A New Aspect of the Arnold Invariant $J^+$ from a Global Viewpoint

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ABSTRACT. In this paper, we study the Arnold invariant $J^+$ for plane and spherical curves. This invariant essentially counts the number of a certain type of local moves called direct self-tangency perestroika in a generic regular homotopy from a standard curve to a given one; the other basic local moves, namely inverse self-tangency perestroika and triple point crossing, do not change the value of $J^+$. Thus, behavior of $J^+$ under local moves is rather obvious. However, it is less understood how $J^+$ behaves in the space of curves on a global scale. We study this problem using Legendrian knots, and give infinitely many regular homotopic curves with the same $J^+$ that cannot be mutually related by inverse self-tangency perestroika and triple point crossing.

1. INTRODUCTION

In this paper, we study the images of generic immersions (i.e., immersions such that all of the self-intersections are transverse double points) from the circle $S^1$ to the plane $\mathbb{R}^2$ and the sphere $S^2$, which we call plane curves and spherical curves, respectively. Two curves are said to be regular homotopic if these are connected by a homotopy that is an immersion all the time. Whitney [14] proved that two plane curves are regular homotopic if and only if these curves have the same rotation number. It is known that any regular homotopy can be changed (by a small perturbation) into a finite sequence of three types of basic homotopies: called a $dR2$-move, an $iR2$-move, and an $R3$-move (cf. Figure 2.2; we give a quick review on these moves in Section 2.)

Arnold [3] introduced numerical invariants $J^+$, $J^-$, and $St$ for plane curves by carefully analyzing the space of immersions from $S^1$ to $\mathbb{R}^2$, and further modified
these invariants in order to obtain invariants $J^+_S$, $J^-_S$ and $St_S$ for spherical curves (see [5]). The Arnold invariants attracted much interest to a new aspect of curves, and have been studied a lot for the last two decades: several explicit formulae of the invariants were given in [10, 11, 13], higher-order versions were obtained in [2, 6, 13], generalizations to fronts were studied in [1, 4, 10], and so on.

As we will see in Section 2, $J^+$ (and thus the modified version $J^+_S$) is invariant under ambient isotopies, iR2-moves and R3-moves, but the values of $J^+$ differ by 2 between two curves if these curves are related by a dR2-move. Thus, for two curves $C_1$ and $C_2$, the difference $J^+(C_1) - J^+(C_2)$ gives a lower bound of the number of dR2-moves in a generic regular homotopy between $C_1$ and $C_2$. The invariants $J^-$ and $St$ (and the modified versions $J^-_S$ and $St_S$) have similar properties (see Section 2). In particular, the Arnold invariants might be “obstructions” to the existence of a generic regular homotopy consisting of specific types of moves for a given two curves (cf. [3, p. 34]).

In order to further study the “obstructions” above, we introduce some rather unusual equivalence relations of curves: for a set $R$ of types of moves for curves, two curves are said to be equivalent under $R$ if one curve can be obtained from the other by successive applications of moves in $R$. Several equivalence relations of curves defined by restricted homotopies were studied in [7–9]. In this paper, we especially focus on the equivalence relation under $\{iR2, R3\}$. This equivalence relation is appropriate for Legendrian theory: dR2-moves are not allowed in Legendrian regular homotopies between fronts. The rotation number of an oriented plane curve is invariant under iR2-moves and R3-moves. For an oriented plane curve $C$, we denote the rotation number of $C$ by $\text{rot}(C) \in \mathbb{Z}$. By the property in the previous paragraph, $J^+$ is also invariant under iR2-moves and R3-moves. We will prove the following theorem.

**Theorem 1.1.** For any $r \in \mathbb{Z}$ and $j^+ \in 2\mathbb{Z}$, there exists an infinite family of plane curves $\{C_i\}$ that satisfies the following properties:

- $\text{rot}(C_i) = r$ and $J^+(C_i) = j^+$ for any $i$.
- For any $i, j$ with $i \neq j$, $C_i$ and $C_j$ are not equivalent under $\{iR2, R3\}$.

For spherical curves, we can define the rotation number modulo 2, which is invariant under regular homotopies, in particular under iR2-moves and R3-moves. We denote the rotation number of a spherical curve $C$ by $\text{rot}_S(C) \in \mathbb{Z}/2\mathbb{Z}$. By the definition of $J^+_S$ (see Section 2), for a spherical curve $C$ the value $J^+_S(C)$ is contained in $V_0 = \mathbb{Z}$ if the rotation number $\text{rot}_S(C)$ is equal to 0, and is contained in $V_1 = (\frac{1}{2}\mathbb{Z}) \setminus \mathbb{Z}$ otherwise.

**Theorem 1.2.** For any $r \in \mathbb{Z}/2\mathbb{Z}$ and any $j^+ \in V_r$, there exists an infinite family of spherical curves $\{C_i\}$ which satisfies the following properties:

- $\text{rot}_S(C_i) = r$ and $J^+_S(C_i) = j^+$ for any $i$.
- For any $i, j$ with $i \neq j$, $C_i$ and $C_j$ are not equivalent under $\{iR2, R3\}$.

The proofs of Theorems 1.1 and 1.2 will be given in Section 5. In order to detect difference between two curves up to equivalence under $\{iR2, R3\}$, we
pay attention to isotopy classes of Legendrian knots in the unit tangent bundle $UTS^2$ of $S^2$ (i.e., the set of the tangent vectors of length 1 for some Riemannian metric of $S^2$) associated with oriented spherical curves. In the proofs of Theorems 1.1 and 1.2, we will give infinite families of oriented curves such that all the curves in a family have the same rotation number and the Arnold invariant, but where the associated Legendrian knots are mutually not isotopic. (This strategy for proving the main theorem should be compared with that in [8], in which the authors associated equivalence classes of curves under R1-moves and weak R3-moves with knots in $S^3$ by positive resolution.) Note that we never make use of contact structures of the unit tangent bundle for proving the main theorems: we merely study isotopy classes of Legendrian knots in the proof. Since $UTS^2$ is diffeomorphic to the real projective space $\mathbb{RP}^3$, $UTS^2$ has the surgery diagram consisting of a $(+2)$-framed unknot. In Section 3, we explain how to obtain a knot diagram of the Legendrian knot associated with a given curve, which is drawn in the surgery diagram of $UTS^2$. The unit tangent bundle $UTS^2$ has the universal double cover from $S^3$. By the covering homotopy property, if two links in $UTS^2$ are isotopic, the preimages of them under the universal cover are also isotopic. In Section 4, we give an algorithm to obtain a link diagram of the preimage of a link in $UTS^2$ under the universal cover, which enables us to detect the difference between two Legendrian knots up to isotopy.

2. Preliminaries

In this section, we set up basic notation and terminology that will be used in this paper. Throughout the paper, we assume that manifolds are oriented and connected, and maps between manifolds are smooth unless otherwise noted.

2.1. Framed links and surgery diagrams of 3-manifolds. By a link we mean the image of an embedding from a disjoint union of oriented circles into a closed 3-manifold. A link $L$ is called a knot if $L$ is connected. For a knot $K$ in $M$, we denote the closure of a tubular neighborhood of $K$ by $\nu K \subset M$. The isotopy class of a circle in $\partial \nu K$ intersecting the meridian of $K$ in one point is called a framing, and a knot with a framing is called a framed knot. A framing of a link $L$ is a disjoint union of framings of each component of $L$. We call a link with a framing a framed link. A knot $K \subset S^3$ has the canonical framing: the framing $K'$ such that the linking number between $K$ and $K'$ is 0. We denote this framing by $\ell$, and the positive meridian of $K$ by $m$. Since any framing of $K$ is uniquely determined by its homology class in $\partial \nu K$, we can describe a framing of $K$ by an integer: the framing corresponding to the homology class $p[m] + [\ell]$ is denoted by $p \in \mathbb{Z}$, which is called a framing coefficient.

We identify the sphere $S^3$ with the one-point compactification $\mathbb{R}^3 \cup \{\infty\}$. We are interested in properties of links invariant under ambient isotopies of $S^3$. Since two links in $S^3$ are isotopic in $S^3$ if and only if these are isotopic in $S^3 \setminus \{\infty\}$, we can assume that any link in $S^3$ is away from $\infty$ without loss of generality. We can
further assume that any link is in general position with respect to the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$; that is, all the self-intersections of the image of a link under $p$ are transverse double points. For a knot $K \subset S^3$, we call the image of $K$ under $p$ a knot projection of $K$. We can add information of overpasses and underpasses to each double point of a knot projection. A knot projection with such information is called a knot diagram. Examples of a knot projection and knot diagrams are shown in Figure 2.1.

![Knot diagrams](image)

Figure 2.1. Above are a knot projection (a), and knot diagrams (b) and (c). The knot described in (b) is isotopic to that described in (c).

Let $M$ be a 3-manifold. A framed knot $K \subset M$ gives rise to a new 3-manifold $M_K$ in the following way: we take a diffeomorphism $\Phi : S^1 \times \partial D^2 \to \partial \nu K$ such that $\Phi$ maps the circle $\{ * \} \times \partial D^2$ to the given framing of $K$ (up to isotopy). Define a 3-manifold $M_K$ as follows:

$$M_K = (M \setminus \text{Int}(\nu K)) \cup \Phi : S^1 \times D^2.$$

It is not hard to see that the diffeomorphism type of $M_K$ does not depend on the choice of $\Phi$. The manifold $M_K$ is called a 3-manifold obtained by Dehn surgery along $K$. We can define Dehn surgery along a framed link $L \subset M$ in a similar manner. It is known that any closed 3-manifold can be obtained by Dehn surgery along a framed link $L \subset S^3$. For a 3-manifold $M$, a link diagram (with framing coefficients) of a framed link $L$ that satisfies $M_L = M$ is called a surgery diagram of $M$.

Let $M$ be a 3-manifold and $L_0$ a framed link in $S^3$ that satisfies $M_{L_0} = M$. We can regard the complement $S^3 \setminus \text{Int}(\nu L_0)$ as a subset of $M$. It is easy to see that any link $L$ in $M$ can be moved by an isotopy so that $L$ is contained in $S^3 \setminus \text{Int}(\nu L_0)$. In particular, we can draw link diagrams $L$ and $L_0$ simultaneously. Such a diagram is called a link diagram of $L$ in a surgery diagram of $M$.

### 2.2. Regular homotopies of curves and the Arnold invariants.

In this subsection, we give a quick review for generic homotopies between curves and the Arnold invariants. The reader can refer to [3, 4] for details on this subject.

For a curve $C$, we introduce three types of local moves, which are shown in Figure 2.2.
The left move in Figure 2.2 can be realized by a regular homotopy which experiences a self-tangency at an intermediate time. In this homotopy, the two direction vectors have the same orientation at the self-tangency. This move is called a direct self-tangency perestroika or simply a dR2-move. The middle move in Figure 2.2 can be also realized by a regular homotopy with a self-tangency, but the orientations of the two direction vectors do not coincide at the self-tangency. This move is called an inverse self-tangency perestroika or an iR2-move. A dR2-move or an iR2-move is said to be positive (respectively, negative) if the move increases (respectively, decreases) the number of double points by 2.

The right move in Figure 2.2 is called a triple point crossing or an R3-move. This move can be realized by a regular homotopy that experiences a triple point. Although we can define positive and negative R3-moves, we will not discuss them in this paper. For this reason, we omit the details of positivity and negativity for R3-moves (the reader can refer to [3]).

It is known that any regular homotopy can be changed (by a small perturbation) into a finite sequence of the three homotopies above together with ambient isotopies. We call such a homotopy a generic regular homotopy.

Since two plane curves are regular homotopic if and only if these curves have the same rotation number (see [14]), any plane curve with the rotation number \( \pm i \) is regular homotopic to the plane curve \( e_i \) (up to orientations) shown in Figure 2.3.

**Theorem 2.1 ([3]).** Denote by \( n_i \) the number of double points of \( e_i \). For a plane curve \( C \subset \mathbb{R}^2 \) with the rotation number \( i \), we take a generic regular homotopy \( H : S^1 \times [0, 1] \to \mathbb{R}^2 \) from \( e_i \) to \( C \). Using \( H \), we assign three integers \( J^+(C), J^-(C) \) and \( St(C) \) to \( C \) as follows:

\[
J^+(C) = \min\{0, -2(|i| - 1)\} + 2(d_+ - d_-),
\]

\[
J^-(C) = \min\{0, -2(|i| - 1)\} - n_i - 2(i_+ - i_-),
\]

\[
St(C) = \max\{0, |i| - 1\} + t_+ - t_-,
\]
where \(d_+\), \(i_-\) and \(t_-\) are the numbers of positive/negative dR2-moves, iR2-moves, and R3-moves in \(H\), respectively. Then, the integers \(J^\pm(C)\) and \(St(C)\) do not depend on the choice of a generic regular homotopy \(H\).

By Theorem 2.1 the integers \(J^\pm(C)\) and \(St(C)\) are invariants of (an isotopy class of) a plane curve \(C\). These invariants are called the Arnold invariants for plane curves.

We can also define similar invariants for spherical curves. For a spherical curve \(C \subset S^2\), we take a point \(x \in S^2 \setminus C\) and denote by \(C_x \subset \mathbb{R}\) the image of \(C\) under the stereographic projection from \(x\). We assign integers \(J^\pm_S(C)\) and \(St_S(C)\) as follows:

\[
J^+_S(C) = J^+(C_x) + \frac{\operatorname{rot}(C_x)^2}{2}, \\
J^-_S(C) = J^-(C_x) + \frac{\operatorname{rot}(C_x)^2}{2}, \\
St_S(C) = St(C_x) - \frac{\operatorname{rot}(C_x)^2}{4}.
\]

It is known that the integers \(J^\pm_S(C)\) and \(St_S(C)\) do not depend on the choice of a point \(x\) (cf [5]). For this reason, these are invariants for (isotopy classes of) spherical curves, which are also called the Arnold invariants for spherical curves.

3. Legendrian Knots in the Unit Tangent Bundle of \(S^2\)

Denote by \(UTS^2 \subset TS^2\) the unit tangent bundle of \(S^2\). In this section, we give an algorithm to obtain a diagram of the Legendrian knot in \(UTS^2\) associated with an oriented curve.

We begin by reviewing Legendrian knots. For an oriented spherical curve \(C\), we take a generic immersion \(f : S^1 \to S^2\) so that the image of \(f\) is \(C\) and so that the orientation of \(C\) induced by that of \(S^1\) coincides with the given orientation. We denote the derivative of \(f\) by \(df : TS^1 \to TS^2\). Since \(f\) is an immersion, \(df(p)\) is everywhere non-zero, and we can compose the projection \(\pi : TS^2 \setminus S^2 \to UTS^2\) to \(df\), where we identify \(S^2\) with the 0-section of \(TS^2\). Since \(f\) is generic, the composition \(\pi \circ df : S^1 \to UTS^2\) is an embedding, where we identify the set of unit positive vectors with \(S^1\) so that we can regard \(S^1\) as a subset of \(TS^1\). The image of the composition \(\pi \circ df\) is a knot in \(UTS^2\). This knot, which is denoted by \(K_C\), is called the Legendrian knot associated with \(C\).

Remark 3.1 (The canonical framing of \(K_C\) and its relation with \(J^+\)). Since \(K_C\) is a Legendrian knot with respect to the canonical contact structure of \(UTS^2\), this knot has the canonical framing that is induced by a vector field everywhere transverse to the contact plane. This framing can be obtained in the following way: we can take a vector field \(V\) on \(C \subset S^2\) so that, for any point \(p \in C\), \(V_p\) and the unit positive tangent vector of \(C\) at \(p\) span the tangent space of \(S^2\). This vector field can be lifted to that of \(K_C\) via \(d\pi\). A parallel shift of \(K_C\) along this
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lift is a framing of \( K_C \). It is easy to see that this framing does not depend on the choice of \( V \) and its lift, and coincides with the canonical framing of \( K_C \).

Arnold [3, 4] proved that, for a plane curve \( C \), the invariant \( J^+(C) \) is equal to \( 1 - \beta_1(K_C) \), where \( \beta_1(K_C) \) is the Bennequin-Tabachnikov number of the Legendrian knot \( K_C \) introduced in [12]. This invariant is defined as follows: for a Legendrian knot \( L \) in \( UT \mathbb{R}^2 \), we take points \( p_1, \ldots, p_s \in \mathbb{R}^2 \) sufficiently far from the origin and a 2-chain \( \Delta \) bounding the union \( L \cup \{ \bigcup_i p^{-1}(p_i) \} \), where \( p : UT \mathbb{R}^2 \to \mathbb{R}^2 \) is the projection. Then, \( \beta(L) \) is defined to be the intersection number between \( \Delta \) and a parallel shift of \( L \) along the canonical framing.

It is easy to prove the following proposition by the local pictures of \( \mathbb{R}^3 \)-moves and \( \mathbb{iR}^2 \)-moves.

**Proposition 3.2.** If two spherical curves \( C_0, C_1 \subset S^2 \) are equivalent under \( \{ \mathbb{iR}^2, \mathbb{R}^3 \} \), then the corresponding Legendrian knots \( K_{C_0} \) and \( K_{C_1} \) are Legendrian isotopic; in particular, these are isotopic (as framed knots) in \( UTS^2 \).

**Remark 3.3.** It is also easy to prove that a \( \mathbb{dR}^2 \)-move for a curve in \( S^2 \) corresponds with a crossing change to the corresponding Legendrian knot.

The projection \( p : UTS^2 \to S^2 \) is an \( S^1 \)-bundle over \( S^2 \) whose Euler number is 2 (the orientation of \( UTS^2 \) is derived from that of \( S^2 \)). Thus, the manifold \( UTS^2 \) has a surgery diagram which consists of an unknot with framing 2. For a curve \( C \subset \mathbb{R}^2 \subset S^2 \), we take a sufficiently large disk \( D \subset \mathbb{R}^2 \) that contains \( C \). The restriction \( p|_{p^{-1}(D)} : p^{-1}(D) \to D \) is the trivial \( S^1 \)-bundle, and the submanifold \( p^{-1}(D) \) coincides with the complement of the framed unknot (which is a solid torus) in the surgery diagram of \( UTS^2 \). We fix a trivialization \( p^{-1}(D) \cong D \times S^1 \), and call each disk \( D \times \{ * \} \) a sectional disk. We take an identification of \( UTS^2 \) with the manifold described by the diagram \( O^{+2} \), so that the set of horizontal vectors in \( D \subset \mathbb{R}^2 \) oriented from right to left coincides with the sectional disk that contains the infinity \( \infty \in S^3 \). The fiber of \( UTS^2 \) at a point \( p_0 \in D \) is the set of unit tangent vectors at \( p_0 \), and this fiber is oriented counterclockwise since the orientation of \( UTS^2 \) is derived from that of \( S^2 \). We identify this fiber with the unit circle \( S^1 \subset \mathbb{C} \) in the obvious way. We can assume that the set \( \{ \exp(\sqrt{-1}\theta) \in S^1 \mid \theta \in (-\pi + \epsilon, \pi - \epsilon) \} \subset S^1 \cong p^{-1}(p_0) \) is projected to a point \( q_0 \) in the diagram \( O^{+2} \) for a sufficiently small \( \epsilon > 0 \). It is easy to see that, for \( \theta_0, \theta_1 \in (-\pi + \epsilon, \pi - \epsilon) \), a point \( \exp(\sqrt{-1}\theta_0) \) is behind a point \( \exp(\sqrt{-1}\theta_1) \) in the diagram \( O^{+2} \) if \( \theta_0 < \theta_1 \). Eventually, we can obtain a knot diagram of \( K_C \) in the surgery diagram \( O^{+2} \) in the following way:

- A knot projection that is derived from \( K_C \) coincides with a diagram of \( C \subset D \subset \mathbb{R}^2 \) except in neighborhoods of points at which \( C \) has a horizontal direction vector oriented from right to left, and this knot projection is drawn inside the \((+2)\)-framed unknot.
- At each double point \( q \) of the knot projection, the curve with direction vector \( \exp(\sqrt{-1}\theta_0) \) goes behind the other curve with direction vector \( \exp(\sqrt{-1}\theta_1) \)
exp(\sqrt{-1} \theta) if \theta_0, \theta_1 \in (-\pi + \varepsilon, \pi - \varepsilon) and \theta_0 < \theta_1, where we identify the fiber of UTS^2|_D with the unit circle S^1 \subset \mathbb{C} in the same way as above.

- In the preimage (under \(p\)) of a neighborhood of each point at which \(C\) has a horizontal direction vector oriented from right to left, \(K_C\) travels along the fiber of the projection \(p: UTS^2 \to S^2\) once, following the rule shown in Figure 3.1.

![Diagram](image)

**Figure 3.1.** Left: diagrams of \(C\) in \(\mathbb{R}^2\). Right: knot diagrams of \(K_C\) in the surgery diagram \(\circ +^2\).

Note that the framing of \(K_C\) is along sectional disks. An example of a pair of a spherical curve and the corresponding framed knot is shown in Figure 3.2.

By regarding \(\mathbb{R}^2\) as a subset of \(S^2 = \mathbb{R}^2 \cup \{\infty\}\), we can thus obtain a knot \(K_C \subset UTS^2\) and a diagram of \(K_C\) drawn in the surgery diagram \(\circ +^2\). Under the identification given in the previous paragraph, when a curve \(C\) is deformed by a sequence of R3-moves, iR2-moves, and isotopies in \(\mathbb{R}^2\) (namely, isotopies in \(S^2\) fixing the point \(\infty \in S^2\)), the associated knot \(K_C\) is deformed by isotopies in \(UTS^2\) avoiding the core of solid torus attached by the surgery. This observation yields the following proposition.

**Proposition 3.4.** If there are two plane curves \(C_0, C_1 \subset \mathbb{R}^2\) equivalent under \{iR2, R3\}, then there exists an ambient isotopy in \(UTS^2\) that deforms \(K_{C_0}\) to \(K_{C_1}\) and keeps the torus attached by the surgery fixed.

Since the Legendrian knot \(K_C\) is contained in the complement of a regular neighborhood of \((+2)\)-framed unknot, which is regarded as a subset of \(S^3\), we can
regard $K_C$ as a framed knot in $S^3$. From Proposition 3.4, we immediately obtain the following corollary.

**Corollary 3.5.** Under the same assumption as in Proposition 3.4, there exists an ambient isotopy in $S^3$ that deforms $K_{C_0}$ to $K_{C_1}$ and keeps the $(+2)$-framed unknot fixed. In particular, $K_{C_0}$ and $K_{C_1}$ are isotopic as framed knots in $S^3$.

4. **The Universal Cover of $UTS^2$ and Knot Diagrams**

Since the Euler number of the $S^1$-bundle $p : UTS^2 \to S^2$ is 2, and in particular $UTS^2$ is diffeomorphic to $\mathbb{R}P^3$, the universal cover of $UTS^2$ is a double cover from $S^3$, which we denote by $q : S^3 \to UTS^2$. In this section, we explain how to obtain a link diagram of the preimage of a knot $K$ in $UTS^2$ under $q$ from a knot diagram of $K$ in the surgery diagram $\bigcirc^{+2}$.

For a knot $K$ in $UTS^2$, we denote the preimage of $K$ under $q$ by $\tilde{K} \subset S^3$. Note that $\tilde{K}$ is a 2-component link if $K$ is null-homologous in $UTS^2$, and a knot otherwise. Any knot $K$ has a knot diagram in the surgery diagram $\bigcirc^{+2}$ as shown in the left side of Figure 4.1 (in other words, any knot in $UTS^2$ can be obtained by taking band sums between a knot contained in a small ball in $UTS^2$ and some meridians of the framed unknot).

The composition $p \circ q : S^3 \to S^2$ is an $S^1$-bundle over $S^2$ whose Euler number is 1. Since the covering map $q : S^3 \to UTS^2$ preserves fibers, the covering transformation $T : S^3 \to S^3$ acts on each fiber of $p \circ q$ by multiplication of $-1 \in S^1$. Thus, we can obtain a diagram of $\tilde{K}$ as shown in the right side of Figure 4.1, where the shaded box outside the framed unknot contains a diagram of the image of a tangle contained in the shaded box inside the unknot under $T$.

Both the diagram $\bigcirc^{+1}$ and the empty diagram describe $S^3$, and these diagrams are related by blowing down. Thus, in order to obtain a usual link diagram of $\tilde{K}$ (i.e., a diagram of $\tilde{K} \subset S^3$ derived from the projection $S^3 \setminus \{\infty\} \to \mathbb{R}^2$), we have to blow down $\tilde{K}$ along the $(+1)$-framed unknot. In Figure 4.2, we describe
FIGURE 4.1. Left: a diagram of $K$ in $UTS^2$. Right: a diagram of the preimage under $q$.

a diagram of the preimage of the knot in the right side of Figure 3.2 under $q$, obtained by the above procedure.

FIGURE 4.2. An example of the preimage of a knot in $UTS^2$ under $q$.

5. INFINITELY MANY CURVES WITH THE SAME VALUES OF THE INVARIANTS

In this section, we prove Theorems 1.1 and 1.2.
Proof of Theorem 1.1. We first note it is sufficient to prove the statement of Theorem 1.1 for non-negative integer \( r \). Indeed, if a family of oriented plane curves satisfies the desired conditions, the family consisting of the same curves with the opposite orientations also satisfies the conditions in Theorem 1.1. For this reason, we only give a proof for the case where \( r \) is non-negative.

For non-negative integers \( a, b, \) and \( c \), we define a curve \( C(a,b,c) \) by Figure 5.2, where tangles \( (a) \), \( (b) \), and \( (c) \) are defined by the local figures in Figure 5.1, and \( x = a, b, \) or \( c \) is the number of tangles labeled by \( (x) \).

The rotation number of \( C(a,b,c) \) is \( a - 1 \). Moreover, it is easy to see that \( J^+(C(a,b+1,c)) - J^+(C(a,b,c)) \) is equal to \(-2\), while \( J^+(C(a,b,c+1)) - J^+(C(a,b,c)) \) is equal to \(2\) for any \( a, b, c \geq 0 \). Thus, we can find non-negative numbers \( b_0 \) and \( c_0 \) such that the value \( J^+(C(r+1,b_0,c_0)) \) is equal to \( j^+ \). Furthermore, by the observation above, all the curves in the family \( \{C(r+1,b_0+k,c_0+k)\}_{k \geq 0} \) have the same rotation number and the same value of the invariant \( J^+ \).

We can obtain the diagram of \( K_{C(r+1,b_0+k,c_0+k)} \), by the algorithm given in Section 3, as shown in Figure 5.3.

The diagram shows that \( K_{C(r+1,b_0+k,c_0+k)} \) is the \( (2, 2(c_0 + k) + 3) \)-torus knot, where we regard this knot as that in \( \mathbb{R}^3 \) (see Proposition 3.4 and the paragraph preceding it); in particular, \( K_{C(r+1,b_0+k_1,c_0+k_1)} \) and \( K_{C(r+1,b_0+k_2,c_0+k_2)} \) are isotopic if and only if \( k_1 = k_2 \). By Proposition 3.4, we have that any two curves in the family \( \{C(r+1,b_0+k,c_0+k)\}_{k \geq 0} \) are not equivalent under \( \{iR^2, R^3\} \). This completes the proof of Theorem 1.1.
We next give a proof of Theorem 1.2.

Proof of Theorem 1.2. By regarding $S^2$ as the one-point compactification of $\mathbb{R}^2$, we think of the curve $C(a, b, c)$ as a spherical curve. As in the proof of Theorem 1.1, it is easy to verify that $J^+_S(C(a, b + 1, c)) - J^-_S(C(a, b, c))$ is equal to $-2$, while $J^+_S(C(a, b, c + 1)) - J^-_S(C(a, b, c))$ is equal to $2$ for any $a, b, c \geq 0$.

We first prove the statement for the case $r = 0$. By the observation above, we can find non-negative integers $b_1$ and $c_1$ such that the value $J^+_S(C(1, b_1, c_1))$ is equal to $j^+$. Furthermore, the value of the invariant $J^+_S$ is the same for all the curves in the family $\{C(1, b_1 + k, c_1 + k)\}_{k \geq 0}$. The diagram of the preimage $q^{-1}(K_{C(1, b_1 + k, c_1 + k)})$ under the universal cover $q : S^3 \rightarrow UTS^2$ obtained by the algorithm in Section 4 is described in Figure 5.4.

It is easy to see that the linking number between the two components of $q^{-1}(K_{C(1, b_1 + k, c_1 + k)})$ is equal to $-2(k + c_1)$. Thus, $q^{-1}(K_{C(1, b_1 + k_1, c_1 + k_1)})$ and
$q^{-1}(K_{C(1,b_1+k_2,c_1+k_2)})$ are isotopic if and only if $k_1 = k_2$. Since any isotopy between two links in $UTS^2$ can be lifted to that of the preimage of them under $q$, $K_{C(1,b_1+k_1,c_1+k_1)}$ and $K_{C(1,b_1+k_2,c_1+k_2)}$ are isotopic if and only if $k_1 = k_2$. By Proposition 3.2, any two spherical curves in the family $\{C(1, b_1 + k, c_1 + k)\}_{k \geq 0}$ are not equivalent under $\{iR2, R3\}$.

We next prove the statement for the case $r = 1$. In the same way as in the previous paragraph, we can find non-negative integers $b_2$ and $c_2$ such that the value $J_q(C(2, b_2 + k, c_2 + k))$ is equal to $j^+$ for any $k \geq 0$. The diagram of the preimage $q^{-1}(K_{C(2,b_2+k,c_2+k)})$ obtained by the algorithm in Section 4 is described in Figure 5.5. We can deduce from this diagram that $q^{-1}(K_{C(2,b_2+k,c_2+k)})$ is the

![Figure 5.5. A diagram of the preimage $q^{-1}(K_{C(2,b_2+k,c_2+k)})$.](image)

connected sum of two $(2, 2(c_2 + k) + 3)$-torus knots. The Jones polynomial of this knot is equal to the square of the polynomial of the $(2, 2(c_2 + k) + 3)$-torus knot, which is equal to $t^{-(c_2+k)-1} - t^{-(c_2+k)-4} - t^{3(c_2+k)-5} + t^{5(c_2+k)-6}$.

In particular, $q^{-1}(K_{C(2,b_2+k_1,c_1+k_1)})$ and $q^{-1}(K_{C(2,b_2+k_2,c_2+k_2)})$ are isotopic if and only if $k_1 = k_2$. Thus, any two curves in the family $\{C(2,b_2+k,c_2+k)\}_{k \geq 0}$ are not equivalent under $\{iR2, R3\}$. This completes the proof of Theorem 1.2. □

**Remark 5.1.** For non-negative integers $a, b, c$, the Arnold invariants of the plane curve $C(a, b, c)$ are calculated as follows:

\[
\begin{align*}
J^+(C(a, b, c)) &= -2b + 2c, \\
J^-(C(a, b, c)) &= -4b - a - 4, \\
St(C(a, b, c)) &= b + 1.
\end{align*}
\]

In particular, any two curves in the family $\{C(r+1, b_0+k, c_0+k)\}_{k \geq 0}$ (constructed in the proof of Theorem 1.1) have different values of the invariants $J^-$ and $St$.

As explained in Section 3, we can regard $K_C$ as an oriented knot in $S^3$ for a plane curve $C$. We denote the framing coefficient of the canonical framing of $K_C$ by $fr(C)$, and the linking number of $K_C$ with the $(+2)$-framed unknot by $lk(C)$. These numbers are clearly invariants of equivalent classes of plane curves under $\{iR2, R3\}$.
Corollary 5.2. As an invariant of equivalent classes of plane curves under \( \{ iR^2, R^3 \} \), the pair \((fr(C), lk(C), V_K^C)\) is stronger than \((\text{rot}, J^+)\), where \(V_K^C\) is the Jones polynomial of \(K^C\).

Proof. It is sufficient to prove that \(fr(C)\) and \(lk(C)\) are equal to \(\beta(K^C)\) and \(\text{rot}(C)\), respectively, where \(\beta(K^C)\) is the Bennequin-Tabachnikov number of \(K^C\) (see Remark 3.1). We take a Seifert surface \(\Sigma \subset S^3\) of \(K^C\) intersecting the \((+2)\)-framed unknot \(K_0\) transversely. The framing coefficient \(fr(K^C)\) is equal to the intersection number \(\tilde{K}^C \cdot \Sigma\), where \(\tilde{K}^C\) is a parallel shift of \(K^C\) along the canonical framing. On the other hand, by the definition, we have that \(\beta(K^C)\) is equal to \(\tilde{K}^C \cdot (\Sigma \setminus (\bigsqcup_q D_q))\), where \(D_q\) is a small disk neighborhood of \(q \in \Sigma \cap K_0\) in \(\Sigma\). Thus, \(fr(C)\) is equal to \(\beta(K^C)\).

Let \(\pi : \mathbb{R}^2 \to \mathbb{R}\) be the projection onto the vertical axis. We denote by \(L_M\) (respectively, \(L_m\)) the set of local maxima (respectively, minima) of the restriction \(\pi|_C\) at which \(C\) goes from right to left with respect to the given orientation. It is not hard to see that the rotation number of \(C\) is equal to \(\#L_M - \#L_m\). According to the algorithm, to obtain the knot \(K^C\) from \(C\) given in Section 3, \(lk(C)\) is also equal to \(\#L_M - \#L_m\) (see Figure 3.1). \(\square\)

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References

A New Aspect of the Arnold Invariant $J^+$ From a Global Viewpoint

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