A Remark to a Division Algorithm in the Proof of Oka's First Coherence Theorem

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Abstract

The problem is the locally finite generation of a relation sheaf $\mathscr{R}(\tau_1, \ldots, \tau_q)$ in $\mathcal{O}_{\mathbf{C}^n}$. After τ_j reduced to Weierstrass' polynomials in z_n, it is the key for applying an induction on n to show that elements of $\mathscr{R}(\tau_1, \ldots, \tau_q)$ are expressed as a finite linear sum of z_n-polynomial-like elements of degree at most $p = \max_j \deg_{z_n} \tau_j$ over $\mathcal{O}_{\mathbf{C}^n}$. In that proof one is used to use a division by τ_j of the *maximum degree*, $\deg_{z_n} \tau_j = p$ (Oka '48, Cartan '50, L. Hörmander '66, R. Narasimhan '66, T. Nishino '96, ...). Here we shall confirm that the division above works by making use of τ_k of the *minimum degree*, $\min_j \deg_{z_n} \tau_j$. This proof is naturally compatible with the simple case when some τ_j is a unit, and gives some improvement in the degree estimate of generators.

1 Introduction and results

It will be of no necessity to mention the importance of Oka's First Coherence Theorem that the sheaf $\mathcal{O}_{\mathbf{C}^n}$ (also denoted simply by $\mathcal{O}_n$) of germs of holomorphic functions over n-dimensional complex vector space $\mathbf{C}^n$ (Oka [7], [8])[2]. Let $\Omega \subset \mathbf{C}^n$ be an open set and let $\tau_j \in \mathcal{O}(\Omega) := \Gamma(\Omega, \mathcal{O}_n)$, $1 \leq j \leq q$. Oka's First Coherence Theorem claims that the relation sheaf $\mathscr{R}(\tau_1, \ldots, \tau_q)$ defined by

$$f_1 \underline{\tau_1}_z + \cdots + f_q \underline{\tau_q}_z = 0, \quad f_j \in \mathcal{O}_{n,z},\ z \in \Omega.$$

is locally finite in Ω, where $\underline{*}_z$ stands for the germ at z. The problem is local, so that we consider in a neighborhood of a point $a \in \Omega$; further we may assume $a = 0$ with complex coordinate system $(z_1, \ldots, z_n)$.

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[2]There are some differences in these two versions of Oka VII.

By Weierstrass' Preparation Theorem τ_j are reduced to Weierstrass' polynomials $P_j \in \mathcal{O}(\mathrm{P}\Delta_{n-1})[z_n]$ about 0, where $\mathrm{P}\Delta_{n-1}$ is a small polydisk in $z' = (z_1, \ldots, z_{n-1}) \in \mathbf{C}^{n-1}$. Set

$$\begin{aligned} \mathscr{R} &= \mathscr{R}(P_1, \ldots, P_q), \\ p &= \max_{1 \le j \le q} \deg_{z_n} P_j, \\ p' &= \min_{1 \le j \le q} \deg_{z_n} P_j \,. \end{aligned}$$

We call $f \in \mathcal{O}_{n-1,b'}[z_n]$ (resp. $f \in \mathcal{O}(\mathrm{P}\Delta_{n-1})[z_n]$) a *$z_n$-polynomial-like* germ (resp. function) and denote by $\deg_{z_n} f$ its degree in variable z_n; for convention, "$\deg_{z_n} f < 0$" means "$f = 0$". We also call an element $(f_j) \in (\mathcal{O}_{n,(b',b_n)})^q$ (resp. $(f_j) \in (\mathcal{O}(\mathrm{P}\Delta_{n-1} \times \mathbf{C}))^q$) with $f_j \in \mathcal{O}_{\mathrm{P}\Delta_{n-1},b'}[z_n]$ (resp. $f_j \in \mathcal{O}(\mathrm{P}\Delta_{n-1})[z_n]$) a *$z_n$-polynomial-like* element (resp. section), and $\deg_{z_n}(f_j) = \max_j \deg_{z_n} f_j$ the degree of (f_j).

The proof of the local finiteness of $\mathscr{R}$ relies on the induction on n, and the key which makes the induction to work is:

Lemma A. *Every element of $\mathscr{R}_b$ at $b = (b', b_n)$ with $b' \in \mathrm{P}\Delta_{n-1}$ is expressed as a finite linear sum of z_n-polynomial-like elements of $\mathscr{R}_b$ of degree at most p with coefficients in $\mathcal{O}_b$.*

There is some structure in the generator system with respect to the degree in z_n. For $1 \le i < j \le q$ there are sections of $\mathscr{R}$ given by

$$T_{i,j} = (0, \ldots, 0, \overset{i\text{-th}}{P_j}, 0, \ldots, 0, \overset{j\text{-th}}{-P_i}, 0, \ldots, 0),$$

which we call the *trivial solutions*, and are z_n-polynomial-like sections of $\deg_{z_n} T_{i,j} \le p$. Without loss of generality we may assume that

$$\begin{aligned} p_1 &= p', \\ p_q &= p, \end{aligned}$$

and set

$$T_j = T_{1,j}, \quad 2 \le j \le q.$$

In the proof of Lemma A a division algorithm is applied; in the original proof of Oka as well as in many references such as H. Cartan [1], R. Narasimhan [4], L. Hörmander [3], T. Nishino [5], J. Noguchi [6],... etc., the division algorithm by P_q of the maximum degree is used to conclude the existence of a finite generator system consisting of $T_{i,q}$ of degree $\leq p$, $1 \leq i \leq q-1$, and a finite number of z_n-polynomial-like elements α of degree $< p$. In case $p' = 0$, it is immediate that the trivial solutions T_j with $2 \leq j \leq q$ form already a generator system, while by the original proof one still needs elements α of degree $< p$.

The aim of this note is to confirm that Oka's original proof still works with the division algorithm by P_1 of the minimum degree in z_n:

Lemma 1.1. *Let the notation be as above. Then an element of $\mathscr{R}_b$ is written as a finite linear sum of the trivial solutions, T_j, $2 \leq j \leq q$, and z_n-polynomial-like elements $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q)$ of $\mathscr{R}_b$ with coefficients in $\mathscr{O}_{n,b}$ such that*

$$
\begin{aligned}
&\deg_{z_n} \alpha_1 \leq p - 1, \\
&\deg_{z_n} \alpha_j \leq p' - 1, \quad 2 \leq j \leq q.
\end{aligned}
\tag{1.2}
$$

N.B. If $p' = 0$, then there is no term of α, and if $p' = 1$. α_j are constants for $2 \leq j \leq q$.

To decrease $p - 1$ in (1.2) one needs to transform the relation sheaf $\mathscr{R}(P_1, P_2, \ldots, P_q)$ with dividing P_j $(2 \leq j \leq q)$ by P_1 (here we use an idea from Hironaka's proof, cf. [2]). Set

$$
\begin{aligned}
P_j = &Q_j P_1 + R_j, \quad Q_j, R_j \in \mathscr{O}_{n-1}(\mathrm{P}\Delta_{n-1})[z_n], \\
&\deg_{z_n} R_j \leq p' - 1, \ 2 \leq j \leq q.
\end{aligned}
$$

Then for $(f_j) \in (\mathcal{O}_{n,z})^q$ we have

$$\sum_{j=1}^{q} f_j \underline{P_j}_z = \left(f_1 + \sum_{j=2}^{q} f_j \underline{Q_j}_z \right) \underline{P_1}_z + \sum_{j=2}^{q} f_j \underline{R_j}_z \tag{1.3}$$
$$= h_1 \underline{P_1}_z + \sum_{j=2}^{q} f_j \underline{R_j}_z,$$

where $h_1 = f_1 + \sum_{j=2}^{q} f_j \underline{Q_j}_z$. Thus the locally finite generation of $\mathscr{R}(P_1, \ldots, P_q)$ is equivalent to that of $\mathscr{R}(P_1, R_2, \ldots, R_q)$. Let

$$T'_j = (R_j, 0, \ldots, 0, \overset{j\text{-th}}{-P_1}, 0, \ldots, 0), \quad 2 \leq j \leq q$$

be the trivial solutions of $\mathscr{R}(P_1, R_2, \ldots, R_q)$, which are z_n-polynomial-like sections of $\deg_{z_n} T'_j = p'$.

Lemma 1.4. *Set $\mathscr{R}' := \mathscr{R}(P_1, R_2, \ldots, R_q)$ be as above. Then an element of $\mathscr{R}'_b$ is written as a finite linear sum of the trivial solutions, T'_j, $2 \leq j \leq q$, of degree p' and z_n-polynomial-like elements $\alpha' = (\alpha'_1, \alpha_2, \ldots, \alpha_q)$ of $\mathscr{R}'_b$ with coefficients in $\mathcal{O}_{n,b}$ such that*

$$\begin{aligned} &\deg_{z_n} \alpha'_1 \leq p' - 2, \\ &\deg_{z_n} \alpha_j \leq p' - 1, \quad 2 \leq j \leq q. \end{aligned} \tag{1.5}$$

N.B. If $p' = 0$, then there is no term of α', and if $p' = 1$. then $\alpha'_1 = 0$ and α'_j are constants for $2 \leq j \leq q$.

2 Proofs of Lemmas

(1)(Lemma 1.1) By making use of Weierstrass' Preparation Theorem at $b = (b', b_n)$ with $b' \in \mathrm{P}\Delta_{n-1}$ we decompose P_1 to a unit u and a Weierstrass polynomial Q:

$$P_1(z', z_n) = u \cdot Q(z', z_n - b_n), \qquad \deg_{z_n} Q = d \leq p_1.$$

Here and in the sequel we abbreviate $\underline{Q}_z$ to Q for the sake of notational simplicity; there will be no confusion.

It follows that $u \in \mathcal{O}_{n-1,b'}[z_n]$, and then

$$\deg_{z_n} u = p_1 - d. \tag{2.1}$$

Take an arbitrary $f = (f_1, \ldots, f_q) \in \mathscr{R}_b$. By Weierstrass' Preparation Theorem we divide f_i by Q:

$$\begin{aligned} f_i =& c_i Q + \beta_i, \quad 1 \leq i \leq q, \\ & c_i \in \mathcal{O}_{n,b}, \quad \beta_i \in \mathcal{O}_{n-1,b'}[z_n], \\ & \deg_{z_n} \beta_i < d. \end{aligned} \tag{2.2}$$

Since $u \in \mathcal{O}_{n,b}$ is a unit, with $\tilde{c}_i := c_i u^{-1}$ we get the division of f_i by P_1:

$$f_i = \tilde{c}_i P_1 + \beta_i, \qquad 1 \leq i \leq q. \tag{2.3}$$

By making use of this we have

$$\begin{aligned} & (f_1, \ldots, f_q) + \tilde{c}_2 T_2 + \cdots + \tilde{c}_q T_q \\ &= (\tilde{c}_1 P_1 + \beta_1, \tilde{c}_2 P_1 + \beta_2, \ldots, \tilde{c}_q P_1 + \beta_q) \\ &\quad + (\tilde{c}_2 P_2, -\tilde{c}_2 P_1, 0, \ldots, 0) \\ &\quad + \ \cdots \\ &\quad + (\tilde{c}_q P_q, 0, \ldots, 0, -\tilde{c}_q P_1) \\ &= \left(\sum_{i=1}^{q} \tilde{c}_i P_i + \beta_1, \beta_2, \ldots, \beta_q \right) \\ &= (g_1, \beta_2, \ldots, \beta_q). \end{aligned} \tag{2.4}$$

Here we put $g_1 = \sum_{i=1}^{q} \tilde{c}_i P_i + \beta_1 \in \mathcal{O}_{n,b}$. Note that $\beta_i \in \mathcal{O}_{n-1,b'}[z_n]$, $2 \leq i \leq q$. Since $(g_1, \beta_2, \ldots, \beta_q) \in \mathscr{R}_b$,

$$g_1 P_1 = -\beta_2 P_2 - \cdots - \beta_q P_q \in \mathcal{O}_{n-1,b'}[z_n]. \tag{2.5}$$

It should be noticed that if $p_1 = 0$, then $P_1 = 1$, $\beta_i = 0$, $1 \le i \le q$, and hence $g_1 = 0$; the proof is finished in this case.

In general, it follows from the expression of the above right-hand side of (2.5) that $g_1 P_1 \in \mathcal{O}_{n-1,b'}[z_n]$ and

$$\deg_{z_n} g_1 P_1 \le \max_{2\le i\le q} \deg_{z_n} \beta_i + \max_{2\le i\le q} \deg_{z_n} P_i \le d + p - 1.$$

On the other hand, $g_1 P_1 = g_1 u Q$ and Q is a Weierstrass' polynomial at b. We see that

$$\begin{aligned} \alpha_1 &:= g_1 u \in \mathcal{O}_{n-1,b'}[z_n], \\ \deg_{z_n} \alpha_1 &= \deg_{z_n} g_1 P_1 - \deg_{z_n} Q \\ &\le d + p - 1 - d = p - 1. \end{aligned} \tag{2.6}$$

Set $\alpha_i = u\beta_i$ for $2 \le i \le q$. Then, by (2.1) and (2.2) we have

$$\deg_{z_n} \alpha_i \le p_1 - d + d - 1 = p_1 - 1 = p' - 1, \quad 2 \le i \le q, \tag{2.7}$$

and by (2.9) that

$$f = -\sum_{i=2}^{q} \tilde{c}_i T_i + u^{-1}(\alpha_1, \alpha_2, \ldots, \alpha_q). \tag{2.8}$$

$\square$

(2)(Lemma 1.4) First note that $(f_1, \ldots, f_q)$ and $(h_1, f_2, \ldots, f_q)$ with $h_1 = f_1 + \sum_{j=2}^{q} f_j Q_j$ as defined in (1.3) are related by

$$\begin{pmatrix} h_1 \\ f_2 \\ \vdots \\ f_q \end{pmatrix} = \begin{pmatrix} 1 & Q_2 & \cdots & Q_q \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_q \end{pmatrix},$$

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_q \end{pmatrix} = \begin{pmatrix} 1 & -Q_2 & \cdots & -Q_q \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ f_2 \\ \vdots \\ f_q \end{pmatrix}.$$

Therefore, the locally finite generation of $\mathscr{R}$ is equivalent to that of $\mathscr{R}'$.

The proof is similar to the above except for some degree estimates. Now we have for $(f_j) \in (\mathcal{O}_{n,b})^q$

$$(2.9) \qquad \begin{aligned} &(f_1, \ldots, f_q) + \tilde{c}_2 T_2' + \cdots + \tilde{c}_q T_q' \\ &= \left(\tilde{c}_1 P_1 + \beta_1 + \sum_{i=2}^{q} \tilde{c}_i R_i, \beta_2, \ldots, \beta_q \right) \\ &= (h_1, \beta_2, \ldots, \beta_q) \,. \end{aligned}$$

Here we put $h_1 = \tilde{c}_1 P_1 + \beta_1 + \sum_{i=2}^{q} \tilde{c}_i R_i \in \mathcal{O}_{n,b}$. In stead of (2.5) we have

$$(2.10) \qquad h_1 P_1 = -\beta_2 R_2 - \cdots - \beta_q R_q \in \mathcal{O}_{n-1,b'}[z_n].$$

From this we obtain

$$\deg_{z_n} h_1 P_1 \leq d - 1 + p' - 1 = d + p' - 2.$$

With $\alpha_1' := h_1 u$ we have $h_1 P_1 = h_1 u Q = \alpha_1' Q$ and so

$$\deg_{z_n} \alpha_1' \leq d + p' - 2 - d = p' - 2.$$

For $\alpha_i = u\beta_i$, $2 \leq i \leq q$ we have the same estimate:

$$\deg_{z_n} \alpha_i \leq p' - 1.$$

With the above defined we have

$$f = -\sum_{i=2}^{q} \tilde{c}_i T_i' + u^{-1}(\alpha_1', \alpha_2, \ldots, \alpha_q).$$

$\square$

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