A New Introductory Lecture of S.C.V.
– Oka Theory

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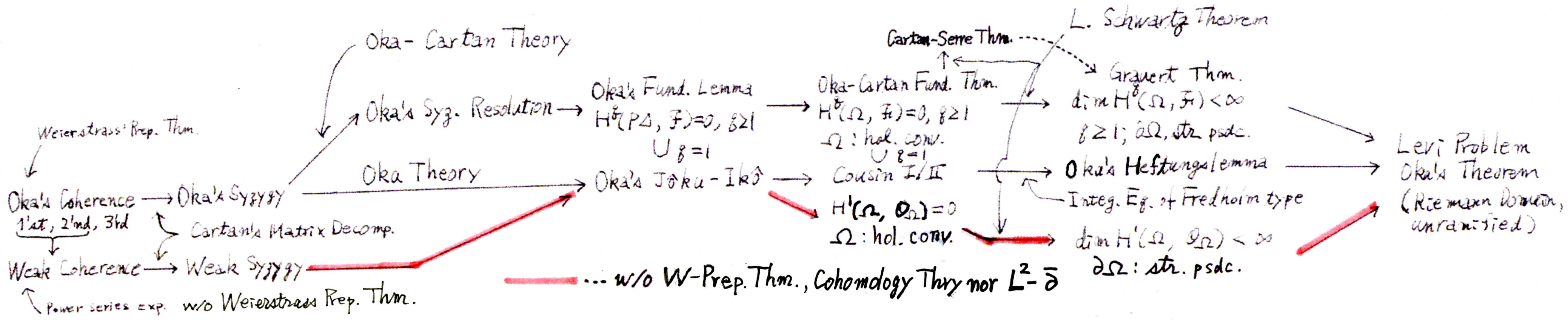
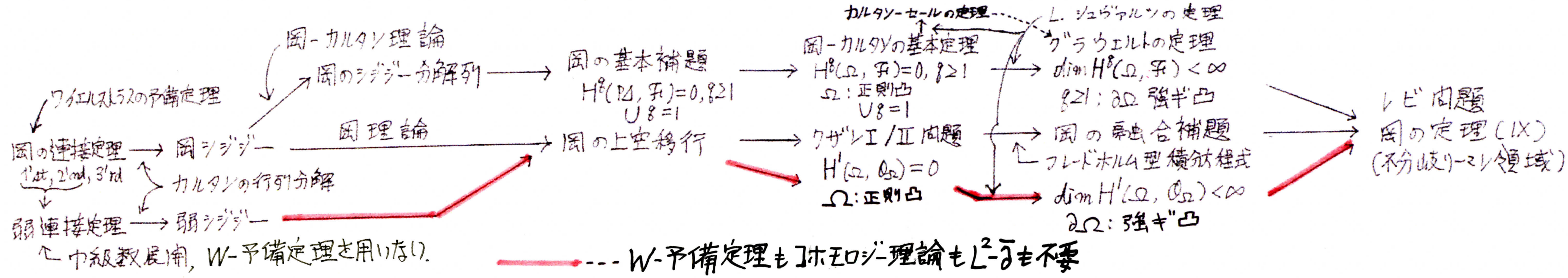
Workshop on Holomorphic Maps, Pluripotentials and Complex Geometry

26 (26–29) Mar. 2019

Mandala of Oka and Oka-Cartan Theories by J. Noguchi (2018)



岡理論及び岡-カルタン理論の曼荼羅図 (野口潤次郎, 2018)



§1 Introduction.

The big 3 problems of Behnke–Thullen (1934):

1. Levi (Hartogs' Inverse) Problem (Chap. IV).
2. Cousin I/II Problem (Chap. V).
3. Approximation (Development) Problem (Chap. VI).

Kiyoshi Oka solved all 3 in the opposite order (1936–1953).

Reference:

1. [AFT] N–, Analytic Function Theory of Several Variables—Elements of Oka's Coherence, Springer, 2016.
2. N–, 多変数解析関数論—学部生へおくる岡の連接定理, 朝倉書店, 東京, 2013:
   Analytic Function Theory in Several Variables (in Japanese), Asakurashoten, Tokyo, 2013.
3. N–, A brief chronicle of the Levi (Hartogs' Inverse) Problem, Coherence and an open problem, to appear in Notices Intern. Cong. Chin. Math., Intern. Press.
4. N–, A weak coherence theorem and remarks to the Oka theory, to appear in Kodai Math. J.

For Levi (Hartogs' Inverse) Problem it generally referred as:

1. Univalent domains of dim $= 2$ by Oka VI (1942).
2.
3.
4. Unramified Riemann domains of dim ≥ 2 by Oka (1953).
5. Univalent domains of dim ≥ 2 by F. Norguet and H.J. Bremermann (1954).

But, in fact, historically,

1. Univalent domains of dim $= 2$ by Oka VI (1942).
2. Unramified Riemann domains of dim ≥ 2 by Oka's unpublished papers (1943).
3. Univalent domains of dim ≥ 2 by S. Hitotsumatsu (1949).
4. Unramified Riemann domains of dim ≥ 2 by Oka (1953).
5. Univalent domains of dim ≥ 2 by F. Norguet and H.J. Bremermann (1954).

Used methods:

Weil's integral to solve an integral equation in 1, 3 and 5:

Jôku-Ikô with estimate and Cousin Integral to solve an integral equation in 2, and 4 with Coherence.

The present approach was inspired by Oka's unpublished papers 2.

Cousin I(/II) Problem:

Let

$\Omega \subset \mathbf{C}^n$ be a domain,

$\mathcal{O}(U)$ be the set of all holomorphic functions in an open $U \subset \Omega$,

$\Omega = \bigcup U_\alpha$ be an open covering, and

$f_\alpha \in \mathscr{M}(U_\alpha)$ $(/\mathscr{M}^*(U_\alpha))$ be (/non-zero) merom. funct's. in U_α.

Call $\{(U_\alpha, f_\alpha)\}$ a Cousin I(/II) data if

$$f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta) \ (/f_\alpha \cdot f_\beta^{-1} \in \mathcal{O}^*(U_\alpha \cap U_\beta)), \quad {}^\forall \alpha, \beta.$$

Find $F \in \mathscr{M}(\Omega)$ $(/\mathscr{M}^*(\Omega))$ **such that**

$$F - f_\alpha \in \mathcal{O}(U_\alpha) \ (/F \cdot f_\alpha^{-1} \in \mathcal{O}^*(U_\alpha)), \quad {}^\forall \alpha.$$

Cousin Integral (Cousin decomposition)

Let $E' \times E_1$ and $E' \times E_2$ be adjacent cuboids with open neighborhoods U_1 and U_2. Let

$\{(U_j, f_j)\}_{j=1,2}$ be a Cousin data, and

$g = f_2 - f_1 \in \mathcal{O}(U_1 \cap U_2)$.

Cousin Integral: $\varphi(z', z_n) = \dfrac{1}{2\pi i} \displaystyle\int_\ell \frac{g(z', \zeta)}{\zeta - z_n} d\zeta.$

On E_α $(\alpha = 1, 2)$, $\varphi_\alpha(z', z_n) = \varphi(z', z_n) = \dfrac{1}{2\pi i} \displaystyle\int_{\ell_\alpha} \frac{g(z', \zeta)}{\zeta - z_n} d\zeta.$

By Cauchy,

$$\varphi_1 - \varphi_2 = g = f_2 - f_1 \text{ on } E_1 \cap E_2.$$

$$F = f_1 + \varphi_1 = f_2 + \varphi_2 \in \mathscr{M}(E_1 \cup E_2), \text{ Solution.}$$

It was Oka's great idea to reduce the general case to the above simple one by **Jôku-Ikô**: Ideal theoretic Jôku-Ikô = Coherence.

Theorem 1.1

The Cousin I/II Problems are always solvable on a polydisk $\mathrm{P}\Delta$.

Proof. Since $\mathrm{P}\Delta \cong$ an open cuboid($\subset \mathbf{C}^n$),

$^{\exists}$closed cuboids $E_\nu \nearrow \mathrm{P}\Delta, \nu = 1, 2, \ldots$.

Using **Cousin Integral** inductively, we have solutions F_ν on E_ν.

Using the **Approximation (Function Developement in $\mathrm{P}\Delta$)**, modify F_ν so that

$$\text{(sup-norm) } \|F_{\nu+1} - F_\nu\|_{E_\nu} < \frac{1}{2^\nu}.$$

$F = F_1 + \sum_{\nu=1}^{\infty}(F_{\nu+1} - F_\nu)$, a solution. □

N.B. This is the prototype method to obtain a solution.

§2 Hartogs domains

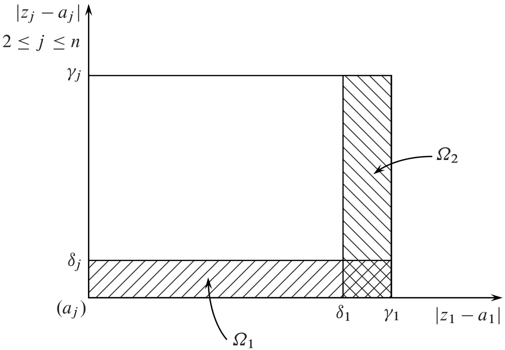
Let $n \geq 2$, $a = (a_j) \in \mathbf{C}^n$, $0 < \delta_j < \gamma_j$, $1 \leq j \leq n$, $\gamma = (\gamma_j)$. Set

$$\mathrm{P}\Delta(a, \gamma) = \{z = (z_j) \in \mathbf{C}^n : |z_j| < \gamma_j, {}^\forall j\},$$

$$\Omega_1 = \{z = (z_j) \in \mathrm{P}\Delta(a, \gamma) : |z_j - a_j| < \delta_j,\ j \geq 2\},$$

$$\Omega_2 = \{z \in \mathrm{P}\Delta(a, \gamma) : \delta_1 < |z_1 - a_1| < \gamma_1\},$$

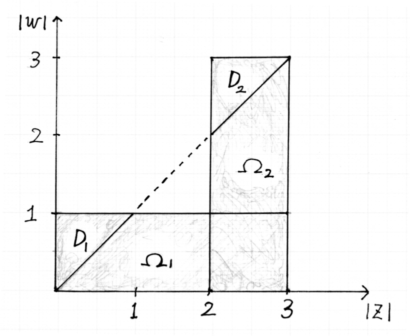
$$\Omega_{\mathrm{H}}(a; \gamma) = \Omega_1 \cup \Omega_2 \quad \text{(Fig. 11).}$$



Hartogs's phenomenon:

$$\mathcal{O}(\Omega_{\mathrm{H}}(a;\gamma)) \cong \mathcal{O}(\mathrm{P}\Delta(a,\gamma)).$$

- $^{\exists}$*Non-solvable Cousin I/II data on* $\Omega_{\mathrm{H}}(a;\gamma)$.



Poles $\frac{1}{z-w}|_{\Omega_1}$ for I (/ Zeros $(z-w)|_{\Omega_1}$ for II). If F is a solution, think $(z-w)F|_{\{z=w\}}$ (/ $F|_{\{z=w\}}$).

Holomorphic Convexity

For $K \subset \Omega$ we define the **holomorphic convex hull** of K by

$$\hat{K}_\Omega = \hat{K}_{\mathcal{O}(\Omega)} = \left\{ z \in \Omega : |f(z)| \leq \sup_K |f|, \ {}^\forall f \in \mathcal{O}(\Omega) \right\}.$$

Ω is said to be **holomorphically convex** if for all $K \Subset \Omega$,

$$\hat{K}_{\mathcal{O}(\Omega)} \Subset \Omega.$$

N.B. Hartogs domains are not holomorphically convex.

Problem: Is Cousin I/II Problem solvable on holomorphically convex domains?

§3 Weak Coherence

Let $\Omega \subset \mathbf{C}^n$ be a domain, $a \in \Omega$,

f be a holomorphic function about a

$\mathscr{O}_a = \{\underline{f}_a = \sum c_\nu (z-a)^\nu$: conv. power series, germs$\}$ (a ring),

$\mathscr{O}_\Omega = \bigcup_{a \in \Omega} \mathscr{O}_a$ (sheaf as sets), $\quad \mathscr{O}_n = \mathscr{O}_{\mathbf{C}^n}$.

Consider:

$\mathscr{O}_\Omega^q$ ($q \in \mathbf{N}$), naturally an $\mathscr{O}_\Omega$-module,

$\mathscr{S} \subset \mathscr{O}_\Omega^q$, an $\mathscr{O}_\Omega$-submodule.

For an open subset $U \subset \Omega$, put

$\mathscr{S}(U) = \left\{ (f_j) \in \mathscr{O}(U)^q : \left(\underline{f_j}_a \right) \in \mathscr{S}_a, \ {}^\forall a \in U \right\}$ (sections).

Definition 3.1

An $\mathscr{O}_\Omega$-submodule $\mathscr{S}$ is locally finite if for ${}^\forall a \in \Omega$, ${}^\exists U \ni a$, a neighborhood, and finitely many $\sigma_k \in \mathscr{S}(U), 1 \leq k \leq \ell$ such that

$$\mathscr{S}_z = \sum_{k=1}^{\ell} \mathscr{O}_z \cdot \underline{\sigma_k}_z, \quad {}^\forall z \in U.$$

$\{\sigma_k\}_{1 \leq k \leq \ell}$ is called a finite generator system of $\mathscr{S}$ on U.

Let $V \subset \Omega$ be an open subset, $\tau_k \in \mathscr{S}(V), 1 \leq k \leq N(< \infty)$, $\mathscr{R}(\tau_1, \ldots, \tau_N) \subset \mathscr{O}_V^N$ be the **relation sheaf** defined by

$$\mathscr{R}(\tau_j) = \bigcup_{a \in V} \left\{ \left(\underline{f_j}_a \right) \in \mathscr{O}_a^N : \sum_j \underline{f_j}_a \cdot \underline{\tau_j}_a = 0 \right\}.$$

For a subset $S \subset \Omega$, define the **ideal sheaf** of S by

$$\mathscr{I}\langle S \rangle = \bigcup_{a \in \Omega} \left\{ \underline{f}_a \in \mathscr{O}_a : f|_S = 0 \right\}.$$

Theorem 3.2 (Weak Coherence)

Let $S \subset \Omega$ be a complex submanifold, possibly non-connected.

1. *The ideal sheaf $\mathscr{I}\langle S\rangle$ is locally finite.*
2. *Let $\{\sigma_j \in \mathscr{I}\langle S\rangle(\Omega) : 1 \le j \le N\}$ be a finite generator system of $\mathscr{I}\langle S\rangle$ on Ω.*

   Then, the relation sheaf $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ is locally finite.

Proof.

1. Locally, $S = \{z_1 = \cdots = z_q = 0\}$ in $U \subset \Omega$. Then,

$$\mathscr{I}\langle S\rangle = \sum_{j=1}^{q} \mathcal{O}_U \cdot z_j.$$

2. This is immediately reduced to the local finiteness of the relation sheaf defined

(3.3) $$\underline{f_1}_z \cdot z_1 + \cdots + \underline{f_q}_z \cdot z_q = 0.$$

Induction on q:

$q = 1$: Trivially $\mathscr{R}(z_1) = 0$, locally finite.

Suppose it up to $q - 1$ ($q \geq 2$) valid. For q, write

$$f_j = \sum_{\nu} c_\nu z^\nu = g_j(z_1, z')z_1 + h_j(z'), \quad z' = (z_2, \ldots, z_n).$$

Then, (3.3) is rewritten as

(3.4)

$$\underline{(f_1 + g_2 z_2 + \cdots + g_q z_q)}_z \cdot z_1 + \underline{h_2(z')}_z \cdot z_2 + \cdots + \underline{h_q(z')}_z \cdot z_q = 0 :$$

$$f_1 = -g_2 z_2 - \cdots - g_q z_q, \tag{3.5}$$

$$\underline{h_2(z')}_z \cdot z_2 + \cdots + \underline{h_q(z')}_z \cdot z_q = 0 \tag{3.6}$$

In (3.5), $g_2, \ldots, g_q$ are finite number of free variables, i.e., locally finite.

(3.6) is the case "$q-1$"; by the induction hypothesis it is locally finite.

Thus, $\mathscr{R}(z_1, \ldots, z_q)$ is locally finite. $\square$

Let $\Omega \subset \mathbf{C}^n = \mathbf{C}^{n-1} \times \mathbf{C}$ be a domain,

$E', E'' \Subset \Omega$ be two closed cuboids as follows:

a closed cuboid $F \Subset \mathbf{C}^{n-1}$ and two adjacent closed rectangles $E'_n, E''_n \Subset \mathbf{C}$ sharing a side ℓ,

$$E' = F \times E'_n, \quad E'' = F \times E''_n, \quad \ell = E'_n \cap E''_n. \tag{3.7}$$

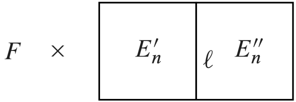


Figure: Adjacent closed cuboids

Lemma 3.8 (Cartan's matrix decomposition)

Let

U be a neighborhood of $F \times \ell$,

$A(z)$ be an invertible (N, N)-matrix valued holomorphic function in U.

Then, $^{\exists}\delta > 0$, sufficiently small such that if $\|A\|_U < \delta$,

$^{\exists}A'(z), A''(z)$, invertible (N, N)-matrix valued holomorphic functions on E', E'', respectively, satisfying

$$A(z) = A'(z)A''(z) \quad \text{on } F \times \ell.$$

Proof. By Cousin Integral with estimate. See Appendix of [AFT]. $\square$

Consider a closed cuboid $E \subset \mathbf{C}^n$, possibly degenerate with some edges of length 0. Define

$\dim E =$ the number of edges of positive lengths:

$$0 \leq \dim E \leq 2n.$$

Lemma 3.9 (Oka Syzygy)

Let $E \Subset \mathbf{C}^n$ be a closed cuboid.

1. *Every locally finite submodule $\mathscr{S}(\subset \mathcal{O}_n^N)$ defined on E (i.e., in a neighborhood of E) has a finite generator system on E.*
2. *Let $\mathscr{S}$ be a submodule on E with a finite generator system $\{\sigma_j\}_{1 \leq j \leq N}$ on E such that $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ is locally finite. Then for ${}^\forall\sigma \in \mathscr{S}(E)$, ${}^\exists a_j \in \mathcal{O}(E)$, $1 \leq j \leq N$, such that*

$$\sigma = \sum_{j=1}^{N} a_j \cdot \sigma_j \quad (\text{on } E). \tag{3.10}$$

Proof.

Double **Cuboid Induction** on $\dim E$: $[1_{q-1}, 2_{q-1}] \Rightarrow 1_q \Rightarrow 2_q$

(a) $\dim E = 0$: 1, 2 Trivial by definition.

(b) Suppose them up to $\dim E = q-1, q \geq 1$, valid.

$\dim E = q$:

1. $2_{q-1}+$ Cartan's matrix decomposition.
2. Write with $T > 0, \theta \geq 0$:

$$E = F \times \{z_n = t + iy_n : 0 \leq t \leq T, |y_n| \leq \theta\},$$
$$\dim F = \begin{cases} q-1, & \theta = 0; \\ q-2, & \theta > 0. \end{cases}$$

Apply the induction hypothesis 2_{q-1} to

$E_t = F \times \{t + iy_n : |y_n| \leq \theta\}$ with $t \in [0, T]$. We then have

$$\sigma = \sum_{j=1}^{N} a_j \cdot \sigma_j \quad \text{(in a nbd. of) } E_t.$$

Let

$$\sigma = \sum_{j=1}^{N} a'_j \cdot \sigma_j, \quad \sigma = \sum_{j=1}^{N} a''_j \cdot \sigma_j$$

be such expressions in adjacent cubes E', E'' with $E' \cap E'' = E_t$.

By 1_q, ${}^{\exists}$a generator system $\{\tau_k = (\tau_{kj})_j\}_k$ of $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ on E.

Since $\sum_{j=1}^{N}(a'_j - a''_j) \cdot \sigma_j = 0$ on E_t, we apply the induction hypothesis 2_{q-1} for $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ to get

$$(a'_j - a''_j) = \sum_k b_k \cdot (\tau_{kj}) \text{ on } E_{x_n}, \ \ b_k \in \mathcal{O}(E_t).$$

Apply Cousin Integral to $b_k = b'_k - b''_k$:

$$\left(a'_j - \sum_k b'_k \tau_{kj}\right) = \left(a''_j - \sum_k b''_k \tau_{kj}\right) = (a'''_j) \in \mathcal{O}(E' \cup E'')^N.$$

$$\sigma = \sum_j a_j''' \cdot \sigma_j, \text{ on } E' \cup E''.$$

Repeat this. $\square$

N.B. We apply this for $\mathscr{I}\langle S\rangle$ of a complex submanifold $S \subset \mathrm{P}\Delta$.

§4 Oka's Jôku-Ikô

Let

$\mathrm{P} \subset \mathbf{C}^n$ be an open cuboid,

$S \subset \mathrm{P}$ be a complex submanifold.

Lemma 4.1 (Oka's Jôku-Ikô)

Let $E \Subset \mathrm{P}$ be a closed cuboid. Then for
${}^\forall g \in \mathscr{O}(E \cap S)$ $(E \cap S \Subset S)$, ${}^\exists G \in \mathscr{O}(E)$ satisfying

$$G|_{E\cap S} = g|_{E\cap S}.$$

Proof. By

Weak Coherence of $\mathscr{I}\langle S\rangle$+Oka Syzygy + Cuboid Induction. □

Approximation

An analytic polyhedron $\mathrm{P} \Subset \Omega$ is a finite union of relatively compact connected components of

$$\{z \in \Omega : |\psi_j(z)| < 1,\ 1 \le j \le L\},\ \psi_j \in \mathcal{O}(\Omega), L < \infty.$$

Theorem 4.2 (Runge–Weil–Oka)

Every holomorphic function on $\bar{\mathrm{P}}$ is uniformly approximated on $\bar{\mathrm{P}}$ by functions of $\mathcal{O}(\Omega)$.

Proof. Let $f \in \mathcal{O}(\bar{\mathrm{P}})$. By Oka map,

$$\Psi : z \in \bar{\mathrm{P}} \hookrightarrow (z, \psi_1(z), \ldots, \psi_L(z)) \in \overline{\mathrm{P}\Delta} \subset \mathbf{C}^{n+L},$$

$\bar{\mathrm{P}}$ is a complex submanifold of $\mathrm{P}\Delta$.

By Oka's Jôku-Ikô, extend f to $F \in \mathcal{O}(\overline{\mathrm{P}\Delta})$.

F is developed to a power series, and hence f is developed to a power series in z and (ψ_j). $\square$

§5 Continuous Cousin Problem

Let $\Omega = \bigcup_\alpha U_\alpha$ be an open covering and $\phi_\alpha \in C(U_\alpha)$, continuous functions.

Definition 5.1

$\{(U_\alpha, \phi_\alpha)\}$ is continuous Cousin data if

$$\phi_\alpha - \phi_\beta \in \mathcal{O}(U_\alpha \cap \beta), \ {}^\forall \alpha, \beta.$$

Continuous Cousin Problem: Find a solution $\Phi \in C(\Omega)$ such that $\Phi - \phi_\alpha \in \mathcal{O}(U_\alpha), \ {}^\forall \alpha.$

The following 3 problems are deduced from Continuous Cousin Problem:

1. Cousin I Problem.
2. Cousin II Problem.
3. Problem of $\bar{\partial} u = f, \bar{\partial} f = 0$ for functions u.

$(\because)$ 1. May assume $\{U_\alpha\}$ locally finite.
Take open $V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, and $\chi_\alpha \in C(\Omega)$ such that $\chi_\alpha \geq 0$; $\chi_\alpha(z) > 0, z \in V_\alpha$; $\chi_\alpha(z) = 0, z \notin U_\alpha$; $\sum_\alpha \chi_\alpha = 1$.
For a Cousin I data (U_α, f_α), set

$$\phi_\alpha = \sum_\gamma (f_\alpha - f_\gamma)\chi_\gamma \in C(U_\alpha).$$

Then, $\phi_\alpha - \phi_\beta = f_\alpha - f_\beta$: $\quad f_\alpha - \phi_\alpha = f_\beta - \phi_\beta$.

Let Φ be a solution of $\{(U_\alpha, \phi_\alpha)\}$. Then

$$f_\alpha \underbrace{-\phi_\alpha + \Phi}_{\text{hol.}} = f_\beta \underbrace{-\phi_\beta + \Phi}_{\text{hol.}}.$$

2. By the assumption of the Oka principle.
3. By Dolbeault's Lemma, locally there are solutions,

$$u_\alpha \in C^\infty(U_\alpha),\ \bar{\partial} u_\alpha = f,\ \bigcup_\alpha U_\alpha = \Omega.$$

Since $\bar{\partial}(u_\alpha - u_\beta) = 0$, $(u_\alpha - u_\beta) \in \mathcal{O}(U_\alpha \cap U_\beta)$. The rest is the same as in 1.

Theorem 5.2

On a holomorphically convex domain every Continuous Cousin Problem is solvable.

Proof. Let $\Omega \subset \mathbf{C}^n$ be a holomorphically convex domain, and $\{(U_\alpha, \phi_\alpha)\}$ be a continuous Cousin data on Ω.

Take $\mathrm{P}_\nu \nearrow \Omega$, increasing analytic polyhedra, and the Oka maps $\bar{\mathrm{P}}_\nu \hookrightarrow \overline{\mathrm{P}\Delta}_{(\nu)}$.

Step 1. Obtain a solution Φ_ν on each $\bar{\mathrm{P}}_\nu \hookrightarrow \overline{\mathrm{P}\Delta}_{(\nu)}$.

By Cuboid Induction + Oka's Jôku-Ikô + Cousin Integral.

Step 2. Since $\Phi_{\nu+1} - \Phi_\nu \in \mathcal{O}(\bar{\mathrm{P}}_\nu)$, applying the Approximation of Runge-Weil-Oka, modify Φ_ν so that

$$\|\Phi_{\nu+1} - \Phi_\nu\|_{\bar{\mathrm{P}}_\nu} < \frac{1}{2^\nu}, \ \nu = 1, 2, \ldots.$$

We have a solution,

$$\Phi = \Phi_1 + \sum_{\nu=1}^{\infty} (\Phi_{\nu+1} - \Phi_\nu).$$

$\square$

§7 Interpolation

In the same way as in the previous section we have

Theorem 6.1 (Interpolation)

Let $\Omega \subset \mathbf{C}^n$ be a holomorphically convex domain and $S \subset \Omega$ be a complex submanifolds.

Then, $f \in \mathcal{O}(\Omega) \to f|_S \in \mathcal{O}(S) \to 0$ (surjective).

If particular, for ${}^\forall\{a_\nu\}$, a discrete sequence of Ω and ${}^\forall c_\nu \in \mathbf{C}$, ${}^\exists F \in \mathcal{O}(\Omega)$ with $F(a_\nu) = c_\nu, {}^\forall \nu$. Conversely, if it holds for Ω, Ω is holomorphically convex.

Proof. Excercise. □

§8 Levi (Hartogs' Inverse) Problem

If a domain $\Omega \subset \mathbf{C}^n$ is maximal with respect to Hartogs phenomenon, Ω is called a **domain of holomorphy**.

Theorem 7.1 (Cartan–Thullen, 1932)

A domain is holomophically convex iff it is a domain of holomorphy.

Let $\mathrm{P}\Delta \subset \mathbf{C}^n$ be any fixed polydisk with center at 0, and $\Omega \subset \mathbf{C}^n$ be a domain. Put

$$\delta_{\mathrm{P}\Delta}(z, \partial\Omega) = \sup\{r > 0 : z + r \cdot \mathrm{P}\Delta \subset \Omega\}, \ z \in \Omega.$$

Theorem 7.2 (Oka)

If Ω is holomorphically convex, $-\log \delta_{\mathrm{P}\Delta}(z, \partial\Omega)$ is plurisubharmonic in $z \in \Omega$.

We call Ω a **pseudoconvex domain** if $-\log \delta_{\mathrm{P}\Delta}(z, \partial\Omega)$ is plurisubharmonic near $\partial\Omega$. Levi (Hartogs' Inverse) Problem: Is a pseudoconvex domain holomorphically convex?

A bounded domain $\Omega \subset \mathbf{C}^n$ is said to be **strongly pseudoconvex** if for ${}^\forall a \in \partial\Omega$, ${}^\exists U \ni a$, a neighborhood and $\varphi \in C^2(U)$ such that $U \cap \Omega = \{\varphi < 0\}$ and

$$i\partial\bar{\partial}\varphi(z) \gg 0, \ z \in U.$$

- If Ω is pseudoconvex, ${}^\exists\Omega_\nu \nearrow \Omega$ with strongly pseudoconvex Ω_ν.

The 1st cohomology $H^1(\Omega, \mathcal{O})$

Let $\Omega = \bigcup U_\alpha$, $\mathscr{U} = \{U_\alpha\}$. Define

$Z^1(\mathscr{U}, \mathcal{O})$, 1-cycle space,

$\delta : C^0(\mathscr{U}, \mathcal{O}) \to B^1(\mathscr{U}, \mathcal{O})$, a boundary operator,

$H^1(\mathscr{U}, \mathcal{O}) = Z^1(\mathscr{U}, \mathcal{O})/B^1(\mathscr{U}, \mathcal{O})$,

$H^1(\Omega, \mathcal{O}) = \varinjlim_{\mathscr{U}} H^1(\mathscr{U}, \mathcal{O}) \hookleftarrow H^1(\mathscr{U}, \mathcal{O})$.

- $H^1(\Omega, \mathcal{O}) = 0 \iff {}^{\forall}$Cont. Cousin Problem is solvable on Ω.

Theorem 7.3

1. *If Ω is holomorphically convex, $H^1(\Omega, \mathcal{O}) = 0$.*
2. *For $\mathscr{U} = \{U_\alpha\}$ an open covering of Ω with ${}^{\forall}U_\alpha$, holomorphically convex,*

$$H^1(\mathscr{U}, \mathcal{O}) \cong H^1(\Omega, \mathcal{O}).$$

L. Schwartz Theorem

Let E be a Hausdorff topological complex vector space with countably many semi-norms. If the asoociated norm on E is complete, E is called a **Fréchet space**.

If E satisfies Baire's Category Theorem, E is called a **Baire space**.

Theorem 7.4 (Open Map)

Let E (resp. F) be a Fréchet (resp. Baire) vector space. If $A : E \to F$ is a continuous linear surjection, then A is an open map.

Theorem 7.5 (L. Schwartz's Finiteness Theorem)

Let E (resp. F) be a Fréchet (resp. Baire) vector space. Let $A : E \to F$ be a continuous linear surjection, and $B : E \to F$ be a compact operator. Then $(A + B)E$ is closed, and $\dim \mathrm{Coker}(A + B)(:= F/(A + B)E) < \infty$.

Proof. Let U be a neighborhood of $0 \in E$ such that $\overline{B(U)}$ is compact. Since $A(U)$ is oepn, ${}^{\exists}b_j \in \overline{B(U)},\ 1 \le j \le N < \infty$, such that

$$\overline{B(U)} \subset \bigcup_j \left(b_j + \frac{1}{2}A(U) \right).$$

By the Open Map Theorem we have

$$F = (A+B)E + \langle b_1, \ldots, b_N \rangle_{\mathbf{C}}, \text{ algebraically.}$$

Modify $\{b_j\}$ so that b_j are linearly independent and

$$F = (A+B)E \oplus \langle b_1, \ldots, b_N \rangle_{\mathbf{C}}.$$

$$(E/\ker(A+B)) \oplus \langle b_1, \ldots, b_N \rangle_{\mathbf{C}} \ni ([x], y) \mapsto (A+B)x \oplus y \in F$$

is a topological isomorphism again by Open Map Theorem. Therefore, $(A+B)E$ is closed and $\dim \mathrm{Coker}(A+B) = N < \infty$. $\square$

Theorem 7.6 (Grauert)

Let Ω be a strongly pseudoconvex domain. Then, $\dim H^1(\Omega, \mathcal{O}) < \infty$.

Proof (Grauert's bumping method).

$\Omega = {}^{\exists}\bigcup_{\text{finite}} V_\alpha$ with V_α, hol. convex,

bumped open ${}^{\exists}\tilde{U}_\alpha \Supset V_\alpha$ with $\tilde{U}_\alpha$, hol. convex,

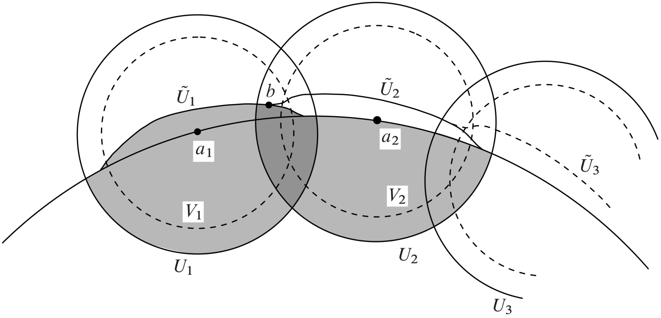


Figure: Boundary bumping method

$\mathscr{V} = \{V_\alpha\}$, bumped covering $\tilde{\mathscr{U}} = \{\tilde{U}_\alpha\}$ ($\Supset \Omega$), so that

$$\tilde{U}_\alpha \cap \tilde{U}_\beta \Supset V_\alpha \cap V_\beta,$$

$$\Psi : \xi \oplus \eta \in Z^1(\tilde{\mathscr{U}}, \mathcal{O}) \oplus C^0(\mathscr{V}, \mathcal{O}) \to \rho(\xi) + \delta\eta \in Z^1(\mathscr{V}, \mathcal{O}) \to 0,$$

where ρ is the restriction map from the bumped $\tilde{\mathscr{U}}$ to $\mathscr{V}$.

Note that $Z^1(\tilde{\mathscr{U}}, \mathcal{O}) \oplus C^0(\mathscr{V}, \mathcal{O})$ and $Z^1(\mathscr{V}, \mathcal{O})$ are Fréchet (in particular, the latter is Baire).

Since ρ is compact (Montel), L. Schwartz applied to $\Psi - \rho$ yields that $\mathrm{Coker}(\Psi - \rho) \cong H^1(\mathscr{V}, \mathcal{O}) \cong H^1(\Omega, \mathcal{O})$ is finite dimensional. $\square$

Theorem 7.7 (Oka)

A strongly pseudoconvex domain is holomorphically convex.

Proof. Let φ be a defining function of $\partial\Omega$ such that $\Omega = \{\varphi < 0\}$, φ is strongly plurisubharmonic in a neighborhood of $\partial\Omega$.

Take a point $b \in \partial\Omega$. By a translation, we may put $b = 0$. Set

$$Q(z) = 2\sum_{j=1}^{n} \frac{\partial\varphi}{\partial z_j}(0)z_j + \sum_{j,k} \frac{\partial^2\varphi}{\partial z_j \partial z_k}(0)z_j z_k.$$

${}^{\exists}\varepsilon, \delta > 0$ satisfying

$$\varphi(z) \geq \Re\, Q(z) + \varepsilon\|z\|^2, \quad \|z\| \leq \delta,$$

$$\inf\{\varphi(z); Q(z) = 0, \|z\| = \delta\} \geq \varepsilon\delta^2 > 0.$$

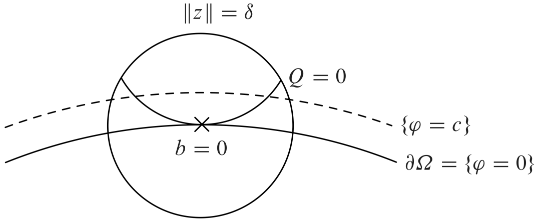


Figure: $\Omega' = \{\varphi < c\}, U_0$

Let $U_1 = \Omega' \setminus \{Q = 0\}$. Then $\mathscr{U} = \{U_0, U_1\}$ is an open covering of Ω', which is stronglly pseudoconvex.

We set

$$f_{01}(z) = \frac{1}{Q(z)}, \quad z \in U_0 \cap U_1,$$
$$f_{10}(z) = -f_{01}(z), \quad z \in U_1 \cap U_0.$$

Then, a 1-cocyle $f = (f_{01}(z), f_{10}(z)) \in Z^1(\mathscr{U}, \mathcal{O})$ is obtained. For $k \in \mathbf{N}$ we define

$$f_{01}^{[k]}(z) = (f_{01}(z))^k, \quad z \in U_0 \cap U_1,$$
$$f_{10}^{[k]}(z) = -f_{01}^{[k]}(z), \quad z \in U_1 \cap U_0.$$

Then $(f^{[k]}) \in Z^1(\mathscr{U}, \mathcal{O})$. Thus we obtain cohomology classes,

$$[f^{[k]}] \in H^1(\mathscr{U}, \mathcal{O}) \hookrightarrow H^1(\Omega', \mathcal{O}), \quad k \in \mathbf{N}.$$

Since Ω' is strongly pseudoconvex, Grauert's Theorem implies $\dim H^1(\Omega', \mathcal{O}) < \infty$.

Therefore, for N large, there is a non-trivial linear relation,

$$\sum_{k=1}^{N} c_k [f^{[k]}] = 0 \in H^1(\mathscr{U}, \mathcal{O}_{\Omega'}) \quad (c_k \in \mathbf{C}).$$

We may suppose that $c_N \neq 0$. Then there exists elements $g_i \in \mathcal{O}(U_i)$, $i = 0, 1$, such that

$$\sum_{k=1}^{N} \frac{c_k}{Q^k(z)} = g_1(z) - g_0(z), \quad z \in U_0 \cap U_1.$$

Therefore,

$$g_0(z) + \sum_{k=1}^{N} \frac{c_k}{Q^k(z)} = g_1(z), \quad z \in U_0 \cap U_1, \quad c_N \neq 0.$$

$^\exists F \in \mathscr{M}(\Omega')$ with poles of order N on $\{Q = 0\}$.

Since $\{Q = 0\} \cap \Omega = \emptyset$, $F|_{\Omega} \in \mathcal{O}(\Omega)$ and $\lim_{z \to 0} |F(z)| = \infty$.

Thus, Ω is holomorphically convex. $\square$

Theorem 7.8 (Oka)

A pseudoconvex domain is holomorphically convex.

Proof. There are strongly psudoconvex domains $\Omega_{\nu} \nearrow \Omega$. Since Ω_{ν} are holomorphically convex, so is the limit Ω (Behnke–Stein). $\square$

Furthermore, we have

Theorem 7.9 (Oka)

A pseudoconvex unramified Riemann domain over $\mathbf{C}^n$ *is holomorphically convex and holomorphically separable; i.e., a Stein manifold.*