A Discussion on the Definition of Holomorphic Functions of One and Several Variables

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This is a note to discuss a natural and comprehensive uniform definition of *holomorphy* of functions of one and several complex variables (or n variables, in short). P. Montel [Mon] discussed the holomorphy of functions of one complex variable in terms of the total differentials of two real variables; it lead to the Looman–Menchoff Theorem (see Remark 2.22). Here we would like to discuss it in the combination of one and n variables.

There seem to be mainly the following types of the definition of holomorphy from (i) to (vi), while (vii) based on total differentials (cf. P. Montel [Mon]) is rare to find, in particular at the textbook level:

(i) In one variable, complex differentiability by $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$; [Ahl], [Car] in one variable, [Con], [Nog1], [Nog2] in one variable.
(ii) The existence of a function $\Delta(z)$, continuous at z_0 such that $f(z) = f(z_0) + (z - z_0)\cdot\Delta(z)$, and the extended form in n variables by a vector-valued function; [FrGr], [Has] in one variable, [Rem].
(iii) In n variables, $f(z_0 + h) - f(z_0) = \sum_{j=1}^{n} \alpha_j h_j + o(\|h\|)$; [Hit1], [Hit2] [SSST].
(iv) C^1-regularity and $df = \partial f$ or Cauchy–Riemann equations; [Car] in n variables, [Hör], [Has] in n variable, [Nog2] in n variables,
(v) Development to convergent power series; [Hur] in one variable, [GuR] in n variables, .
(vi) In n variables, continuity and separate holomorphy; [Kod], [Nis], [Dem] in n variables.
(vii) The **C**-linearity of the (real) total differential; [Mon], [Dem] with C^1-regularity, both in one variable, [For] in n variables: These are not introductory textbooks.

In one variable, it seems to be the most common to use (i) with mentioning (iii) with $n = 1$ (it is very rare to take (iii) with $n = 1$ for the definition), but in several variables there seems to be no standard definition of holomorphy. The reason may be because there is no direct extension of the expression (i) in several variables, in which (iv)—(vi) are mainly used.

(v) is the strongest, but the disadvantage is that the Cauchy Integral Theorem is reduced to a trivial statement because of the local existence of the primitive of a holomorphic function everywhere it is defined. The C^1-regularity of (iv) is also excessive for the Cauchy Integral Theorem (Goursat's proof): Hence the excellent flavor of the Cauchy Integral Theorem is dismissed. In (ii) $\Delta(z)$ is unique in one variable, but in the case of $n \geq 2$ the value of the n-vector-valued function $\Delta(z)$ is not uniquely determined except for at $z = z_0$. (vi) is very dependent on the coordinate system and is not preserved even under linear changes of coordinates. The expressions of (iii) and (vii) are very close, but there is yet a difference in using the real total differential in (vii). It is an advantage of (vii) that the complex structure in the real structure is expilicitly clarified in terms of complex linearity.

After all, they are equivalent, but the author finds that the definition by (vii) might be the most reasonable and consistent in one and several complex variables. It is the point of the present note to look at the *complex structure of the tangent space of (the graph of) the given function* in terms of the total differential; also, it gives a smooth introduction from elementary calculus.

However, it should be noted that the concept of the "*differential*" of a function, even of one real variable, may be unfamiliar to the students in the introductory course; this aspect requires a special attention and a careful presentation in the course.

Key words: complex analysis, holomorphy, holomorphic functions. analytic functions, total differential.

1 Differentiation in Real Variables

1.1 Derivatives and Differentials in One Real Variable

First, we briefly recall the notions of derivative and differetial for a real-valued function $f : I \to \mathbf{R}$ defined on an open interval $I \subset \mathbf{R}$. If the limit

$$\alpha := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \quad a \in I, \tag{1.1}$$

exists, f is said to be differentiable at a and α is called the *derivative* of f at a, written $f'(a)$ (; if f is differentiable at every point of I, we obtain a function $\frac{df}{dx}(x) = f'(x)$ of $x \in I$, also called the *derivative* of f). Then f is continuous at a, and the ***differential*** $df_a : \mathbf{R} \to \mathbf{R}$ of f at a is defined to be a **linear map** given by

$$df_a : h \in \mathbf{R} \longrightarrow f'(a)h \in \mathbf{R}. \tag{1.2}$$

The real line in $\mathbf{R}^2$

$$(a, f(a)) + \{(h, df_a(h)) : h \in \mathbf{R}\} \tag{1.3}$$

is called the tangent line of the graph of $y = f(x)$ at $(a, f(a))$ (cf. Fig. 1).

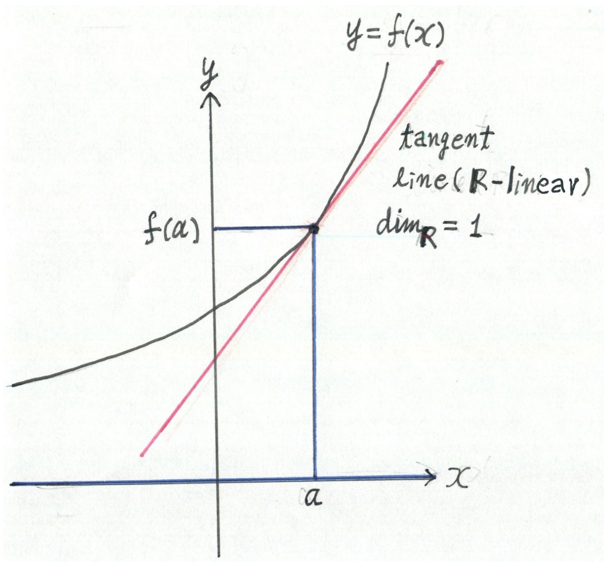

Figure 1: Real 1 variable

1.2 Partial Derivatives

Now we consider complex-valued function of several varaibles. Let U be an open set of $\mathbf{R}^n$ with the natural coordinate system $x = (x_1, \ldots, x_n)$. Let $f : U \to \mathbf{C}$ be a function in U. We consider the following limit at $a = (a_j) \in U$:

$$\lim_{h \to 0} \frac{f(a_1, \ldots, a_j + h, \ldots, a_n) - f(a_1, \ldots, a_j, \ldots, a_n)}{h}. \tag{1.4}$$

If this limit exists, $f(x)$ is said to be *partially differentiable* at a with respect to x_j, and the limit is denoted by $\frac{\partial f}{\partial x_j}(a)$, called the *partial derivative* of f at a (with respect to x_j). The operation to get $\frac{\partial f}{\partial x_j}(a)$ is called the *partial differentiation* (at a with respect to x_j). If it exists at every point $x \in U$, $f(x)$ is said to be *partially differentiable* in U with respect to x_j, and the function

$$\frac{\partial f}{\partial x_j} : x \in U \longrightarrow \frac{\partial f}{\partial x_j}(x) \in \mathbf{C}$$

is called the *partial derivative* of f (with respect to x_j).

Theorem 1.5 *Let U be a domain in $\mathbf{R}^n$, and let $f : U \to \mathbf{C}$ be a partially differentiable function with respet to every x_j in U. If $\frac{\partial f}{\partial x_j}(x) = 0$ $(1 \leq \forall j \leq n)$ everywhere in U, then $f(x)$ is a constant function.*

Proof. Assume $\frac{\partial f}{\partial x_j}(x) = 0$ $(x \in U,\ 1 \leq j \leq n)$. Let $a \in U$ and put $\alpha = f(a)$. Set $E = \{x \in U : f(x) = \alpha\} \neq \emptyset$. Let $b = (b_1, \ldots, b_n) \in E$ be an arbitrary point. Take a $\delta > 0$ such that $V := \{b + h : h = (h_j) \in \mathbf{R}^n, |h_j| < \delta\} \subset U$. Let $b + h \in V$ with $h = (h_1, \ldots, h_n) \in \mathbf{R}^n$ be any point. By the assumption $f(x_1, b_2, \ldots, b_n)$ is constant in x_1 with $|x_1 - b_1| < \delta$, and so $f(b_1 + h_1, b_2, \ldots, b_n) = \alpha$. Similarly to x_2, we have $f(b_1 + h_1, b_2 + h_2, b_3 \ldots, b_n) = \alpha$. Repeating this, we have $f(b + h) = \alpha$, and $b + h \in E$; hence, $V \subset E$. Therefore E is open.

Let $c \in \bar{E}$ be an accumulation point of E in U. There are a point $b' \in E$ and a $\delta > 0$ such that $c \in \{b' + (h_j) : |h_j| < \delta\} \subset U$. By the same arguments as above we see that $f(c) = \alpha$, and so $c \in E$; thus, E is closed. Since U is connected, $E = U$, and $f(x)$ is constant in U. □

1.3 Total Differentials

Even if $f(x_1, \ldots, x_n)$ is partially differentiable with respect all x_j, $f(x_1, \ldots, x_n)$ is *not* necessarily continuous in general $(n \geq 2)$: A well-known example is:

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0, 0), \\ 0, & (x_1, x_2) = (0, 0). \end{cases} \tag{1.6}$$

This $f(x_1, x_2)$ is partially differentiable in $\mathbf{R}^2$ with respect to x_1 and x_2, but not continuous at 0: With $y = kx$ $(k \in \mathbf{R})$, $f(x, kx) = k/(1 + k^2)$, so that as $x \to 0$, the value depends on $k \in \mathbf{R}$, whereas $f(0, 0) = 0$ by definition.

We need a little stronger differentiability for $f(x_1, \ldots, x_n)$:

Definition 1.7 (i) $f(x)$ is *totally differentiable* at $a \in U$ if there are constants $\alpha_j \in \mathbf{C}$, $1 \leq j \leq n$, such that as $h = (h_1, \ldots, h_n)\ (\in \mathbf{R}^n) \to 0$,

$$f(a + h) - f(a) = \sum_{j=1}^{n} \alpha_j h_j + \varepsilon(h), \quad \lim_{h \to 0} \frac{\varepsilon(h)}{\|h\|} = 0. \tag{1.8}$$

(ii) If $f(x)$ is totally differentiable at every point of U, $f(x)$ is said to be *totally differentiable* in U.

In some references, a totally differentiable function is simply called a *differentiable function.*

Remark 1.9 The term "totally" is used to distinguish the notion from "partially". Note that if $f(x)$ is totally differentiable at a as above, then

(i) $f(x)$ *is continuous at* a;
(ii) $f(x)$ is partially differentiable with respect to all x_j at a, so that $\alpha_j = \frac{\partial f}{\partial x_j}(a)$ *are uniquely determined.*

Due to Landau's symbol, such $\varepsilon(h)$ above is commonly written "$o(\|h\|)$", with which (1.8) is written

$$f(a+h) - f(a) = \sum_{j=1}^{n} \alpha_j h_j + o(\|h\|). \tag{1.10}$$

We may consider (1.10) to be the *first order approximation* of $f(x) - f(a)$ at a by a $\mathbf{R}$-linear function. Taking out the main part of the approximation, we have a generalization of differential (1.2) in one variable:

Definition 1.11 The ***total differential*** df_a of $f(x)$ at a is the **R-linear map** defined by

$$df_a : (h_1, \ldots, h_n) \in \mathbf{R}^n \longrightarrow \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a) h_j \in \mathbf{C} \cong \mathbf{R}^2. \tag{1.12}$$

In some references, the total differential is referred to as the *total derivative.*

Remark 1.13 Whereas in (1.10), $h \in \mathbf{R}^n$ is taken close to 0, $h = (h_j)$ of (1.12) is not neccessary near 0, but just an arbitrary point of the real vector space $\mathbf{R}^n$.

The real affine space

$$(a, f(a)) + \{(h, df_a(h)) : h \in \mathbf{R}^n\} \subset \mathbf{R}^n \times \mathbf{C} \cong \mathbf{R}^n \times \mathbf{R}^2$$

is called the *tangent space* of the graph of $y = f(x)$ at $(a, f(a))$, which is a generalization of the tangent line (1.3) in one variable: Cf. Fig. 2, Fig. 1.

Let $f(x), g(x)$ be totally differentiable functions at $a \in \mathbf{R}^n$, and let $\alpha, \beta \in \mathbf{C}$ be constants. Then

$$\alpha f(x) + \beta g(x), \quad f(x) \cdot g(x)$$

are totally differentiable at a.

We consider a composed function of totally differentiable functions. Let $f_k(x)$ $(1 \le k \le m)$ be real-valued functions in an open set $U \subset \mathbf{R}^n$, and let $f = (f_k) : U \to V$ with an open set $V \subset \mathbf{R}^m$. Let $G(y)$ be a complex-valued function of $y \in V$.

Theorem 1.14 *Let the notation be as above. Assume that all* $f_k(x)$ $(1 \le k \le m)$ *are totally differentiable at* $a \in U$*, and that* $G(y)$ *is totally differentiable at* $b = f(a) \in V$*. Then the composed function* $G \circ f(x)$ *is totally differentiable at* a *and satisfies*

$$G \circ f(a+h) - G \circ f(a) = \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{\partial G}{\partial y_k}(b) \cdot \frac{\partial f_k}{\partial x_j}(a) h_j + o(\|h\|) \tag{1.15}$$

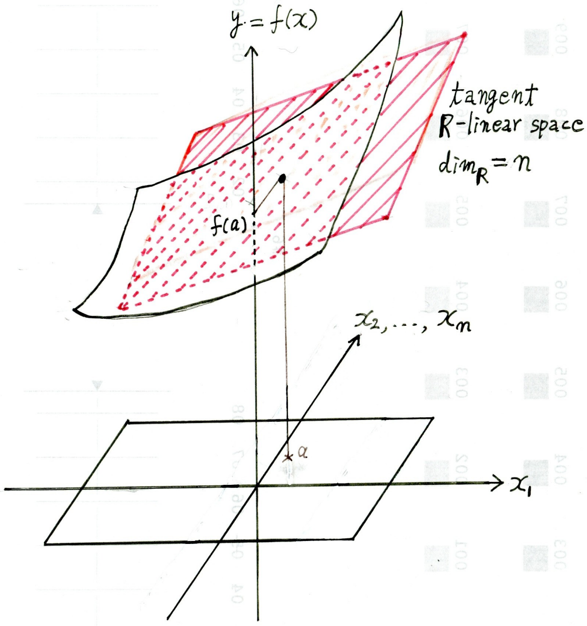

Figure 2: Real n variables

for small $h \in \mathbf{R}^n$. In particular,

$$\frac{\partial (G \circ f)}{\partial x_j}(a) = \sum_{k=1}^{m} \frac{\partial G}{\partial y_k}(f(a)) \cdot \frac{\partial f_k}{\partial x_j}(a). \tag{1.16}$$

Proof. For small $\eta := (\eta_k) \in \mathbf{R}^m$ and $h := (h_j) \in \mathbf{R}^n$ we have by definition

$$\begin{aligned} G(b+\eta) - G(b) &= \sum_{k=1}^{m} \frac{\partial G}{\partial y_k}(b)\eta_k + \varepsilon(\eta), & \lim_{\|\eta\| \to 0} \frac{\varepsilon(\eta)}{\|\eta\|} &= 0, \\ f_k(a+h) - f_k(a) &= \sum_{j=1}^{n} \frac{\partial f_k}{\partial x_j}(a)h_j + \varepsilon_k(h), & \lim_{\|h\| \to 0} \frac{\varepsilon_k(h)}{\|h\|} &= 0. \end{aligned} \tag{1.17}$$

Set $\eta_k = f_k(a+h) - f_k(a)$ and $\eta = (\eta_k)$. We substitute these to (1.17). There is a constant $C > 0$ with $\|\eta\| \le C\|h\|$, so that $\varepsilon(\eta) = o(\|h\|)$; moreover,

$$\sum_{k=1}^{m} \frac{\partial G}{\partial y_k}(b)\varepsilon_k(h) = o(\|h\|).$$

Therefore we deduce (1.15). $\square$

Proposition 1.18 *A function of C^1-class is totally differentiable.*

Proof. Let $f(x)$ be a function of C^1-class in a neighborhood of $a \in \mathbf{R}^n$. We may assume that $f(x)$ is real-valued. The claim derives from the following computaion by making use of the Mean Value Theorem and the continuity of the partial derivatives:

$$\begin{aligned} f(a+h) - f(a) = {} & f(a + (h_1, h_2, \ldots, h_n)) - f(a + (0, h_2, \ldots, h_n)) \\ & + f(a + (0, h_2, h_3, \ldots, h_n)) - f(a + (0, 0, h_3, \ldots, h_n)) \\ & \vdots \end{aligned}$$

$$
\begin{aligned}
&+ f(a + (0, , \dots, 0, h_n)) - f(a) \\
&= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a + (0, \dots, \theta_j h_j, h_{j+1}, \dots, h_n))h_j \quad (0 < \exists\, \theta_j < 1) \\
&= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a)h_j + o(\|h\|). \qquad \square
\end{aligned}
$$

Remark 1.19 The relation among the differentiability of functions of several variables is as follows:

C^1-class $\Rightarrow$ Totally differentiable $\Rightarrow$ Partially differentiable.

2 Holomorphic Functions of One Variable

2.1 Total Differential in a Complex Coordinate

Let $D \subset \mathbf{C}$ be an open set. Let $f(z)$ be a complex-valued function in D. With $z = x + iy \in \mathbf{C}$ in the real and imaginary parts we recall the total differentiability of $f(z) = f(x, y)$ at $z_0 = x_0 + iy_0$, which we rewrite in the complex coordinate (cf. Definition 1.7): There are complex constants α_1, α_2 such that

$$
f(z_0 + h) - f(z_0) = \alpha_1 h_1 + \alpha_2 h_2 + o(|h|) \tag{2.1}
$$

for $h = h_1 + ih_2$ with $h_1, h_2 \in \mathbf{R}$. If they exist, $\alpha_1 = \frac{\partial f}{\partial x}(z_0)$ and $\alpha_2 = \frac{\partial f}{\partial y}(z_0)$ (cf. Remark 1.9). Since $h_1 = \frac{1}{2}(h + \bar{h})$ and $h_2 = \frac{1}{2i}(h - \bar{h})$, (2.1) is equivalent to

$$
\begin{aligned}
f(z_0 + h) - f(z_0) &= \frac{\alpha_1}{2}(h + \bar{h}) + \frac{\alpha_2}{2i}(h - \bar{h}) + o(|h|) \\
&= \frac{1}{2}\left(\alpha_1 + \frac{1}{i}\alpha_2\right) h + \frac{1}{2}\left(\alpha_1 - \frac{1}{i}\alpha_2\right) \bar{h} + o(|h|) \\
&= \beta h + \gamma \bar{h} + o(|h|),
\end{aligned} \tag{2.2}
$$

where $\beta = \frac{1}{2}\left(\alpha_1 + \frac{1}{i}\alpha_2\right)$ and $\gamma = \frac{1}{2}\left(\alpha_1 - \frac{1}{i}\alpha_2\right)$. Therefore we have

Lemma 2.3 *$f(z)$ is totally differentiable at z_0 if and only if there are constants β, γ such that*

$$
f(z_0 + h) - f(z_0) = \beta h + \gamma \bar{h} + o(|h|). \tag{2.4}
$$

We put

$$
\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right). \tag{2.5}
$$

Let $\bar{f}(z) := \overline{f(z)}$ denote the complex conjugate. Note that

$$
\begin{aligned}
\overline{\left(\frac{\partial f}{\partial z}\right)} &= \frac{\partial \bar{f}}{\partial \bar{z}}, \qquad \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = \frac{\partial \bar{f}}{\partial z}, \\
\frac{\partial z}{\partial z} &= 1, \qquad \frac{\partial z}{\partial \bar{z}} = 0.
\end{aligned} \tag{2.6}
$$

Then β, γ in (2.4) are unique (cf. Remark 1.9 (ii)) and given by

$$
\beta = \frac{\partial f}{\partial z}(z_0), \quad \gamma = \frac{\partial f}{\partial \bar{z}}(z_0),
$$

and so (2.4) takes the following form:

$$f(z_0+h)-f(z_0)=\frac{\partial f}{\partial z}(z_0)h+\frac{\partial f}{\partial \bar z}(z_0)\bar h+o(|h|). \tag{2.7}$$

If $f(z)$ is totally differentiable at $z_0\in D$, then $f(z)$ is continuous at z_0, and the total differential of $f(z)$ at z_0 in the complex coordinate is written

$$df_{z_0}: h\in \mathbf{C}\longrightarrow \frac{\partial f}{\partial z}(z_0)h+\frac{\partial f}{\partial \bar z}(z_0)\bar h\in \mathbf{C}. \tag{2.8}$$

This is **R**-linear, but not **C**-linear in general; *df_{z_0} is **C**-linear if and only if $\frac{\partial f}{\partial \bar z}(z_0)=0$.*

Let $w_0=f(z_0)$ and let $G(w)$ be a function defined in a neighborhood of w_0.

Proposition 2.9 *Assume that $f(z)$ (resp. $G(w)$) is totally differentiable at z_0 (resp. w_0). Then the composed function $G\circ f(z)$ is totally differentiable at z_0 and satisfies*

$$\frac{\partial (G\circ f)}{\partial z}(z_0)=\frac{\partial G}{\partial w}(f(z_0))\frac{\partial f}{\partial z}(z_0)+\frac{\partial G}{\partial \bar w}(f(z_0))\frac{\partial \bar f}{\partial z}(z_0), \tag{2.10}$$

$$\frac{\partial (G\circ f)}{\partial \bar z}(z_0)=\frac{\partial G}{\partial w}(f(z_0))\frac{\partial f}{\partial \bar z}(z_0)+\frac{\partial G}{\partial \bar w}(f(z_0))\frac{\partial \bar f}{\partial \bar z}(z_0). \tag{2.11}$$

Proof. This follows from Theorem 1.14, or rather from its poof combined with the expression of (2.7). In fact we have for small h and η

$$\begin{aligned} f(z_0+h)-f(z_0)&=\frac{\partial f}{\partial z}(z_0)h+\frac{\partial f}{\partial \bar z}(z_0)\bar h+o(|h|),\\ G(w_0+\eta)-G(w_0)&=\frac{\partial G}{\partial w}(w_0)\eta+\frac{\partial G}{\partial \bar w}(w_0)\bar\eta+o(|\eta|). \end{aligned}$$

With setting $\eta=f(z_0+h)-f(z_0)$ we obtain

$$\begin{aligned} G\circ f(z_0+h)-G\circ f(z_0)=&\left(\frac{\partial G}{\partial w}(w_0)\frac{\partial f}{\partial z}(z_0)+\frac{\partial G}{\partial \bar w}(w_0)\frac{\partial \bar f}{\partial z}(z_0)\right)h\\ &+\left(\frac{\partial G}{\partial w}(w_0)\frac{\partial f}{\partial \bar z}(z_0)+\frac{\partial G}{\partial \bar w}(w_0)\frac{\partial \bar f}{\partial \bar z}(z_0)\right)\bar h+o(|h|). \end{aligned}$$

We thus deduce (2.10) and (2.11). □

2.2 Complex Differentiability and Holomorphic Functions of One Variable

We keep the notation above. Let $D\subset \mathbf{C}$ be a domain with the natural coordinate $z=x+iy$. We define the complex total differentiability as a special case of the total differentiability, and holomorphic functions:

Definition 2.12 (**Holomorphic function**) (i) A function $f: D\to \mathbf{C}$ is *complex differentiable* at $z_0\in D$ if $f(z)$ $(=f(x,y))$ is totally differentiable at z_0 and the total differential $df_{z_0}:\mathbf{C}\to\mathbf{C}$ is **C**-linear; i.e., $df_{z_0}(h)=\alpha h$ with $\alpha:=df_{z_0}(1)$, or

$$f(z_0+h)-f(z_0)=\alpha h+o(|h|)\quad \textit{as } h\to 0.$$

In this case, $f(z)$ is continuous at z_0.

(ii) $f(z)$ is *holomorphic* in D if $f(z)$ is complex differentiable at every point of D; then, the function

$$f'(z)=\frac{df}{dz}(z)=\frac{d}{dz}f(z),\quad z\in D$$

is called the *derivative* of $f(z)$ (or f).

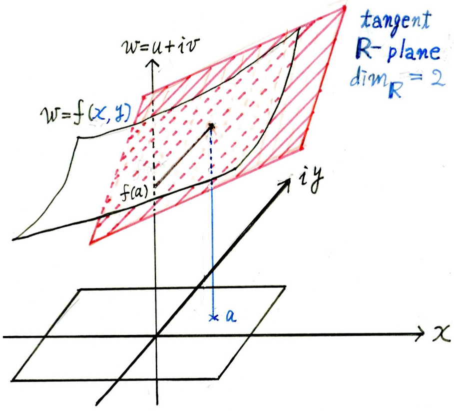

Figure 3: Real 2 variables $(x, y) = x + iy$,
$w = f(x, y)$ totally differentiable

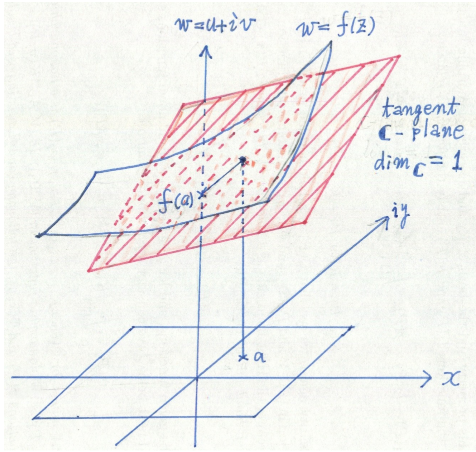

Figure 4: Complex 1 variable $z = x + iy$,
$w = f(z)$ holomorphic

Remark 2.13 (i) Note here that we do not assume the continuity of the derivative $f'(z)$, while the infinite differentiability of $f(z)$ is deduced from *the Cauchy Integral Formula after Goursat's proof of Cauchy's Integral Theorem.*

(ii) The **C**-linearity of the total differential df_a means, in other words, that *the tangent space of the graph of* $w = f(z)$ *at* $(a, f(a))$ *is a complex plane*; this is the point of the observation of the present note. Cf. Fig. 3, Fig. 4, Remark 1.13.

Proposition 2.14 *Let* $f : D \to \mathbf{C}$ *be a function and let* $z_0 \in D$. *The following are mutually equivalent:*

(i) $f(z)$ *is complex differentiable at* z_0.
(ii) $f(z)$ *is totally differentiable at* z_0 *and satisfies the so-called Cauchy–Riemann equation*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0. \tag{2.15}$$

(iii) $\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$ *exists.*

Proof. Immediate by the definition and (2.8). □

The above (iii) is formally the same as in the case of real one variable, but here h moves around in the complex plane (the real two dimensional space); this is a great difference.

Remark 2.16 With $f(z) = u(x, y) + iv(x, y)$ in the real and imaginary parts and by the comparison of the real and imaginary parts, (2.15) is equivalent to

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0), \quad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0). \tag{2.17}$$

These are called the *Cauchy–Riemann equations* (or *relations*).

Proposition 2.18 **(**Derivative of composed functions **)** *If* $G(w)$ *and* $w = f(z)$ *are two holomorphic functions such that a composition* $G \circ f(z) = G(f(z))$ *is defined, then* $G \circ f(z)$ *is holomorphic and satisfies*

$$\frac{d}{dz} G \circ f(z) = \frac{dG}{dw}(f(z)) \cdot \frac{df}{dz}(z). \tag{2.19}$$

Proof. This is immediate from Proposition 2.9. □

Theorem 2.20 *Let D be a domain, and let $f : D \to \mathbf{C}$ be a holomorphic function. If $f'(z) = 0$ everywhere in D, then $f(z)$ is a constant.*

Proof. Assume that $f'(z) = 0$ in U. Since $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}$ and $\frac{\partial f}{\partial y} = i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)$, the assumption implies that $\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0$ everywhere in D. By Theorem 1.5, $f(x, y)$ is constant in D. □

Proposition 2.21 *A function $f(z)$ of C^1-class is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$.*

Proof. Immediate by Proposition 1.18 and the definition. □

Remark 2.22 (i) One may take a C^1 function satisfying $\frac{\partial f}{\partial \bar{z}} = 0$ for the definition of holomorphic functions. But it is an excellent point of Cauchy's Integral Theorem which holds without the continuity of the partial derivatives of the holomorphic function (Goursat's proof). The C^1 assumption dismisses the excellence of Cauchy's Integral Theorem.

(ii) In Proposition 2.21 one may wonder to what extent the regularity condition of C^1-class can be decreased. The best knowlege may be the following:
Looman–Menchoff Theorem: *Let $f : D \to \mathbf{C}$ be a continuous function of which partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist everywhere in D. Then f is complex differentiable (holomorphic) in D if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ in D.*
The proof is very involved (cf. [Nar] Chap. 1 §6).

3 Holomorphic Functions of Several Variables

3.1 Total Differential in Complex Coordinates

We denote by $\mathbf{C}^n$ the complex vector space of n $(\in \mathbf{N})$ product of the complex plane $\mathbf{C}$. For the coordinates we write $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ and $z_j = x_j + iy_j$ with real variables x_j, y_j. By the natural correspondence

$$\mathbf{C} \ni (z_1, \dots, z_n) \mapsto (x_1, y_1, \dots, x_n, y_n) \in \mathbf{R}^{2n}$$

we identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$.

Let $D \subset \mathbf{C}^n$ be an open set. Let $f(z)$ be a complex-valued function in D. With $z = (z_j) = (x_j + iy_j) \in \mathbf{C}^n$ in real and imaginary parts we recall the total differentiability of $f(z) = f(x_1, y_1, \dots, x_n, y_n)$ at $z_0 = (z_{0j}) = (x_{0j} + iy_{0j})$, which we rewrite in the complex coordinates as in §2.1. By definition $f(z)$ is totally differentiable at z_0 if for $h = (h_j) = (h_{j1} + ih_{j2})$ with small $h_{j1}, h_{j2} \in \mathbf{R}$

$$f(z_0 + h) - f(z_0) = \sum_{j=1}^{n} (\alpha_{j1} h_{j1} + \alpha_{j2} h_{j2}) + o(\|h\|), \tag{3.1}$$

where α_{j1} and α_{j2} are complex constants. If it is the case, $\alpha_{j1} = \frac{\partial f}{\partial x_j}(z_0)$ and $\alpha_{j2} = \frac{\partial f}{\partial y_j}(z_0)$. As in the case of one variable (§2.1), (3.1) is written

$$f(z_0 + h) - f(z_0) = \sum_{j=1}^{n} \beta_j h_j + \sum_{j=1}^{n} \gamma_j \bar{h}_j + o(\|h\|), \tag{3.2}$$

where $\beta_j = \frac{1}{2}\left(\alpha_{j1} + \frac{1}{i}\alpha_{j2}\right)$ and $\gamma_j = \frac{1}{2}\left(\alpha_{j1} - \frac{1}{i}\alpha_{j2}\right)$. Therefore we have

Lemma 3.3 *$f(z)$ is totally differentiable at z_0 if and only if there are constants β_j, γ_j $(1 \le j \le n)$ such that*

$$f(z_0+h) - f(z_0) = \sum_{j=1}^{n} \beta_j h_j + \sum_{j=1}^{n} \gamma_j \bar{h}_j + o(\|h\|). \tag{3.4}$$

Here β, γ are uniquely determined if they exist.

We define partial differential operators as follows:

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \frac{1}{i}\frac{\partial}{\partial y_j}\right), \quad 1 \le j \le n, \\ \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{1}{i}\frac{\partial}{\partial y_j}\right), \quad 1 \le j \le n. \end{aligned} \tag{3.5}$$

Note that (2.6) holds for each z_j.

Then β_j, γ_j in (3.4) are given by

$$\beta_j = \frac{\partial f}{\partial z_j}(z_0), \quad \gamma_j = \frac{\partial f}{\partial \bar{z}_j}(z_0),$$

and so (3.4) is reduced to

$$f(z_0+h) - f(z_0) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z_0) h_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}(z_0) \bar{h}_j + o(\|h\|). \tag{3.6}$$

If $f(z)$ is totally differentiable at $z_0 \in D$, the total differential of $f(z)$ at z_0 is written

$$df_{z_0} : h \in \mathbf{C}^n \longrightarrow \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z_0) h_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}(z_0) \bar{h}_j \in \mathbf{C}. \tag{3.7}$$

This is $\mathbf{R}$-linear, but not $\mathbf{C}$-linear in general; df_{z_0} is $\mathbf{C}$-linear if and only if $\frac{\partial f}{\partial \bar{z}_j}(z_0) = 0$, $1 \le j \le n$.

Let $F(z) = (F_1(z), \ldots, F_m(z))$ be an m-vector valued function in a neighborhood of $z_0 \in \mathbf{C}^n$, and let $w_0 = F(z_0) \in \mathbf{C}^m$. Let $G(w)$ be a function defined in a neighborhood of w_0.

Proposition 3.8 *Assume that $F_k(z)$ $(1 \le k \le m)$ and $G(w)$ are totally differentiable at z_0 and at w_0, respectively. Then the composed function $G \circ F(z)$ is totally differentiable at z_0 and satisfies*

$$\frac{\partial (G\circ F)}{\partial z_j}(z_0) = \sum_{k=1}^{m} \frac{\partial G}{\partial w_k}(F(z_0)) \frac{\partial F_k}{\partial z_j}(z_0) + \sum_{k=1}^{m} \frac{\partial G}{\partial \bar{w}_k}(F(z_0)) \frac{\partial \bar{F}_k}{\partial z_j}(z_0), \tag{3.9}$$

$$\frac{\partial (G\circ F)}{\partial \bar{z}_j}(z_0) = \sum_{k=1}^{m} \frac{\partial G}{\partial w_k}(F(z_0)) \frac{\partial F_k}{\partial \bar{z}_j}(z_0) + \sum_{k=1}^{m} \frac{\partial G}{\partial \bar{w}_k}(F(z_0)) \frac{\partial \bar{F}_k}{\partial \bar{z}_j}(z_0). \tag{3.10}$$

Proof. By making use of (3.6) the poof is similar to the one variable case (cf. Proof of Proposition 2.9). □

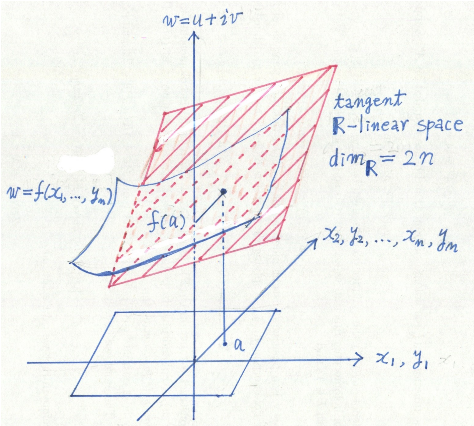

Figure 5: $\mathbf{C}^n \cong \mathbf{R}^{2n}$, real $2n$ variables, $w = f(x_1, \ldots, y_n)$ totally differentiable

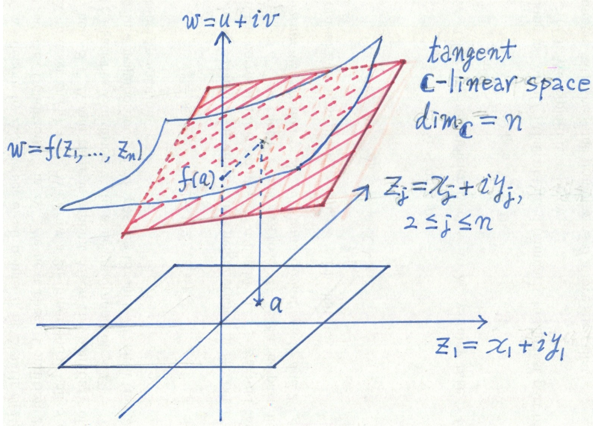

Figure 6: $\mathbf{C}^n$, complex n variables, $w = f(z_1, \ldots, z_n)$ holomorphic

3.2 Complex Total Differentiability and Holomorphic Functions of Several Variables

We keep the notation. Let $D \subset \mathbf{C}^n$ be a domain with the natural coordinate system $z = (z_1, \ldots, z_n)$ as above. Let $f : D \to \mathbf{C}$ be a function. We define the complex total differentiability as a special case of the total differentiability, and holomorphic functions of several complex variables:

Definition 3.11 (**Holomorphic function**) (i) $f(z)$ is *complex totally differentiable* at $z_0 \in D$ if $f(z)$ is totally differentiable at z_0 and the total differential $df_{z_0} : \mathbf{C}^n \to \mathbf{C}$ is $\mathbf{C}$-linear; i.e.,

$$df_{z_0}(h) = \sum_{j=1}^{n} \alpha_j h_j, \qquad h = (h_j) \in \mathbf{C}^n \tag{3.12}$$

with $\alpha_j := df_{z_0}(0, \ldots, \overset{j\text{th}}{1}, \ldots, 0)$ $(1 \leq j \leq n)$, or

$$f(z_0 + h) - f(z_0) = \sum_{j=1}^{n} \alpha_j h_j + o(\|h\|) \quad \text{as } h = (h_1, \ldots, h_n) \to 0$$

In this case, $f(z)$ is continuous at z_0.

(i) $f(z)$ is *holomorphic* in D if $f(z)$ is complex totally differentiable at every point of D.

Remark 3.13 (i) Note here that $f(z)$ is not assumed to be of C^1-class, while the infinite differentiability of $f(z)$ will be proved later.

(ii) As in the case of one variable (Remark 2.13 (ii)), the $\mathbf{C}$-linearity of the total differential df_a means that ***the tangent space of the graph of*** $w = f(z)$ ***at*** $(a, f(a)$ ***is a complex linear space***: Cf. Fig. 5, Fig. 6.

Proposition 3.14 (i) *Let $f(z)$ $(z \in D)$ be totally differentiable at z_0. Then df_{z_0} is $\mathbf{C}$-linear if and only if the so-called Cauchy–Riemann equations*

$$\frac{\partial f}{\partial \bar{z}_j}(z_0) = 0, \quad 1 \leq j \leq n,$$

are satisfied.

(ii) *A function* $f(z)$ *of* C^1*-class in* D *is holomorphic if and only if* $\frac{\partial f}{\partial \bar{z}_j}(z) = 0$ $(1 \le j \le n)$ *in* D.

Proof. (i) By the definition and (3.7).
(ii) By Proposition 1.18 and (i) above. □

We call $\frac{\partial}{\partial z_j}$ (resp. $\frac{\partial}{\partial \bar{z}_j}$) the (resp. *anti-*) *holomorphic partial differential operator.*

Theorem 3.15 *Let* D *be a domain of* $\mathbf{C}^n$*, and let* $f : D \to \mathbf{C}$ *be a holomorphic function. If* $\frac{\partial f}{\partial z_j}(z) = 0$ $(1 \le j \le n)$ *everywhere in* D*, then* $f(z)$ *is a constant.*

Proof. Assume that $\frac{\partial f}{\partial z_j} = 0$ $(1 \le j \le n)$ in D. As in the proof of Theorem 2.20 we see that $\frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial y_j} = 0$ $(1 \le j \le n)$ in D. By Theorem 1.5, $f(z)$ is constant in D. □

For the composition of holomorphic functions we have:

Proposition 3.16 *Let* $F(z) = (F_1(z), \ldots, F_m(z))$ *be an* m*-vector valued holomorphic function in a neighborhood of* $z_0 \in \mathbf{C}^n$*, and let* $w_0 = F(z_0) \in \mathbf{C}^m$*. Let* $G(w)$ *be a holomorphic function defined in a neighborhood of* w_0.
Then the composed function $G \circ F(z)$ *is holomorphic and satisfies*

$$\frac{\partial (G \circ F)}{\partial z_j}(z_0) = \sum_{k=1}^{m} \frac{\partial G}{\partial w_k}(F(z_0)) \frac{\partial F_k}{\partial z_j}(z_0). \tag{3.17}$$

Proof. By Proposition 3.8. □

Proposition 3.18 *Let* $f(z)$ *be a function in* D*. Then the following are equivalent:*

(i) $f(z)$ *is holomorphic in* D.
(ii) $f(z)$ *is continuous and separately holomorphic in* D*; i.e.,* (separate holomorphy) *for an arbitrarily fixed point* $a = (a_j) \in D$ *and* j *with* $1 \le j \le n$*, the function* $f(a_1, \ldots, z_j, \ldots, a_n)$ *is a holomorphic function of one variable* $z_j \in \{z_j \in \mathbf{C} : (a_1, \ldots, z_j, \ldots, a_n) \in D\}$.

Proof. (i)⇒(ii): Clear.
(ii)⇒(i): Because of the condition, we have Cauchy's Integral Formula for $f(z)$ over closed polydisks in D by multiple path integrals, with which the proof is immediate. □

Remark 3.19 (i) One may take (ii) above for the definition of holomorphic functions, but the separate holomorphy is very dependent on the coordinates. Even after a linear transformation of the coordinates (z_j), the property of separate holomorphy is not preserved, so that the definition by (ii) loses the sense. The above Definition 3.11 is preserved by linear transformations of (z_j), and more in deed by the changes of holomorphic local coordinates, which is easily confirmed through the partial derivative formulae of composed functions (Proposition 3.8).

(ii) If a function $f(z)$ in an open set D $(\subset \mathbf{C}^n)$ is continuous and separately holomorphic, then $f(z)$ *is an analytic function, i.e.,* $f(z)$ *is developed to a convergent power series about every point of* D. There is a style to take this property for the definition of holomorphy (K.T. Weierstrass); in this case, however as mentioned in [Car], it is non-tirivial and must be proved that *a power series convergent in a polydisk* $\mathrm{P}\Delta(a; r)$ *is a holomorphic function there.*

It is also noted that the definition of holomorphy by analyticity is so strong that Cauchy's Integral Theorem (in one variable) is reduced to an almost trivial statement by the existence of the local primitives everywhere in the domain; here, in fact, there are many arguments historically since Weierstrass.

(iii) (Hartogs Theorem) In the implication (ii)⇒(i) of Proposition 3.18 the 'continuity' of $f(z)$ is superflurous. But the proof is not at the level of an introductory book (cf., e.g., [Hit2], [Hör], [Nis], [Nog3]).

The following are just samples:

References

[Ahl] L. Ahlfors, Complex Analysis, 3rd ed. McGrraw-Hill Internat. Book Co., 1979.

[Car] H. Cartan, Théorie Élémentaire de Fonctions Analytiques d'Une ou Plusieures Variables Complexes, Hermann, Paris, 1961; Japanese Translation by R. Takahashi, Iwanami Shoten, Tokyo, 1965.

[Con] J.B. Conway, Functions of One Complex Variables, GTM 11, Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[Dem] J.-P. Demailly, Complex Analytic and Differentiable Geometry, 2012, www-fourier.ujf-grenoble.fr/~demailly/ .

[For] F. Forstnerič, Stein Manifolds and Holomorphic Mappings, Ergeb. Math. Grenzgeb. 3, Vol. 56, Springer-Verlag, 2011; 2nd Edition, 2017.

[FrGr] K. Fritzsche and H. Grauert, From Holomorphic Functions to Complex Manifolds, GTM 213, Springer-Verlag, New York, 2002.

[GuR] R.C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice-Hall, 1965.

[Has] F. Haslinger, Complex Analysis—A functional analytic approach, Walter de Gruyter, Berlin-Boston, 2018.

[Hit1] S. Hitotumatu, On the definition of holomorphic functions of several complex variables (in Japanese), Math. Soc. Jpn., Sugaku 8-1 (1956), 25-25.

[Hit2] S. Hitotsumatsu, Analytic Function Theory in Several Variables (in Japanese), Baifukan, Tokyo, 1960.

[Hör] L. Hörmander, Introduction to Complex Analysis in Several Variables, 3rd Ed., North-Holland, 1989.

[Hur] A. Hurwitz, Vorlesungen über Allgemeinentheorie und Elliptische Funktionen; herausgegeben und erhanzt durch einen Abschnitt über Geometrische Funktionentheorie von R. Courant, Grundl. Math. Wissen. 3, Verl. J. Springer, Berlin, 1929; First Ed., 1922.

[Kod] K. Kodaira, Complex Manifolds and Deformation of Complex Structures, Classics in Math., Springer, Berlin–Heidelberg, 1986/2005.

[Mon] P. Montel, Sur les différentielles totales et les fonctions monogènes, C.R. Acad. Sci. Paris **156** (1913), 1820–1822.

[Nar] R. Narasimhan, Complex Analysis in One Variable (Part I), 2nd edition with Y. Nievergelt, Springer Sci.+Bus. Media, New York, 2001; Originally publ. by Birkh.user, Boston, 1985.

[Nis] T. Nishino, Function Theory in Several Complex Variables, transl. by N. Levenberg and H. Yamaguchi, Amer. Math. Soc. Providence, R.I., 2001: the original Japanese ed., The University of Tokyo Press, Tokyo, 1996.

[Nog1] J. Noguchi, Introduction to Complex Analysis, MMONO **168**, Amer. Math. Soc. Rhode Island, 1997: The Japanese Ed., Sugakusensho 12, Shokabo, Tokyo, 1993.

[Nog2] J. Noguchi, Analytic Function Theory of Several Variables–Elements of Oka's Coherence, Springer, Singapore, 2016: The Japanese Ed., Asakura Shoten, Tokyo, 1st. 2013, 2nd. 2019.

[Nog3] J. Noguchi, Basic Oka Theory in Several Complex Variables, Springer UTX, 2024.

[Rem] R. Remmert, Theory of Complex Functions, transl. by R.B. Burckel, GTM 122, Springer, New York, 1991: the original ed. Funktionentheorie I, 2nd Ed., 1989.

[SSST] G. Della Sala, A. Saracco, A. Simioniuc and G. Tomassini, Lectures on complex analysis and analytic geometry, Appunti, Scuola normale superiore 3, 2006.