain $\Omega_{\mathrm{H}}(a;\gamma)$

Remark 2.2 (Example of Hartogs' phenomenon)

$\mathcal{O}(\Omega_{\mathrm{H}}(a;\gamma)) \cong \mathcal{O}(\mathrm{P}\Delta(a,\gamma))$; $\Omega_{\mathrm{H}}(a;\gamma)$ is *not* hol. convex, and $\mathrm{P}\Delta(a,\gamma)$ is hol. convex or a domain of hol.

§3 Cousin I(/ II) Problem:

Let

$\Omega \subset \mathbf{C}^n$ be a domain,

$\Omega = \bigcup U_\alpha$ be an open covering, and

$f_\alpha \in \mathscr{M}(U_\alpha)$ $(/\, \mathscr{M}^*(U_\alpha))$ be (/ non-zero) merom. funct's. in U_α.

Call $\{(U_\alpha, f_\alpha)\}$ a Cousin I(/ II) data if

$$f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta) \ (/\, f_\alpha \cdot f_\beta^{-1} \in \mathcal{O}^*(U_\alpha \cap U_\beta)), \quad {}^\forall \alpha, \beta.$$

Cousin I(/ II) Problem: **Assume that Ω is a domain of hol. Find $F \in \mathscr{M}(\Omega)$ $(/\, \mathscr{M}^*(\Omega))$ such that**

$$\mathrm{I}: F - f_\alpha \in \mathcal{O}(U_\alpha) \quad \big(/\, \mathrm{II}: F \cdot f_\alpha^{-1} \in \mathcal{O}^*(U_\alpha)\big), \quad {}^\forall \alpha.$$

- ${}^{\exists}$*Non-solvable Cousin I/ II data on* $\Omega_{\mathrm{H}}(a;\gamma)$.

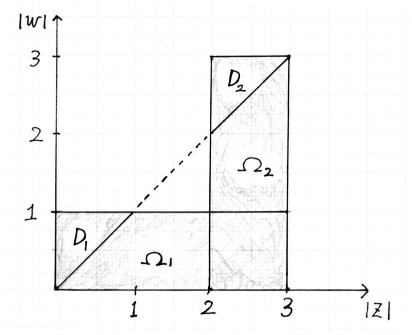


Figure: 2. Hartogs domain with non-solv. C-data

Poles $\frac{1}{z-w}|_{\Omega_1}$ for I (/ Zeros $(z-w)|_{\Omega_1}$ for II). If F is a solution, think $(z-w)F|_{\{z=w\}}$ (/ $(z-w)(F|_{\{z=w\}})^{-1}$).

We consider $(\mathbf{C}^*)^2$, which is a domain of holomorphy. With $\eta \in \mathbf{C}, \Im\eta \neq 0$ we let D be a zero locus locally defined by

$$w = z^\eta, \quad (z, w) \in (\mathbf{C}^*)^2.$$

Then the Cousin data given by D is not solvable; i.e. there is no holomorphic function on $(\mathbf{C}^*)^2$, having zeros only on D.

Cousin Integral (Cousin decomposition)

Let $E' \times E_1, E' \times E_2 \Subset \mathbf{C}^n$ be adjacent closed cuboids with open neighborhoods U_1 and U_2 (cf. Fig. 3). Let

$\{(U_\alpha, f_\alpha)\}_{\alpha=1,2}$ be a Cousin I data, and

$g = f_2 - f_1 \in \mathcal{O}(U_1 \cap U_2)$.

Cousin Integral: $\varphi(z', z_n) = \dfrac{1}{2\pi i} \displaystyle\int_\ell \dfrac{g(z', \zeta)}{\zeta - z_n} d\zeta$.

By Cauchy, on E_α $(\alpha = 1, 2)$,

$$\varphi_\alpha(z', z_n) = \varphi(z', z_n) = \frac{1}{2\pi i} \int_{\ell_\alpha} \frac{g(z', \zeta)}{\zeta - z_n} d\zeta,$$

$$\varphi_1 - \varphi_2 = g = f_2 - f_1 \text{ on } E_1 \cap E_2,$$

$$F = f_1 + \varphi_1 = f_2 + \varphi_2 \in \mathscr{M}(E_1 \cup E_2), \text{ Solution.}$$

It is Oka's great idea to reduce the general case to the above simple one by **Jôku-Ikô (上空移行)**: Ideal theoretic Jôku-Ikô = Coherence (連接).

Cousin Integral with estimate:

$E' \times$

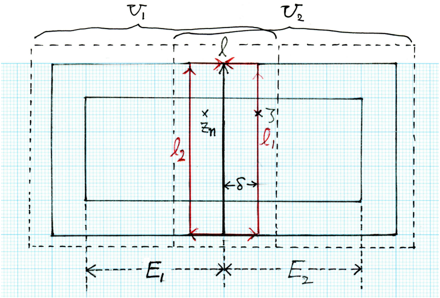


Figure: 3. Adjacent cuboid

With the notation above, let E_α $(\alpha = 1, 2)$ be as in Fig. 3. we have

$$\sup_{E_\alpha} |\varphi_\alpha| \leq \frac{\text{length of } \ell_\alpha}{2\pi\delta} \sup_{\ell_\alpha} |g| = \frac{C}{\delta} \sup_{\ell_\alpha} |g|, \quad \alpha = 1, 2,$$

where $\delta > 0$ comes from the estimate of the Cauchy kernel and $C > 0$ is a constant determined by the shape of the domain.

Theorem 3.1

The Cousin I/II Problem is always solvable on a polydisk $\mathrm{P}\Delta$.

Proof. C-I: Since $\mathrm{P}\Delta \cong$ an open cuboid($\subset \mathbf{C}^n$),

$^{\exists}$closed cuboids $E_\nu \nearrow \mathrm{P}\Delta, \nu = 1, 2, \ldots$.

Using **Cousin Integral** inductively, we have solutions F_ν on E_ν.

Using the **Approximation (Function Development in $\mathrm{P}\Delta$)**, modify F_ν so that

$$\text{(sup-norm) } \|F_{\nu+1} - F_\nu\|_{E_\nu} < \frac{1}{2^\nu}.$$

$F = F_1 + \sum_{\nu=1}^{\infty}(F_{\nu+1} - F_\nu)$, is a solution.

C-II: Similar with infinite product. □

N.B. This is the prototype method to obtain a solution.

§4 Weak Coherence

4.1 Weak Coherence Theorem. Let $a \in \mathbf{C}^n$,

f be a holomorphic function about a

$\mathcal{O}_a = \{\underline{f}_a = \sum c_\nu (z-a)^\nu$: conv. power series, germs$\}$ (a ring),

$\Omega \subset \mathbf{C}^n$ be a domain,

$\mathcal{O}_\Omega = \bigsqcup_{a \in \Omega} \mathcal{O}_a$ (sheaf as sets, no topology), $\quad \mathcal{O}_n = \mathcal{O}_{\mathbf{C}^n}$.

Consider: $\mathcal{O}_\Omega^q = \bigsqcup_{a \in \Omega} \mathcal{O}_a^q$ ($q \in \mathbf{N}$), naturally an $\mathcal{O}_\Omega$-module,

$\mathscr{S}_a \subset \mathcal{O}_a^q$, an $\mathcal{O}_a$-submodule.

$\mathscr{S} = \bigsqcup_{a \in \Omega} \mathscr{S}_a \subset \mathcal{O}_\Omega^q$, an $\mathcal{O}_\Omega$-submodule, an *analytic submodule*.

For an open subset $U \subset \Omega$, put

$\mathscr{S}(U) = \left\{ (f_j) \in \mathcal{O}(U)^q : \left(\underline{f_j}_a \right) \in \mathscr{S}_a, \ {}^\forall a \in U \right\}$ (sections).

Definition 4.1

An analy. submodule $\mathscr{S}$ over Ω is locally finite if for ${}^\forall a \in \Omega$, ${}^\exists U \ni a$, a nbd., and finitely many $\sigma_k \in \mathscr{S}(U), 1 \leq k \leq \ell$, such that

$$\mathscr{S}_z = \sum_{k=1}^{\ell} \mathcal{O}_z \cdot \underline{\sigma_k}_z, \quad {}^\forall z \in U.$$

$\{\sigma_k\}_{1 \leq k \leq \ell}$ is called a finite generator system of $\mathscr{S}$ on U.

Let $V \subset \Omega$ be an open subset, $\tau_k \in \mathscr{S}(V), 1 \leq k \leq N(< \infty)$, $\mathscr{R}(\tau_1, \ldots, \tau_N) \subset \mathcal{O}_V^N$ be the **relation sheaf** defined by

$$\mathscr{R}(\tau_j) = \bigsqcup_{a \in V} \left\{ \left(\underline{f_j}_a\right) \in \mathcal{O}_a^N : \sum_j \underline{f_j}_a \cdot \underline{\tau_j}_a = 0 \right\} \text{ (analy. submodule).}$$

For a subset $S \subset \Omega$, define the **ideal sheaf** of S by

$$\mathscr{I}\langle S \rangle = \bigsqcup_{a \in \Omega} \left\{ \underline{f}_a \in \mathcal{O}_a : f|_S = 0 \right\}.$$

Theorem 4.2 (Weak Coherence)

Let $S \subset \Omega$ be a complex submanifold, possibly non-connected.

1. *The ideal sheaf $\mathscr{I}\langle S \rangle$ is locally finite.*
2. *Let $\{\sigma_j \in \mathscr{I}\langle S \rangle(\Omega) : 1 \leq j \leq N\}$ be a finite generator system of $\mathscr{I}\langle S \rangle$ on Ω.*

   Then, the relation sheaf $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ is locally finite.

Remark. For the notion K. Oka used "*idéal de domaines indetérminées*", later termed "coherence" by H. Cartan.

What is the difference? I think:

"Idéal de domaines indetérminées" represents a way of thinking.

"Coherence" represents the formed object (concept).

Proof.

1. Locally, $S = \{z_1 = \cdots = z_q = 0\}$ in $U \subset \Omega$. Then,

$$\mathscr{I}\langle S\rangle = \sum_{j=1}^{q} \mathcal{O}_U \cdot z_j.$$

2. This is immediately reduced to the local finiteness of the relation sheaf defined

$$(4.3) \qquad \underline{f_1}_z \cdot z_1 + \cdots + \underline{f_q}_z \cdot z_q = 0.$$

Induction on q: (1) $q = 1$: Trivially $\mathscr{R}(z_1) = 0$, locally finite.
(2) Suppose it up to $q-1$ ($q \geq 2$) valid. For q, write

$$f_j = \sum_{\nu} c_\nu z^\nu = g_j(z_1, z')z_1 + h_j(z'), \quad z' = (z_2, \ldots, z_n).$$

Then, (4.3) is rewritten as

(4.4)

$$\underline{(f_1 + g_2 z_2 + \cdots + g_q z_q)}_z \cdot z_1 + \underline{h_2(z')}_z \cdot z_2 + \cdots + \underline{h_q(z')}_z \cdot z_q = 0 :$$

$$f_1 = -g_2 z_2 - \cdots - g_q z_q, \tag{4.5}$$

$$\underline{h_2(z')}_z \cdot z_2 + \cdots + \underline{h_q(z')}_z \cdot z_q = 0 \tag{4.6}$$

In (4.5), $g_2, \ldots, g_q$ are finite number of free variables, i.e., f_1 finitely generated.

(4.6) is the case "$q-1$"; by the induction hypothesis $(\underline{h_2}_z, \ldots, \underline{h_q}_z)$ is locally finitely generated.

Thus, $\mathscr{R}(z_1, \ldots, z_q)$ is locally finite. $\square$

4.2 Cartan's matrix decomposition.

Let A be an (N, N)-matrix with operator norm $\|A\|$. If $\|A\| < 1$,

$$(\mathbf{1}_N - A)^{-1} = \mathbf{1}_N + A + A^2 + \cdots .$$

If $A = A' + A''$ with $\|A'\|, \|A''\| < 1/2$, we have by the above:

$$\begin{aligned}
&(\mathbf{1}_N - (A' + A'')) = (\mathbf{1}_N - A') \\
&\cdot \underbrace{(\mathbf{1}_N - A')^{-1}(\mathbf{1}_N - (A' + A''))(\mathbf{1}_N - A'')^{-1}} \cdot (\mathbf{1}_N - A'') \\
&= (\mathbf{1}_N - A') \cdot (\mathbf{1}_N - N(A', A'')) \cdot (\mathbf{1}_N - A'').
\end{aligned}$$

Lemma 4.7

For $\|A'\|, \|A''\| \leq 1/2$, $\|N(A', A'')\| \leq 4 \max\{\|A'\|^2, \|A''\|^2\}$.

Let $\Omega \subset \mathbf{C}^n = \mathbf{C}^{n-1} \times \mathbf{C}$ be a domain,

$E', E'' \Subset \Omega$ be two closed cuboids as follows:

a closed cuboid $F \Subset \mathbf{C}^{n-1}$ and two adjacent closed rectangles $E'_n, E''_n \Subset \mathbf{C}$ sharing a side ℓ,

$$E' = F \times E'_n, \quad E'' = F \times E''_n, \quad \ell = E'_n \cap E''_n. \tag{4.8}$$

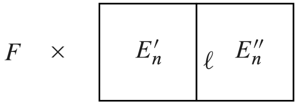


Figure: 4. Adjacent closed cuboids

Lemma 4.9 (Cartan's matrix decomposition)

Let

U be a neighborhood of $F \times \ell$,

$B(z)$ be an invertible (N, N)-matrix valued hol. function in U.

Then, $^{\exists}\epsilon_0 > 0$, sufficiently small such that if $\|\mathbf{1}_N - B\|_U < \epsilon_0$,

$^{\exists}B'(z), B''(z)$, invertible (N, N)-matrix valued holomorphic functions on E', E'', respectively, satisfying

$$B(z) = B'(z)B''(z) \quad \text{on } F \times \ell.$$

Proof.

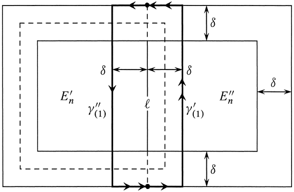


Figure: 5. δ-closed nbh. of closed cubes

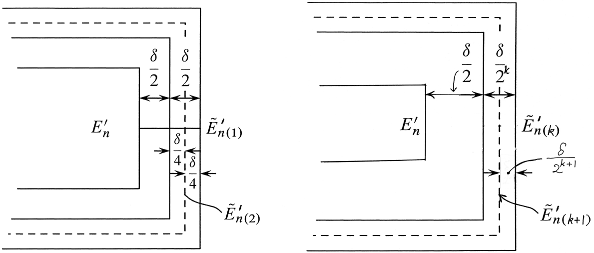


Figure: 6. $\frac{\delta}{2^k}$-closed nbh. of cubes

Using Cousin Integ. with estimate, we have a totological sequence

$$\begin{aligned} B &= (\mathbf{1} - A) = (\mathbf{1} - (A_1' + A_1'')) \\ &= (\mathbf{1} - A_1')(\mathbf{1} - N(A_1', A_1''))(\mathbf{1} - A_1'') \\ &= (\mathbf{1} - A_1') \cdots (\mathbf{1} - A_k')(\mathbf{1} - N(A_k', A_k''))(\mathbf{1} - A_k'') \cdots (\mathbf{1} - A_1'') \end{aligned}$$

such that: If $\|A\| \leq \epsilon_0$ is sufficiently small, we have

$$\|A_k'\|, \|A_k''\| \leq \frac{C}{2^k}, \quad C = C'\epsilon_0.$$

Inductively,

$$\begin{aligned} \|A_{k+1}'\|, \|A_{k+1}''\| \leq C' 2^{k+1} 4 \frac{C^2}{2^{2k}} &= 16 C' C \frac{C}{2^{k+1}} \\ &= 16 {C'}^2 \epsilon_0 \frac{C}{2^{k+1}} < \frac{C}{2^{k+1}}, \end{aligned}$$

where $\epsilon_0 > 0$ is taken so that $16 {C'}^2 \epsilon_0 < 1$.

$$B = (\mathbf{1} - A) = \prod_{k=1}^{\infty} (\mathbf{1} - A'_k) \cdot \prod_{k=\infty}^{1} (\mathbf{1} - A''_k) = B' \cdot B''.$$

See Appendix of [AFT]. $\square$

Lemma 4.10 (Generator merging, H. Cartan)

Let $E' \cup E''$ be adjacent cuboids, and
$\mathscr{S} \subset \mathcal{O}^N_{E' \cup E''}$ be an analy. submodule,
$\sigma' = \{\sigma'_j\}_{j=1}^{p'}$ (/ $\sigma'' = \{\sigma''_k\}_{k=1}^{p''}$) be a generator system of $\mathscr{S}$ over E' (/ E'') such that on $E' \cap E''$

$$\sigma' = A\sigma'', \quad \sigma'' = B\sigma'.$$

Then, $^\exists$ a generator system on $E' \cup E''$.

Proof. Put $\tilde{\sigma}' = {}^t(\sigma', 0_{p''})$, $\tilde{\sigma}'' = {}^t(0_{p'}, \sigma'')$,

$$\tilde{A} = \left(\begin{array}{c|c} \mathbf{1}_{p'} & A \\ \hline -B & \mathbf{1}_{p''} - BA \end{array}\right).$$

Then,

$$\tilde{\sigma}' = \tilde{A}\tilde{\sigma}'',$$

$$\tilde{A}^{-1} = \left(\begin{array}{c|c} \mathbf{1}_{p'} & -A \\ \hline 0 & \mathbf{1}_{p''} \end{array}\right) \left(\begin{array}{c|c} \mathbf{1}_{p'} & 0 \\ \hline B & \mathbf{1}_{p''} \end{array}\right).$$

Approximate A, B by polynomial elements and put them into the right-hand-side of $\tilde{A}^{-1}$, resulting R, such that $\|\mathbf{1} - \tilde{A}R\|$ is sufficiently small. By Cartan's matrix decomp., $\tilde{A}R = S'S''$; hence,

$$\tilde{\sigma}' = S'S''R^{-1}\tilde{\sigma}'', \quad S'^{-1}\tilde{\sigma}' = S''R^{-1}\tilde{\sigma}''.$$

$\square$

4.3 Oka Syzygy

Consider a closed cuboid $E \subset \mathbf{C}^n$, possibly degenerate with some edges of length 0. Define

dim E = the # of edges of positive lengths: $0 \leq \dim E \leq 2n$.

Lemma 4.11 (Oka Syzygy)

Let $E \Subset \mathbf{C}^n$ be a closed cuboid.

1. *Every locally finite submodule $\mathscr{S}(\subset \mathcal{O}_n^N)$ defined on E (i.e., in a neighborhood of E) has a finite generator system on E.*
2. *Let $\mathscr{S}$ be a submodule on E with a finite generator system $\{\sigma_j\}_{1 \leq j \leq L}$ on E such that $\mathscr{R}(\sigma_1, \ldots, \sigma_L)$ is locally finite. Then for $^\forall \sigma \in \mathscr{S}(E)$, $^\exists a_j \in \mathcal{O}(E)$, $1 \leq j \leq L$, such that*

$$\sigma = \sum_{j=1}^{L} a_j \cdot \sigma_j \quad (\text{on } E). \tag{4.12}$$

Proof.

Double **Cuboid Induction** on $\dim E$: $[1_{q-1}, 2_{q-1}] \Rightarrow 1_q \Rightarrow 2_q$

(a) $\dim E = 0$: 1, 2 Trivial by definition.

(b) Suppose them up to $\dim E = q-1, q \geq 1$, valid.

$\dim E = q$:

1. $2_{q-1}+$ Cartan's matrix decomposition.
2. Write with $T > 0, \theta \geq 0$:

$$E = F \times \{z_n = t + iy_n : 0 \leq t \leq T, |y_n| \leq \theta\},$$

$$\dim F = \begin{cases} q-1, & \theta = 0; \\ q-2, & \theta > 0. \end{cases}$$

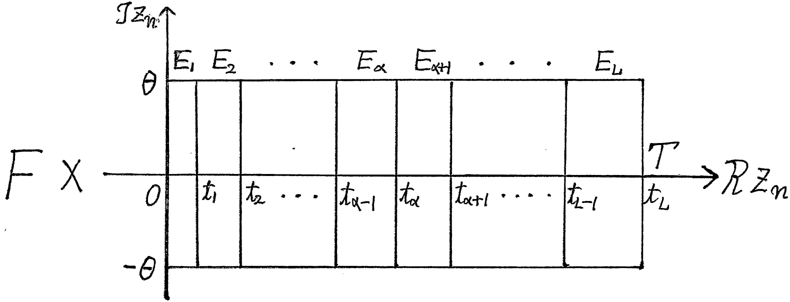


Figure: 7. $E_t \subset E$

Apply the induction hypothesis 2_{q-1} to

$E_t = F \times \{t + iy_n : |y_n| \leq \theta\}$ with $t \in [0, T]$. We then have

$$\sigma = \sum_{j=1}^{L} a_j \cdot \sigma_j \quad \text{(in a nbd. of) } E_t.$$

Let

$$\sigma = \sum_{j=1}^{L} a'_j \cdot \sigma_j, \quad \sigma = \sum_{j=1}^{L} a''_j \cdot \sigma_j$$

be such expressions in adjacent cuboids E', E'' with $E' \cap E'' = E_t$.
By 1_q, $^{\exists}$a generator system $\{\tau_k = (\tau_{kj})_j\}_k$ of $\mathscr{R}(\sigma_1, \ldots, \sigma_L)$ on E.
Since $\sum_{j=1}^{L}(a'_j - a''_j) \cdot \sigma_j = 0$ on E_t, we apply the induction hypothesis 2_{q-1} for $\mathscr{R}(\sigma_1, \ldots, \sigma_L)$ to get

$$(a'_j - a''_j) = \sum_k b_k \cdot (\tau_{kj}) \text{ on } E_{x_n}, \ b_k \in \mathcal{O}(E_t).$$

Apply Cousin Integral to $b_k = b'_k - b''_k$:

$$\left(a'_j - \sum_k b'_k \tau_{kj}\right) = \left(a''_j - \sum_k b''_k \tau_{kj}\right) = (a'''_j) \in \mathcal{O}(E' \cup E'')^L.$$

$$\sigma = \sum_j a'''_j \cdot \sigma_j, \text{ on } E' \cup E''.$$

Repeat this. $\square$

N.B. We apply this for $\mathscr{I}\langle S \rangle$ of a complex submanifold $S \subset \mathrm{P}\Delta$.

§5 Oka's Jôku-Ikô

Let

$\mathrm{P} \subset \mathbf{C}^n$ be an open cuboid,

$S \subset \mathrm{P}$ be a complex submanifold.

Lemma 5.1 (Oka's Jôku-Ikô)

Let $E \Subset \mathrm{P}$ be a closed cuboid. Then for
${}^{\forall} g \in \mathcal{O}(E \cap S)$ $(E \cap S \Subset S)$, ${}^{\exists} G \in \mathcal{O}(E)$ *satisfying*

$$G|_{E \cap S} = g|_{E \cap S}.$$

Proof. By

Weak Coherence of $\mathscr{I}\langle S\rangle$+Oka Syzygy + Cuboid Induction. $\square$

Approximation

An *analytic polyhedron* $\mathrm{P} \Subset \Omega$ is a finite union of relatively compact connected components of

$$\{z \in \Omega : |\psi_j(z)| < 1,\ 1 \le j \le L\},\ \psi_j \in \mathcal{O}(\Omega), L < \infty.$$

Theorem 5.2 (Runge–Weil–Oka)

Every holomorphic function on $\bar{\mathrm{P}}$ is uniformly approximated on $\bar{\mathrm{P}}$ by functions of $\mathcal{O}(\Omega)$.

Proof. Let $f \in \mathcal{O}(\bar{\mathrm{P}})$. By Oka map,

$$\Psi : z \in \bar{\mathrm{P}} \hookrightarrow (z, \psi_1(z), \ldots, \psi_L(z)) \in \overline{\mathrm{P}\Delta} \subset \mathbf{C}^{n+L},$$

$\bar{\mathrm{P}}$ is a complex submanifold of $\mathrm{P}\Delta$.

By Oka's Jôku-Ikô, extend f to $F \in \mathcal{O}(\overline{\mathrm{P}\Delta})$.

F is developed to a power series, and hence f is developed to a power series in z and (ψ_j). $\square$

§6 Continuous Cousin Problem

Let $\Omega = \bigcup_\alpha U_\alpha$ be an open covering and $\phi_\alpha \in C(U_\alpha)$, continuous functions.

Definition 6.1

$\{(U_\alpha, \phi_\alpha)\}$ is a continuous Cousin data if

$$\phi_\alpha - \phi_\beta \in \mathcal{O}(U_\alpha \cap \beta), \ {}^\forall \alpha, \beta.$$

Continuous Cousin Problem: Find a solution $\Phi \in C(\Omega)$ such that $\Phi - \phi_\alpha \in \mathcal{O}(U_\alpha)$, ${}^\forall \alpha$.

The following 3 problems are deduced from Cont. Cousin Problem:

1. Cousin I Problem.
2. Cousin II Problem.
3. Problem to solve $\bar{\partial} u = f$ with $\bar{\partial} f = 0$ for functions u.

$\because$) 1. May assume $\{U_\alpha\}$ locally finite.
Take open $V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, covering Ω, and $\chi_\alpha \in C(\Omega)$ such that $\chi_\alpha \geq 0$; $\chi_\alpha(z) > 0, z \in V_\alpha$; $\chi_\alpha(z) = 0, z \notin U_\alpha$; $\sum_\alpha \chi_\alpha = 1$.
For a Cousin I data (U_α, f_α), set

$$\phi_\alpha = \sum_\gamma (f_\alpha - f_\gamma)\chi_\gamma \in C(U_\alpha).$$

Then, $\phi_\alpha - \phi_\beta = f_\alpha - f_\beta$: $\quad f_\alpha - \phi_\alpha = f_\beta - \phi_\beta$.

Let Φ be a solution of $\{(U_\alpha, \phi_\alpha)\}$. Then

$$f_\alpha \underbrace{-\phi_\alpha + \Phi}_{\text{hol.}} = f_\beta \underbrace{-\phi_\beta + \Phi}_{\text{hol.}}.$$

2. Oka Principle: Let $\{(f_\alpha, U_\alpha)\}$ be a Cousin II data on Ω.

Oka Principle: *Assume the existence of a continuous solution $F \in C(\Omega)$ satisfying*

$$g_\alpha := F/f_\alpha \in C(U_\alpha), \quad \textit{nowhere vanishing.}$$

May assume that ${}^\forall U_\alpha$ is simply connected. Then, $h_\alpha := {}^\exists \log g_\alpha \in C(U_\alpha)$, and

$$h_\alpha - h_\beta = \log f_\beta / f_\alpha \in \mathcal{O}(U_\alpha \cap U_\beta);$$

the pairs (h_α, U_α) form a Continuous Cousin data.

Let $H \in C(\Omega)$ be a solution of $\{(h_\alpha, U_\alpha)\}$. Then,

$$G := e^{h_\alpha - H} \cdot f_\alpha = e^{h_\beta - H} \cdot f_\beta \in \mathscr{M}^*(\Omega) \quad \text{(solution)}.$$

3. **Dolbeault's Lemma:** *Let $f = \sum_{j=1}^{n} f_j d\bar{z}_j$ be a $(0,1)$-form of C^1-class defined in a neighborhood of a closed polydisk $\overline{\mathrm{P}\Delta}$ such that $\bar{\partial} f = 0$. Then there is a C^2 function u in $\mathrm{P}\Delta$ such that $\bar{\partial} u = f$.*

Let f be a $(0,1)$-form of C^1-class in Ω with $\bar{\partial} f = 0$.

Locally there are solutions,

$$u_\alpha \in C^\infty(U_\alpha),\ \bar{\partial} u_\alpha = f,\ \bigcup_\alpha U_\alpha = \Omega.$$

Since $\bar{\partial}(u_\alpha - u_\beta) = 0$, $(u_\alpha - u_\beta) \in \mathcal{O}(U_\alpha \cap U_\beta)$. The rest is the same as in 1.

Theorem 6.2

On a holomorphically convex domain every Continuous Cousin Problem is solvable.

Proof. Let $\Omega \subset \mathbf{C}^n$ be a holomorphically convex domain, and $\{(U_\alpha, \phi_\alpha)\}$ be a Continuous Cousin data on Ω.

Take $\mathrm{P}_\nu \nearrow \Omega$, increasing analytic polyhedra, and the Oka maps $\psi_\nu : \bar{\mathrm{P}}_\nu \hookrightarrow \overline{\mathrm{P}\Delta}_{(\nu)}$.

Step 1. Obtain a solution Φ_ν on each $\bar{\mathrm{P}}_\nu \hookrightarrow \overline{\mathrm{P}\Delta}_{(\nu)}$.

By Cuboid Induction + Oka's Jôku-Ikô + Cousin Integral.

$E \subset$ nbd of $\overline{\mathrm{P}\Delta}_{(\nu)}$, a closed cuboid with $q = \dim E$.

Claim. $^{\exists}$ Continuous solution f in a nbd. of $S = \psi_\nu(\bar{\mathrm{P}}_\nu) \cap E$ in $\psi_\nu(\bar{\mathrm{P}}_\nu) \cong \bar{\mathrm{P}}_\nu \Subset \Omega$.

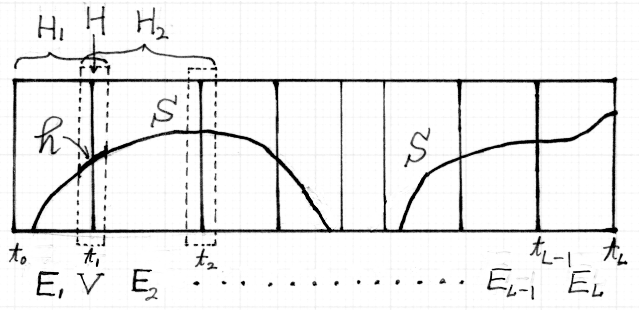


Figure: 8. Cuboid Induction

Step 2. Since $\Phi_{\nu+1} - \Phi_\nu \in \mathcal{O}(\bar{\mathrm{P}}_\nu)$, applying the Approximation of Runge-Weil-Oka, modify Φ_ν so that

$$\|\Phi_{\nu+1} - \Phi_\nu\|_{\bar{\mathrm{P}}_\nu} < \frac{1}{2^\nu},\ \nu = 1, 2, \ldots.$$

We have a solution,

$$\Phi = \Phi_1 + \sum_{\nu=1}^{\infty} (\Phi_{\nu+1} - \Phi_\nu).$$

$\square$

§7 Interpolation

In the same way as in the previous section we have

Theorem 7.1 (Interpolation)

Let $\Omega \subset \mathbf{C}^n$ be a holomorphically convex domain and
$S \subset \Omega$ be a complex submanifold.
Then, $f \in \mathcal{O}(\Omega) \to f|_S \in \mathcal{O}(S) \to 0$ (surjective).
If particular, for $^\forall\{a_\nu\}$, a discrete sequence of Ω and $^\forall c_\nu \in \mathbf{C}$, $^\exists F \in \mathcal{O}(\Omega)$ with $F(a_\nu) = c_\nu, ^\forall \nu$. Conversely, if it holds for Ω, Ω is holomorphically convex.

Proof. Jôku-Ikô. $\square$

§8 Levi (Hartogs' Inverse) Problem

Let $\mathrm{P}\Delta \subset \mathbf{C}^n$ be any fixed polydisk with center at 0, and $\Omega \subset \mathbf{C}^n$ be a domain. Put

$$\delta_{\mathrm{P}\Delta}(z, \partial\Omega) = \sup\{r > 0 : z + r \cdot \mathrm{P}\Delta \subset \Omega\}, \ z \in \Omega.$$

Theorem 8.1 (Oka)

If Ω is holomorphically convex (domain of holomorphy), $-\log \delta_{\mathrm{P}\Delta}(z, \partial\Omega)$ is plurisubharmonic in $z \in \Omega$.

We call Ω a **pseudoconvex domain** if $-\log \delta_{\mathrm{P}\Delta}(z, \partial\Omega)$ is plurisubharmonic near $\partial\Omega$.

Levi (Hartogs' Inverse) Problem: Is a pseudoconvex domain holomorphically convex?

A bounded domain $\Omega \subset \mathbf{C}^n$ is said to be **strongly pseudoconvex** if for ${}^{\forall}a \in \partial\Omega$, ${}^{\exists}U \ni a$, a neighborhood and $\varphi \in C^2(U)$ such that $U \cap \Omega = \{\varphi < 0\}$ and

$$i\partial\bar{\partial}\varphi(z) \gg 0, \ z \in U.$$

- If Ω is pseudoconvex, ${}^{\exists}\Omega_\nu \nearrow \Omega$ with strongly pseudoconvex Ω_ν.

The 1st cohomology $H^1(\Omega, \mathcal{O})$ as a $\mathbf{C}$-vector space.

Definition. Let $\Omega = \bigcup U_\alpha$, $\mathscr{U} = \{U_\alpha\}$. Define

$Z^1(\mathscr{U}, \mathcal{O})$, 1-cycle space,

$\delta : C^0(\mathscr{U}, \mathcal{O}) \to B^1(\mathscr{U}, \mathcal{O})$, a boundary operator,

$H^1(\mathscr{U}, \mathcal{O}) = Z^1(\mathscr{U}, \mathcal{O})/B^1(\mathscr{U}, \mathcal{O})$,

$H^1(\Omega, \mathcal{O}) = \lim\limits_{\substack{\rightarrow \\ \mathscr{U}}} H^1(\mathscr{U}, \mathcal{O}) \hookleftarrow H^1(\mathscr{U}, \mathcal{O})$.

- $H^1(\Omega, \mathcal{O}) = 0 \iff {}^\forall$Cont. Cousin Problem is solvable on Ω.

Theorem 8.2

1. *If Ω is holomorphically convex, $H^1(\Omega, \mathcal{O}) = 0$.*
2. *For $\mathscr{U} = \{U_\alpha\}$ an open covering of Ω with ${}^\forall U_\alpha$, holomorphically convex,*

$$H^1(\mathscr{U}, \mathcal{O}) \cong H^1(\Omega, \mathcal{O}).$$

L. Schwartz Theorem

Let E be a Hausdorff topological complex vector space with at most countably many semi-norms;

E is **Fréchet**, if the associated distance on E is complete;

E is **Baire**, if E satisfies Baire's Category Theorem.

Theorem 8.3 (Open Map)

Let E (resp. F) be a Fréchet (resp. Baire) vector space.

If $A : E \to F$ is a continuous linear surjection,

then A is an open map.

Theorem 8.4 (L. Schwartz's Finiteness Theorem)

Let E (resp. F) be a Fréchet (resp. Baire) vector space. Let $A : E \to F$ be a continuous linear surjection, and $B : E \to F$ be a compact operator. Then $(A+B)E$ is closed, and

$$\dim \mathrm{Coker}(A+B)(:= F/(A+B)E) < \infty.$$

Proof. Heuristic: With $C := A + B$ we have

$$CE + BE = F.$$

Taking a quotient by CE, one gets

$$BE/CE = F/CE = \mathrm{Coker}\, C.$$

Since B is a compact operator, BE/CE is a locally compact topological vector space: Hence it is finite dimensional.

But, the closedness of CE is not known.

All these are proved at once by showing

$$F = (A+B)E + {}^{\exists}\langle b_1, \ldots, b_N \rangle_{\mathbf{C}}, \ b_j \in F, \ N < \infty \quad \text{(algebraically).}$$

So, how to find b_j?

(Demailly's idea) Let U be a neighborhood of $0 \in E$ such that $\overline{B(U)}$ is compact. Since $A(U)$ is open (Open Map Thm.), ${}^{\exists}b_j \in \overline{B(U)}, \ 1 \le j \le N < \infty$, such that

$$\overline{B(U)} \subset \bigcup_j \left(b_j + \frac{1}{2} A(U) \right).$$

Modify $\{b_j\}$ so that b_j are linearly independent and

$$(E / \ker(A+B)) \oplus \langle b_1, \ldots, b_N \rangle_{\mathbf{C}} \ni ([x], y) \mapsto (A+B)x \oplus y \in F$$

is a topological isomorphism again by Open Map Thm. Therefore, $(A+B)E$ is closed and $\dim \mathrm{Coker}(A+B) = N < \infty$. □

Theorem 8.5 (Grauert)

Let Ω be a strongly pseudoconvex domain. Then, $\dim H^1(\Omega, \mathcal{O}) < \infty$.

Proof (Grauert's bumping method).

$\Omega = {}^{\exists}\bigcup_{\text{finite}} V_\alpha$ with V_α, hol. convex,

bumped open ${}^{\exists}\tilde{U}_\alpha \Supset V_\alpha$ with $\tilde{U}_\alpha$, hol. convex,

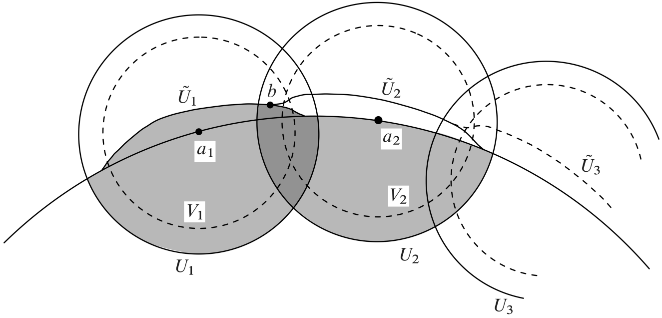


Figure: Boundary bumping method

$\mathscr{V} = \{V_\alpha\}$, bumped covering $\tilde{\mathscr{U}} = \{\tilde{U}_\alpha\}$ ($\Supset \Omega$), so that

$$\tilde{U}_\alpha \cap \tilde{U}_\beta \Supset V_\alpha \cap V_\beta,$$

$$\Psi : \xi \oplus \eta \in Z^1(\tilde{\mathscr{U}}, \mathcal{O}) \oplus C^0(\mathscr{V}, \mathcal{O}) \to \rho(\xi) + \delta\eta \in Z^1(\mathscr{V}, \mathcal{O}) \to 0,$$

where ρ is the restriction map from the bumped $\tilde{\mathscr{U}}$ to $\mathscr{V}$.

Note that $Z^1(\tilde{\mathscr{U}}, \mathcal{O}) \oplus C^0(\mathscr{V}, \mathcal{O})$ and $Z^1(\mathscr{V}, \mathcal{O})$ are Fréchet (in particular, the latter is Baire).

Since ρ is compact (Montel), L. Schwartz applied to Ψ and $-\rho$ yields that $\mathrm{Coker}(\Psi - \rho) \cong H^1(\mathscr{V}, \mathcal{O}) \cong H^1(\Omega, \mathcal{O})$ is finite dimensional. $\square$

Theorem 8.6 (Oka)

A strongly pseudoconvex domain is holomorphically convex.

Proof. Let φ be a defining function of $\partial\Omega$ such that $\Omega = \{\varphi < 0\}$, φ is strongly plurisubharmonic in a neighborhood of $\partial\Omega$.

Take a point $b \in \partial\Omega$. By a translation, we may put $b = 0$. Set

$$Q(z) = 2\sum_{j=1}^{n} \frac{\partial\varphi}{\partial z_j}(0) z_j + \sum_{j,k} \frac{\partial^2\varphi}{\partial z_j \partial z_k}(0) z_j z_k.$$

${}^{\exists}\varepsilon, \delta > 0$ satisfying

$$\varphi(z) \geq \Re\, Q(z) + \varepsilon\|z\|^2, \quad \|z\| \leq \delta,$$

$$\inf\{\varphi(z); Q(z) = 0, \|z\| = \delta\} \geq \varepsilon\delta^2 > 0.$$

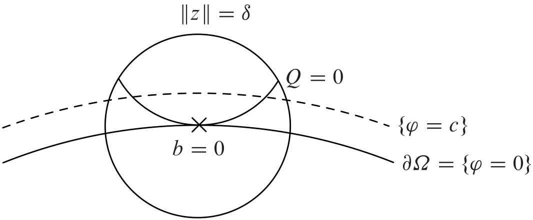


Figure: $\Omega' = \{\varphi < c\}, U_0$

Let $\Omega' = \{\varphi < c\}$ with very small $c > 0$, $U_1 = \Omega' \setminus \{Q = 0\}$. Then $\mathscr{U} = \{U_0, U_1\}$ is an open covering of Ω', which is strongly pseudoconvex.

We set

$$f_{01}(z) = \frac{1}{Q(z)}, \quad z \in U_0 \cap U_1,$$
$$f_{10}(z) = -f_{01}(z), \quad z \in U_1 \cap U_0.$$

Then, a 1-cocyle $f = (f_{01}(z), f_{10}(z)) \in Z^1(\mathscr{U}, \mathcal{O})$ is obtained. For $k \in \mathbf{N}$ we define

$$f_{01}^{[k]}(z) = (f_{01}(z))^k, \quad z \in U_0 \cap U_1,$$
$$f_{10}^{[k]}(z) = -f_{01}^{[k]}(z), \quad z \in U_1 \cap U_0.$$

Then $(f^{[k]}) \in Z^1(\mathscr{U}, \mathcal{O})$. Thus we obtain cohomology classes,

$$[f^{[k]}] \in H^1(\mathscr{U}, \mathcal{O}) \hookrightarrow H^1(\Omega', \mathcal{O}), \quad k \in \mathbf{N}.$$

Since Ω' is strongly pseudoconvex, Grauert's Theorem implies $\dim H^1(\Omega', \mathcal{O}) < \infty$.

Therefore, for N large, there is a non-trivial linear relation,

$$\sum_{k=1}^{N} c_k [f^{[k]}] = 0 \in H^1(\mathscr{U}, \mathcal{O}_{\Omega'}) \quad (c_k \in \mathbf{C}).$$

We may suppose that $c_N \neq 0$. Then there exists elements $g_i \in \mathcal{O}(U_i)$, $i = 0, 1$, such that

$$\sum_{k=1}^{N} \frac{c_k}{Q^k(z)} = g_1(z) - g_0(z), \quad z \in U_0 \cap U_1.$$

Therefore,

$$g_0(z) + \sum_{k=1}^{N} \frac{c_k}{Q^k(z)} = g_1(z), \quad z \in U_0 \cap U_1, \quad c_N \neq 0.$$

(8.7) $\quad {}^{\exists}F \in \mathscr{M}(\Omega')$ with poles of order N on $\{Q = 0\}$.

Since $\{Q = 0\} \cap \Omega = \emptyset$, $F|_\Omega \in \mathcal{O}(\Omega)$ and $\lim_{z \to 0} |F(z)| = \infty$.
Thus, Ω is holomorphically convex.

Theorem 8.8 (Oka)

A pseudoconvex domain is holomorphically convex.

Proof. There are strongly pseudoconvex domains $\Omega_\nu \nearrow \Omega$. Since Ω_ν are holomorphically convex, so is the limit Ω (Behnke–Stein). □

Furthermore, we have

Theorem 8.9 (Oka)

A pseudoconvex unramified Riemann domain over $\mathbf{C}^n$ is holomorphically convex and holomorphically separable (i.e., a Stein manifold).

Proof.

Let $\pi : \Omega \to \mathbf{C}^n$ be an unramified Riemann domain. Assume that $-\log \delta_{\mathrm{P}\boldsymbol{\Delta}}(x, \partial\Omega)$ is plurisubharmonic near $\partial\Omega$.

Step 1°: Construct a (continuous) plurisubharmonic exhaustion $\lambda : \Omega \to \mathbf{R}$.

Step 2°: Show that $\Omega_c = \{\lambda < c\}$ with ${}^\forall c \in \mathbf{R}$ is holomorphically convex. We may enlarge a little bit Ω_c to a strongly pseudoconvex domain Ω'_c. Then apply the same argument as in the case of univalent domains.

Step 3° (Hol. Separability): Take two distinct points $Q_1, Q_2 \in \Omega'_c$.
We may assume: $\pi(Q_1) = \pi(Q_2) = a \in \mathbf{C}^n$.
Let $\phi(t), t \geq 0$, be any affine linear curve with $\phi(0) = a$.
Then lifting ${}^{\exists 1}\phi_j(t) \in \Omega'_c$ of $\phi(t)$ such that $\phi_j(0) = Q_j\ (j = 1, 2)$.
Since Ω'_c is relatively compact, $\phi_j(t)$ hits the boundary $\partial\Omega'_c$.
We may assume that $\phi_1(t)$ hits $\partial\Omega'_c$ first with $t = T \in \mathbf{R}$, so that $\phi_j([0, T]) \subset \bar{\Omega}'_c\ (j = 1, 2)$ and $\phi_1(T) \in \partial\Omega'_c$.
Note that $\phi_1(T) \neq \phi_2(T)$.
With setting $b = \phi_1(T)$ we have by (8.7) a meromorphic function F_b in $\Omega'_{c\epsilon} \Supset \Omega'_c$ which is holomorphic in Ω'_c.

Consider the Taylor expansions of F_b at Q_1 and Q_2 in $(z_1, \ldots, z_n)$. Since F_b has a pole at $\phi_1(T)$ and no pole at $\phi_2(T)$, those two expansions must be different. Therefore, there is some partial differential operator $\partial^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}$ with a multi-index α such that

$$\partial^\alpha F_b(Q_1) \neq \partial^\alpha F_b(Q_2).$$

Since $\partial^\alpha F_b$ is holomorphic in Ω'_c, this finishes the proof of hol. separation..

Step 4°: For every pair $c < c' (\in \mathbf{R})$, $\Omega_c \Subset \Omega_{c'}$ is a Runge pair (by Jôku-Ikô). $\square$