A New Introductory Lecture of S.C.V. – the Oka Theory

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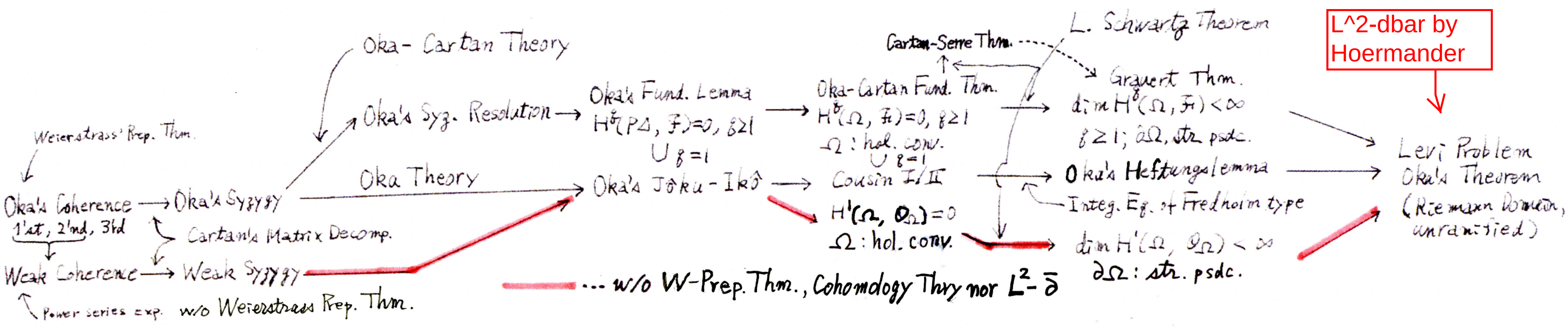
Montreal

10 May 2019

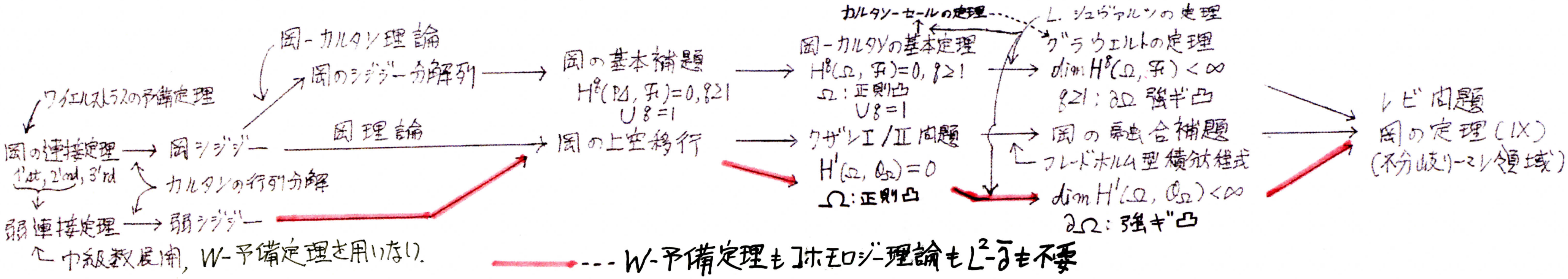

日本民族は情の民族である。
人と人との間によく心が通い合うし、
人と自然との間にもよく心が通い合う。
この心を情というのである。
日本民族は情によって
つながっているのである。
岡潔
岡 潔
Oka Kiyoshi
日本数学史上最高の世界的数学者
最初の橋本市名誉市民
1901年 紀見村出身の父寛治 母八重の長男として誕生
1925年 京都帝国大学理学部数学科を卒業と同時に同大学講師に就任
1929年 フランスに留学 生涯の研究テーマとなる「多変数解析函数論」に出会う
後年 その分野において 世界中の数学者が解けなかった「三大問題」を
たった一人ですべて解決した
京都帝国大学のほか 広島文理科大学 北海道大学
奈良女子大学 京都産業大学で教鞭を執る
1960年 文化勲章受賞
1961年 橋本市名誉市民となる
1963年 随筆集「春宵十話」を刊行 以後「風蘭」「紫の火花」「春風夏雨」「月影」
「日本のこころ」「一葉舟」等を著し
日本人は何を学ぶべきか世に問い続けた
1978年 逝去
この岡潔先生の教えが永く そして広く引き継がれ 日本の青少年
の将来に夢と希望を与えることを祈り 郷土のためにこれを建つ
2013年11月吉日
寄贈 Rotary 橋本ロー
2016/01/31


Sur les fonctions analytiques de plusieurs variables
VIII — Lemme fondamental
Par Kiyoshi Oka
Introduction.
finis sans points critiques.
岡 潔
人を先にして自分を後にせよ
岡 潔
Rotary
橋本ロータリークラブ

Mandala of Oka and Oka-Cartan Theories by J. Noguchi (2018)



岡理論及び岡-カルタン理論の曼荼羅図（野口潤次郎, 2018）



§1 Introduction.

The **Big 3 Problems** of Behnke–Thullen (1934):

1. Levi (Hartogs' Inverse) Problem (Chap. IV).
2. Cousin I/II Problem (Chap. V).
3. Approximation (Development) Problem (Chap. VI).

Kiyoshi Oka solved all 3 in the opposite order (1936–1953).

Their difficulties commented by H. Cartan:

From H. Cartan's "Sur l'Œuvre de Kiyoshi Oka";
in KIYOSHI OKA COLLECTED PAPERS, Ed. R. Remmert,
translated by R. Narasimhan, Springer 1984:

La publication, en 1934, de la monographie de BEHNKE–THULLEN faisant le point sur l'état de la théorie des fonctions analytiques de plusieurs variables complexes à un moment crucial de son développement, et mettant en évidence les principaux problèmes ouverts à cette époque, semble avoir joué un rôle déterminant dans l'orientation des recherches d'Oka: il se fixa pour tâche de résoudre ces problèmes difficiles, tâche quasi-surhumaine. On peut dire qu'il y réussit, surmontant l'un après l'autre les obstacles redoutables qui se trouvaient sur sa route.

Mais il faut avouer que les aspects techniques de ses démonstrations et le mode de présentation de ses résultats rendent difficile la tâche du lecteur, et que ce n'est qu'au prix d'un réel effort que l'on parvient à saisir la portée de ses résultats, qui est considérable. C'est pourquoi il est peut-être encore utile aujourd'hui, en hommage au grand créateur que fut Kiyoshi OKA, de présenter l'ensemble de son œuvre.

[English translation]:

The publication in 1934 of the monograph of Behnke–Thullen surveying the point of the state of the theory of analytic functions of several complex variables at a crucial moment of its development, and highlighting the main open problems at the time, seems to have played a determining role in the orientation of Oka's researches: he settled his task to solve these difficult problems, the task quasi-superhumane. One can say that he succeeded there, surmounting one after the other the formidable obstructions on his way.

But we must admit that the technical aspects of his proofs and the mode of presentation of his results make it difficult to read, and that it is possible only at the cost of a real effort to grasp the scope of its results, which is considerable. That is why it may still be useful today, in tribute to the great creator that was Kiyoshi OKA, to present his œvre.

(translated by Noguchi, 2019)

Reference:

1. [AFT] N–, Analytic Function Theory of Several Variables—Elements of Oka's Coherence, Springer, 2016.
2. N–, 多変数解析関数論—学部生へおくる岡の連接定理, 朝倉書店, 東京, 2013:
   Analytic Function Theory in Several Variables (in Japanese), Asakurashoten, Tokyo, 2013.
3. N–, A brief chronicle of the Levi (Hartogs' Inverse) Problem, Coherence and an open problem, to appear in Notices Intern. Cong. Chin. Math., Intern. Press.
4. N–, A weak coherence theorem and remarks to the Oka theory, to appear in Kodai Math. J.

For Levi (Hartogs' Inverse) Problem it is generally referred as:

1. Univalent domains of dim $= 2$ by Oka VI (1942).
2.
3.
4. Unramified Riemann domains of dim ≥ 2 by Oka IX (1953).
5. Univalent domains of dim ≥ 2 by F. Norguet and H.J. Bremermann (1954).

But, in fact, historically,

1. Univalent domains of dim $= 2$ by Oka VI (1942):

   Weil's integral.

2. Unramified Riemann domains of dim ≥ 2 (in Japanese) by Oka's unpublished papers, pp. 109 (1943): Semi-Coherence + Jôku-Ikô + Cauchy's (Cousin's) Integral.

3. Univalent domains of dim ≥ 2 (in Japanese) by S. Hitotsumatsu (1949): Weil's Integral.

4. Unramified Riemann domains of dim ≥ 2 by Oka IX (1953): Coherence + Jôku-Ikô + Cauchy's (Cousin's) Integral.

5. Univalent domains of dim ≥ 2 by F. Norguet and H.J. Bremermann (1954): Weil's Integral.

The present approach was inspired by Oka's unpublished papers 2.

Cousin I(/II) Problem:

Let

$\Omega \subset \mathbf{C}^n$ be a domain,

$\mathcal{O}(U)$ be the set of all holomorphic functions in an open $U \subset \Omega$,

$\Omega = \bigcup U_\alpha$ be an open covering, and

$f_\alpha \in \mathscr{M}(U_\alpha)$ $(/\mathscr{M}^*(U_\alpha))$ be (/non-zero) merom. funct's. in U_α.

Call $\{(U_\alpha, f_\alpha)\}$ a Cousin I(/II) data if

$$f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta)\ (/f_\alpha \cdot f_\beta^{-1} \in \mathcal{O}^*(U_\alpha \cap U_\beta)), \quad {}^\forall \alpha, \beta.$$

Find $F \in \mathscr{M}(\Omega)$ $(/\mathscr{M}^*(\Omega))$ **such that**

$$F - f_\alpha \in \mathcal{O}(U_\alpha)\ (/F \cdot f_\alpha^{-1} \in \mathcal{O}^*(U_\alpha)), \quad {}^\forall \alpha.$$

Cousin Integral (Cousin decomposition)

Let $E' \times E_1$ and $E' \times E_2$ be adjacent cuboids with open neighborhoods U_1 and U_2. Let

$\{(U_j, f_j)\}_{j=1,2}$ be a Cousin data, and

$g = f_2 - f_1 \in \mathcal{O}(U_1 \cap U_2)$.

Cousin Integral: $\varphi(z', z_n) = \dfrac{1}{2\pi i} \displaystyle\int_{\ell} \frac{g(z',\zeta)}{\zeta - z_n} d\zeta.$

On E_α $(\alpha = 1, 2)$, $\varphi_\alpha(z', z_n) = \varphi(z', z_n) = \dfrac{1}{2\pi i} \displaystyle\int_{\ell_\alpha} \frac{g(z',\zeta)}{\zeta - z_n} d\zeta.$

By Cauchy,

$$\varphi_1 - \varphi_2 = g = f_2 - f_1 \text{ on } E_1 \cap E_2.$$

$$F = f_1 + \varphi_1 = f_2 + \varphi_2 \in \mathscr{M}(E_1 \cup E_2), \text{ Solution.}$$

It was Oka's great idea to reduce the general case to the above simple one by **Jôku-Ikô**: Ideal theoretic Jôku-Ikô = Coherence.

Theorem 1.1

The Cousin I/II Problems are always solvable on a polydisk $\mathrm{P}\Delta$*.*

Proof. Since $\mathrm{P}\Delta \cong$ an open cuboid($\subset \mathbf{C}^n$),
$^{\exists}$closed cuboids $E_\nu \nearrow \mathrm{P}\Delta, \nu = 1, 2, \ldots$.
Using **Cousin Integral** inductively, we have solutions F_ν on E_ν.
Using the **Approximation (Function Developement in $\mathrm{P}\Delta$)**, modify F_ν so that

$$\text{(sup-norm) } \|F_{\nu+1} - F_\nu\|_{E_\nu} < \frac{1}{2^\nu}.$$

$F = F_1 + \sum_{\nu=1}^{\infty}(F_{\nu+1} - F_\nu)$, is a solution. □

N.B. This is the prototype method to obtain a solution.

§2 Hartogs domains

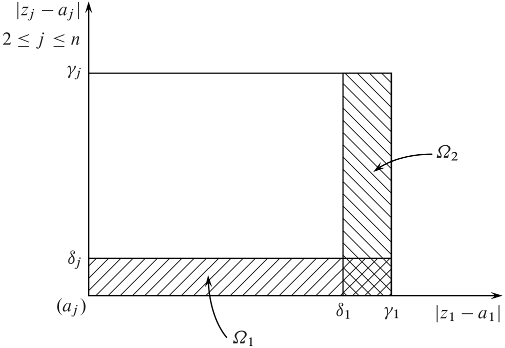
Let $n \geq 2$, $a = (a_j) \in \mathbf{C}^n$, $0 < \delta_j < \gamma_j$, $1 \leq j \leq n$, $\gamma = (\gamma_j)$. Set

$$\mathrm{P}\Delta(a, \gamma) = \{z = (z_j) \in \mathbf{C}^n : |z_j - a_j| < \gamma_j, {}^\forall j\},$$

$$\Omega_1 = \{z = (z_j) \in \mathrm{P}\Delta(a, \gamma) : |z_j - a_j| < \delta_j,\ j \geq 2\},$$

$$\Omega_2 = \{z \in \mathrm{P}\Delta(a, \gamma) : \delta_1 < |z_1 - a_1| < \gamma_1\},$$

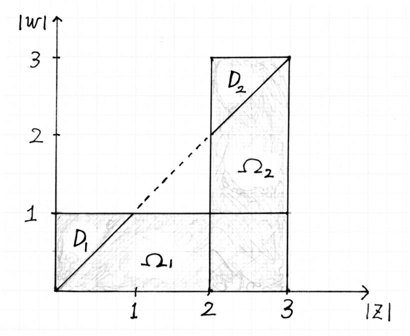
$$\Omega_{\mathrm{H}}(a; \gamma) = \Omega_1 \cup \Omega_2 \quad \text{(Fig. 16)}.$$



Hartogs's phenomenon:

$$\mathcal{O}(\Omega_{\mathrm{H}}(a;\gamma)) \cong \mathcal{O}(\mathrm{P}\Delta(a,\gamma)).$$

- $^{\exists}$*Non-solvable Cousin I/II data on* $\Omega_{\mathrm{H}}(a;\gamma)$.



Poles $\dfrac{1}{z-w}|_{\Omega_1}$ for I (/ Zeros $(z-w)|_{\Omega_1}$ for II). If F is a solution, think $(z-w)F|_{\{z=w\}}$ (/ $F|_{\{z=w\}}$).

Holomorphic Convexity

For $K \subset \Omega$ we define the **holomorphic convex hull** of K by

$$\hat{K}_\Omega = \hat{K}_{\mathcal{O}(\Omega)} = \left\{ z \in \Omega : |f(z)| \leq \sup_K |f|,\ {}^\forall f \in \mathcal{O}(\Omega) \right\}.$$

Ω is said to be **holomorphically convex** if for all $K \Subset \Omega$,

$$\hat{K}_{\mathcal{O}(\Omega)} \Subset \Omega.$$

N.B. Hartogs domains are not holomorphically convex.

Problem: Is Cousin I/II Problem solvable on holomorphically convex domains?

§3 Weak Coherence

Let $\Omega \subset \mathbf{C}^n$ be a domain, $a \in \Omega$,

f be a holomorphic function about a

$\mathcal{O}_a = \{\underline{f}_a = \sum c_\nu (z-a)^\nu$: conv. power series, germs$\}$ (a ring),

$\mathcal{O}_\Omega = \bigsqcup_{a \in \Omega} \mathcal{O}_a$ (sheaf as sets), $\quad \mathcal{O}_n = \mathcal{O}_{\mathbf{C}^n}$.

Consider:

$\mathcal{O}_\Omega^q = \bigsqcup_{a \in \Omega} \mathcal{O}_a^q$ ($q \in \mathbf{N}$), naturally an $\mathcal{O}_\Omega$-module,

$\mathscr{S} = \bigsqcup_{a \in \Omega} \mathscr{S}_a \subset \mathcal{O}_\Omega^q$, an $\mathcal{O}_\Omega$-submodule.

For an open subset $U \subset \Omega$, put

$\mathscr{S}(U) = \left\{ (f_j) \in \mathcal{O}(U)^q : \left(\underline{f_j}_a \right) \in \mathscr{S}_a, \ {}^\forall a \in U \right\}$ (sections).

Definition 3.1

An $\mathscr{O}_\Omega$-submodule $\mathscr{S}$ is locally finite if for ${}^\forall a \in \Omega$, ${}^\exists U \ni a$, a nbd., and finitely many $\sigma_k \in \mathscr{S}(U), 1 \leq k \leq \ell$, such that

$$\mathscr{S}_z = \sum_{k=1}^{\ell} \mathscr{O}_z \cdot \underline{\sigma_k}_z, \quad {}^\forall z \in U.$$

$\{\sigma_k\}_{1 \leq k \leq \ell}$ is called a finite generator system of $\mathscr{S}$ on U.

Let $V \subset \Omega$ be an open subset, $\tau_k \in \mathscr{S}(V), 1 \leq k \leq N(< \infty)$, $\mathscr{R}(\tau_1, \ldots, \tau_N) \subset \mathscr{O}_V^N$ be the **relation sheaf** defined by

$$\mathscr{R}(\tau_j) = \bigsqcup_{a \in V} \left\{ \left(\underline{f_j}_a \right) \in \mathscr{O}_a^N : \sum_j \underline{f_j}_a \cdot \underline{\tau_j}_a = 0 \right\}.$$

For a subset $S \subset \Omega$, define the **ideal sheaf** of S by

$$\mathscr{I}\langle S \rangle = \bigsqcup_{a \in \Omega} \{ \underline{f}_a \in \mathscr{O}_a : f|_S = 0 \}.$$

Theorem 3.2 (Weak Coherence)

Let $S \subset \Omega$ be a complex submanifold, possibly non-connected.

1. *The ideal sheaf $\mathscr{I}\langle S \rangle$ is locally finite.*
2. *Let $\{\sigma_j \in \mathscr{I}\langle S \rangle(\Omega) : 1 \le j \le N\}$ be a finite generator system of $\mathscr{I}\langle S \rangle$ on Ω.*

   Then, the relation sheaf $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ is locally finite.

Proof.

1. Locally, $S = \{z_1 = \cdots = z_q = 0\}$ in $U \subset \Omega$. Then,

$$\mathscr{I}\langle S \rangle = \sum_{j=1}^{q} \mathcal{O}_U \cdot z_j.$$

2. This is immediately reduced to the local finiteness of the relation sheaf defined

$$\underline{f_1}_z \cdot z_1 + \cdots + \underline{f_q}_z \cdot z_q = 0. \tag{3.3}$$

Induction on q:

$q = 1$: Trivially $\mathscr{R}(z_1) = 0$, locally finite.

Suppose it up to $q-1$ ($q \geq 2$) valid. For q, write

$$f_j = \sum_\nu c_\nu z^\nu = g_j(z_1, z')z_1 + h_j(z'), \quad z' = (z_2, \ldots, z_n).$$

Then, (3.3) is rewritten as

(3.4)

$$\underline{(f_1 + g_2 z_2 + \cdots + g_q z_q)}_z \cdot z_1 + \underline{h_2(z')}_z \cdot z_2 + \cdots + \underline{h_q(z')}_z \cdot z_q = 0 :$$

$$f_1 = -g_2 z_2 - \cdots - g_q z_q, \tag{3.5}$$

$$\underline{h_2(z')}_z \cdot z_2 + \cdots + \underline{h_q(z')}_z \cdot z_q = 0 \tag{3.6}$$

In (3.5), $g_2, \ldots, g_q$ are finite number of free variables, i.e., locally finite.

(3.6) is the case "$q-1$"; by the induction hypothesis it is locally finite.

Thus, $\mathscr{R}(z_1, \ldots, z_q)$ is locally finite. $\square$

Let $\Omega \subset \mathbf{C}^n = \mathbf{C}^{n-1} \times \mathbf{C}$ be a domain,

$E', E'' \Subset \Omega$ be two closed cuboids as follows:

a closed cuboid $F \Subset \mathbf{C}^{n-1}$ and two adjacent closed rectangles $E'_n, E''_n \Subset \mathbf{C}$ sharing a side ℓ,

$$E' = F \times E'_n, \quad E'' = F \times E''_n, \quad \ell = E'_n \cap E''_n. \tag{3.7}$$

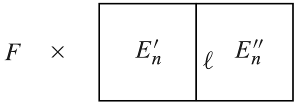


Figure: Adjacent closed cuboids

Lemma 3.8 (Cartan's matrix decomposition)

Let

U be a neighborhood of $F \times \ell$,

$A(z)$ be an invertible (N, N)-matrix valued hol. function in U.

Then, ${}^{\exists}\delta > 0$, sufficiently small such that if $\|A\|_U < \delta$,

${}^{\exists}A'(z), A''(z)$, invertible (N, N)-matrix valued holomorphic functions on E', E'', respectively, satisfying

$$A(z) = A'(z)A''(z) \quad \text{on } F \times \ell.$$

Proof. H. Cartan used $(\mathbf{1}_N + A)^{-1} = e^{-\log(\mathbf{1}_N + A)}$.

We simply use

$$(\mathbf{1}_N - A)^{-1} = \mathbf{1}_N + A + A^2 + \cdots,$$

and Cousin Integral with estimate. See Appendix of [AFT]. □

Consider a closed cuboid $E \subset \mathbf{C}^n$, possibly degenerate with some edges of length 0. Define

$\dim E =$ the number of edges of positive lengths:

$$0 \le \dim E \le 2n.$$

Lemma 3.9 (Oka Syzygy)

Let $E \Subset \mathbf{C}^n$ be a closed cuboid.

1. *Every locally finite submodule $\mathscr{S} (\subset \mathcal{O}_n^N)$ defined on E (i.e., in a neighborhood of E) has a finite generator system on E.*
2. *Let $\mathscr{S}$ be a submodule on E with a finite generator system $\{\sigma_j\}_{1 \le j \le N}$ on E such that $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ is locally finite. Then for ${}^\forall \sigma \in \mathscr{S}(E)$, ${}^\exists a_j \in \mathcal{O}(E)$, $1 \le j \le N$, such that*

$$\sigma = \sum_{j=1}^{N} a_j \cdot \sigma_j \quad (\text{on } E). \tag{3.10}$$

Proof.

Double **Cuboid Induction** on $\dim E$: $[1_{q-1}, 2_{q-1}] \Rightarrow 1_q \Rightarrow 2_q$

(a) $\dim E = 0$: 1, 2 Trivial by definition.

(b) Suppose them up to $\dim E = q-1, q \geq 1$, valid.

$\dim E = q$:

1. $2_{q-1}+$ Cartan's matrix decomposition.
2. Write with $T > 0, \theta \geq 0$:

$$E = F \times \{z_n = t + iy_n : 0 \leq t \leq T, |y_n| \leq \theta\},$$

$$\dim F = \begin{cases} q-1, & \theta = 0; \\ q-2, & \theta > 0. \end{cases}$$

Apply the induction hypothesis 2_{q-1} to

$E_t = F \times \{t + iy_n : |y_n| \leq \theta\}$ with $t \in [0, T]$. We then have

$$\sigma = \sum_{j=1}^{N} a_j \cdot \sigma_j \quad \text{(in a nbd. of) } E_t.$$

Let

$$\sigma = \sum_{j=1}^{N} a'_j \cdot \sigma_j, \quad \sigma = \sum_{j=1}^{N} a''_j \cdot \sigma_j$$

be such expressions in adjacent cuboids E', E'' with $E' \cap E'' = E_t$.

By 1_q, $^\exists$a generator system $\{\tau_k = (\tau_{kj})_j\}_k$ of $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ on E.

Since $\sum_{j=1}^{N}(a'_j - a''_j) \cdot \sigma_j = 0$ on E_t, we apply the induction hypothesis 2_{q-1} for $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ to get

$$(a'_j - a''_j) = \sum_k b_k \cdot (\tau_{kj}) \text{ on } E_{x_n},\ \ b_k \in \mathcal{O}(E_t).$$

Apply Cousin Integral to $b_k = b'_k - b''_k$:

$$\left(a'_j - \sum_k b'_k \tau_{kj}\right) = \left(a''_j - \sum_k b''_k \tau_{kj}\right) = (a'''_j) \in \mathcal{O}(E' \cup E'')^N.$$

$$\sigma = \sum_j a_j''' \cdot \sigma_j, \text{ on } E' \cup E''.$$

Repeat this. $\square$

N.B. We apply this for $\mathscr{I}\langle S\rangle$ of a complex submanifold $S \subset \mathrm{P}\Delta$.

§4 Oka's Jôku-Ikô

Let

$\mathrm{P} \subset \mathbf{C}^n$ be an open cuboid,

$S \subset \mathrm{P}$ be a complex submanifold.

Lemma 4.1 (Oka's Jôku-Ikô)

Let $E \Subset \mathrm{P}$ be a closed cuboid. Then for
${}^{\forall} g \in \mathcal{O}(E \cap S)$ $(E \cap S \Subset S)$, ${}^{\exists} G \in \mathcal{O}(E)$ satisfying

$$G|_{E \cap S} = g|_{E \cap S}.$$

Proof. By

Weak Coherence of $\mathscr{I}\langle S \rangle$+Oka Syzygy + Cuboid Induction. $\square$

Approximation

An analytic polyhedron $\mathrm{P} \Subset \Omega$ is a finite union of relatively compact connected components of

$$\{z \in \Omega : |\psi_j(z)| < 1,\ 1 \leq j \leq L\},\ \psi_j \in \mathcal{O}(\Omega), L < \infty.$$

Theorem 4.2 (Runge–Weil–Oka)

Every holomorphic function on $\bar{\mathrm{P}}$ is uniformly approximated on $\bar{\mathrm{P}}$ by functions of $\mathcal{O}(\Omega)$.

Proof. Let $f \in \mathcal{O}(\bar{\mathrm{P}})$. By Oka map,

$$\Psi : z \in \bar{\mathrm{P}} \hookrightarrow (z, \psi_1(z), \ldots, \psi_L(z)) \in \overline{\mathrm{P}\Delta} \subset \mathbf{C}^{n+L},$$

$\bar{\mathrm{P}}$ is a complex submanifold of $\mathrm{P}\Delta$.

By Oka's Jôku-Ikô, extend f to $F \in \mathcal{O}(\overline{\mathrm{P}\Delta})$.

F is developed to a power series, and hence f is developed to a power series in z and (ψ_j). $\square$

§5 Continuous Cousin Problem

Let $\Omega = \bigcup_\alpha U_\alpha$ be an open covering and $\phi_\alpha \in C(U_\alpha)$, continuous functions.

Definition 5.1

$\{(U_\alpha, \phi_\alpha)\}$ is continuous Cousin data if

$$\phi_\alpha - \phi_\beta \in \mathcal{O}(U_\alpha \cap \beta),\ {}^\forall \alpha, \beta.$$

Continuous Cousin Problem: Find a solution $\Phi \in C(\Omega)$ such that $\Phi - \phi_\alpha \in \mathcal{O}(U_\alpha),\ {}^\forall \alpha.$

The following 3 problems are deduced from Cont. Cousin Problem:

1. Cousin I Problem.
2. Cousin II Problem.
3. Problem of $\bar{\partial} u = f, \bar{\partial} f = 0$ for functions u.

$\because$) 1. May assume $\{U_\alpha\}$ locally finite.

Take open $V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, covering Ω, and $\chi_\alpha \in C(\Omega)$ such that $\chi_\alpha \geq 0$; $\chi_\alpha(z) > 0, z \in V_\alpha$; $\chi_\alpha(z) = 0, z \notin U_\alpha$; $\sum_\alpha \chi_\alpha = 1$.

For a Cousin I data (U_α, f_α), set

$$\phi_\alpha = \sum_\gamma (f_\alpha - f_\gamma)\chi_\gamma \in C(U_\alpha).$$

Then, $\phi_\alpha - \phi_\beta = f_\alpha - f_\beta$: $\quad f_\alpha - \phi_\alpha = f_\beta - \phi_\beta$.

Let Φ be a solution of $\{(U_\alpha, \phi_\alpha)\}$. Then

$$f_\alpha \underbrace{-\phi_\alpha + \Phi}_{\text{hol.}} = f_\beta \underbrace{-\phi_\beta + \Phi}_{\text{hol.}}.$$

2. By the assumption of the Oka principle.
3. By Dolbeault's Lemma, locally there are solutions,

$$u_\alpha \in C^\infty(U_\alpha),\ \bar{\partial} u_\alpha = f,\ \bigcup_\alpha U_\alpha = \Omega.$$

Since $\bar{\partial}(u_\alpha - u_\beta) = 0$, $(u_\alpha - u_\beta) \in \mathcal{O}(U_\alpha \cap U_\beta)$. The rest is the same as in 1.

Theorem 5.2

On a holomorphically convex domain every Continuous Cousin Problem is solvable.

Proof. Let $\Omega \subset \mathbf{C}^n$ be a holomorphically convex domain, and $\{(U_\alpha, \phi_\alpha)\}$ be a continuous Cousin data on Ω.

Take $\mathrm{P}_\nu \nearrow \Omega$, increasing analytic polyhedra, and the Oka maps $\bar{\mathrm{P}}_\nu \hookrightarrow \overline{\mathrm{P}\Delta}_{(\nu)}$.

Step 1. Obtain a solution Φ_ν on each $\bar{\mathrm{P}}_\nu \hookrightarrow \overline{\mathrm{P}\Delta}_{(\nu)}$.

By Cuboid Induction + Oka's Jôku-Ikô + Cousin Integral.

Step 2. Since $\Phi_{\nu+1} - \Phi_\nu \in \mathcal{O}(\bar{\mathrm{P}}_\nu)$, applying the Approximation of Runge-Weil-Oka, modify Φ_ν so that

$$\|\Phi_{\nu+1} - \Phi_\nu\|_{\bar{\mathrm{P}}_\nu} < \frac{1}{2^\nu}, \ \nu = 1, 2, \ldots .$$

We have a solution,

$$\Phi = \Phi_1 + \sum_{\nu=1}^{\infty} (\Phi_{\nu+1} - \Phi_\nu).$$

$\square$

§7 Interpolation

In the same way as in the previous section we have

Theorem 6.1 (Interpolation)

Let $\Omega \subset \mathbf{C}^n$ be a holomorphically convex domain and

$S \subset \Omega$ be a complex submanifold.

Then, $f \in \mathcal{O}(\Omega) \to f|_S \in \mathcal{O}(S) \to 0$ (surjective).

If particular, for $^\forall\{a_\nu\}$, a discrete sequence of Ω and $^\forall c_\nu \in \mathbf{C}$,

$^\exists F \in \mathcal{O}(\Omega)$ with $F(a_\nu) = c_\nu, ^\forall \nu$. Conversely, if it holds for Ω, Ω is

holomorphically convex.

Proof. Excercise. $\square$

§8 Levi (Hartogs' Inverse) Problem

Let $\Omega \subset \mathbf{C}^n$ be a domain.

If $\Omega \subset \mathbf{C}^n$ is maximal with respect to Hartogs phenomenon, Ω is called a **domain of holomorphy**.

Theorem 7.1 (Cartan–Thullen, 1932)

A domain is holomophically convex iff it is a domain of holomorphy.

Let $\mathrm{P}\Delta \subset \mathbf{C}^n$ be any fixed polydisk with center at 0, and $\Omega \subset \mathbf{C}^n$ be a domain. Put

$$\delta_{\mathrm{P}\Delta}(z, \partial\Omega) = \sup\{r > 0 : z + r \cdot \mathrm{P}\Delta \subset \Omega\}, \ z \in \Omega.$$

Theorem 7.2 (Oka)

If Ω is holomorphically convex, $-\log \delta_{\mathrm{P}\Delta}(z, \partial\Omega)$ is plurisubharmonic in $z \in \Omega$.

We call Ω a **pseudoconvex domain** if $-\log \delta_{\mathrm{P}\Delta}(z, \partial\Omega)$ is plurisubharmonic near $\partial\Omega$.

Levi (Hartogs' Inverse) Problem: Is a pseudoconvex domain holomorphically convex?

A bounded domain $\Omega \subset \mathbf{C}^n$ is said to be **strongly pseudoconvex** if for ${}^\forall a \in \partial\Omega$, ${}^\exists U \ni a$, a neighborhood and $\varphi \in C^2(U)$ such that $U \cap \Omega = \{\varphi < 0\}$ and

$$i\partial\bar{\partial}\varphi(z) \gg 0,\ z \in U.$$

- If Ω is pseudoconvex, ${}^\exists\Omega_\nu \nearrow \Omega$ with strongly pseudoconvex Ω_ν.

The 1st cohomology $H^1(\Omega, \mathcal{O})$

Let $\Omega = \bigcup U_\alpha$, $\mathscr{U} = \{U_\alpha\}$. Define

$Z^1(\mathscr{U}, \mathcal{O})$, 1-cycle space,

$\delta : C^0(\mathscr{U}, \mathcal{O}) \to B^1(\mathscr{U}, \mathcal{O})$, a boundary operator,

$H^1(\mathscr{U}, \mathcal{O}) = Z^1(\mathscr{U}, \mathcal{O})/B^1(\mathscr{U}, \mathcal{O})$,

$H^1(\Omega, \mathcal{O}) = \varinjlim_{\mathscr{U}} H^1(\mathscr{U}, \mathcal{O}) \hookleftarrow H^1(\mathscr{U}, \mathcal{O})$.

- $H^1(\Omega, \mathcal{O}) = 0 \iff {}^\forall$Cont. Cousin Problem is solvable on Ω.

Theorem 7.3

1. *If Ω is holomorphically convex, $H^1(\Omega, \mathcal{O}) = 0$.*
2. *For $\mathscr{U} = \{U_\alpha\}$ an open covering of Ω with ${}^\forall U_\alpha$, holomorphically convex,*

$$H^1(\mathscr{U}, \mathcal{O}) \cong H^1(\Omega, \mathcal{O}).$$

L. Schwartz Theorem

Let E be a Hausdorff topological complex vector space with at most countably many semi-norms;

E is **Fréchet**, if the asoociated norm on E is complete;

E is **Baire**, if E satisfies Baire's Category Theorem.

Theorem 7.4 (Open Map)

Let E (resp. F) be a Fréchet (resp. Baire) vector space.

If $A : E \to F$ is a continuous linear surjection,

then A is an open map.

Theorem 7.5 (L. Schwartz's Finiteness Theorem)

Let E (resp. F) be a Fréchet (resp. Baire) vector space. Let $A : E \to F$ be a continuous linear surjection, and $B : E \to F$ be a compact operator. Then $(A + B)E$ is closed, and

$$\dim \mathrm{Coker}(A + B)(:= F/(A + B)E) < \infty.$$

Proof. Heurestic: With $C := A + B$ we have

$$CE + BE = F.$$

Taking a quotien by CE, one gets

$$BE/CE = F/CE = \mathrm{Coker}\, C.$$

Since B is a copmact operaor, BE/CE is a locally compact topological vector space: Hence it is finite dimensional.

But, the closedness of CE is not known.

All these are proved at once by showing

$$F = (A+B)E + \langle b_1, \ldots, b_N \rangle_{\mathbf{C}}, \ b_j \in F, \ \text{(alg'ly)}.$$

So, how to find b_j?

(Demailly's idea) Let U be a neighborhood of $0 \in E$ such that $\overline{B(U)}$ is compact. Since $A(U)$ is open (Open Map Thm.), ${}^{\exists} b_j \in \overline{B(U)}, \ 1 \leq j \leq N < \infty$, such that

$$\overline{B(U)} \subset \bigcup_j \left(b_j + \frac{1}{2} A(U) \right).$$

Modify $\{b_j\}$ so that b_j are linearly independent and

$$(E/\ker(A+B)) \oplus \langle b_1, \ldots, b_N \rangle_{\mathbf{C}} \ni ([x], y) \mapsto (A+B)x \oplus y \in F$$

is a topological isomorphism again by Open Map Thm. Therefore, $(A+B)E$ is closed and $\dim \mathrm{Coker}(A+B) = N < \infty$. $\square$

Theorem 7.6 (Grauert)

Let Ω be a strongly pseudoconvex domain. Then, $\dim H^1(\Omega, \mathcal{O}) < \infty$.

Proof (Grauert's bumping method).

$\Omega = {}^{\exists}\bigcup_{\text{finite}} V_\alpha$ with V_α, hol. convex,

bumped open ${}^{\exists}\tilde{U}_\alpha \Supset V_\alpha$ with $\tilde{U}_\alpha$, hol. convex,

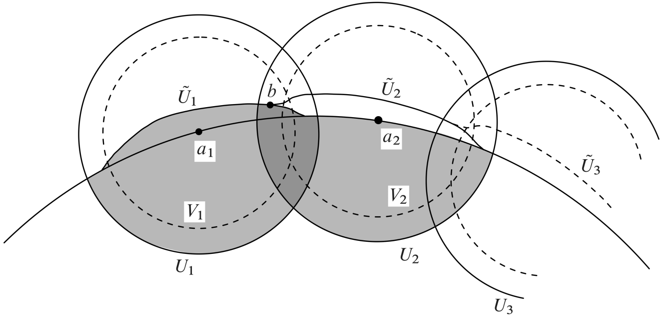


Figure: Boundary bumping method

$\mathscr{V} = \{V_\alpha\}$, bumped covering $\tilde{\mathscr{U}} = \{\tilde{U}_\alpha\}$ ($\Supset \Omega$), so that

$$\tilde{U}_\alpha \cap \tilde{U}_\beta \Supset V_\alpha \cap V_\beta,$$

$$\Psi : \xi \oplus \eta \in Z^1(\tilde{\mathscr{U}}, \mathcal{O}) \oplus C^0(\mathscr{V}, \mathcal{O}) \to \rho(\xi) + \delta\eta \in Z^1(\mathscr{V}, \mathcal{O}) \to 0,$$

where ρ is the restriction map from the bumped $\tilde{\mathscr{U}}$ to $\mathscr{V}$.

Note that $Z^1(\tilde{\mathscr{U}}, \mathcal{O}) \oplus C^0(\mathscr{V}, \mathcal{O})$ and $Z^1(\mathscr{V}, \mathcal{O})$ are Fréchet (in particular, the latter is Baire).

Since ρ is compact (Montel), L. Schwartz applied to Ψ and $-\rho$ yields that $\mathrm{Coker}(\Psi - \rho) \cong H^1(\mathscr{V}, \mathcal{O}) \cong H^1(\Omega, \mathcal{O})$ is finite dimensional. $\square$

Theorem 7.7 (Oka)

A strongly pseudoconvex domain is holomorphically convex.

Proof. Let φ be a defining function of $\partial\Omega$ such that $\Omega = \{\varphi < 0\}$, φ is strongly plurisubharmonic in a neighborhood of $\partial\Omega$.

Take a point $b \in \partial\Omega$. By a translation, we may put $b = 0$. Set

$$Q(z) = 2\sum_{j=1}^{n} \frac{\partial\varphi}{\partial z_j}(0) z_j + \sum_{j,k} \frac{\partial^2\varphi}{\partial z_j \partial z_k}(0) z_j z_k.$$

${}^{\exists}\varepsilon, \delta > 0$ satisfying

$$\varphi(z) \geq \Re\, Q(z) + \varepsilon \|z\|^2, \quad \|z\| \leq \delta,$$

$$\inf\{\varphi(z); Q(z) = 0, \|z\| = \delta\} \geq \varepsilon\delta^2 > 0.$$

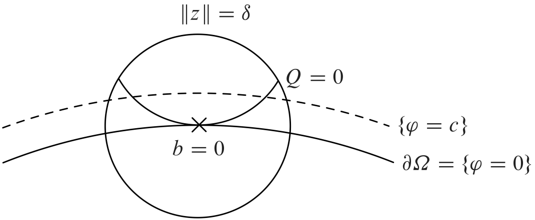


Figure: $\Omega' = \{\varphi < c\}, U_0$

Let $\Omega' = \{\varphi < c\}$ with very small $c > 0$, $U_1 = \Omega' \setminus \{Q = 0\}$. Then $\mathscr{U} = \{U_0, U_1\}$ is an open covering of Ω', which is stronglly pseudoconvex.

We set

$$f_{01}(z) = \frac{1}{Q(z)}, \quad z \in U_0 \cap U_1,$$
$$f_{10}(z) = -f_{01}(z), \quad z \in U_1 \cap U_0.$$

Then, a 1-cocyle $f = (f_{01}(z), f_{10}(z)) \in Z^1(\mathscr{U}, \mathcal{O})$ is obtained. For $k \in \mathbf{N}$ we define

$$f_{01}^{[k]}(z) = (f_{01}(z))^k, \quad z \in U_0 \cap U_1,$$
$$f_{10}^{[k]}(z) = -f_{01}^{[k]}(z), \quad z \in U_1 \cap U_0.$$

Then $(f^{[k]}) \in Z^1(\mathscr{U}, \mathcal{O})$. Thus we obtain cohomology classes,

$$[f^{[k]}] \in H^1(\mathscr{U}, \mathcal{O}) \hookrightarrow H^1(\Omega', \mathcal{O}), \quad k \in \mathbf{N}.$$

Since Ω' is strongly pseudoconvex, Grauert's Theorem implies $\dim H^1(\Omega', \mathcal{O}) < \infty$.

Therefore, for N large, there is a non-trivial linear relation,

$$\sum_{k=1}^{N} c_k [f^{[k]}] = 0 \in H^1(\mathscr{U}, \mathcal{O}_{\Omega'}) \quad (c_k \in \mathbf{C}).$$

We may suppose that $c_N \neq 0$. Then there exists elements $g_i \in \mathcal{O}(U_i)$, $i = 0, 1$, such that

$$\sum_{k=1}^{N} \frac{c_k}{Q^k(z)} = g_1(z) - g_0(z), \quad z \in U_0 \cap U_1.$$

Therefore,

$$g_0(z) + \sum_{k=1}^{N} \frac{c_k}{Q^k(z)} = g_1(z), \quad z \in U_0 \cap U_1, \quad c_N \neq 0.$$

(7.8) $\qquad {}^{\exists}F \in \mathscr{M}(\Omega')$ with poles of order N on $\{Q = 0\}$.

Since $\{Q = 0\} \cap \Omega = \emptyset$, $F|_\Omega \in \mathcal{O}(\Omega)$ and $\lim_{z\to 0} |F(z)| = \infty$. Thus, Ω is holomorphically convex.

Theorem 7.9 (Oka)

A pseudoconvex domain is holomorphically convex.

Proof. There are strongly psudoconvex domains $\Omega_\nu \nearrow \Omega$. Since Ω_ν are holomorphically convex, so is the limit Ω (Behnke–Stein).

$\square$

Furthermore, we have

Theorem 7.10 (Oka)

A pseudoconvex unramified Riemann domain over $\mathbf{C}^n$ is holomorphically convex and holomorphically separable; i.e., a Stein manifold.

Proof.

Let $\pi : \Omega \to \mathbf{C}^n$ be an unramified Riemann domain. Assume that $-\log \delta_{\mathrm{P}\boldsymbol{\Delta}}(x, \partial\Omega)$ is plurisubharmonic near $\partial\Omega$.

Step 1°: Construct a (continuous) plurisubharmonic exhaustion $\lambda : \Omega \to \mathbf{R}$.

Step 2°: Show that $\Omega_c = \{\lambda < c\}$ with ${}^\forall c \in \mathbf{R}$ is holomorphically convex. We may enlarge a little bit Ω_c to a strongly pseudoconvex domain Ω'_c. Then apply the same argument as in the case of univalent domains.

Step 3° (Hol. Separability): Take two distinct points $Q_1, Q_2 \in \Omega'_c$.

We may assume: $\pi(Q_1) = \pi(Q_2) = a \in \mathbf{C}^n$.

Let $\phi(t), t \geq 0$, be any affine linear curve with $\phi(0) = a$.

Then lifting ${}^{\exists 1}\phi_j(t) \in \Omega'_c$ of $\phi(t)$ such that $\phi_j(0) = Q_j$ $(j = 1, 2)$.

Since Ω'_c is relatively compact, $\phi_j(t)$ hits the boundary $\partial\Omega'_c$.

We may assume that $\phi_1(t)$ hits $\partial\Omega'_c$ first with $t = T \in \mathbf{R}$, so that $\phi_j([0, T]) \subset \bar{\Omega}'_c$ $(j = 1, 2)$ and $\phi_1(T) \in \partial\Omega'_c$.

Note that $\phi_1(T) \neq \phi_2(T)$.

With setting $b = \phi_1(T)$ we have by (7.8) a meromorphic function F_b in $\Omega'_{c\epsilon} \Supset \Omega'_c$ which is holomorphic in Ω'_c.

Consider the Taylor expansions of F_b at Q_1 and Q_2 in $(z_1, \ldots, z_n)$. Since F_b has a pole at $\phi_1(T)$ and no pole at $\phi_2(T)$, those two expansions must be different. Therefore, there is some partial differential operator $\partial^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}$ with a multi-index α such that

$$\partial^\alpha F_b(Q_1) \neq \partial^\alpha F_b(Q_2).$$

Since $\partial^\alpha F_b$ is holomorphic in Ω'_c, this finishes the proof of hol. separation..

Step 4°: For every pair $c < c' (\in \mathbf{R})$, $\Omega_c \Subset \Omega_{c'}$ is a Runge pair (by Jôku-Ikô). $\square$