Some remarks on basic materials in several complex variables

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1. 序：学部向けの多変数教科書

- 証明をきちんと付ける。読めば解る。そのまま、板書できる。証明で、引用はしない。

- 局所理論が基礎で、他分野でも多用される。中でも、
  連接定理 (Coherence Theorems)
  (1) 第 1: $\mathscr{O}_{\mathbf{C}^n}$ (Oka VII, 1948/'50).
  (2) 第 2: $\mathscr{I}\langle A\rangle$, 幾何学的イデアル (Oka VIII, 1951/H. Cartan 1950).
  (3) 第 3: $\hat{\mathscr{O}}_X$, 正規化 (Oka VIII, 1951).

  第 2 $\Rightarrow$ 解析的集合の特異点集合の解析性。

- Hartogs 現象、正則包。正則（凸）領域。

- 近似、Cousin I+II・岡原理。

- 擬凸問題。Levi 問題。$n \geq 2$; $n = 1$, 不存在、自明。($\mathbf{R}^n$ 同様)
  (1) Grauert 法 - - - 多い。L. Schwartz の Fredholm 定理がネック。
  (2) L^2-$\bar\partial$ 法、 Hörmander 流。別流。
  (3) 岡のオリジナル法 - - - Fredholm の第 2 種積分方程式。- - -少ない。

・“coherence, cohérence”, 首尾一貫した。“連接” とは、ニュアンスが異なる。

教科書の例：

(1) 一松本 (1960): 連接第 1 のみ。L. Schwartz 証明では、Bourbaki, Espace vectorial topologique を引用。$u : E \to F \to 0$, $v : E \to F$ compact $\Rightarrow$ $(u+v)(E)$ 閉。

(2) Gunning–Rossi (1965): 連接第1・2 (Chap. IV, 中頃)。L. Schwartz 証明、附録、Dunford–Schwartz 引用。

(3) Hörmander (1966): 連接は後回し (Chap. 6), 第1・2。第2章 (入口基礎) が難しい......岡理論を表に出さずにやっているので展開に無理がある（大事な所で、後でやる定理を引くなど)。

(4) Grauert–Remmert の3部作 - - - 局所理論、連接1・2・3・4、スタイン理論 (Oka–Cartan):

   (a) Analytische Stellenalgabra, GL176 (1971), 240pp.

   (b) Theorie der Steinschen Räume, GL236 (1977); Theory of Stein Spaces, GL265 (1984), 249pp.

   (c) Coherent Analytic Sheaves, GL265 (1984), 249pp. 【計、約 750pp.】

   しかし、不思議なことに**擬凸（Levi）問題は扱っていない !!**

(5) 西野本 (1996, 318pp.): 連接第1・2・3 (7・8章)、擬凸問題（岡のオリジナル法; 唯一)。層コホモロジー理論はなし。

   “岡数学” の再現を目標、難しい。

(6) J.-P. Demailly, Web-book, 455pp.; p. 425, $H^q(X, \mathscr{F}) = 0,\ q \geq 1$, **Oka–Cartan 基本定理** (NG ‘Theorem B’), 他色々やってるが。

(7) 野口

   (a) 多変数解析関数論 (2013, 357pp.) 和・英: 連接1 (2章)・2・3、層コホモロジー、上空移行 (Joku-Iko)、近似、Cousin I+II, 擬凸問題 (Levi 問題は Grauert 法) - - - スタンダード。

   読めば解る。そのまま板書、証明で引用なし。

   (b) 岡理論新入門 (2021, 238pp.) 和・英: 連接第1 (2章)、上空移行、近似、Cousin

I+II(岡原理)、擬凸、岡のオリジナル法、Grauert 法の両方。
層コホモロジー理論なし。

(c) 複素解析 – 一変数・多変数の関数（相原共著）(2024, 383pp.): 実数、複素数〜コーシー 〜 岡原理。
コーシー、留数、部分分数展開、無限積、楕円関数、リーマンの写像定理、ピカールの定理、連接第1 (6章)、上空移行、近似、Cousin I+II(岡原理) (8章) (Mittag-Leffler, Weierstrass を含む)。
コホモロジー・擬凸はなし。

Reviews of 7a, English ver.:

- By F. Haslinger: "Opposite to the order in existing books on complex analysis this book begins with Oka's First Coherence Theorem .... It is remarkable that the approach to Levi's Problem and to the Cousin I and II Problems here is very different to the one for instance in Hörmander's book on several complex variables ...... Many classical proofs are improved and simplified making the book accessible for beginning graduate students. In the appendix the reader finds an interesting biography of Kiyoshi Oka, enhancing a better understanding of the development of the important general concepts of complex analysis." (Monatsch. Math. 2018)
- By V. Tosati: "This book gives a comprehensive introduction to the theory of coherence of analytic sheaves as developed by Oka, Cartan, Grauert and many others. The exposition is aimed at beginning graduate students, is based on lectures given by the author in Tokyo, and follows an unusual order in presenting the

material, which should particulary benefit beginners. .... Overall the book is extremely readable and yet detailed and clear. Its novel viewpoint on Oka ’ s theory, essentially inverting the order in which the main theorems are proved, ... This book is destined to become a classic on the topic of coherence in complex analysis." (zbMath.)

- By E.S. Zeron: "This is a really interesting book that serves as a self-contained text for a course on the theory of Oka's coherence and the Oka-Cartan fundamental theorems, and some of their applications. It is based on the lectures that the author has delivered in the last ten years; this aspect is evident after analysing the creative and interesting order in which the main results are presented. .... We think that the material in this book is so well organised that it can be used as the basic notes for teaching two or three consecutive courses: ...." (MR MathSciNet, AMS)

今日の Points： 分かっていることなんだが、

(1) どういうつもりで書くのか？　What is the idea to write?

(2) 定理をどう述べるのか。How to state Theorems.

(3) どう証明するのか。証明は本文で。How to prove them.

2. 多変数の特徴; Speciality of $n \geq 2$

2.1. Poincaré

(Better to presect after Hartog's phenomenon. Because of a convenience today, the order is altered.)

Thm. 2.1. Polydisk $\Delta^n \not\cong B^n$ Ball ($\Subset \mathbf{C}^n$), $n \geq 2$.

Pf. (Only by Montel; due to M. Range (Remmert); Nog. UTX; 相原・野口, 演習問題)
$n = 2$. Suppose $\exists\, f : \Delta^2 \to B^2$, bihol. As $w \to w_0 \in \partial\Delta$ ($|w_0| = 1$), by Montel

$$f(z, w_\nu) \to f_{w_0} : \Delta \to \partial B^2.$$

∂B^2 is strictly psedoconvex, so that $f_{w_0} \equiv const.$; $\frac{d}{dz} f_{w_0} \equiv 0$.
With $\forall\, z$ fixed, $\frac{\partial f}{\partial z}(z, w)$ has the value 0 on $|w| = 1$, so that $\frac{\partial f}{\partial z}(z, w) \equiv 0$: 矛盾。 □

N.B. 通常 $\mathrm{Aut}(*)$, needs more preparations. No Riemann's mapping theorem in $n \geq 2$.

2.2. Hartogs 現象

Hartogs 現象 (一斉解析接続): 領域 $\Omega \subsetneqq \Omega'$, $\mathscr{O}(\Omega) = \mathscr{O}(\Omega')$.

Hartogs 領域 $\Omega \subsetneqq \Delta(1)^n$ $(n \geq 2)$, $\mathscr{O}(\Omega) = \mathscr{O}(\Delta(1)^n)$.

Thm. 2.2 (Hartogs–Osgood). $\Omega \subset \mathbf{C}^n$ $(n \geq 2)$, $K \Subset \Omega$ compact s.t. $\Omega \setminus K$ connected.
$\Rightarrow \mathscr{O}(\Omega \setminus K) = \mathscr{O}(\Omega)$.

Pf. (1) 初めは、$\Omega = \mathbf{C}^n$ の場合をやる。$n = 2$ として

$$f(z, w) = \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{f(z, \zeta)}{\zeta - w} d\zeta, \quad R \gg 0.$$

(2) 一般は、"連続 Cousin" (後出) の後。$\Omega \Subset \mathbf{C}^n$ に帰着。$U_0 = \Omega$, $U_1 = \mathbf{C}^n \setminus K$, $U_0 \cup U_1 = \mathbf{C}^n$.
$\chi \in C^0(\mathbf{C}^n)$, $\chi|_{\mathrm{nbd}\, K} \equiv 1$, $\mathrm{Supp}\, \chi \Subset U_0$.

$$f_0 = (\chi - 1) f \in C^0(U_0), \quad f_1 = \chi f \in C^0(U_1),$$

(2.3) $$f_1 - f_0 = f \in O(U_0 \cap U_1).$$

"連続 Cousin の解" $F \in C^0(\mathbf{C}^n)$ s.t. $F - f_j \in \mathscr{O}(U_j)$, $j = 0, 1$. (2.3) より

$$(f_1 - F) - (f_0 - F) = f.$$

$g_0 := (f_0 - F) \in \mathscr{O}(\Omega)$, $g_1 := (f_1 - F) \in \mathscr{O}(\mathbf{C}^n \setminus K) = \mathscr{O}(\mathbf{C}^n)$ ($\because$ (1)).
よって、 $f = g_1 - g_0 \in \mathscr{O}(\Omega)$. $\square$

N.B. Hörm 本：$\bar\partial$–compact support (Ehrenpreis)... 一般的。

Thm. 2.4. $\Omega \Subset \mathbf{C}^n$ 正則領域ならば、 $\partial\Omega$ 連結。特に、(Levi 問題解決後) $\partial\Omega$ は強擬凸ならば、連結。

Ref. Czarnecki–Kulcycki–Lubawski, Ann. Polon. Math. 103 (2021). Cf. Nog. [16].

Pf. By Thm. 2.2, $\mathbf{C}^n \setminus \bar\Omega$ is connected. Assume that $\Gamma := \partial\Omega$ is not connected. We may form a homotopically non-trivial continuous map $\Phi : \mathbf{C}^n \to S^1$: 矛盾。 $\square$

強擬凸 CR 問題: $\Omega \Subset \mathbf{C}^n$ 強擬凸領域、$K = \hat{K}_\Omega \Rightarrow \Omega \setminus K$ 連結。$\mathscr{O}(\Omega \setminus K) = \mathscr{O}(\Omega)$.
$K \nearrow \Omega \Rightarrow \Omega \setminus K \to \partial\Omega$ **連結**, $\mathscr{O}(\partial\Omega) = \mathscr{O}(\Omega)$**?**: Tangential CR, CR-Extension Problem, S. Bochner/H. Kneser by M. Range [18], 2002 (Hörm, Kohn–Rossi).

For $\Omega \subset \mathbf{C}^n$, a domain, **正則包、holomorphic hull** $\tilde\Omega \supset \Omega$, maximal with $\mathscr{O}(\tilde\Omega) = \mathscr{O}(\Omega)$.
If $\tilde\Omega = \Omega$, Ω **正則領域、 a domain of holomorphy**..... Extrinsic.

• **必然的問題** 一般に in general, $\Omega \subset \mathbf{C}^n$ 単葉、正則包 $\tilde\Omega \to \mathbf{C}^n$ **無限複葉** multivalent, ∞-sheeted, must be dealt with..

・**正則包を求める問題：** 数学だけでなく、理論物理、N.N. Bogolyubov, H. Araki (A generalization of Bochers theorem, Helvetica Phy. Acta 36 (1963)), H. Epstein–A. Martin (arXiv 2019, hep-th; "domain of analyticity"=domain of holomorphy).

For $K \subset \Omega$, **正則凸包** $\hat{K}_\Omega := \{z \in \Omega : |f(z)| \le \sup_K |f|, \ \forall f \in \mathscr{O}(\Omega)\}$.

Def. 2.5. (Intrinsic) Ω, **正則凸** $\Leftrightarrow$ $\hat{K}_\Omega \Subset \Omega$ for $\forall K \Subset \Omega$.

Thm. 2.6 (**Cartan–Thullen** (1932)). 一般に、Ω 正則凸領域 $\Rightarrow$ 正則領域。
逆は、Ω が単葉ならば成立：

$$\text{単葉正則領域} \Leftrightarrow \text{単葉正則凸領域。}$$

Def. 2.7. 複素多様体 (空間) X が Stein とは：

(1) 第 2 可算。
(2) 正則凸。
(3) 正則分離：For $p \ne q \in X$, $\exists f \in \mathscr{O}(X)$ s.t. $f(p) \ne f(q)$.
(4) 局所座標条件：$\forall p \in X$, $\exists \varphi_1, \ldots, \varphi_n \in \mathscr{O}(X)$, p の周りで正則局所座標。

Thm. 2.8 (**岡の擬凸問題の解決** $n = 2$ 単葉 1941/'42; $n \ge 2$ 一般複葉 1943/1953).
一般複葉領域について

(1) 正則領域 $\Rightarrow$ 擬凸領域。
(2) 擬凸 $\Rightarrow$ Stein. (Hartogs の逆問題、途上に Levi 問題の解決)
(3) 正則領域 $\Leftrightarrow$ Stein 領域 (正則凸)。

N.B. 正則領域 ⇒(直接) 正則凸領域 (Stein) は無限葉では、無い！

問題。領域が擬凸とは？ (本の中で、どう必然性をもたすか？)

「 局所 Stein ⇒ 大域 Stein」も一つ。

・ “擬凸の数 = 人の数”。

3. 三大問題

連接以前。

三大問題： 近似、Cousin I+II、擬凸。

初めの二つは、正則領域で問われた。

1. 近似：$K(=\hat{K}_\Omega) \Subset \Omega$ 上の正則関数を $\mathscr{O}(\Omega)$ の関数で一様近似可能?
2. Cousin I: Mittag-Leffler の定理の多変数版。

   Cousin II: Weierstrass の定理の多変数版。

4. 擬凸問題：後出。

・三大問題解決の為には、第 1 連接定理だけで十分。Oka IX で Oka VIII(第 2・3) を引くところがあるが、第 2・3 連接とは別もの。

・解析層も、$\mathscr{F} \subset \mathscr{O}_\Omega^N$, 集合として、位相は不要。切断は “ベクトル値正則関数”、で十分。

岡理論の要：**Joku-Iko Principle 上空移行の原理** (Oka I, 1936 ∼ IX, 1953),

a **“methodological principle”**:

解析的多面体 $\mathrm{P} = \{z \in \Omega : |f_j(z)| < 1, f_j \in \mathscr{O}(\Omega), 1 \leq j \leq l\} \Subset \Omega$

Oka map

$\Phi_{\mathrm{P}} : z \in \mathrm{P} \to (z, f_1(z), \ldots, f_l(z)) \in \mathrm{P}\Delta_N \Subset \mathbf{C}^N,\ N = n+l; \quad S = \Phi_{\mathrm{P}}(\mathrm{P}) \subset \mathrm{P}\Delta_N$ (submfd.).

Thm. 3.1 (Oka Extension). $\mathrm{P}\Delta_N \Subset \mathbf{C}^N$, polydisk 多重円板。$S \subset \mathrm{P}\Delta_N$, cx. submfd. 複素部分多様体。$\Rightarrow$

$$\mathscr{O}(\mathrm{P}\Delta_N) \ni f \mapsto f|_S \in \mathscr{O}(S) \to 0.$$

連続 Cousin 問題: $\Omega = \bigcup U_\alpha,\ f_\alpha \in C^0(U_\alpha)$ s.t. $f_\alpha - f_\beta \in \mathscr{O}(U_\alpha \cap U_\beta)$.
$\Rightarrow \exists F \in C^0(\Omega)$ s.t. $F - f_\alpha \in \mathscr{O}(U_\alpha)$?

・連続 Cousin $\Rightarrow$ 同時に、 Cousin I, II 岡原理、$\bar{\partial}$–solution.

Thm. 3.2 (岡原理、Cousin II). Ω、正則領域。$\Omega = \bigcup U_\alpha,\ f_\alpha \in \mathscr{M}(U_\alpha)\ s.t.,\ f_\alpha/f_\beta \in \mathscr{O}^*(U_\alpha \cap U_\beta)$.
仮定、$\exists G \in C^0(\Omega')$, *an open dense* $\Omega' \subset \Omega$ *s.t.* $G/f_\alpha \in C^{0^*}(U_\alpha), \forall \alpha$ (位相解の存在)
$\Rightarrow \exists F \in \mathscr{M}(\Omega)$, s.t. $F/f_\alpha \in \mathscr{O}^*(U_\alpha), \forall \alpha$ (解析解)

・“bayalable”, line bundle, Chern class は不要。

4. 第 2 連接定理

任意部分集合 $A \subset \Omega \subset \mathbf{C}^n$. For $a \in \Omega$, define the **ideal sheaf** of A by

$$\mathscr{I}_a\langle A\rangle = \{\underline{f}_a \in \mathscr{O}_a : f \in \mathscr{O}(U), a \in U, f|_{U\cap A} = 0\}, \quad \mathscr{I}\langle A\rangle = \bigsqcup_{a\in\Omega} \mathscr{I}_a\langle A\rangle \subset \mathscr{O}_\Omega.$$

$f|_{U\cap A} = 0 \Leftrightarrow f|_{U\cap \bar{A}} = 0$. 故に、$\mathscr{I}\langle A\rangle = \mathscr{I}\langle \bar{A}\rangle$.

To think of $\mathscr{I}\langle A\rangle$, it is reasonable to assume $A = \bar{A}$, closed.

Thm. 4.1 (岡の第 2 連接定理、Oka 1948/'51, H. Cartan 1950). 閉集合 $A \subset \Omega$.

$$A\text{、解析的} \Longleftrightarrow \mathscr{I}\langle A\rangle\text{、連接的。}$$

Cf. 野口 [14], [16], 相原・野口 [1]. 通常は、$\Rightarrow$ のみ。

・"$\Leftarrow$" reveals the goodness of the wording, **連接** in 漢字, better than 'coherence'.

5. 擬凸問題

・Condition, $\mathbf{R}$-valued function $\Rightarrow$ Holomorphic functions.

初めの解決 Oka (Proc. 1941/VI 1942): 単葉領域、$n = 2$ 次元。

方法：Weil–Oka 積分+Fredholm 積分方程式

"$n \geq 3$ でも同様に成立すると考える。"

Weil 積分：Weil 条件、 $f(z) \in \mathscr{O}(\Omega)$

$$f(z) - f(w) = \sum_{j=1}^{n} Q_j(z_j - w_j), \quad Q_j \in \mathscr{O}(\Omega \times \Omega).$$

Oka's modification, Weil–Oka (Oka V, 1941): $(z_0, w_0) \in \Omega^2$ $(n = 2)$ given, $\exists\, R(z, w) \in \mathscr{O}(\Omega^2)$ s.t. $R(z_0.w_0) = 1$, $\forall\, f \in \mathscr{O}(\Omega)$

$$R \cdot (f(z) - f(w)) = \sum_{j=1}^{2} Q_j(z_j - w_j), \ \ Q_j \in \mathscr{O}(\Omega^2).$$

Thm. 5.1 (H. Hefer, Münster 学位論文 1940/Math. Ann. 1950*). $n \geq 2$, $R \equiv 1$ で $O.K.$

*) Footnote: Der Verfasser ist 1941 im Osten gefallen. (H. Behnke, K. Stein).

その後の解決：

(1) 1943 K. Oka, 108pp., 日本語, 未発表、 $n \geq 2$, 一般不分岐領域。
"**a Primitive Coherence Theorem**"+Joku-Iko+Fredholm Integral Eq. Cf. Nog. [15].
⇒ "**Coherence Theorem**"+Joku-Iko+Fredholm Integral Eq., Oka IX 1953.

(2) 1949 S. Hitotsumatsu, 数学 1, $n \geq 3$, 単葉領域。
Weil-Oka 積分表示+Fredholm 積分方程式。

(3) 1953 Oka IX, $n \geq 2$, 不分岐複葉領域、
第 1 連接定理+Fredholm 積分方程式。中途報告 (本命は分岐領域)。

(4) 1954, F. Norguet, $n \geq 3$, 単葉領域。
Weil-Oka 積分表示+Fredholm 積分方程式。

(5) 1954, H.J. Bremermann, $n \geq 3$, 単葉領域。
Hefer+Weil 積分表示+Fredholm 積分方程式。

(6) 1958 C.B. Morrey, Levi 問題、 Abstract Levi Problem by method of PDE.

(7) 1958 H. Grauert, Levi 問題、the Bumping Method + L. Schawartz's Fredholm Theorem.

(8) 1965 L. Hörmander, L^2–$\bar{\partial}$. 本 1966.

• **(2) S. Hitotsumatsu 1949 を引用すべし。**

6. 多重劣調和関数

Jensen 公式の導出。

Thm. 6.1 (Jensen). $\varphi \in C^2(\Delta(R))$ $(\Delta(R) \subset \mathbf{C})$. For $0 < r < R$,

$$\frac{1}{2\pi}\int_0^r \varphi(re^{i\theta})d\theta - \varphi(0) = \frac{1}{2\pi}\int_0^r \frac{dt}{t}\int_{\Delta(t)} \Delta\varphi \, dxdy, \quad z = x + iy.$$

Pf. 通常 Stokes で左辺から. ここでは逆に、右辺 Laplacian Δ から：

$$\begin{aligned}
\text{右辺} &= \frac{1}{2\pi}\int_0^r \frac{dt}{t}\int_{\Delta(t)}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi\rho d\rho d\theta \\
&= \frac{1}{2\pi}\int_0^r \frac{dt}{t}\int_0^t \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\rho d\rho \int_0^{2\pi} \varphi(\rho e^{i\theta})d\theta \\
&= \int_0^r \frac{dt}{t}\int_0^t \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\rho d\rho \frac{1}{2\pi}\int_0^{2\pi} \varphi(\rho e^{i\theta})d\theta \\
&= \int_0^r \frac{dt}{t}\int_0^t \left(\rho\frac{\partial^2}{\partial\rho^2} + \frac{\partial}{\partial\rho}\right)M(\rho)d\rho \quad \left[M(\rho) := \frac{1}{2\pi}\int_0^{2\pi}\varphi(\rho e^{i\theta})d\theta\right] \\
&= \int_0^r \frac{dt}{t} t\frac{d}{dt}M(t) = M(r) - M(0).
\end{aligned}$$

□

Thm. 6.2 (Hartogs 分離正則性). $\Omega \subset \mathbf{C}^n$, $f : \Omega \to \mathbf{C}$ が、変数毎に正則 $\Rightarrow f \in \mathscr{O}(\Omega)$.

Pf. 一次変換+poly-Poisson Integral. Cf. [16]. $z_0 \in \mathrm{P}\Delta(r) \subset \mathrm{P}\Delta(2r) \subset \Omega$. □

Thm. 6.3. $\Omega \subset \mathbf{C}^n$, domain of holomorphy (or Stein mfd., space). $\varphi \in \mathrm{PSH}(\Omega)$. $\Rightarrow$ $\{\varphi < c\}$ Stein & Runge in Ω.

Pf. Cousin I. Cf. Nog. [15], [16]. □

・これは、**擬凸問題以前**。次も同様：

Cor. 6.4. Ω domain of holomorphy (Stein), $K \Subset \Omega \Rightarrow$

$$\hat{K}_{\mathrm{PSH}(\Omega)} := \left\{ z \in \Omega : \varphi(z) \le \sup_K \varphi,\ \forall \varphi \in \mathrm{PSH}(\Omega) \right\} = \hat{K}_\Omega \ (\Subset \Omega).$$

N.B. Hörm 本、 Thm. 4.3.4 by L^2–$\bar\partial$. この流儀では、擬凸以前と見えてこない。

擬凸問題**要の定理**：

Thm. 6.5 (Oka VI/IX). $\Omega/\mathbf{C}^n$、不分岐正則領域 $\Rightarrow -\log d(p, \partial\Omega)$、PSH 多重劣調和.
$\Rightarrow \exists \varphi \in \mathrm{PSH}(\Omega) \cap C^0(\Omega)$, 階位関数 exh. ftn.、階位集合 $\Omega_c = \{\varphi < c\} \Subset \Omega$.

Claim 6.6. Ω_c, Stein.

Lem. 6.7 (Levi 問題). $\Omega_0 \Subset \Omega$, $\partial\Omega_0$ C^2-class, 強擬凸。 $\Rightarrow \Omega_0$ Stein.

Oka's Method. $a < b$,
$\Omega_1 = \{z = (z_j) \in \Omega_0 : \Re z_1 < b\}$,
$\Omega_2 = \{z = (z_j) \in \Omega_0 : \Re z_1 > a\}$,
Ω_j $(j = 1, 2)$ Stein $\Rightarrow \Omega_0$ Stein.

$\Omega_3 := \Omega_1 \cap \Omega_2$.
For a given $f \in \mathscr{O}(\Omega_3)$, solve Cousin I: $f_j \in \mathscr{O}(\Omega_j)$ with $f_1 - f_2 = f$:
Here $\tilde{g}$ is an Oka extension of g.

$\exists\, K(\tilde{g})$ $(g \in \mathscr{O}(\Omega_3))$ compact integral operator s.t. $g - K(\tilde{g}) = g_1 - g_2$, $g_j \in \mathscr{O}(\Omega_j)$.

Find $g \in \mathscr{O}(\Omega_3)$ satisfying the Fredholm Integral Equation of 2nd kind

$$g - \lambda K(\tilde{g}) = f, \quad \lambda \in \mathbf{C};$$

here $\lambda = 1$ (Oka VI).

Thm. 6.8 (**Oka** (1941 (Proc.), 1942, 1943 (unpublished) and 1953)). This is solvable.

Pf. By the Oka Extention with estimate & a succesive asymptotic approximation method. □

- **Cousin I is solvable mod a compact operator.**

Grauert's Bumping Method. $\Omega = \Omega_1 \cup \Omega_2$ 開被覆。$\mathscr{V} = \{\Omega_1, \Omega_2\}$,
$\Omega_j \Subset \exists\, U_j$ slightly, strictly larger, $\mathscr{U} = \{U_1, U_2\}$ s.t.

$$\Phi : \xi \oplus \eta \in Z^1(\mathscr{U}, \mathscr{O}) \oplus C^0(\mathscr{V}, \mathscr{O}) \to \rho(\xi) + \delta\eta \in Z^1(\mathscr{V}, \mathscr{O}) \to 0.$$

$\rho =$ restriction operator in $U_1 \cap U_2 \Supset \Omega_1 \cap \Omega_2$, compact.

- **Cousin I is solvable mod a compact operator. But no equation.**

Thm. 6.9 (**L. Schwartz's Fredholm**). E: Frechét top. vect. space,
F: Baire top. vect. space.
$\phi : E \to F \to 0$ continuous surjective homomorphism,
$\psi : E \to F$ compact operator.
$\Rightarrow \dim \mathrm{Coker}(\phi + \psi) < \infty$.

Pf. Cf. Textbooks of Nog. [12] 2nd. Ed., [13], [14], [15]. □

Apply this to $\phi = \Phi$, $\psi = -\rho$. Then

Thm. 6.10 (Grauert). $\mathrm{Coker}(\Phi - \rho) \cong H^1(\mathscr{V}, \mathscr{O})$, finite dimensional.

• **Cousin I is solvable mod a finite dimensional vector space.**

7. Fredholm Alternative Theorem and Questions

$[a, b] \subset \mathbf{R}$: an interval, $\lambda \in \mathbf{C}$.
$K(s, t) \in C^0([a, b]^2)$, $\mathbf{C}$-valued.
Fredholm Integral Eq. of 2nd kind: For $f(s) \in C^0([a, b])$

$$\text{(FIE)} \qquad \varphi(s) - \lambda \int_a^b K(s, t)\varphi(t)dt = f(s).$$

Thm. 7.1 (Fredholm Alternative). Either,

(1) for $\forall f \in C^0([a, b])$, (FIE) has a unique solution, $\exists 1 \varphi \in C^0([a, b])$,

or

(2) equation, $\varphi(s) - \lambda \int_a^b K(s, t)\varphi(t)dt = 0$
has finitely many linearly independent solutions, $\varphi_j \in C^0([a, b]), 1 \leq j \leq k$;
λ called an eigenvalue or a spectrum.

In the case (2), in order to (FIE) being solvable, $f(s)$ must satisfy the conditon

$$\int_a^b f(s)\varphi_j(s)ds = 0, \quad 1 \leq j \leq k;$$

in this case, with a special solution $\psi(s)$

$$\varphi(s) = \psi(s) + \sum_{j=1}^{k} c_j \varphi_j(s), \quad c_j \in \mathbf{C}.$$

・Grauert's method makes sense even in the non-solvable case:

Thm. 7.2 (Grauert). $\Omega \Subset X$, 強擬凸 $\Rightarrow \dim H^1(\Omega, \mathscr{O}) < \infty$.

Quest. 7.3. What happens in this case for Oka's method? : If possible, specrum theory?

・Extension of Oka's method (solvable case):

- T. Nishino, Sur les espaces analytiques holomorphiquement complets, J. Math. Kyoto Univ. **1**(2) (1962).
- A. Andreotti and R. Narasimhan, Oka's Heftungslemma and the Levi problem for complex spaces, Trans. Amer. Math. Soc. **111** (1964).

・**Method by Integral Representation**

- G.M. Henkin and J. Leiterer, Theory of Functions on Compex Manifolds, Birkhäuser, 1984.
- R.M. Range, Holomorphic Functions and Integral Representations in Several Complex Variables, GTM108, Springer, 1986.

- C. Laurent-Thiébaut, Holomorphic Function Theory in Several Variables : an introduction, UTX, Springer, 2011.

Laurent-Thiébaut, Chap. VII, Bochner-Martinelli–Koppelman kernel:
$\Omega \Subset X$ (cx. mfd.), 強擬凸, $f \in C^0_{(p,q)}(\bar{\Omega})$ with $\bar{\partial} f \in C^0_{(p,q+1)}(\bar{\Omega})$,

$$f = \bar{\partial} T^p_q f + T^p_{q+1} \bar{\partial} f + K^p_q f,$$

wherer T^p_q is the solution operator by integral kernel, and
K^p_q is a **compact operator.** $\Rightarrow \dim H^{(p.q)}(\bar{\Omega}) < \infty$.

Quest. 7.4. Spectrum theory?

・分岐領域、Ramified domain $/\mathbf{C}^n$.

X, cx. space, $\pi : X \to \mathbf{C}^n$, locally finite.

Probl. 7.5. $\forall z \in \mathbf{C}^n$, $\exists$nbd $U \ni z$ in $\mathbf{C}^n$, Stein s.t. $\pi^{-1}U$ is Stein (rel. locally Stein) $\Rightarrow$ X, Stein?

Counter Example of Fornæss (1978). $\exists$smooth $X \xrightarrow{\pi} \mathbf{C}^2$, ramified, 2-sheeted, rel. locally Stein, but not holomorphically convex.

Quest. 7.6. $\exists f \in \mathscr{O}(X)$ separating the sheets of $\pi : X \to \mathbf{C}^2$?

Probl. 7.7. Obstruction? Necessary/sufficient condition?

Cf. Nog., Math. Ann. 367 (2017).

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