Weights of mod p automorphic forms and partial Hasse invariants

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(with an appendix by Wushi Goldring)

Abstract

For a connected, reductive group G over a finite field endowed with a cocharacter μ, we define the zip cone of (G, μ) as the cone of all possible weights of mod p automorphic forms on the stack of G-zips. This cone is conjectured to coincide with the cone of weights of characteristic p automorphic forms for Hodge-type Shimura varieties of good reduction. We prove in full generality that the cone of weights of characteristic 0 automorphic forms is contained in the zip cone, which gives further evidence to this conjecture. Furthermore, we determine exactly when the zip cone is generated by the weights of partial Hasse invariants, which is a group-theoretical generalization of a result of Diamond–Kassaei and Goldring–Koskivirta.

1 Introduction

This paper is aimed at understanding automorphic forms in characteristic p. They are sections of certain automorphic vector bundles over Shimura varieties. The second-named author and W. Goldring have illustrated in several papers (e.g. [GK19a, GK18]) that Shimura varieties share many geometric properties with the stack of G-zips of Moonen–Wedhorn and Pink–Wedhorn–Ziegler ([MW04, PWZ11]). In this paper, we study various cones generated by weights of some classes of automorphic forms coming from this stack.

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum and $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$ the corresponding Shimura variety with level K over a number field $\mathbf{E}$ (the reflex field). Let $\mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ be a cocharacter attached to $\mathbf{X}$, and $\mathbf{L} \subset \mathbf{G}_{\mathbb{C}}$ the Levi subgroup centralizing μ. Choose a Borel pair $(\mathbf{B}, \mathbf{T})$ such that $\mathbf{B}$ is contained in the parabolic $\mathbf{P}$ with Levi $\mathbf{L}$ defined by μ. Write Φ for the set of $\mathbf{T}$-roots and Φ^+ for the positive roots (with respect to the opposite Borel $\mathbf{B}^+$). Denote by Δ the set of simple roots and let $I := \Delta_{\mathbf{L}}$ be the simple roots of $\mathbf{L}$. For any $\mathbf{L}$-dominant character $\lambda \in X^*(\mathbf{T})$, we can attach a vector bundle $\mathcal{V}_I(\lambda)$ (called automorphic vector bundle) on $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X})$, modeled on the $\mathbf{L}$-representation $\mathbf{V}_I(\lambda) := \mathrm{Ind}_{\mathbf{B}}^{\mathbf{P}}(\lambda)$ induced from λ. When $(\mathbf{G}, \mathbf{X})$ is of Hodge-type and p is a prime of good reduction, we have an integral model $\mathscr{S}_K$ over $\mathcal{O}_{\mathbf{E}_\mathfrak{p}}$ (where $\mathfrak{p}|p$) by works of Kisin and Vasiu. Furthermore, $\mathcal{V}_I(\lambda)$ extends to a vector bundle over $\mathscr{S}_K$. In this paper, we are interested in the question: For which $\lambda \in X^*(T)$ does $\mathcal{V}_I(\lambda)$ admit nonzero global sections?

Set $S_K := \mathscr{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_\mathfrak{p}}} \overline{\mathbb{F}}_p$. When $F = \mathbb{C}$ (resp. $F = \overline{\mathbb{F}}_p$), denote by $C_K(F)$ the cone of $\lambda \in X^*(\mathbf{T})$ such that $\mathcal{V}_I(\lambda)$ admits nonzero sections on $\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) \otimes_{\mathbf{E}} \mathbb{C}$ (resp. S_K). For a cone $C \subset X^*(\mathbf{T})$, define the saturation (or saturated cone) of C as the set of $\lambda \in X^*(\mathbf{T})$ such that some positive multiple of λ lies in C. We always denote the saturation with a calligraphic letter $\mathcal{C}$. For example, write $\mathcal{C}_K(F)$ for the saturation of $C_K(F)$. The set $C_K(F)$ depends on the level K, but one can show that the saturated cone $\mathcal{C}_K(F)$ does not ([Kos19, Corollary 1.5.3]). Therefore, we may denote it simply by $\mathcal{C}(F)$.

We first consider the case $F = \mathbb{C}$. Griffiths–Schmid introduced in [GS69] the set:

$$\mathcal{C}_{\mathsf{GS}} = \left\{ \lambda \in X^*(\mathbf{T}) \;\middle|\; \begin{array}{ll} \langle \lambda, \alpha^\vee \rangle \geq 0 & \text{for } \alpha \in I, \\ \langle \lambda, \alpha^\vee \rangle \leq 0 & \text{for } \alpha \in \Phi^+ \setminus \Phi^+_{\mathbf{L}} \end{array} \right\}.$$

The following conjecture is expected, but we could not find a reference for it.

Conjecture 1. *One has* $\mathcal{C}(\mathbb{C}) = \mathcal{C}_{\mathsf{GS}}$.

The inclusion $\mathcal{C}(\mathbb{C}) \subset \mathcal{C}_{\mathsf{GS}}$ is proved for general Hodge-type Shimura varieties in [GK22b, Theorem 2.6.4]. The opposite inclusion should follow by studying the Lie algebra cohomology appearing in the cohomology of Shimura varieties.

Regarding $\mathcal{C}(\overline{\mathbb{F}}_p)$, very little is known. Diamond–Kassaei ([DK17, DK23]) and Goldring–Koskivirta ([GK18]) have shown in the case of Hilbert–Blumenthal Shimura varieties that $\mathcal{C}(\overline{\mathbb{F}}_p) = \mathcal{C}_{\mathsf{pHa}}$, the cone generated by the weights of partial Hasse invariants on S_K. One goal of this paper is to discuss possible generalizations of this result to other cases. For general groups, we seek a description or an approximation of the cone $\mathcal{C}(\overline{\mathbb{F}}_p)$. Our approach uses the stack of G-zips of Moonen–Wedhorn and Pink–Wedhorn–Ziegler. Let G be a reductive group over a finite field $\mathbb{F}_q$ and $\mu \colon \mathbb{G}_{\mathrm{m},k} \to G_k$ a cocharacter over $k = \overline{\mathbb{F}}_q$ (in the context of Shimura varieties, we always take $q = p$). The stack of G-zips of type μ is denoted by $G\text{-}\mathtt{Zip}^\mu$. After possibly conjugating μ, we may choose a Borel pair (B, T) over $\mathbb{F}_q$ such that B is contained in the parabolic subgroup P defined by μ (see §2.2). Write $L \subset G_k$ for the centralizer of μ and define $I := \Delta_L$. The vector bundles $\mathcal{V}_I(\lambda)$ for $\lambda \in X^*(T)$ can also be defined on $G\text{-}\mathtt{Zip}^\mu$. We attach to (G, μ) a cone $C_{\mathsf{zip}} \subset X^*(T)$, defined as the set of λ such that $\mathcal{V}_I(\lambda)$ admits nonzero sections on $G\text{-}\mathtt{Zip}^\mu$. It is a group-theoretical version of $C_K(\overline{\mathbb{F}}_p)$ and can be interpreted in terms of representation theory of reductive groups (see §2.4). When (G, μ) arises by reduction from an abelian-type Shimura datum, there is a natural smooth map $\zeta \colon S_K \to G\text{-}\mathtt{Zip}^\mu$ by [Zha18] and [IKY], which is known to be surjective. The map ζ induces by pullback of sections an inclusions $C_{\mathsf{zip}} \subset C_K(\overline{\mathbb{F}}_p)$ and $\mathcal{C}_{\mathsf{zip}} \subset \mathcal{C}(\overline{\mathbb{F}}_p)$. Goldring and the second-named author have conjectured

Conjecture 2 ([GK18, Conjecture 2.1.6])**.** *One has* $\mathcal{C}(\overline{\mathbb{F}}_p) = \mathcal{C}_{\mathsf{zip}}$.

In the case of Hilbert–Blumenthal Shimura varieties one has $\mathcal{C}_{\mathsf{zip}} = \mathcal{C}_{\mathsf{pHa}}$, hence Conjecture 1 is compatible with the result of Diamond–Kassaei mentioned above. Aside from this case, Goldring and the second-named author showed this conjecture for Picard modular surfaces at a split prime and Siegel threefolds ([GK18, Theorem D]). They also treat the case of Siegel modular varieties attached to $\mathrm{GSp}(6)$ and unitary Shimura varieties of signature (r, s) with $r + s \leq 4$ at split or inert primes (with the exception of $r = s = 2$ and p inert) in the paper [GK22a].

We now describe our results more precisely. We defined in [GK19a] the stack of G-zip flags, denoted by $G\text{-}\mathtt{ZipFlag}^\mu$, which is a group-theoretical analogue of the flag space of Ekedahl–van der Geer ([EvdG09]). There is a natural projection $\pi \colon G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$ whose fibers are flag varieties isomorphic to P/B. The stack $G\text{-}\mathtt{ZipFlag}^\mu$ carries a family of line bundles $\mathcal{V}_{\mathsf{flag}}(\lambda)$ for $\lambda \in X^*(T)$ such that $\pi_*(\mathcal{V}_{\mathsf{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$. In particular, we can identify $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda))$ and $H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{V}_{\mathsf{flag}}(\lambda))$. Moreover, $G\text{-}\mathtt{ZipFlag}^\mu$ admits a stratification $(\mathcal{F}_w)_{w \in W}$ analogous to the Bruhat decomposition, where $W = W(G, T)$ is the Weyl group of G. By [IK24], there exists a family of partial Hasse invariants $\{h_\alpha\}_{\alpha \in \Delta}$ (where Δ is the set of simple roots). Specifically, h_α is a section of $\mathcal{V}_{\mathsf{flag}}(\lambda_\alpha)$ (for some $\lambda_\alpha \in X^*(T)$) whose vanishing locus is the closure of a single codimension one stratum in $G\text{-}\mathtt{ZipFlag}^\mu$ (and each such stratum is cut out by exactly one of the h_α). The cone generated by the $(\lambda_\alpha)_{\alpha \in \Delta}$ is called the partial Hasse invariant cone C_{pHa} (Definition 3.6.1).

One has by construction $C_{\mathsf{pHa}} \subset C_{\mathsf{zip}}$. As an analogue of [DK23, Corollary 8.3], we ask whether $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}_{\mathsf{zip}}$ holds in general. Let $w_{0,L}$ be the longest element in the Weyl group $W_L = W(L,T)$. Let σ denote the action of Frobenius on the based root datum of (G,B,T). We show:

Theorem 1 (Theorem 4.3.1)**.** *The following are equivalent:*
(i) *One has* $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}_{\mathsf{zip}}$.
(ii) *One has* $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{pHa}}$.
(iii) L *is defined over* $\mathbb{F}_q$ *and* σ *acts on* Δ_L *by* $-w_{0,L}$.

Pairs (G,μ) satisfying condition (iii) are called of Hasse-type. For a Shimura variety S_K as above, we always have $\mathcal{C}_{\mathsf{pHa}} \subset \mathcal{C}_{\mathsf{zip}} \subset \mathcal{C}(\overline{\mathbb{F}}_p)$. We deduce that a necessary condition for $\mathcal{C}(\overline{\mathbb{F}}_p)$ to be generated by partial Hasse invariants is that (G,μ) is of Hasse-type. A classification of Hasse-type cases is given in an appendix by Wushi Goldring (see §A). For example, orthogonal Shimura varieties give rise to pairs (G,μ) of Hasse-type (see §7.2). Condition (ii) has also an interpretation for Shimura varieties. One can show in general that $C_K(\mathbb{C}) \subset C_K(\overline{\mathbb{F}}_p)$ ([Kos19, Proposition 1.8.3]) and hence $\mathcal{C}(\mathbb{C}) \subset \mathcal{C}(\overline{\mathbb{F}}_p)$. Since it is expected that $\mathcal{C}(\mathbb{C}) = \mathcal{C}_{\mathsf{GS}}$, Condition (ii) is necessary for $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}(\overline{\mathbb{F}}_p)$ to hold. From Conjecture 1 and Conjecture 2, we expect that the containment $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{zip}}$ should hold in general, which is now a purely group-theoretical statement. We confirm this expectation:

Theorem 2 (Theorem 6.4.3)**.** *For general* (G,μ)*, we have* $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{zip}}$.

This theorem gives further evidence for Conjecture 2. In [Kos19, Corollary 3.5.6], Theorem 2 was proved only when P is defined over $\mathbb{F}_q$. We now explain the proof of Theorem 2. The proof uses a general technique that makes it possible to reduce questions pertaining to $\mathcal{C}_{\mathsf{zip}}$ to the case of a split group. In the split case, Theorem 2 is already known by [Kos19, Corollary 3.5.6]. We explain how we can reduce to the case of a split group. Denote by $L_0 \subset L$ the largest algebraic subgroup defined over $\mathbb{F}_q$. It is a Levi subgroup of L containing T. There is a cocharacter μ_0 with centralizer L_0, and we consider the pair $(G_{\mathbb{F}_{q^r}}, \mu_0)$, where $r \geq 1$ is such that $G_{\mathbb{F}_{q^r}}$ is split. Denote by $C_{\mathsf{zip}}(G_{\mathbb{F}_{q^r}}, \mu_0)$ the zip cone of $(G_{\mathbb{F}_{q^r}}, \mu_0)$ and $\mathcal{C}_{\mathsf{zip}}(G_{\mathbb{F}_{q^r}}, \mu_0)$ for its saturation. Let $w_{0,L}$ and w_{0,L_0} be the longest elements in the Weyl groups of L and L_0 respectively. Write $X^*_{+,L}(T)$ for the set of L-dominant characters. We show the following.

Theorem 3 (Theorem 6.4.1)**.** *We have*

$$X^*_{+,L}(T) \cap \Big(w_{0,L} w_{0,L_0} \mathcal{C}_{\mathsf{zip}}(G_{\mathbb{F}_{q^r}}, \mu_0) \Big) \subset \mathcal{C}_{\mathsf{zip}}.$$

This theorem is useful in general to reduce questions on $\mathcal{C}_{\mathsf{zip}}$ to the case of a split group, as explained in Remark 6.4.2. In particular, Theorem 3 reduces Theorem 2 to the case of a split group, for which it is already known. The proof of Theorem 3 relies on a closer study of the case when G is a Weil restriction (see §6).

Our final result is the construction of natural mod p automorphic forms attached to the highest weight vectors of the representations $V_I(\lambda)$. Let λ be an L-dominant character and let $f_\lambda \in V_I(\lambda)$ denote the highest weight vector of $V_I(\lambda)$. There is a natural way of defining the norm $\mathbf{f}_\lambda := \mathrm{Norm}_{L_\varphi}(f_\lambda)$ of f_λ. Here L_φ is a certain finite (generally non-smooth) subgroup of L containing $L_0(\mathbb{F}_q)$. There is an integer $m \geq 0$ determined by L_φ, such that the norm $\mathrm{Norm}_{L_\varphi}(f_\lambda)$ is a section of $\mathcal{V}_I(d\lambda)$ (where $d = q^m |L_0(\mathbb{F}_q)|$) over the μ-ordinary locus $\mathcal{U}_\mu$ of $G\text{-}\mathtt{Zip}^\mu$ (see §3.5 for details). For $\alpha \in \Delta$, let r_α be the smallest integer $r \geq 1$ such that $\sigma^r(\alpha) = \alpha$.

Theorem 4 (Proposition 3.5.1)**.** *The section* $\mathbf{f}_\lambda$ *extends to* G-$\mathtt{Zip}^\mu$ *if and only if for all* $\alpha \in \Delta \setminus \Delta_L$ *one has*

$$\sum_{w \in W_{L_0}(\mathbb{F}_q)} \sum_{i=0}^{r_\alpha - 1} q^{i+\ell(w)} \langle w\lambda, \sigma^i(\alpha^\vee)\rangle \leq 0. \tag{1}$$

Let $\mathcal{C}_{\mathsf{hw}}$ be the set of L-dominant characters λ satisfying the above inequality (1). Theorem 4 shows that $\mathcal{C}_{\mathsf{hw}} \subset \mathcal{C}_{\mathsf{zip}}$, which provides another natural subcone of $\mathcal{C}_{\mathsf{zip}}$. We obtain a family of interesting automorphic forms $(\mathbf{f}_\lambda)_{\lambda \in \mathcal{C}_{\mathsf{hw}}}$ in characteristic p of weight $d\lambda$ (by pullback via ζ). There is also an analogue of Theorem 4 for the lowest weight vector (§5.2), and we define the lowest weight cone $\mathcal{C}_{\mathsf{lw}}$ similarly. When P is defined over $\mathbb{F}_q$, one has $\mathcal{C}_{\mathsf{lw}} = \mathcal{C}_{\mathsf{hw}}$ but in general $\mathcal{C}_{\mathsf{hw}} \subset \mathcal{C}_{\mathsf{lw}}$.

The motivation for introducing the family $(\mathbf{f}_\lambda)_\lambda$ is the following. As mentioned above, Diamond–Kassaei showed in [DK17] that the weight of any Hilbert modular form in characteristic p is spanned by the weights of partial Hasse invariants. This is also true for the Siegel-type Shimura variety $\mathcal{A}_2$, but it fails for $\mathcal{A}_n$ when $n \geq 3$. In the case $n = 3$, Goldring and the second-named author showed that the weight of any automorphic form for $\mathcal{A}_3$ is spanned by the weights of partial Hasse invariants and of the forms $(\mathbf{f}_\lambda)_{\lambda \in \mathcal{C}_{\mathsf{hw}}}$. Therefore, these forms seem to have some significance for more general groups. Moreover, the vanishing locus of $\mathbf{f}_\lambda$ is an interesting subvariety stable by Hecke operators, that we plan to investigate in future papers.

We briefly explain the content of each section. In §2 we review the stack of G-zips, vector bundles thereon and the connection with Shimura varieties. Section 3 is dedicated to the study of the cone C_{zip}, called the zip cone. We explain the motivation for introducing this set. We define several related subcones which arise naturally. We define automorphic forms on G-$\mathtt{Zip}^\mu$ attached to highest weight vectors. In section 4, we consider pairs (G, μ) of Hasse-type and we give a complete characterization in terms of C_{zip}. In section 5, similarly to the highest weight vectors, we show that the lowest weight vectors give rise naturally to certain automorphic forms on G-$\mathtt{Zip}^\mu$. In section 6, we study pairs (G, μ) where G is the Weil restriction of a reductive group defined over an extension. This machinery makes it possible to reduce several questions to the case of a split group. Using this, we can check in full generality the expectation that $\mathcal{C}_{\mathrm{GS}} \subset \mathcal{C}_{\mathrm{zip}}$. Finally, in the last section, we illustrate the results in the case of a unitary group $U(2,1)$ and for odd orthogonal groups. In the appendix by Wushi Goldring, we give an exhaustive classification of pairs (G, μ) of Hasse-type.

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2 Preliminaries and reminders on the stack of G-zips

2.1 Notation

Throughout the paper, p is a prime number, q is a power of p and $\mathbb{F}_q$ is the finite field with q elements. We write $k = \overline{\mathbb{F}}_q$ for an algebraic closure of $\mathbb{F}_q$. The notation G will always denote a connected reductive group over $\mathbb{F}_q$. For a k-scheme X, we denote by $X^{(q)}$ its q-th power Frobenius twist and by $\varphi \colon X \to X^{(q)}$ its relative Frobenius morphism. Write

$\sigma \in \mathrm{Gal}(k/\mathbb{F}_q)$ for the q-power Frobenius. We will always write (B,T) for a Borel pair of G, i.e. $T \subset B \subset G$ are a maximal torus and a Borel subgroup in G. We do not assume that T is split over $\mathbb{F}_q$. Let B^+ be the Borel subgroup of G opposite to B with respect to T (i.e. the unique Borel subgroup B^+ of G such that $B^+ \cap B = T$). We will use the following notations:

- As usual, $X^*(T)$ (resp. $X_*(T)$) denotes the group of characters (resp. cocharacters) of T. The group $\mathrm{Gal}(k/\mathbb{F}_q)$ acts naturally on these groups. Let $W = W(G_k, T)$ be the Weyl group of G_k. Similarly, $\mathrm{Gal}(k/\mathbb{F}_q)$ acts on W. Furthermore, the actions of $\mathrm{Gal}(k/\mathbb{F}_q)$ and W on $X^*(T)$ and $X_*(T)$ are compatible in a natural sense. We write $W(\mathbb{F}_q)$ for the $\mathrm{Gal}(k/\mathbb{F}_q)$-fixed subgroup of W.
- $\Phi \subset X^*(T)$: the set of T-roots of G.
- $\Phi^+ \subset \Phi$: the system of positive roots with respect to B^+ (i.e. $\alpha \in \Phi^+$ when the α-root group U_α is contained in B^+). This convention may differ from other authors. We use it to match the conventions of previous publications [GK19a], [Kos19].
- $\Delta \subset \Phi^+$: the set of simple roots.
- For $\alpha \in \Phi$, let $s_\alpha \in W$ be the corresponding reflection. The system $(W, \{s_\alpha \mid \alpha \in \Delta\})$ is a Coxeter system. We write $\ell\colon W \to \mathbb{N}$ for the length function. Hence $\ell(s_\alpha) = 1$ for all $\alpha \in \Delta$. Let w_0 denote the longest element of W.
- For a subset $K \subset \Delta$, let W_K denote the subgroup of W generated by $\{s_\alpha \mid \alpha \in K\}$. Write $w_{0,K}$ for the longest element in W_K.
- Let ${}^KW$ (resp. W^K) denote the subset of elements $w \in W$ which have minimal length in the coset $W_K w$ (resp. wW_K). Then ${}^KW$ (resp. W^K) is a set of representatives of $W_K \backslash W$ (resp. W/W_K). The map $g \mapsto g^{-1}$ induces a bijection ${}^KW \to W^K$. The longest element in the set ${}^KW$ is $w_{0,K}w_0$.
- $X^*_+(T)$ denotes the set of dominant characters, i.e. characters $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$.
- For a subset $I \subset \Delta$, let $X^*_{+,I}(T)$ denote the set of characters $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in I$. We call them I-dominant characters.
- Let $P \subset G_k$ be a parabolic subgroup containing B and let $L \subset P$ be the unique Levi subgroup of P containing T. Then we define a subset $I_P \subset \Delta$ as the unique subset such that $W(L,T) = W_{I_P}$. For an arbitrary parabolic subgroup $P \subset G_k$ containing T, we define $I_P \subset \Delta$ as $I_P := I_{P'}$ where P' is the unique conjugate of P containing B.
- For a parabolic $P \subset G_k$, write $\Delta^P := \Delta \setminus I_P$.
- For all $\alpha \in \Phi$, choose an isomorphism $u_\alpha\colon \mathbb{G}_\mathrm{a} \to U_\alpha$ so that $(u_\alpha)_{\alpha\in\Phi}$ is a realization in the sense of [Spr98, 8.1.4]. In particular, we have
$$t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t)x), \quad \forall x \in \mathbb{G}_\mathrm{a},\ \forall t \in T. \tag{2.1.1}$$
- Let $\phi_\alpha\colon \mathrm{SL}_2 \to G$ denote the map attached to α, as in [Spr98, 9.2.2]. It satisfies
$$\phi_\alpha\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = u_\alpha(x), \quad \phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = u_{-\alpha}(x).$$
- Fix a B-representation (V,ρ). For $j \in \mathbb{Z}$ and $\alpha \in \Phi$, we define a map $E_\alpha^{(j)}\colon V \to V$ as follows. Let $V = \bigoplus_{\nu \in X^*(T)} V_\nu$ be the weight decomposition of V. For $v \in V_\nu$, we can write uniquely
$$u_\alpha(x)v = \sum_{j \geq 0} x^j E_\alpha^{(j)}(v), \quad \forall x \in \mathbb{G}_\mathrm{a},$$
for elements $E_\alpha^{(j)}(v) \in V_{\nu + j\alpha}$ ([IK21, Lemma 3.3.1]). Extend $E_\alpha^{(j)}$ by additivity to a map $V \to V$. For $j < 0$, put $E_\alpha^{(j)} = 0$.

2.2 The stack of G-zips

We recall some facts about the stack of G-zips of Pink–Wedhorn–Ziegler in [PWZ11].

2.2.1 Definitions

Let G be a connected reductive group over $\mathbb{F}_q$. In this paper, a zip datum is a tuple $\mathcal{Z} := (G, P, L, Q, M, \varphi)$ consisting of the following objects:
(i) $P \subset G_k$ and $Q \subset G_k$ are parabolic subgroups of G_k.
(ii) $L \subset P$ and $M \subset Q$ are Levi subgroups such that $L^{(q)} = M$.
For an algebraic group H, denote by $R_{\mathrm{u}}(H)$ the unipotent radical of H. If $P' \subset G_k$ is a parabolic subgroup with Levi subgroup $L' \subset P'$, any $x \in P'$ can be written uniquely as $x = \overline{x}u$ with $\overline{x} \in L'$ and $u \in R_{\mathrm{u}}(P')$. We denote by $\theta^{P'}_{L'} : P' \to L'$ the map $x \mapsto \overline{x}$. Since $M = L^{(q)}$, we have a Frobenius isogeny $\varphi \colon L \to M$. Put

$$E := \{(x, y) \in P \times Q \mid \varphi(\theta^P_L(x)) = \theta^Q_M(y)\}.$$

Equivalently, E is the subgroup of $P \times Q$ generated by $R_{\mathrm{u}}(P) \times R_{\mathrm{u}}(Q)$ and elements of the form $(a, \varphi(a))$ with $a \in L$. Let $G \times G$ act on G by $(a, b) \cdot g := agb^{-1}$, and let E act on G by restricting this action to E. The stack of G-zips of type $\mathcal{Z}$ ([PWZ11],[PWZ15]) can be defined as the quotient stack

$$G\text{-}\mathtt{Zip}^{\mathcal{Z}} = [E \backslash G_k] \,.$$

2.2.2 Cocharacter datum

A *cocharacter datum* is a pair (G, μ) where G is a reductive connected group over $\mathbb{F}_q$ and $\mu \colon \mathbb{G}_{\mathrm{m},k} \to G_k$ is a cocharacter. One can attach to (G, μ) a zip datum $\mathcal{Z}_\mu$, defined as follows. First, denote by $P_+(\mu)$ (resp. $P_-(\mu)$) the unique parabolic subgroup of G_k such that $P_+(\mu)(k)$ (resp. $P_-(\mu)(k)$) consists of the elements $g \in G(k)$ satisfying that the map

$$\mathbb{G}_{\mathrm{m},k} \to G_k;\ t \mapsto \mu(t)g\mu(t)^{-1} \quad (\text{resp. } t \mapsto \mu(t)^{-1}g\mu(t))$$

extends to a morphism of varieties $\mathbb{A}^1_k \to G_k$. We obtain a pair of parabolics $(P_+(\mu), P_-(\mu))$ in G_k whose intersection $P_+(\mu) \cap P_-(\mu) = L(\mu)$ is the centralizer of μ (it is a common Levi subgroup of $P_+(\mu)$ and $P_-(\mu)$). Set $P := P_-(\mu)$, $Q := (P_+(\mu))^{(q)}$, $L := L(\mu)$ and $M := (L(\mu))^{(q)}$. The tuple $\mathcal{Z}_\mu := (G, P, L, Q, M, \varphi)$ is a zip datum, which we call the zip datum attached to the cocharacter datum (G, μ). We write simply $G\text{-}\mathtt{Zip}^\mu$ for $G\text{-}\mathtt{Zip}^{\mathcal{Z}_\mu}$. We always consider zip data of this form.

Remark 2.2.1. A general zip datum (G, P, L, Q, M, φ) is of the form $\mathcal{Z}_\mu$ for a cocharacter $\mu \colon \mathbb{G}_{\mathrm{m},k} \to G_k$ if and only if $\sigma(P)$ and Q are opposite parabolic subgroups with common Levi $M = \sigma(L)$.

Remark 2.2.2. If μ is defined over $\mathbb{F}_q$, then so are P and Q. In this case, we have $L = M$ and P, Q are opposite parabolic subgroups with common Levi subgroup L.

2.2.3 Frames

Let $\mathcal{Z} = (G, P, Q, L, M)$ be a zip datum. In this paper, a frame for $\mathcal{Z}$ is a triple (B, T, z) where (B, T) is a Borel pair of G_k defined over $\mathbb{F}_q$ satisfying
(i) One has the inclusion $B \subset P$.

(ii) $z \in W$ is an element satisfying the conditions

$$ {}^{z}B \subset Q \quad \text{and} \quad B \cap M = {}^{z}B \cap M. $$

We put $B_M := B \cap M$. Other papers ([PWZ11, PWZ15, KW18]) use the convention $B \subset Q$ instead of $B \subset P$. A frame (as defined here) may not always exist. However, if (G, μ) is a cocharacter datum and $\mathcal{Z}_\mu$ is the associated zip datum by §2.2.2, then there exists a $G(k)$-conjugate $\mu' = \mathrm{ad}(g) \circ \mu$ (with $g \in G(k)$) such that $\mathcal{Z}_{\mu'}$ admits a frame by Lemma 2.2.3 below. Hence, it is harmless to assume that a frame exists, and we only consider zip data that admit frames. With respect to the Borel pair (B, T), we define subsets I, J, Δ^P of Δ as follows:

$$ I := I_P, \quad J := I_Q, \quad \Delta^P = \Delta \setminus I. $$

Lemma 2.2.3 ([GK19b, Lemma 2.3.4])**.** *Let $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ be a cocharacter, and let $\mathcal{Z}_\mu$ be the attached zip datum. Assume that (B, T) is a Borel pair defined over $\mathbb{F}_q$ such that $B \subset P$. Define the element*

$$ z := w_0 w_{0,J} = \sigma(w_{0,I}) w_0. $$

Then (B, T, z) is a frame for $\mathcal{Z}_\mu$.

2.2.4 Parametrization of the E-orbits in G

By [PWZ11, Proposition 7.1], there are finitely many E-orbits in G. The E-orbits are smooth and locally closed in G, and the Zariski closure of an E-orbit is a union of E-orbits. We review the parametrization of E-orbits following [PWZ11]. For $w \in W$, fix a representative $\dot{w} \in N_G(T)$, such that $(w_1 w_2)^{\cdot} = \dot{w}_1 \dot{w}_2$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ (this is possible by choosing a Chevalley system, [ABD+66, XXIII, §6]). For $w \in W$, define G_w as the E-orbit of $\dot{w}\dot{z}^{-1}$. If no confusion occurs, we write w instead of $\dot{w}$. For $w, w' \in {}^I W$, write $w' \preccurlyeq w$ if there exists $w_1 \in W_I$ such that $w' \leq w_1 w \sigma(w_1)^{-1}$. This defines a partial order on ${}^I W$ ([PWZ11, Corollary 6.3]).

Theorem 2.2.4 ([PWZ11, Theorem 7.5, Theorem 11.2, Theorem 11.3, Theorem 11.5])**.** *We have two bijections:*

$$ {}^I W \longrightarrow \{E\text{-orbits in } G_k\}, \qquad w \mapsto G_w \tag{2.2.1} $$
$$ W^J \longrightarrow \{E\text{-orbits in } G_k\}, \qquad w \mapsto G_w. \tag{2.2.2} $$

For $w \in {}^I W \cup W^J$, one has $\dim(G_w) = \ell(w) + \dim(P)$ and the Zariski closure of G_w is

$$ \overline{G}_w = \bigsqcup_{w' \in {}^I W, \ w' \preccurlyeq w} G_{w'} $$

for $w \in {}^I W$, and

$$ \overline{G}_w = \bigsqcup_{w' \in W^J, \ w' \preccurlyeq w} G_{w'} $$

for $w \in W^J$.

In particular, there is a unique open E-orbit $U_{\mathcal{Z}} \subset G$ corresponding to the longest elements $w_{0,I} w_0 \in {}^I W$ via (2.2.1) and to $w_0 w_{0,J} \in W^J$ via (2.2.2). The E-orbit $U_{\mathcal{Z}}$ is dense in G. If $\mathcal{Z} = \mathcal{Z}_\mu$ (see §2.2.2), write $U_\mu = U_{\mathcal{Z}_\mu}$. In this case, we can choose $z = w_0 w_{0,J} = \sigma(w_{0,I}) w_0$ (Lemma 2.2.3), hence (2.2.2) shows that $1 \in U_\mu$. We put $\mathcal{U}_\mu := [E \backslash U_\mu]$, which we call the μ-ordinary locus.

2.3 Vector bundles on the stack of G-zips

2.3.1 Representation theory

For an algebraic group G over a field K, denote by $\mathrm{Rep}(G)$ the category of algebraic representations of G on finite-dimensional K-vector spaces. We denote a representation $\rho\colon G \to \mathrm{GL}_K(V)$ by (V,ρ), or sometimes simply ρ or V. For an algebraic group G over $\mathbb{F}_q$, a G_k-representation (V,ρ) and an integer m, we denote by $(V^{[m]},\rho^{[m]})$ the representation such that $V^{[m]} = V$ and

$$\rho^{[m]}\colon G_k \xrightarrow{\varphi^m} G_k \xrightarrow{\rho} \mathrm{GL}(V).$$

Let H be a split connected reductive K-group and choose a Borel pair (B_H, T) defined over K. If K has characteristic zero, $\mathrm{Rep}(H)$ is semisimple. In characteristic p however, this is no longer true in general. For $\lambda \in X^*_+(T)$, let $\mathcal{L}_\lambda$ be the line bundle attached to λ on the flag variety H/B_H by the usual associated sheaf construction ([Jan03, §5.8]). Define an H-representation $V_H(\lambda)$ by

$$V_H(\lambda) := H^0(H/B_H, \mathcal{L}_\lambda). \tag{2.3.1}$$

In other words, one has $V_H(\lambda) = \mathrm{Ind}_{B_H}^H \lambda$. The representation $V_H(\lambda)$ is of highest weight λ. If $\mathrm{char}(K) = 0$, the representation $V_H(\lambda)$ is irreducible. We view elements of $V_H(\lambda)$ as regular maps $f\colon H \to \mathbb{A}^1$ satisfying

$$f(hb) = \lambda(b^{-1})f(h), \quad \forall h \in H,\ \forall b \in B_H. \tag{2.3.2}$$

For dominant characters λ, λ', there is a natural surjective map

$$V_H(\lambda) \otimes V_H(\lambda') \to V_H(\lambda + \lambda'). \tag{2.3.3}$$

In the description given by (2.3.2), this map is $f \otimes f' \mapsto ff'$ (for $f \in V_H(\lambda)$, $f' \in V_H(\lambda')$). Denote by $W_H := W(H,T)$ the Weyl group and $w_{0,H} \in W_H$ the longest element. Then $V_H(\lambda)$ has a unique B_H-stable line, which is a weight space for the weight $w_{0,H}\lambda$.

2.3.2 Vector bundles on quotient stacks

For an algebraic stack $\mathcal{X}$, write $\mathfrak{VB}(\mathcal{X})$ for the category of vector bundles on $\mathcal{X}$. Let X be a k-scheme and H an affine k-group scheme acting on X. If $\rho\colon H \to \mathrm{GL}(V)$ is an algebraic representation of H, it gives rise to a vector bundle $\mathcal{V}_{H,X}(\rho)$ on the stack $[H\backslash X]$. This vector bundle can be defined geometrically as $[H\backslash(X \times_k V)]$ where H acts diagonally on $X \times_k V$. We obtain a functor

$$\mathcal{V}_{H,X}\colon\ \mathrm{Rep}(H) \to \mathfrak{VB}([H\backslash X]). \tag{2.3.4}$$

Similarly to the usual associated sheaf construction [Jan03, §5.8, equation (1)], the global sections of $\mathcal{V}_{H,X}(\rho))$ are given by

$$H^0([H\backslash X], \mathcal{V}_{H,X}(\rho)) = \{f\colon X \to V \mid f(h\cdot x) = \rho(h)f(x),\ \forall h \in H,\ \forall x \in X\}. \tag{2.3.5}$$

2.3.3 Vector bundles on G-$\mathtt{Zip}^\mu$

Fix a cocharacter datum (G,μ), let $\mathcal{Z} = (G,P,L,Q,M,\varphi)$ be the attached zip datum. Fix a frame (B,T) as in §2.2.3. By (2.3.4), we have a functor $\mathcal{V}_{E,G}\colon\ \mathrm{Rep}(E) \to \mathfrak{VB}(G\text{-}\mathtt{Zip}^\mu)$, that we simply denote by $\mathcal{V}$. For $(V,\rho) \in \mathrm{Rep}(E)$, the global sections of $\mathcal{V}(\rho)$ are

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) = \{f\colon G_k \to V \mid f(\epsilon\cdot g) = \rho(\epsilon)f(g),\ \forall \epsilon \in E,\ \forall g \in G_k\}.$$

Since G admits an open dense E-orbit (see discussion below Theorem 2.2.4), the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ is finite-dimensional ([Kos19, Lemma 1.2.1]). The first projection $p_1\colon E \to P$ induces a functor $p_1^*\colon \operatorname{Rep}(P) \to \operatorname{Rep}(E)$. If $(V,\rho) \in \operatorname{Rep}(P)$, we write again $\mathcal{V}(\rho)$ for $\mathcal{V}(p_1^*(\rho))$. In this paper, we only consider E-representations coming from P in this way. Let $\theta_L^P\colon P \to L$ be the natural projection modulo $R_{\mathrm{u}}(P)$, as in §2.2.1. It induces a fully faithful functor

$$(\theta_L^P)^*\colon \operatorname{Rep}(L) \to \operatorname{Rep}(P)$$

whose image is the full subcategory of $\operatorname{Rep}(P)$ of P-representations trivial on $R_{\mathrm{u}}(P)$. Hence, we view $\operatorname{Rep}(L)$ as a full subcategory of $\operatorname{Rep}(P)$. If $(V,\rho) \in \operatorname{Rep}(L)$, write again $\mathcal{V}(\rho) := \mathcal{V}((\theta_L^P)^*\rho)$. For $\lambda \in X^*(T)$, write $B_L := B \cap L$ and define an L-representation $(V_I(\lambda), \rho_{I,\lambda})$ as follows

$$V_I(\lambda) = \operatorname{Ind}_{B_L}^L \lambda, \quad \rho_{I,\lambda}\colon L \to \operatorname{GL}(V_I(\lambda)).$$

This is the representation defined in (2.3.1) for $H = L$ and $B_H = B_L$. Let $\mathcal{V}_I(\lambda)$ be the vector bundle on $G\text{-}\mathtt{Zip}^\mu$ attached to $V_I(\lambda)$, and call it an *automorphic vector bundle* on $G\text{-}\mathtt{Zip}^\mu$ associated to λ. This terminology stems from Shimura varieties (see §2.6 below for further details). For $\lambda \in X^*(L)$, viewing λ as an element of $X^*(T)$ by restriction, the vector bundle $\mathcal{V}_I(\lambda)$ is a line bundle. Note that if $\lambda \in X^*(T)$ is not I-dominant, then $V_I(\lambda) = 0$ and thus $\mathcal{V}_I(\lambda) = 0$.

2.4 Global sections over $G\text{-}\mathtt{Zip}^\mu$

We review some results of [IK21] regarding the global sections of $\mathcal{V}(\rho)$ for a P-representation ρ. We start with sections over the open substack $\mathcal{U}_\mu \subset G\text{-}\mathtt{Zip}^\mu$. Recall that $\mathcal{U}_\mu = [E\backslash U_\mu]$ and $1 \in U_\mu$ (see §2.2.4). By (2.3.5), an element of $H^0(\mathcal{U}_\mu, \mathcal{V}(\rho))$ can be viewed a map $h\colon G \to V$ satisfying $h(agb^{-1}) = \rho(a)h(g)$ for all $(a,b) \in E$ and all $g \in G$. Since the E-orbit of 1 is open dense in G, the map $h \mapsto h(1)$ is an injection

$$\operatorname{ev}_1\colon H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) \to V. \tag{2.4.1}$$

We give the image of this map. Let L_φ be the scheme-theoretical stabilizer subgroup of 1 in E. By definition, one has

$$L_\varphi = E \cap \{(x,x) \mid x \in G_k\}, \tag{2.4.2}$$

which is a 0-dimensional algebraic group (in general non-smooth). The first projection $E \to P$ induces a closed immersion $L_\varphi \to P$. Identify L_φ with its image and view it as a subgroup of P. Denote by $L_0 \subset L$ the largest algebraic subgroup defined over $\mathbb{F}_q$. In other words,

$$L_0 = \bigcap_{n\geq 0} L^{(q^n)}. \tag{2.4.3}$$

Lemma 2.4.1 ([KW18, Lemma 3.2.1])**.**
(1) *One has* $L_\varphi \subset L$.
(2) *The group* L_φ *can be written as a semidirect product* $L_\varphi = L_\varphi^\circ \rtimes L_0(\mathbb{F}_q)$ *where* L_φ° *is the identity component of* L_φ. *Furthermore,* L_φ° *is a finite unipotent algebraic group.*
(3) *Assume that* P *is defined over* $\mathbb{F}_q$. *Then* $L_0 = L$ *and* $L_\varphi = L(\mathbb{F}_q)$, *viewed as a constant algebraic group.*

Lemma 2.4.2 ([IK21, Corollary 3.2.3])**.** *The map* (2.4.1) *induces an identification*

$$H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) = V^{L_\varphi}.$$

Here, the notation V^{L_φ} denotes the space of scheme-theoretical invariants, i.e. the set of $v \in V$ such that for any k-algebra R, one has $\rho(x)v = v$ in $V \otimes_k R$ for all $x \in L_\varphi(R)$. We now consider the space of global sections over $G\text{-}\mathtt{Zip}^\mu$. Restriction of sections to $\mathcal{U}_\mu \subset G\text{-}\mathtt{Zip}^\mu$ induces an injective map $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) \to H^0(\mathcal{U}_\mu, \mathcal{V}(\rho)) = V^{L_\varphi}$. For simplicity, we assume here that P is defined over $\mathbb{F}_q$ (for the general result, see [IK21, Theorem 3.4.1]). We will need the general version in the proof of Proposition 3.4.1, but in the simple setting when ρ is a character $L \to \mathbb{G}_{\mathrm{m}}$. For $\alpha \in \Phi$, choose a realization $(u_\alpha)_{\alpha\in\Phi}$ (see §2.1). Fix a P-representation (V, ρ) and let $V = \bigoplus_{\nu \in X^*(T)} V_\nu$ be its T-weight decomposition. Define the Brylinski–Kostant filtration (*cf.* [XZ19, (3.3.2)]) indexed by $c \in \mathbb{R}$ on V_ν by:

$$\mathrm{Fil}_c^\alpha V_\nu = \bigcap_{j>c} \mathrm{Ker}\left(E_\alpha^{(j)} \colon V_\nu \to V_{\nu+j\alpha}\right)$$

where the map E_α was defined in §2.1. For $\chi \in X^*(T)_{\mathbb{R}}$ and $\nu \in X^*(T)$, set also

$$\mathrm{Fil}_\chi^P V_\nu = \bigcap_{\alpha \in \Delta^P} \mathrm{Fil}^{-\alpha}_{\langle \chi, \alpha^\vee \rangle} V_\nu.$$

The Lang torsor morphism $\wp \colon T \to T$, $g \mapsto g\varphi(g)^{-1}$ induces isomorphisms:

$$\begin{aligned} \wp^* \colon X^*(T)_{\mathbb{R}} &\xrightarrow{\sim} X^*(T)_{\mathbb{R}};\ \lambda \mapsto \lambda \circ \wp = \lambda - q\sigma^{-1}(\lambda) \\ \wp_* \colon X_*(T)_{\mathbb{R}} &\xrightarrow{\sim} X_*(T)_{\mathbb{R}};\ \delta \mapsto \wp \circ \delta = \delta - q\sigma(\delta). \end{aligned} \tag{2.4.4}$$

Theorem 2.4.3 ([IK21, Corollary 3.4.2])**.** *Assume that P is defined over $\mathbb{F}_q$. For all $(V, \rho) \in \mathrm{Rep}(P)$, the map ev_1 induces an identification*

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) = V^{L(\mathbb{F}_q)} \cap \bigoplus_{\nu \in X^*(T)} \mathrm{Fil}^P_{\wp^{*-1}(\nu)} V_\nu.$$

In the general case of an arbitrary parabolic P, $V^{L(\mathbb{F}_q)}$ is replaced by V^{L_φ} and $\mathrm{Fil}_c^\alpha V_\nu$ is replaced by a generalized Brylinski–Kostant filtration (see [IK21, Theorem 3.4.1]). In the special case when ρ is trivial on $R_{\mathrm{u}}(P)$, Theorem 2.4.3 simplifies greatly. Set $\delta_\alpha := \wp_*^{-1}(\alpha^\vee)$ and define a subspace $V_{\geq 0}^{\Delta^P} \subset V$ by

$$V_{\geq 0}^{\Delta^P} = \bigoplus_{\langle \nu, \delta_\alpha \rangle \geq 0,\ \forall \alpha \in \Delta^P} V_\nu. \tag{2.4.5}$$

If T is split over $\mathbb{F}_q$, then $\delta_\alpha = -\alpha^\vee/(q-1)$, and $V_{\geq 0}^{\Delta^P}$ is the direct sum of the weight spaces V_ν for those $\nu \in X^*(T)$ satisfying $\langle \nu, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Delta^P$.

Corollary 2.4.4. *Assume that P is defined over $\mathbb{F}_q$ and furthermore that $(V, \rho) \in \mathrm{Rep}(P)$ is trivial on $R_{\mathrm{u}}(P)$. Then one has*

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) = V^{L(\mathbb{F}_q)} \cap V_{\geq 0}^{\Delta^P}.$$

2.5 The stack of G-zip flags

2.5.1 Definition

Let (G, μ) be a cocharacter datum with attached zip datum $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ (§2.2.2). Fix a frame (B, T, z) with $z = \sigma(w_{0,I})w_0 = w_0 w_{0,J}$ (Lemma 2.2.3). The stack of zip flags ([GK19a, Definition 2.1.1]) is defined as

$$G\text{-}\mathtt{ZipFlag}^\mu = [E \backslash (G_k \times P/B)]$$

where the group E acts on the variety $G_k\times(P/B)$ by the rule $(a,b)\cdot(g,hB):=(agb^{-1},ahB)$ for all $(a,b)\in E$ and all $(g,hB)\in G_k\times P/B$. The first projection $G_k\times P/B\to G_k$ is E-equivariant, and yields a natural morphism of stacks

$$\pi\colon\ G\text{-}\mathtt{ZipFlag}^{\mu}\to G\text{-}\mathtt{Zip}^{\mu}\,.$$

Set $E':=E\cap(B\times G_k)$. Then the injective map $G_k\to G_k\times P/B;\ g\mapsto(g,B)$ yields an isomorphism of stacks $[E'\backslash G_k]\simeq G\text{-}\mathtt{ZipFlag}^{\mu}$ (see [GK19a, (2.1.5)]). We recall the stratification of $G\text{-}\mathtt{ZipFlag}^{\mu}$. First, define the Schubert stack as the quotient stack

$$\mathrm{Sbt}:=[B\backslash G_k/B].$$

This stack is finite and smooth. Its topological space is isomorphic to W, endowed with the topology induced by the Bruhat order on W. This follows easily from the Bruhat decomposition of G. One can show that $E'\subset B\times{}^zB$. In particular, there is a natural projection map $[E'\backslash G_k]\to[B\backslash G_k/{}^zB]$. Composing with the isomorphism $[B\backslash G_k/{}^zB]\to[B\backslash G_k/B]$ induced by $G_k\to G_k;\ g\mapsto gz$, we obtain a smooth, surjective map

$$\psi\colon\ G\text{-}\mathtt{ZipFlag}^{\mu}\to\mathrm{Sbt}\,.$$

For $w\in W$, put $\mathrm{Sbt}_w:=[B\backslash BwB/B]$, it is a locally closed substack of Sbt. The *flag strata* of $G\text{-}\mathtt{ZipFlag}^{\mu}$ are defined as fibers of the map ψ. They are locally closed substacks (endowed with the reduced structure). Concretely, let $w\in W$ and put

$$F_w:=B(wz^{-1}){}^zB=BwBz^{-1},$$

which is the $B\times{}^zB$-orbit of wz^{-1}. The set F_w is locally closed in G_k, and one has $\dim(F_w)=\ell(w)+\dim(B)$. Then, via the isomorphism $G\text{-}\mathtt{ZipFlag}^{\mu}\simeq[E'\backslash G_k]$, the flag strata of $G\text{-}\mathtt{ZipFlag}^{\mu}$ are the locally closed substacks

$$\mathcal{F}_w:=[E'\backslash F_w],\quad w\in W. \tag{2.5.1}$$

The set $F_{w_0}\subset G_k$ is open in G_k and similarly the stratum $\mathcal{F}_{w_0}$ is open in $G\text{-}\mathtt{ZipFlag}^{\mu}$. The $B\times{}^zB$-orbits of codimension 1 are $F_{s_\alpha w_0}$ for $\alpha\in\Delta$. The Zariski closure $\overline{F}_w$ is normal ([RR85, Theorem 3]) and coincides with $\bigcup_{w'\leq w}F_{w'}$, where $\leq$ is the Bruhat order of W.

2.5.2 Vector bundles on $G\text{-}\mathtt{ZipFlag}^{\mu}$

Let $\rho\colon B\to\mathrm{GL}(V)$ be an algebraic representation, and view ρ as a representation of E' via the first projection $E'\to B$. Via the isomorphism $G\text{-}\mathtt{ZipFlag}^{\mu}\simeq[E'\backslash G_k]$, we obtain a vector bundle $\mathcal{V}_{\mathrm{flag}}$ on $G\text{-}\mathtt{ZipFlag}^{\mu}$. Let $(V,\rho)\in\mathrm{Rep}(P)$ and let $\mathcal{V}(\rho)$ be the attached vector bundle on $G\text{-}\mathtt{Zip}^{\mu}$. Then one has

$$\pi^*(\mathcal{V}(\rho))=\mathcal{V}_{\mathsf{flag}}(\rho|_B).$$

Note that the rank of $\mathcal{V}_{\mathsf{flag}}(\rho)$ is the dimension of ρ. In particular, if $\lambda\in X^*(B)$, then $\mathcal{V}_{\mathsf{flag}}(\lambda)$ is a line bundle. For $(V,\rho)\in\mathrm{Rep}(B)$, consider the P-representation $\mathrm{Ind}_B^P(\rho)$ defined by

$$\mathrm{Ind}_B^P(\rho)=\{f\colon P\to V\mid f(xb)=\rho(b^{-1})f(x),\ \forall b\in B,\ \forall x\in P\}.$$

For $y\in P$ and $f\in\mathrm{Ind}_B^P(\rho)$, the element $y\cdot f$ is the function $x\mapsto f(y^{-1}x)$.

Proposition 2.5.1 ([IK24, Proposition 3.2.1]). *For $(V,\rho) \in \mathrm{Rep}(B)$, we have the identification $\pi_*(\mathcal{V}_{\mathsf{flag}}(\rho)) = \mathcal{V}(\mathrm{Ind}_B^P(\rho))$. In particular $\pi_*(\mathcal{V}_{\mathsf{flag}}(\rho))$ is a vector bundle on $G\text{-}\mathtt{Zip}^\mu$.*

In particular, if ρ is a character $\lambda \in X^*(T)$, then $\mathcal{V}_{\mathsf{flag}}(\lambda)$ is a line bundle and one has:

$$\pi_*(\mathcal{V}_{\mathsf{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$$

where the vector bundle $\mathcal{V}_I(\lambda)$ was defined in §2.3.3. Hence, we have

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) = H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{V}_{\mathsf{flag}}(\lambda)). \tag{2.5.2}$$

If $f\colon G_k \to k$ is a section of the right hand side of (2.5.2), then the corresponding function $f_I\colon G_k \to V_I(\lambda)$ on the left hand side of (2.5.2) is given by

$$(f_I(g))(x) = f((x^{-1}, \varphi(x)^{-1}) \cdot g) = f(x^{-1} g \varphi(x)) \tag{2.5.3}$$

for all $g \in G_k$ and $x \in L$, by the construction of the identification. Note also that the line bundles $\mathcal{V}_{\mathsf{flag}}(\lambda)$ satisfy the following identity:

$$\mathcal{V}_{\mathsf{flag}}(\lambda+\lambda') = \mathcal{V}_{\mathsf{flag}}(\lambda) \otimes \mathcal{V}_{\mathsf{flag}}(\lambda'), \quad \forall \lambda, \lambda' \in X^*(T). \tag{2.5.4}$$

We can also define vector bundles on the stack Sbt as in [IK24, §4]. For our purpose, it is enough to define line bundles on Sbt. Using (2.3.4), we can attach to each $(\chi_1, \chi_2) \in X^*(T) \times X^*(T)$ a line bundle $\mathcal{V}_{\mathrm{Sbt}}(\chi_1,\chi_2)$ on Sbt. One has

$$\psi^* \mathcal{V}_{\mathrm{Sbt}}(\chi_1,\chi_2) = \mathcal{V}_{\mathsf{flag}}(\chi_1 + (z\chi_2)\circ\varphi) = \mathcal{V}_{\mathsf{flag}}(\chi_1 + q\sigma^{-1}(z\chi_2)). \tag{2.5.5}$$

2.5.3 Partial Hasse invariants

We recall some results of [IK24]. By [GK19a, Theorem 2.2.1(a)], the line bundle $\mathcal{V}_{\mathrm{Sbt}}(\chi_1,\chi_2)$ admits a nonzero section over Sbt_{w_0} if and only if $\chi_1 = -w_0\chi_2$. If this condition is satisfied, $H^0(\mathrm{Sbt}_{w_0}, \mathcal{V}_{\mathrm{Sbt}}(\chi_1,\chi_2))$ is one-dimensional. For $\chi \in X^*(T)$, let h_χ be any nonzero element

$$h_\chi \in H^0(\mathrm{Sbt}_{w_0}, \mathcal{V}_{\mathrm{Sbt}}(-w_0\chi, \chi)).$$

By [GK19a, Theorem 2.2.1(c)], h_χ extends to Sbt if and only if χ is a dominant character. Using (2.5.5) and $z = \sigma(w_{0,I})w_0$, we obtain a section

$$\mathrm{Ha}_\chi := \psi^*(h_\chi) \in H^0(\mathcal{F}_{w_0}, \mathcal{V}_{\mathsf{flag}}(-w_0\chi + q w_{0,I} w_0(\sigma^{-1}\chi))),$$

and for $\chi \in X^*_+(T)$ the section Ha_χ extends to $G\text{-}\mathtt{ZipFlag}^\mu$. In particular, let $\alpha \in \Delta$ and suppose χ_α is a character satisfying

$$\begin{cases} \langle \chi_\alpha, \alpha^\vee \rangle > 0 \\ \langle \chi_\alpha, \beta^\vee \rangle = 0 \qquad \text{for all } \beta \in \Delta \setminus \{\alpha\}. \end{cases} \tag{2.5.6}$$

In this case, the section h_{χ_α} vanishes exactly on the codimension one stratum $\overline{\mathrm{Sbt}}_{w_0 s_\alpha}$. Similarly, the section $\mathrm{Ha}_{\chi_\alpha}$ cuts out the Zariski closure of the codimension one stratum $\mathcal{F}_{w_0 s_\alpha}$.

Definition 2.5.2. *For $\alpha \in \Delta$ and χ_α satisfying (2.5.6), we call the section $\mathrm{Ha}_{\chi_\alpha}$ a partial Hasse invariant for the stratum $\mathcal{F}_{w_0 s_\alpha}$.*

2.6 Shimura varieties and G-zips

We explain the connection between the stack of G-zips and Shimura varieties. Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum [Del79, 2.1.1]. Write $\mathbf{E} = E(\mathbf{G}, \mathbf{X})$ for the reflex field of $(\mathbf{G}, \mathbf{X})$ and $\mathcal{O}_{\mathbf{E}}$ for its ring of integers. Given an open compact subgroup $K \subset \mathbf{G}(\mathbf{A}_f)$, write $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_K$ for Deligne's canonical model at level K over $\mathbf{E}$ (see [Del79]). For $K \subset \mathbf{G}(\mathbb{A}_f)$ small enough, $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_K$ is a smooth, quasi-projective scheme over $\mathbf{E}$. Every inclusion $K' \subset K$ induces a finite étale projection $\pi_{K'/K} \colon \mathrm{Sh}(\mathbf{G}, \mathbf{X})_{K'} \to \mathrm{Sh}(\mathbf{G}, \mathbf{X})_K$.

Fix a prime number p, and assume that the level K is of the form $K = K_p K^p$ where $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ is a hyperspecial subgroup and $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ is an open compact subgroup. Then one has $K_p = \mathscr{G}(\mathbb{Z}_p)$ where $\mathscr{G}$ is a reductive group over $\mathbb{Z}_p$ such that $\mathscr{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathbf{G}_{\mathbb{Q}_p}$ and $\mathscr{G} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is connected.

We assume that $(\mathbf{G}, \mathbf{X})$ is of abelian-type. For any place v above p in $\mathbf{E}$, Kisin ([Kis10]) and Vasiu ([Vas99]) constructed a smooth, canonical model $\mathscr{S}_K$ over $\mathcal{O}_{\mathbf{E}_v}$ of $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_K$. Let $\kappa(v)$ denote the residue field of $\mathcal{O}_{\mathbf{E}_v}$ and let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\kappa(v)$. Set $S_K := \mathscr{S}_K \otimes_{\mathcal{O}_{\mathbf{E}_v}} \overline{\mathbb{F}}_p$. We can take a representative $\mu \in \{\mu\}$ defined over $\mathbf{E}_v$ by [Kot84, (1.1.3) Lemma (a)]. We can also assume that μ extends to $\mu \colon \mathbb{G}_{\mathrm{m}, \mathcal{O}_{\mathbf{E}_v}} \to \mathscr{G}_{\mathcal{O}_{\mathbf{E}_v}}$ ([Kim18, Corollary 3.3.11]). It gives rise to a parabolic subgroup $\mathscr{P} \subset \mathscr{G}_{\mathcal{O}_{\mathbf{E}_v}}$, with Levi subgroup $\mathscr{L}$ equal to the centralizer of μ. As explained in [IK21, §2.5], we can assume (after possibly twisting μ) that there is a Borel pair $(\mathscr{B}, \mathscr{T})$ over $\mathbb{Z}_p$ in $\mathscr{G}$ such that $\mathscr{B}_{\mathcal{O}_{\mathbf{E}_v}} \subset \mathscr{P}$. Let G, P, B, T denote the special fibers of $\mathscr{G}, \mathscr{P}, \mathscr{B}, \mathscr{T}$ respectively. By slight abuse of notation, we denote again by μ its mod p reduction $\mu \colon \mathbb{G}_{\mathrm{m},k} \to G_k$. Then (G, μ) is a cocharacter datum, and it yields a zip datum (G, P, L, Q, M, φ) as in §2.2.2, where $q = p$.

By [Zha18, 4.1] and [IKY, §3.5], there exists a natural smooth morphism

$$\zeta \colon S_K \to G\text{-}\mathtt{Zip}^{\mu}. \tag{2.6.1}$$

This map is also surjective by [SYZ21, Corollary 3.5.3(1)].

For each $\mathbf{L}$-dominant character $\lambda \in X^*_{+,I}(\mathbf{T})$, we denote the mod p reduction by the same symbol $\lambda \in X^*_{+,I}(T)$. Then we have a vector bundle $\mathcal{V}_I(\lambda)$ on G-$\mathtt{Zip}^{\mu}$ as in §2.3.3. We denote the pullback of $\mathcal{V}_I(\lambda)$ under ζ by the same symbol. The vector bundle $\mathcal{V}_I(\lambda)$ on S_K is called *the automorphic vector bundles of weight* λ.

The flag space of the Siegel modular variety $\mathcal{A}_n$ was first introduced by Ekedahl–van der Geer in [EvdG09]. It parametrizes pairs $(\underline{A}, \mathcal{F}_\bullet)$ where $\underline{A} = (A, \lambda) \in \mathcal{A}_n$ is a principally polarized abelian variety of relative dimension n and $\mathcal{F}_\bullet \subset H^1_{\mathrm{dR}}(A)$ is a full symplectic flag refining the Hodge filtration of $H^1_{\mathrm{dR}}(A)$. In general, we defined in [GK19a, §9.1] the flag space $\mathrm{Flag}(S_K)$ of S_K as the fiber product

$$\begin{tikzcd} \mathrm{Flag}(S_K) \arrow[r, "\zeta_{\mathsf{flag}}"] \arrow[d, "\pi_K"'] & G\text{-}\mathtt{ZipFlag}^{\mu} \arrow[d, "\pi"] \\ S_K \arrow[r, "\zeta"'] & G\text{-}\mathtt{Zip}^{\mu} \end{tikzcd}$$

The stratification $(\mathcal{F}_w)_{w \in W}$ on G-$\mathtt{ZipFlag}^{\mu}$ induces by pullback via ζ_{flag} a stratification $(\mathrm{Flag}(S_K)_w)_{w \in W}$ of $\mathrm{Flag}(S_K)$ by locally closed, smooth subschemes. Moreover, we obtain a line bundle $\mathcal{V}_{\mathsf{flag}}(\lambda)$ on $\mathrm{Flag}(S_K)$. Since ζ is smooth, pullback and pushforward commute, so we have $\pi_{K,*}(\mathcal{V}_{\mathsf{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$. In particular, the space of automorphic forms $H^0(S_K, \mathcal{V}_I(\lambda))$ identifies with $H^0(\mathrm{Flag}(S_K), \mathcal{V}_{\mathsf{flag}}(\lambda))$.

3 The zip cone

In this section, we will consider several subsets of $X^*(T)$. A *cone* in $X^*(T)$ will be an additive submonoid containing 0. If $C \subset X^*(T)$ is a cone, we define its saturation (or saturated cone) as follows

$$\mathcal{C} = \{\lambda \in X^*(T) \mid \exists N \geq 1,\ N\lambda \in C\}.$$

We say that C is saturated if $C = \mathcal{C}$. Define also $C_{\mathbb{Q}_{\geq 0}}$ as follows

$$C_{\mathbb{Q}_{\geq 0}} = \left\{ \sum_{i=1}^{N} a_i \lambda_i \;\middle|\; N \geq 1,\ a_i \in \mathbb{Q}_{\geq 0},\ \lambda_i \in C \right\}.$$

There is a bijection between saturated cones of $X^*(T)$ and additive submonoids of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ stable by $\mathbb{Q}_{\geq 0}$. The bijection is given by the maps $C \mapsto C_{\mathbb{Q}_{\geq 0}}$ and $C' \mapsto C' \cap X^*(T)$.

3.1 Example: Hilbert–Blumenthal Shimura varieties

We recall some results of Diamond–Kassaei in [DK17] and extended in [DK23] that motivate this paper. We give a short explanation of [DK17, Corollary 5.4]. The authors study Hilbert automorphic forms in characteristic p. Specifically, let $\mathbf{F}/\mathbb{Q}$ be a totally real extension of degree $d = [\mathbf{F} : \mathbb{Q}]$ and let $\mathbf{G}$ be the subgroup of $\mathrm{Res}_{\mathbf{F}/\mathbb{Q}}(\mathrm{GL}_{2,\mathbf{F}})$ defined by

$$\mathbf{G}(R) = \{g \in \mathrm{GL}_2(R \otimes_{\mathbb{Q}} \mathbf{F}) \mid \det(g) \in R^\times\}.$$

Let p be a prime number unramified in $\mathbf{F}$ (in [DK23], p is allowed to be ramified in $\mathbf{F}$). The lattice $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbf{F}} \subset \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbf{F}$ gives rise to a reductive model $\mathscr{G}$ over $\mathbb{Z}_p$. Fix a small enough level $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ outside p and set $K_p := \mathscr{G}(\mathbb{Z}_p)$ and $K = K_p K^p$. Let S_K be the (geometric) special fiber of the corresponding Hilbert–Blumenthal Shimura variety of level K. The scheme S_K is smooth of dimension d over $\overline{\mathbb{F}}_p$. It parametrizes tuples $(A, \lambda, \iota, \overline{\eta})$ of abelian schemes over $\overline{\mathbb{F}}_p$ of dimension d endowed with a principal polarization λ, an action ι of $\mathcal{O}_{\mathbf{F}}$ on A and a K^p-level structure $\overline{\eta}$.

Let $\Sigma := \mathrm{Hom}(\mathbf{F}, \overline{\mathbb{Q}}_p)$ be the set of field embeddings $\mathbf{F} \to \overline{\mathbb{Q}}_p$. Write $(\mathbf{e}_\tau)_\tau$ for the canonical basis of $\mathbb{Z}^\Sigma$. Let σ denote the action of Frobenius on Σ. For each $\tau \in \Sigma$, there is an associated line bundle ω_τ on S_K. For $\mathbf{k} = \sum_\tau k_\tau \mathbf{e}_\tau \in \mathbb{Z}^\Sigma$, define

$$\omega^{\mathbf{k}} := \bigotimes_{\tau \in \Sigma} \omega_\tau^{k_\tau}.$$

Elements of $H^0(X_{\overline{\mathbb{F}}_p}, \omega^{\mathbf{k}})$ are called mod p Hilbert modular forms of weight $\mathbf{k}$. There is an Ekedahl–Oort stratification on S_K given by the isomorphism class of the p-torsion $A[p]$ (with its additional structure given by λ and ι). There is a unique open stratum (on which A is an ordinary abelian variety). The codimension one strata can be labeled as $(S_{K,\tau})_{\tau \in \Sigma}$. Andreatta–Goren ([AG05]) constructed partial Hasse invariants Ha_τ for each $\tau \in \Sigma$. The weight of Ha_τ is given by

$$\mathbf{h}_\tau := e_\tau - p e_{\sigma^{-1}\tau}.$$

Note that the sign of $\mathbf{h}_\tau$ is different in [AG05] and [DK17], due to a different convention of positivity. The main property of Ha_τ is that it vanishes exactly on the Zariski closure of the codimension one stratum $S_{K,\tau}$. It is a special case of the sections Ha_α defined in Definition 2.5.2. Define the partial Hasse invariant cone $\mathcal{C}_{\mathsf{pHa}} \subset \mathbb{Z}^\Sigma$ as the cone of $\mathbf{k} \in \mathbb{Z}^\Sigma$ which are spanned (over $\mathbb{Q}_{\geq 0}$) by the weights $(\mathbf{h}_\tau)_{\tau \in \Sigma}$ defined above.

Theorem 3.1.1 (Diamond–Kassaei, [DK17, Theorem 5.1, Corollary 5.4]).
(1) *Let $f \in H^0(S_K, \omega^{\mathbf{k}})$ and $\tau \in \Sigma$. Assume that $pk_\tau > k_{\sigma^{-1}\tau}$. Then f is divisible by* Ha_τ.
(2) *If $H^0(S_K, \omega^{\mathbf{k}}) \neq 0$, then $\mathbf{k} \in \mathcal{C}_{\mathsf{pHa}}$.*

The authors define a minimal cone $\mathcal{C}_{\min} \subset \mathcal{C}_{\mathsf{pHa}}$ as follows:

$$\mathcal{C}_{\min} = \{\mathbf{k} \in \mathbb{Z}^\Sigma \mid pk_\tau \leq k_{\sigma^{-1}\tau} \ \text{ for all } \tau \in \Sigma\}.$$

Theorem 3.1.1(1) shows that any Hilbert modular form f of weight $\mathbf{k}$ can be written as a product $f = f_{\min}h$, where $f_{\min}$ has weight $\mathbf{k}_{\min} \in \mathcal{C}_{\min}$ and h is a product of partial Hasse invariants. In particular (2) is a direct consequence of (1). One motivation of this paper is to understand the natural setting in which one might expect a generalization to other Shimura varieties of Theorem 3.1.1(2). In [GK22a], Goldring and the second-named author show that (1) also admits a similar generalization for several Hodge-type Shimura varieties.

3.2 General setting

We attempt to give an abstract setting in which Theorem 3.1.1 may generalize. First, by observing the example of Hilbert–Blumenthal varieties, we extract the essential properties of the objects we want to study. Specifically, we consider a stack Y over $k = \overline{\mathbb{F}}_p$ endowed with the following structure:

(a) There is a locally closed stratification $Y = \bigsqcup_{i=1}^N Y_i$ such that the Zariski closure of a stratum is a union of strata.

(b) There is a free, finite-type $\mathbb{Z}$-module Λ and a family of line bundles $(\omega(\lambda))_{\lambda \in \Lambda}$ on Y, such that $\omega(\lambda + \lambda') = \omega(\lambda) \otimes \omega(\lambda')$ for all $\lambda, \lambda' \in \Lambda$.

(c) For each codimension one stratum $Y_i \subset Y$, there is $\lambda_i \in \Lambda$ and $\mathrm{Ha}_i \in H^0(X, \omega(\lambda_i))$ such that the support of $\mathrm{div}(\mathrm{Ha}_i)$ is $\overline{Y}_i$. By analogy, we call Ha_i a partial Hasse invariant for Y_i.

Denote by $I_1 \subset I$ the indices such that Y_i has codimension one. Let $C_{\mathsf{pHa}} \subset \Lambda$ denote the cone generated by the elements $\{\lambda_i \mid i \in I_1\}$, and call it the partial Hasse invariant cone. Put

$$C_Y := \{\lambda \in \Lambda \mid H^0(Y, \omega(\lambda)) \neq 0\}.$$

By definition, one has $C_{\mathsf{pHa}} \subset C_Y$. If Y is connected, then C_Y is a cone (i.e. an additive submonoid) of Λ. Indeed, if $\lambda, \lambda' \in \Lambda$ and f, f' are nonzero sections of $\omega(\lambda)$ and $\omega(\lambda')$ respectively, then ff' is a section of $\omega(\lambda + \lambda')$. Since Y is connected, ff' is nonzero. Write $\mathcal{C}_{\mathsf{pHa}}$ and $\mathcal{C}_Y$ for the saturation of C_{pHa} and C_Y inside Λ, respectively.

Definition 3.2.1. *Let Y be a stack satisfying* (a)*,* (b) *and* (c)*. We say that Y has the Hasse property if $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}_Y$.*

For example, Theorem 3.1.1 (2) shows that the geometric special fiber of the Hilbert–Blumenthal Shimura variety at a place of good reduction satisfies the Hasse property. Let (G, μ) be a cocharacter datum and let G-$\mathtt{Zip}^\mu$ be the attached stack of G-zips. Fix a frame (B, T, z) as in §2.2.3. Then, the stack of zip flags G-$\mathtt{ZipFlag}^\mu$ (§2.5) satisfies all requirements (a), (b) and (c) above. First, we have the flag stratification $G\text{-}\mathtt{ZipFlag}^\mu = \bigsqcup_{w \in W} \mathcal{F}_w$ as in §(2.5.1). Setting $\Lambda := X^*(T)$, we have the family of line bundles $(\mathcal{V}_{\mathsf{flag}}(\lambda))_{\lambda \in X^*(T)}$ satisfying (b) by (2.5.4). Finally, we have partial Hasse invariants (§2.5.3). To be precise, there is an ambiguity in the definition of C_{pHa}, because if f is a partial Hasse invariant for $\mathcal{F}_{w_0 s_\alpha}$ (Definition 2.5.2), then χf^n for $\chi \in X^*(G)$ and $n \geq 1$ is also a partial Hasse invariant for $\mathcal{F}_{w_0 s_\alpha}$. Later, we give an unambiguous definition of C_{pHa} in Definition 3.6.1. In this paper, we give a full answer as to whether $G\text{-}\mathtt{ZipFlag}^\mu$ satisfies the Hasse property.

Similarly, let Y be a scheme endowed with a smooth, surjective morphism $\zeta_Y \colon Y \to G\text{-}\mathtt{ZipFlag}^\mu$. Then Y inherits naturally by pullback all the structure from $G\text{-}\mathtt{ZipFlag}^\mu$, and hence satisfies all required properties (a), (b) and (c) above. In particular, if we start with a scheme X and a smooth, surjective morphism $\zeta \colon X \to G\text{-}\mathtt{Zip}^\mu$, then we can consider the flag space $Y := \mathrm{Flag}(X)$ (similarly to the flag space of S_K defined at the end of §2.6). It is defined as the fiber product

$$Y := X \times_{G\text{-}\mathtt{Zip}^\mu} G\text{-}\mathtt{ZipFlag}^\mu . \tag{3.2.1}$$

The induced map $\zeta_{\mathsf{flag}} \colon Y \to G\text{-}\mathtt{ZipFlag}^\mu$ is again smooth and surjective. Hence, Y inherits the structure as above and satisfies (a), (b) and (c). Denote by $\pi \colon Y \to X$ and $\pi\colon G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$ the natural projections. In both cases, we have $\pi_*(\mathcal{V}_{\mathsf{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$ because ζ is smooth and π is proper. Therefore, the cones C_Y and $C_{G\text{-}\mathtt{ZipFlag}^\mu}$ can also be written as follows:

$$C_Y = \{\lambda \in X^*(T) \mid H^0(X, \mathcal{V}_I(\lambda)) \neq 0\}, \tag{3.2.2}$$

$$C_{\mathsf{zip}} := C_{G\text{-}\mathtt{ZipFlag}^\mu} = \{\lambda \in X^*(T) \mid H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0\}. \tag{3.2.3}$$

We will use the notation C_{zip} (introduced in [Kos19]), instead of $C_{G\text{-}\mathtt{ZipFlag}^\mu}$. When Y is given as (3.2.1) above, we call Y *the flag space* of (X, ζ). Furthermore, we make the slight abuse of saying that (X, ζ) satisfies the Hasse property if $(Y, \zeta_{\mathsf{flag}})$ does. In particular, let $X = S_K$ be the geometric special fiber modulo p of a Hodge-type Shimura variety with good reduction at p. By Zhang's result, there is a smooth, surjective morphism $\zeta \colon X \to G\text{-}\mathtt{Zip}^\mu$, and so we obtain $(Y, \zeta_{\mathsf{flag}})$ as above. Our goal is to investigate which Hodge-type Shimura varieties satisfy the Hasse property.

We now return to a general pair (X, ζ). Since ζ is surjective, pullback by ζ_{flag} induces an inclusion

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) \subset H^0(X, \mathcal{V}_I(\lambda)).$$

Hence we have inclusions $C_{\mathsf{zip}} \subset C_Y$ and $\mathcal{C}_{\mathsf{zip}} \subset \mathcal{C}_Y$. Furthermore, Hasse invariants exist already on $G\text{-}\mathtt{ZipFlag}^\mu$ by section 2.5.3, hence the cone C_{pHa} generated by their weights satisfies $C_{\mathsf{pHa}} \subset C_{\mathsf{zip}}$. Therefore, we have in general

$$\mathcal{C}_{\mathsf{pHa}} \subset \mathcal{C}_{\mathsf{zip}} \subset \mathcal{C}_Y .$$

In particular, if the pair (X, ζ) satisfies the Hasse property, then all three cones above coincide. In other words, a necessary condition for X to satisfy the Hasse property is that $G\text{-}\mathtt{Zip}^\mu$ itself satisfies this property, which is equivalent to the condition $\mathcal{C}_{\mathsf{zip}} = \mathcal{C}_{\mathsf{pHa}}$. This is an obstruction for a potential generalization of Theorem 3.1.1(2) to other Shimura varieties.

Remark 3.2.2. When we start with a pair (X, ζ) and construct $(Y, \zeta_{\mathsf{flag}})$ by fiber product as in (3.2.1), formula (3.2.2) shows immediately that

$$\mathcal{C}_Y \subset X^*_{+,I}(T). \tag{3.2.4}$$

Indeed, this follows simply from the fact that if λ is not I-dominant, then $\mathcal{V}_I(\lambda) = 0$. Thus, in the example of Shimura varieties, we have the inclusion (3.2.4).

3.3 Previous results

We review previous results from [GK18]. Let (X, ζ) be a pair consisting of a k-scheme X and a smooth, surjective morphism of stacks $\zeta \colon X \to G\text{-}\mathtt{Zip}^{\mu}$, and let $(Y, \zeta_{\mathsf{flag}})$ be the flag space of X. We make the following assumption:

Assumption 3.3.1.
(A) *For any $w \in W$ with $\ell(w) = 1$, the closed stratum $\overline{Y}_w = \zeta_{\mathsf{flag}}^{-1}(\overline{\mathcal{F}}_w)$ is pseudo-complete (i.e. any element of $H^0(\overline{Y}_w, \mathcal{O}_{\overline{Y}_w})$ is locally constant on $\overline{Y}_w$ for the Zariski-topology).*
(B) *The restriction ζ to any connected component $X^{\circ} \subset X$ is smooth and surjective.*

For example, Condition (A) is satisfied if X is a proper k-scheme. In general, it can happen that the inclusion $\mathcal{C}_{\mathsf{pHa}} \subset \mathcal{C}_{\mathsf{zip}}$ is strict. In this case, it is impossible for Y to satisfy the Hasse property. However, Goldring and the second-named author conjectured in general:

Conjecture 3.3.2. *Under Assumption 3.3.1, we have $\mathcal{C}_Y = \mathcal{C}_{\mathsf{zip}}$.*

Let S_K be the special fiber of a Hodge-type Shimura variety at a prime p of good reduction. In this case, we write $C_K(\overline{\mathbb{F}}_p)$ for the cone C_Y, i.e.

$$C_K(\overline{\mathbb{F}}_p) := \{\lambda \in X^*(T) \mid H^0(S_K, \mathcal{V}_I(\lambda)) \neq 0\}. \tag{3.3.1}$$

By [Kos19, Corollary 1.5.3], the saturation of $C_K(\overline{\mathbb{F}}_p)$ is independent of K, so we simply denote it by $\mathcal{C}(\overline{\mathbb{F}}_p)$. Let $\zeta \colon S_K \to G\text{-}\mathtt{Zip}^{\mu}$ be the map (2.6.1). We do not know whether the pair (X, ζ) always satisfies condition (A) of Assumption 3.3.1. However, by [GK19a, Theorem 6.2.1], the map $\zeta \colon S_K \to G\text{-}\mathtt{Zip}^{\mu}$ admits an extension to a toroidal compactification

$$\zeta^{\Sigma} \colon S_K^{\Sigma} \to G\text{-}\mathtt{Zip}^{\mu}$$

where Σ is a sufficiently fine cone decomposition. By construction, the pullback $\mathcal{V}_I^{\Sigma}(\lambda) := \zeta^{\Sigma,*}(\mathcal{V}_I(\lambda))$ is the canonical extension of $\mathcal{V}_I(\lambda)$ to S_K^{Σ}. Furthermore, by [And23, Theorem 1.2], the map ζ^{Σ} is smooth. Since ζ is surjective, ζ^{Σ} is also surjective. By [WZ23, Proposition 6.20], any connected component $S^{\circ} \subset S_K^{\Sigma}$ intersects the unique zero-dimensional stratum. Since $\zeta^{\Sigma} \colon S^{\circ} \to G\text{-}\mathtt{Zip}^{\mu}$ is smooth, it has an open image, therefore it must be surjective. In particular, the pair $(S_K^{\Sigma}, \zeta^{\Sigma})$ satisfies Conditions (A) and (B). Furthermore, in most cases Koecher's principle holds by [LS18, Theorem 2.5.11], i.e. we have an equality

$$H^0(S_K^{\Sigma}, \mathcal{V}_I^{\Sigma}(\lambda)) = H^0(S_K, \mathcal{V}_I(\lambda)).$$

In particular, the cone attached to the pair $(S_K^{\Sigma}, \zeta^{\Sigma})$ is the same as the cone attached to (S_K, ζ), namely $C_K(\overline{\mathbb{F}}_p)$. Therefore, by the above discussion, we deduce that Conjecture 3.3.2 applies to Shimura varieties and predicts the following:

Conjecture 3.3.3. *If S_K is the special fiber of a Hodge-type Shimura variety at a prime p of good reduction, we have $\mathcal{C}(\overline{\mathbb{F}}_p) = \mathcal{C}_{\mathsf{zip}}$.*

In [GK18, Theorem D], the authors proved that certain Shimura varieties satisfy the Hasse property. Specifically, they showed the following:

Theorem 3.3.4 ([GK18, Theorem D]). *Let (X, ζ) be a pair which satisfies Assumption 3.3.1 and let $(Y, \zeta_{\mathsf{flag}})$ be the flag space of X. Suppose that (G, μ) is one of the following three pairs:*
(1) *G is an $\mathbb{F}_p$-form of GL_2^n for some $n \geq 1$, and μ is non-trivial on each factor,*

(2) $G = \mathrm{GL}_{3,\mathbb{F}_p}$, *and* $\mu\colon z \mapsto \mathrm{diag}(z,z,1)$,
(3) $G = \mathrm{GSp}(4)_{\mathbb{F}_p}$, *and* $\mu\colon z \mapsto \mathrm{diag}(z,z,1,1)$.
Then (X,ζ) *satisfies the Hasse property. In other words, we have* $\mathcal{C}_Y = \mathcal{C}_{\mathsf{zip}} = \mathcal{C}_{\mathsf{pHa}}$

The above theorem also holds if we change the group G to a group with the same adjoint group. By the above discussion, Theorem 3.3.4 applies to Hilbert–Blumenthal Shimura varieties, Picard surfaces at a split prime, Siegel modular threefolds and shows that Conjecture 3.3.2 holds in each case. Goldring and the second-named author proved Conjecture 3.3.2 in [GK22a] for certain Shimura varieties for which the inclusion $\mathcal{C}_{\mathsf{pHa}} \subset \mathcal{C}_{\mathsf{zip}}$ is strict. Namely, they showed Conjecture 3.3.2 for the Siegel modular variety $\mathcal{A}_3$ as well as unitary Shimura varieties of signature (r,s) with $r+s \leq 4$ at split or inert primes, except when $r=s=2$ and p is inert. With the exception of the case $r=s=2$ and p split, the inclusion $\mathcal{C}_{\mathsf{pHa}} \subset \mathcal{C}_{\mathsf{zip}}$ is strict in each of these cases.

3.4 First properties of C_{zip}

Let (G,μ) be a cocharacter datum over $\mathbb{F}_q$ and $\mathcal{Z}_\mu = (G,P,L,Q,M,\varphi)$ the attached zip datum (§2.2.2). Fix a frame (B,T,z) with $z = \sigma(w_{0,I})w_0$ (see §2.2.2). Let $C_{\mathsf{zip}} \subset X^*(T)$ be the zip cone, defined in (3.2.3). We start with some elementary properties of C_{zip}. As we already noted, we have $C_{\mathsf{zip}} \subset X^*_{+,I}(T)$. Furthermore, the cone C_{zip} has maximal rank in $X^*(T)$, in the sense that $\mathrm{Span}_{\mathbb{Q}}(C_{\mathsf{zip}}) = X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. This was shown in [GK19a, Lemma 3.4.2] (with the notation of *loc. cit.*, $\mathcal{C}_{w_0} \subset C_{\mathsf{zip}}$ and $\mathcal{C}_{w_0}$ has maximal rank). Note that the cocharacter datum is assumed to be Hodge-type in [GK19a, §3.4], but this assumption is unnecessary for [GK19a, Lemma 3.4.2].

Next, we consider line bundles on $G\text{-}\mathtt{Zip}^\mu$. Recall that $\mathcal{V}_I(\lambda)$ is a line bundle if and only if $\lambda \in X^*(L)$ (viewed as a subgroup of $X^*(T)$). Define the following set:

$$X^*_-(L)_{\mathrm{reg}} = \{\lambda \in X^*(L) \mid \langle \lambda, \alpha^\vee \rangle < 0,\ \forall \alpha \in \Delta^P\}. \tag{3.4.1}$$

These characters were termed L-ample in [GK19a, Definition N.5.1]. The notation used in (3.4.1) is more enlightening, since these characters are in particular in $X^*_-(T)$ (the cone of anti-dominant characters). An immediate consequence of [KW18, Theorem 5.1.4] is the inclusion $X^*_-(L)_{\mathrm{reg}} \subset \mathcal{C}_{\mathsf{zip}}$. Set $X^*_-(L) := X^*_-(T) \cap X^*(L)$. The stronger inclusion $X^*_-(L) \subset \mathcal{C}_{\mathsf{zip}}$ is claimed in [Kos19, Proposition 1.6.1] with an incomplete proof, so we give one below:

Proposition 3.4.1. *We have* $X^*_-(L) \subset \mathcal{C}_{\mathsf{zip}}$.

Proof. Let $\lambda \in X^*_-(T) \cap X^*(L)$. Applying [IK21, Theorem 3.4.1] to the one-dimensional L-representation $V_I(\lambda)$, we obtain:

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) = V_I(\lambda)^{L_\varphi} \cap \bigcap_{\alpha \in \Delta^P} \mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V_\lambda.$$

Furthermore, $\mathrm{Fil}_{\delta_\alpha}^{\Xi_\alpha, \mathbf{a}_\alpha, \mathbf{r}_\alpha} V_\lambda = V_\lambda = V_I(\lambda)$ if $\langle \lambda, \delta_\alpha \rangle \geq 0$ and is 0 otherwise. Let $d_\alpha \geq 1$ be an integer such that α is defined over $\mathbb{F}_{q^{d_\alpha}}$. We find that $\delta_\alpha = -\frac{1}{q^{d_\alpha}-1} \sum_{i=0}^{d_\alpha - 1} q^i \sigma^i(\alpha^\vee)$. Since $\lambda \in X^*_-(T)$, we have $\langle \lambda, \sigma^i(\alpha^\vee) \rangle \leq 0$ for all i, hence $\langle \lambda, \delta_\alpha \rangle \geq 0$. We deduce $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) = V_I(\lambda)^{L_\varphi}$. Finally, if we change λ to $N\lambda$ where N divides the order of the finite group scheme L_φ, we obtain $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) = V_I(\lambda)$. In particular, this space is nonzero, and this proves the result. □

3.5 Norm of the highest weight vector

Recall that we always have $1 \in U_\mu$, where $U_\mu \subset G_k$ is the unique open E-orbit. Recall the definition of the finite subgroup $L_\varphi \subset L$ given in (2.4.2). Put $N_\varphi = |L_0(\mathbb{F}_q)| q^m$ where L_0 is the Levi subgroup defined in (2.4.3) and $m \geq 0$ is the smallest integer such that the finite unipotent group L_φ° is annihilated by φ^m. For $\lambda, \lambda' \in X^*(T)$ and $f \in V_I(\lambda)$, $f' \in V_I(\lambda')$, let $ff' \in V_I(\lambda + \lambda')$ be the image of $f \otimes f'$ by the map (2.3.3). For $\lambda \in X^*_{+,I}(T)$ and $f \in V_I(\lambda)$ define

$$\mathrm{Norm}_{L_\varphi}(f) := \left(\prod_{s \in L_0(\mathbb{F}_q)} s \cdot f \right)^{q^m} \in V_I(N_\varphi \lambda). \tag{3.5.1}$$

It is clear that $\mathrm{Norm}_{L_\varphi}(f)$ is L_φ-invariant. In particular, it gives rise to an element in $H^0(\mathcal{U}_\mu, \mathcal{V}(N_\varphi \lambda))$ by Lemma 2.4.2. In general, it is difficult to determine whether $\mathrm{Norm}_{L_\varphi}(f)$ extends to a global section. However, this is possible when f is a highest weight vector, as we now explain.

Let $f_\lambda \in V_I(\lambda)$ be a nonzero element in the highest weight line of $V_I(\lambda)$. The following result generalizes [Kos19, Theorem 2] (where P was assumed to be defined over $\mathbb{F}_p$, here we do not make this assumption). For $\alpha \in \Delta^P$, denote by r_α the smallest integer $r \geq 1$ such that $\sigma^r(\alpha) = \alpha$.

Proposition 3.5.1. *The section* $\mathrm{Norm}_{L_\varphi}(f_\lambda)$ *extends to a global section over* G-$\mathtt{Zip}^\mu$ *if and only if for all* $\alpha \in \Delta^P$*, the following holds:*

$$\sum_{w \in W_{L_0}(\mathbb{F}_q)} \sum_{i=0}^{r_\alpha - 1} q^{i+\ell(w)} \ \langle w\lambda, \sigma^i(\alpha^\vee) \rangle \leq 0. \tag{3.5.2}$$

Before giving the proof, we need to recall some facts from [IK21, §3.1]. First, we have

$$G_k \setminus U_\mu = \bigcup_{\alpha \in \Delta^P} Z_\alpha, \quad Z_\alpha = \overline{E \cdot s_\alpha}$$

where $E \cdot s_\alpha$ denotes the E-orbit of s_α and the bar denotes the Zariski closure. This follows easily from Theorem 2.2.4. For any $\alpha \in \Delta^P$, define an open subset

$$X_\alpha := G_k \setminus \bigcup_{\beta \in \Delta^P,\, \beta \neq \alpha} Z_\beta.$$

Then $U_\mu \subset X_\alpha$ and $X_\alpha \setminus U_\mu$ is irreducible. Choose a realization $(u_\alpha)_{\alpha \in \Phi}$ and let $\phi_\alpha \colon \mathrm{SL}_2 \to G$ be the map attached to α (see §2.1). Set $Y := E \times \mathbb{A}^1$ and $Y_0 := E \times \mathbb{G}_\mathrm{m}$. For $\alpha \in \Delta^P$, define $\psi_\alpha \colon Y \to G$ by

$$\psi_\alpha \colon ((x,y),t) \mapsto x \phi_\alpha\left(A(t)\right) y^{-1} \quad \text{where } A(t) = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_{2,k}.$$

It satisfies $\psi_\alpha((x,y),t) \in X_\alpha$ for all $((x,y),t) \in Y$ and $\psi_\alpha((x,y),t) \in U_\mu$ if and only if $t \neq 0$ (see [IK21, Proposition 3.1.4]).

We now prove Proposition 3.5.1. We use a similar argument as in [Kos19, Theorem 3.5.3]. Set $\mathcal{U}'_\mu := \pi^{-1}(\mathcal{U}_\mu)$, where $\pi \colon G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$ is the natural projection. One has clearly $\mathcal{U}'_\mu \simeq [E' \backslash U_\mu]$ via the isomorphism $G\text{-}\mathtt{ZipFlag}^\mu \simeq [E' \backslash G]$ explained in §2.5. We have an identification

$$H^0(\mathcal{U}_\mu, \mathcal{V}_I(N_\varphi \lambda)) = H^0(\mathcal{U}'_\mu, \mathcal{V}_{\mathsf{flag}}(N_\varphi \lambda))$$

similarly to (2.5.2). In particular, we can view $\mathrm{Norm}_{L_\varphi}(f_\lambda)$ as a function $h\colon U_\mu \to \mathbb{A}^1$ satisfying the relation $h(axb^{-1}) = \lambda(a)h(x)$ for all $(a,b) \in E'$ and $x \in U_\mu$ (using (2.3.5)). Specifically, the function h is given by

$$h(x_1x_2^{-1}) = \mathrm{Norm}_{L_\varphi}(f_\lambda)(\theta_L^P(x_1)^{-1}) \tag{3.5.3}$$

for all $(x_1,x_2) \in E$ using (2.5.3). The function h is well-defined because $\mathrm{Norm}_{L_\varphi}(f_\lambda)$ is L_φ-invariant. Furthermore, $\mathrm{Norm}_{L_\varphi}(f_\lambda)$ extends to $G\text{-}\mathtt{Zip}^\mu$ if and only if h extends to a function $G \to \mathbb{A}^1$. By the strategy explained in [Kos19, §3.2] and in [IK21, §3.1], the function h extends to G if and only if for each $\alpha \in \Delta^P$, the function $h \circ \psi_\alpha\colon Y_0 \to \mathbb{A}^1$ extends to a function $Y \to \mathbb{A}^1$. It remains to compute the t-valuation of the function $h \circ \psi_\alpha$, viewed as an element of $R[t, \frac{1}{t}]$ where $R = k[E]$ is the ring of functions of E. Put

$$m_\alpha = \min\{m \geq 1 \mid \sigma^{-m}(\alpha) \notin I\}, \quad \alpha \in \Delta^P$$

and $t_\alpha = t^{-1}\alpha(\varphi(\delta_\alpha(t)))^{-1} = t\alpha(\delta_\alpha(t))^{-1} \in t^{\mathbb{Q}}$, where t is an indeterminate and $\delta_\alpha = \wp_*^{-1}(\alpha^\vee)$ as defined in §2.4. Set

$$u_{t,\alpha} = \prod_{i=1}^{m_\alpha - 1} \phi_{\sigma^{-i}(\alpha)}\left(\begin{pmatrix} 1 & -t_\alpha^{\frac{1}{q^i}} \\ 0 & 1 \end{pmatrix}\right)$$

where the product is taken in increasing order of indices. By the proof of [IK21, Proposition 3.1.4], for all $(x,y) \in E$ and $t \in \mathbb{G}_{\mathrm{m}}$, we can write $\psi_\alpha((x,y),t) = x_1x_2^{-1}$ with $(x_1,x_2) \in E$ and

$$x_1 = x\phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\right)\delta_\alpha(t)u_{t,\alpha}.$$

By definition of m_α, all the roots $\sigma^{-i}(\alpha)$ (for $1 \leq i \leq m_\alpha - 1$) appearing in the formula of $u_{t,\alpha}$ lie in I. Using (3.5.3), we deduce:

$$\begin{aligned} h \circ \psi_\alpha((x,y),t) &= \mathrm{Norm}_{L_\varphi}(f_\lambda)(u_{t,\alpha}^{-1}\delta_\alpha(t)^{-1}\theta_L^P(x)^{-1}) \\ &= \left(\prod_{s \in L_0(\mathbb{F}_q)} f_\lambda(su_{t,\alpha}^{-1}\delta_\alpha(t)^{-1}\theta_L^P(x)^{-1})\right)^{q^m}. \end{aligned}$$

Consider the element $f_\lambda(su_{t,\alpha}^{-1}\delta_\alpha(t)^{-1}\theta_L^P(x)^{-1})$, which lies in $R[t^{\mathbb{Q}}]$. We can still speak of the t-valuation of this element, which is a rational number. Equivalently, to simplify notation, we change $\theta_L^P(x)^{-1}$ to a generic element $g \in L$ and we compute the t-valuation of $F_s(t,g) := f_\lambda(su_{t,\alpha}^{-1}\delta_\alpha(t)^{-1}g)$, viewed as an element of $k[L][t^{\mathbb{Q}}]$. Let $v_\alpha(s)$ be this valuation. We put $B_L^+ = B^+ \cap L$. Define a parabolic subgroup of L by $Q_0 := L_0B_L^+$. It is clear that $u_{t,\alpha}$ lies in $R_{\mathrm{u}}(Q_0)$, thus for all $s \in L_0(\mathbb{F}_q)$, we have $su_{t,\alpha}^{-1}s^{-1} \in R_{\mathrm{u}}(Q_0)$. Since f_λ is invariant by $R_{\mathrm{u}}(B_L^+)$, we obtain $F_s(t,g) = f_\lambda(s\delta_\alpha(t)^{-1}g)$. Now, the rest of the proof is completely similar to [Kos19, Theorem 3.5.3]. We recall it briefly.

Let $B_{L_0} := B \cap L_0$ and $B_{L_0}^+ := B^+ \cap L_0$. If we change s to bs with $b \in B_{L_0}^+(\mathbb{F}_q)$, then $v_\alpha(bs) = v_\alpha(s)$. Indeed, this follows from $f_\lambda(bs\delta_\alpha(t)^{-1}g) = \lambda(b)^{-1}f_\lambda(s\delta_\alpha(t)^{-1}g)$ since f_λ is a B_L^+-eigenfunction. Similarly, we claim that $v_\alpha(sb) = v_\alpha(s)$ for all $b \in B_{L_0}^+(\mathbb{F}_q)$. By symmetry, it suffices to show $v_\alpha(sb) \geq v_\alpha(s)$. We can write $F_{sb}(t,g) = F_s(t,\Gamma(t)g)$ where

$$\Gamma(t) := \delta_\alpha(t)b\delta_\alpha(t)^{-1}.$$

We view Γ as a map $\Gamma\colon \mathrm{Spec}(k[t^{\mathbb{Q}}]) \to L$. Since $\alpha \in \Delta^P$, the cocharacter $\alpha^\vee$ is anti-L-dominant. It follows that for all $j \in \mathbb{Z}$, $\sigma^j(\alpha^\vee)$ is an anti-L_0-dominant quasi-cocharacter.

It is easy to see that δ_α is explicitly given by the formula

$$\delta_\alpha = -\frac{1}{q^{r_\alpha}-1}\sum_{j=0}^{r_\alpha-1} q^j \sigma^j(\alpha^\vee). \tag{3.5.4}$$

In particular, δ_α is L_0-dominant. We deduce that the function Γ extends to a map $\mathrm{Spec}(k[t^{\mathbb{Q}_{\geq 0}}]) \to L$. This follows simply from the fact that $\delta_\alpha(t)$ acts on the root space U_β (for $\beta \in \Phi$) by $t^{\langle \beta, \delta_\alpha\rangle}$, using (2.1.1). Write $F_s(t,g) = t^{v_\alpha(s)}F_{s,0}(t,g)$ where $F_{s,0}(t,g)$ is an element of $k[L][t^{\mathbb{Q}}]$ whose t-valuation is 0. Then $F_{sb}(t,g) = t^{v_\alpha(s)}F_{s,0}(t,\Gamma(t)g)$ and $F_{s,0}(t,\Gamma(t)g) \in k[L][t^{\mathbb{Q}_{\geq 0}}]$. Hence $v_\alpha(sb) \geq v_\alpha(s)$ as claimed.

Now, consider the Bruhat decomposition of $L_0(\mathbb{F}_q)$:

$$L_0(\mathbb{F}_q) = \bigsqcup_{w \in W_{L_0}(\mathbb{F}_q)} B^+_{L_0}(\mathbb{F}_q) w B^+_{L_0}(\mathbb{F}_q)$$

as in [Kos19, Lemma 3.4.4]. By [Kos19, Lemma 3.4.5], one has

$$|B^+_{L_0}(\mathbb{F}_q) w B^+_{L_0}(\mathbb{F}_q)| = |T(\mathbb{F}_q)| q^{\dim(R_{\mathrm{u}}(B_{L_0}))+\ell(w)}.$$

Thus, we can determine completely v_α from the values $v_\alpha(w)$ for $w \in W_{L_0}(\mathbb{F}_q)$. Similarly to [Kos19, Proposition 3.5.2], we have $v_\alpha(w) = \langle w\lambda, \delta_\alpha\rangle$. We deduce that the t-valuation of $h \circ \psi_\alpha((x,y),t)$ is

$$q^m \sum_{s \in L_0(\mathbb{F}_q)} v_\alpha(s) = q^m |T(\mathbb{F}_q)| q^{\dim(R_{\mathrm{u}}(B_{L_0}))} \sum_{w \in W_{L_0}(\mathbb{F}_q)} q^{\ell(w)} \langle w\lambda, \delta_\alpha\rangle.$$

The statement of Proposition 3.5.1 then follows by replacing δ_α by the expression in (3.5.4).

Definition 3.5.2. *We denote by $\mathcal{C}_{\mathsf{hw}} \subset X^*_{+,I}(T)$ the subset of characters λ satisfying the inequalities* (3.5.2) *and call $\mathcal{C}_{\mathsf{hw}}$ the highest weight cone.*

By construction, for all $\lambda \in \mathcal{C}_{\mathsf{hw}}$, the section $\mathbf{f}_\lambda := \mathrm{Norm}_{L_\varphi}(f_\lambda)$ is a nonzero section of $\mathcal{V}_I(N_\varphi\lambda)$ over $G\text{-}\mathtt{Zip}^\mu$. In particular, we deduce $N_\varphi\lambda \in C_{\mathsf{zip}}$ and hence $\lambda \in \mathcal{C}_{\mathsf{zip}}$. We deduce that $\mathcal{C}_{\mathsf{hw}} \subset \mathcal{C}_{\mathsf{zip}}$. If S_K is the good reduction special fiber of a Hodge-type Shimura variety and $\zeta\colon S_K \to G\text{-}\mathtt{Zip}^\mu$ is the map (2.6.1), we obtain a family of mod p automorphic forms $\zeta^*(\mathbf{f}_\lambda)_{\lambda \in \mathcal{C}_{\mathsf{hw}}}$. We also have by §2.5.3 the family $\zeta^*(\mathrm{Ha}_\chi)_{\chi \in X^*_+(T)}$. The vanishing locus of Ha_χ is a union of Ekedahl–Oort strata of codimension one. On the other hand, the vanishing locus of $\mathbf{f}_\lambda$ is highly nontrivial. It is an interesting closed subvariety stable by Hecke operators.

3.6 Partial Hasse invariant cone, Griffiths–Schmid cone

As mentioned in §3.2, we give an unambiguous definition of C_{pHa}.

Definition 3.6.1 ([Kos19, Definition 1.7.1])**.** *Define C_{pHa} as the image of $X^*_+(T)$ by*

$$h_{\mathcal{Z}}\colon X^*(T) \to X^*(T); \quad \lambda \mapsto \lambda - q w_{0,I}(\sigma^{-1}\lambda).$$

We write $\mathcal{C}_{\mathsf{pHa}}$ for the saturation of C_{pHa}. One has $\mathcal{C}_{\mathsf{pHa}} \subset X^*_{+,I}(T)$ since $-w_{0,I}\sigma^{-1}(\lambda) \in X^*_{+,I}(T)$ for $\lambda \in X^*_+(T)$. If G is split over $\mathbb{F}_q$, we have an equivalence

$$\lambda \in \mathcal{C}_{\mathsf{pHa}} \iff q w_{0,I}\lambda + \lambda \in X^*_-(T).$$

Definition 3.6.2. *Let $\mathcal{C}_{\mathsf{GS}}$ denote the set of characters $\lambda \in X^*(T)$ satisfying*

$$\begin{aligned} &\langle \lambda, \alpha^\vee \rangle \geq 0 \quad \text{for } \alpha \in I, \\ &\langle \lambda, \alpha^\vee \rangle \leq 0 \quad \text{for } \alpha \in \Phi^+ \setminus \Phi^+_L. \end{aligned}$$

One sees easily that $\lambda \in \mathcal{C}_{\mathsf{GS}}$ if and only if $-w_{0,I}\lambda$ is dominant. Clearly $\mathcal{C}_{\mathsf{GS}}$ is a saturated subcone of $X^*(T)$ and contains $X^*_-(L)$. We explain the significance of $\mathcal{C}_{\mathsf{GS}}$. Consider a Hodge-type Shimura variety $\mathrm{Sh}(\mathbf{G}, \mathbf{X})_K$ over the reflex field $\mathbf{E}$, with good reduction at the prime p, as in §2.6. Similarly to (3.3.1), we define a cone $C_K(\mathbb{C})$ by

$$C_K(\mathbb{C}) = \{\lambda \in X^*(T) \mid H^0(\mathrm{Sh}(\mathbf{G}, \mathbf{X})_K \otimes_{\mathbf{E}} \mathbb{C}, \mathcal{V}_I(\lambda)) \neq 0\}.$$

Again, the saturation of $C_K(\mathbb{C})$ is independent of K, so we denote it by $\mathcal{C}(\mathbb{C})$. Based on the results of [GS69], it is expected that $\mathcal{C}(\mathbb{C}) = \mathcal{C}_{\mathsf{GS}}$, but we could not find a reference for this conjecture. The inclusion $\mathcal{C}(\mathbb{C}) \subset \mathcal{C}_{\mathsf{GS}}$ is proved for general Hodge-type Shimura varieties in [GK22b, Theorem 2.6.4].

By reduction modulo p, one can show that $\mathcal{C}(\mathbb{C}) \subset \mathcal{C}(\overline{\mathbb{F}}_p)$ (see [Kos19, Proposition 1.8.3]). Combining the expectation $\mathcal{C}(\mathbb{C}) = \mathcal{C}_{\mathsf{GS}}$ with Conjecture 3.3.3, one should expect an inclusion $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{zip}}$ (at least for groups attached to Shimura varieties). In Theorem 6.4.3, we confirm this expectation and prove $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{zip}}$ in general (this was previously shown in [Kos19] only in the case when P is defined over $\mathbb{F}_p$). This result gives evidence for Conjecture 3.3.3.

3.7 Inclusion relations of cones

Let us briefly summarize in a diagram the cones that appear in our construction:

$$\begin{tikzcd}
& \mathcal{C}_{\mathsf{pHa}} \arrow[dr, hook] & & \\
X^*_-(L) \arrow[r, hook] \arrow[dr, hook] & \mathcal{C}_{\mathsf{hw}} \arrow[r, hook] & \mathcal{C}_{\mathsf{zip}} \arrow[r, hook] & X^*_{+,I}(T) \\
& \mathcal{C}_{\mathsf{GS}} \arrow[r, hook] \arrow[ur, hook] & \mathcal{C}_{\mathsf{lw}} \arrow[u, hook, dotted, "\text{Cond. 5.1.1}"'] &
\end{tikzcd} \tag{3.7.1}$$

All arrows of this diagram are inclusions. The lowest weight cone $\mathcal{C}_{\mathsf{lw}}$ is defined in §5.2. The inclusion $\mathcal{C}_{\mathsf{lw}} \subset \mathcal{C}_{\mathsf{zip}}$ is shown only under Condition 5.1.1 (hence the dotted arrow in the above diagram).

Lemma 3.7.1. *One has $X^*_-(L) \subset \mathcal{C}_{\mathsf{hw}}$.*

Proof. For $\lambda \in X^*_-(L)$, we have $w\lambda = \lambda$ for all $w \in W_L$. Hence $\langle w\lambda, \sigma^i\alpha^\vee \rangle \leq 0$ for all $i \in \mathbb{Z}$, $w \in W_{L_0}(\mathbb{F}_q)$ and $\alpha \in \Delta^P$. Thus $\lambda \in \mathcal{C}_{\mathsf{hw}}$. ☐

We postpone the proof of $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{zip}}$ in the general case, which is quite involved. The following was proved in [Kos19, Corollary 3.5.6]:

Lemma 3.7.2. *Assume that P is defined over $\mathbb{F}_q$. Then one has $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{hw}}$.*

This shows $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{zip}}$ in the case when P is defined over $\mathbb{F}_q$. However, the inclusion $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{hw}}$ is false in general. This happens for example in the case of Picard modular surfaces of signature $(2, 1)$ at an inert prime, where the group G is a unitary group of rank

3 over $\mathbb{F}_p$. In this example, all cones $\mathcal{C}_{\mathsf{pHa}}$, $\mathcal{C}_{\mathsf{GS}}$, $\mathcal{C}_{\mathsf{hw}}$ and $\mathcal{C}_{\mathsf{zip}}$ are distinct and there is no inclusion relation between the first three. These four cones are also distinct for $G = \mathrm{Sp}(6)$ ([Kos19, §5.5]), and more generally for $G = \mathrm{Sp}(2n)$, $n \geq 3$. In particular, in those cases the inclusion $\mathcal{C}_{\mathsf{pHa}} \subset \mathcal{C}_{\mathsf{zip}}$ is strict, hence $G\text{-}\mathtt{ZipFlag}^{\mu}$ does not satisfy the Hasse property. As a consequence, the Siegel-type Shimura variety $\mathcal{A}_n$ does not satisfy the Hasse property for $n \geq 3$.

4 Hasse-type zip data

4.1 Topology of $C_{\mathsf{zip},\mathbb{R}_{\geq 0}}$

Let (G, μ) be a cocharacter datum. We showed $X^*_-(L) \subset \mathcal{C}_{\mathsf{zip}}$ in Proposition 3.4.1. For $X^*_-(L)_{\mathrm{reg}}$ (see (3.4.1)), we have a more precise result ([KW18, Theorem 5.1.4]):

Theorem 4.1.1. *For all $\lambda \in X^*_-(L)_{\mathrm{reg}}$, there is a section $h \in H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}_I(N_\varphi \lambda))$ whose non-vanishing locus is exactly $\mathcal{U}_\mu$.*

Here $N_\varphi \geq 1$ is the integer defined in §3.5. Since $\lambda \in X^*(L)$, the vector bundle $\mathcal{V}_I(\lambda)$ is a line bundle, and thus $\mathcal{V}_I(N_\varphi \lambda) = \mathcal{V}_I(\lambda)^{\otimes N_\varphi}$. A subset of an $\mathbb{R}$-vector space stable under linear combination with coefficients in $\mathbb{R}_{\geq 0}$ will be called an $\mathbb{R}_{\geq 0}$-subcone. We endow $X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$ with the subspace topology of $X^*(T)_{\mathbb{R}}$.

Lemma 4.1.2. *Let $C \subset X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$ be an $\mathbb{R}_{\geq 0}$-subcone and let $\lambda \in C$. Then C is a neighborhood of λ in $X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$ if and only if for all $\lambda' \in X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$, there exists $r \in \mathbb{R}_{>0}$ such that $\lambda' + r\lambda \in C$.*

Proof. First, assume that C is a neighborhood of λ in $X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$. There is an open subset V of $X^*(T)_{\mathbb{R}}$ such that $\lambda \in V \cap X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}} \subset C$. Fix $\lambda' \in X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$. For large $r \in \mathbb{R}_{>0}$, we have $\lambda + \frac{\lambda'}{r} \in V$, and this element is also in $X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$. Thus $\lambda' + r\lambda \in C$.

We prove the converse. We claim that for all $\lambda' \in X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$, there exists $r > 1$ such that $\lambda + \frac{\lambda' - \lambda}{r} \in C$. Indeed, let $r \in \mathbb{R}_{>0}$ such that $\lambda + \frac{\lambda'}{r} \in C$. Then for all $\gamma > 0$, we have $\gamma\lambda + \frac{\gamma\lambda'}{r} = \lambda + \frac{\gamma(\lambda' - \lambda)}{r} + (\gamma - 1 + \frac{\gamma}{r})\lambda \in C$. For $\gamma = \frac{r}{r+1}$, we have $\gamma - 1 + \frac{\gamma}{r} = 0$ hence $\lambda + \frac{\lambda' - \lambda}{r+1} \in C$. Hence, by taking λ as the origin, we are reduced to the following:

> Let $X \subset \mathbb{R}^n$ be an intersection of closed half-spaces containing 0, and $0 \in Y \subset X$ a convex subset satisfying: for all $x \in X$, $\exists r \in \mathbb{R}_{>0}$, $\frac{x}{r} \in Y$. Then Y is a neighborhood of 0 in X.

Taking intersections with a neighborhood of 0 in $\mathbb{R}^n$ which is a convex polytope, we may assume that X is a convex polytope. Since X is the convex hull of finitely many points, there exists $r > 1$ such that $\frac{1}{r}X = \{\frac{x}{r} \mid x \in X\} \subset Y$. Hence, it suffices to show that $\frac{1}{r}X$ is a neighborhood of 0 in X. There are linear forms $u_1, \ldots, u_d$ on $\mathbb{R}^n$ and $m_1, \ldots, m_d \in \mathbb{R}_{\geq 0}$ such that $x \in X$ if and only if $u_i(x) \leq m_i$ for all $i = 1, \ldots, d$. Hence $u = (u_1, \ldots, u_d)$ maps X to $Z = \prod_{i=1}^d]-\infty, m_i]$. For $r > 1$, $\frac{1}{r}Z$ is clearly a neighborhood of 0 in Z, hence $\frac{1}{r}X = u^{-1}(\frac{1}{r}Z)$ is a neighborhood of 0 in X. $\square$

The following Lemma was proved in a slightly restricted setting in [Kos19, Proposition 2.2.1], so we restate it below.

Lemma 4.1.3. *The cone $C_{\mathsf{zip},\mathbb{R}_{\geq 0}}$ is a neighborhood of $X^*_-(L)_{\mathrm{reg}}$ in $X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$.*

Proof. For $\lambda \in X_-^*(L)_{\mathrm{reg}}$, we show that $C_{\mathsf{zip},\mathbb{R}_{\geq 0}}$ is a neighborhood of λ in $X_{+,I}^*(T)_{\mathbb{R}_{\geq 0}}$. By Lemma 4.1.2, it suffices to show that for all $\lambda' \in X_{+,I}^*(T)_{\mathbb{R}_{\geq 0}}$, there is $r \in \mathbb{R}_{>0}$ such that $\lambda' + r\lambda \in C_{\mathsf{zip},\mathbb{R}_{\geq 0}}$. We may assume $\lambda' \in X_{+,I}^*(T)$ by scaling. Let $h \in H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(N_\varphi \lambda))$ be the section provided by Theorem 4.1.1. By Lemma 2.4.2, $H^0(\mathcal{U}_\mu, \mathcal{V}_I(N_\varphi \lambda'))$ is nonzero; let h' be a nonzero element therein. This section may have poles on the complement of $\mathcal{U}_\mu$. However, since h vanishes on the complement of $\mathcal{U}_\mu$, there exists $d \geq 1$ such that $h^d h'$ has no poles. Hence $h^d h' \in H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(N_\varphi \lambda' + dN_\varphi \lambda))$, and thus $N_\varphi(\lambda' + d\lambda) \in C_{\mathsf{zip}}$, hence $\lambda' + d\lambda \in \mathcal{C}_{\mathsf{zip}}$. The result follows. □

Lemma 4.1.4. *$\mathcal{C}_{\mathsf{GS},\mathbb{R}_{\geq 0}}$ and $\mathcal{C}_{\mathsf{hw},\mathbb{R}_{\geq 0}}$ are neighborhoods of $X_-^*(L)_{\mathrm{reg}}$ in $X_{+,I}^*(T)_{\mathbb{R}_{\geq 0}}$.*

Proof. The open subset of $X_{+,I}^*(T)_{\mathbb{R}_{\geq 0}}$ defined by the equations $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Phi^+ \setminus \Phi_L^+$ is contained in $\mathcal{C}_{\mathsf{GS},\mathbb{R}_{\geq 0}}$ and contains $X_-^*(L)_{\mathrm{reg}}$, which proves the first part of the assertion. Replacing $\leq$ by $<$ in the inequalities (3.5.2), we get an open subset of $X_{+,I}^*(T)_{\mathbb{R}_{\geq 0}}$ containing $X_-^*(L)_{\mathrm{reg}}$ (same proof as Lemma 3.7.1), which proves the second part. □

We may ask whether $C_{\mathsf{pHa},\mathbb{R}_{\geq 0}}$ is also a neighborhood of $X_-^*(L)_{\mathrm{reg}}$. The proof of the following result is similar to [Kos19, Lemma 2.3.1], where the cocharacter datum (G, μ) was assumed to be of Hodge-type, but this assumption is superfluous. We reproduce partly the proof to explain the appropriate changes (we replace the character η_ω in [Kos19, Lemma 2.3.1] by the set $X_-^*(L)_{\mathrm{reg}}$). The following holds for an arbitrary cocharacter datum (G, μ):

Proposition 4.1.5. *The following are equivalent:*
(i) The cone $C_{\mathsf{pHa},\mathbb{R}_{\geq 0}}$ is a neighborhood of $X_-^*(L)_{\mathrm{reg}}$ in $X_{+,I}^*(T)_{\mathbb{R}_{\geq 0}}$.
(ii) One has $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{pHa}}$.
(iii) P is defined over $\mathbb{F}_q$ and the Frobenius σ acts on I by $\sigma(\alpha) = -w_{0,I}\alpha$ for all $\alpha \in I$.

Proof. Since $\mathcal{C}_{\mathsf{GS},\mathbb{R}_{\geq 0}}$ is a neighborhood of $X_-^*(L)_{\mathrm{reg}}$ in $X_{+,I}^*(T)_{\mathbb{R}_{\geq 0}}$, we have (ii) $\Rightarrow$ (i). Assume that (i) holds. In particular, $X_-^*(L)_{\mathrm{reg}} \subset \mathcal{C}_{\mathsf{pHa}}$, hence $h_{\mathcal{Z}}^{-1}(X_-^*(L)_{\mathrm{reg}}) \subset X_+^*(T)_{\mathbb{R}_{\geq 0}}$. Let $\lambda \in X_-^*(L)_{\mathrm{reg}}$ and write $\lambda = h_{\mathcal{Z}}(\chi)$ for $\chi \in X_+^*(T)_{\mathbb{R}_{\geq 0}}$. Hence for all $\alpha \in I$, we have $\langle h_{\mathcal{Z}}(\chi), \alpha^\vee \rangle = 0$, which amounts to $\langle \chi, \alpha^\vee \rangle = q \langle \chi, \sigma(w_{0,I}\alpha^\vee) \rangle$. Since $\alpha \in I$, $w_{0,I}\alpha$ is a negative root, and so is $\sigma(w_{0,I}\alpha)$. We deduce that $\langle \chi, \alpha^\vee \rangle = \langle \chi, \sigma(w_{0,I}\alpha^\vee) \rangle = 0$ (in particular $\chi \in X^*(L)$). Since $X_-^*(L)_{\mathrm{reg}}$ generates $X^*(L)$, this shows that $h_{\mathcal{Z}}^{-1}$ maps $X^*(L)_{\mathbb{R}}$ to itself, and all elements in the image satisfy $\langle \chi, \sigma(w_{0,I}\alpha^\vee) \rangle = 0$ for all $\alpha \in I$. For dimension reasons, $h_{\mathcal{Z}}^{-1}(X^*(L)_{\mathbb{R}}) = X^*(L)_{\mathbb{R}}$, hence any character $\chi \in X^*(L)$ is orthogonal to $\sigma(\alpha^\vee)$ for all $\alpha \in I$. Hence we must have $\sigma(I) = I$, thus P is defined over $\mathbb{F}_q$. Next, for $\alpha \in I$, let $\lambda_\alpha \in X_{+,I}^*(T)$ such that $\langle \lambda_\alpha, \beta^\vee \rangle = 0$ for all $\beta \in \Delta \setminus \{\alpha\}$ and $\langle \lambda_\alpha, \alpha^\vee \rangle > 0$. Let $\lambda \in X_-^*(L)_{\mathrm{reg}}$. There exist $r \in \mathbb{R}_{>0}$ and $\chi_\alpha \in X_+^*(T)_{\mathbb{R}_{\geq 0}}$ such that $h_{\mathcal{Z}}(\chi_\alpha) = r\lambda + \lambda_\alpha$. As before, we deduce $\langle \chi_\alpha, \beta^\vee \rangle = \langle \chi_\alpha, \sigma(w_{0,I}\beta^\vee) \rangle = 0$ for all $\beta \in I \setminus \{\alpha\}$. The character χ_α cannot be orthogonal to all $\beta^\vee$ for $\beta \in I$, hence $\langle \chi_\alpha, \alpha^\vee \rangle \neq 0$. Furthermore, since the map $I \to I$, $\beta \mapsto -\sigma(w_{0,I}\beta)$ is a bijection, we must have $-\sigma(w_{0,I}\alpha) = \alpha$. This shows (i) $\Rightarrow$ (iii). Finally, the implication (iii) $\Rightarrow$ (ii) is completely similar to (3) $\Rightarrow$ (4) in the proof of [Kos19, Lemma 2.3.1] (after changing p to q). □

Definition 4.1.6. *We say that a cocharacter datum (G, μ) is of Hasse-type if the equivalent conditions of Proposition 4.1.5 are satisfied.*

The main result of this section is that (i), (ii), (iii) above are also equivalent to the equality $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}_{\mathsf{zip}}$. For the time being, the following is an immediate consequence of Lemma 4.1.3:

Corollary 4.1.7. *Assume that $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}_{\mathsf{zip}}$ holds. Then (G, μ) is of Hasse-type.*

Recall that $\mathcal{C}_{\mathsf{pHa}} = \mathcal{C}_{\mathsf{zip}}$ means by definition that $G\text{-}\mathtt{ZipFlag}^{\mu}$ satisfies the Hasse property (Definition 3.2.1). This shows that Theorem 3.1.1(2) can only potentially generalize to Hodge-type Shimura varieties S_K such that the associated zip datum (G,μ) is of Hasse-type. Indeed, if the flag space of S_K satisfies the Hasse property, then so does $G\text{-}\mathtt{ZipFlag}^{\mu}$, and hence (G,μ) must be of Hasse-type by Corollary 4.1.7. In Theorem 3.3.4, all three cases (1), (2) and (3) are of Hasse-type.

4.2 Maximal flag stratum

We prove some technical results used in the proof of Theorem 4.3.1. Let (G,μ) be an arbitrary cocharacter datum, and let (B,T,z) be a frame with $z=\sigma(w_{0,I})w_0$ (Remark 2.2.2). Recall that $H^0(G\text{-}\mathtt{Zip}^{\mu},\mathcal{V}_I(\lambda))$ identifies with $H^0(G\text{-}\mathtt{ZipFlag}^{\mu},\mathcal{V}_{\mathsf{flag}}(\lambda))$ by (2.5.2). Via the isomorphism $G\text{-}\mathtt{ZipFlag}^{\mu}\simeq [E'\backslash G]$ (see §2.5) and (2.3.5), an element of the space $H^0(G\text{-}\mathtt{ZipFlag}^{\mu},\mathcal{V}_{\mathsf{flag}}(\lambda))$ can be viewed as a function $f\colon G\to\mathbb{A}^1$ satisfying

$$f(agb^{-1})=\lambda(a)f(g),\quad \forall (a,b)\in E',\ \forall g\in G. \tag{4.2.1}$$

Recall that $G\text{-}\mathtt{ZipFlag}^{\mu}$ admits a stratification $(\mathcal{F}_w)_{w\in W}$ (§2.5) where $\mathcal{F}_w:=[E'\backslash F_w]$ and $F_w=BwBz^{-1}$ is the $B\times {}^zB$-orbit of wz^{-1}. The unique open stratum is $\mathcal{U}_{\max}=\mathcal{F}_{w_0}$. Write also $U_{\max}:=F_{w_0}=Bw_0Bz^{-1}$ (the $B\times {}^zB$-orbit of $w_0z^{-1}=\sigma(w_{0,I})^{-1}$). The codimension one $B\times {}^zB$-orbits are the $F_{s_\alpha w_0}$ for $\alpha\in\Delta$. Define $\mathcal{U}'_\mu:=\pi^{-1}(\mathcal{U}_\mu)\simeq[E'\backslash U_\mu]$.

Lemma 4.2.1.
(1) *The stabilizer of $\sigma(w_{0,I})^{-1}$ in $B\times {}^zB$ is $S:=\{(t,\sigma(w_{0,I})t\sigma(w_{0,I})^{-1})\mid t\in T\}$.*
(2) *The map $B_M\to U_{\max}$, $b\mapsto \sigma(w_{0,I})b^{-1}$ induces an isomorphism $[B_M/T]\simeq\mathcal{U}_{\max}$, where T acts on B_M on the right by the action $B_M\times T\to B_M$, $(b,t)\mapsto \varphi(t)^{-1}b\sigma(w_{0,I})t\sigma(w_{0,I})^{-1}$.*
(3) *Assume that P is defined over $\mathbb{F}_q$. Then $U_{\max}\subset U_\mu$, and $\mathcal{U}_{\max}\subset\mathcal{U}'_\mu$.*

Proof. We prove (1). Let $(x,y)\in B\times {}^zB$ such that $x\sigma(w_{0,I})^{-1}y^{-1}=\sigma(w_{0,I})^{-1}$. Write $y=zy'z^{-1}$ with $y'\in B$. Since $z=\sigma(w_{0,I})w_0$, we obtain $xw_0y'^{-1}w_0^{-1}\sigma(w_{0,I})^{-1}=\sigma(w_{0,I})^{-1}$, hence $x=w_0y'w_0^{-1}$. It follows that $x\in B\cap w_0Bw_0^{-1}=T$. We can write $y=\sigma(w_{0,I})x\sigma(w_{0,I})^{-1}$, which proves (1). To show (2), note that the map $B\times {}^zB\to U_{\max}$; $(x,y)\mapsto x\sigma(w_{0,I})y^{-1}$ induces an isomorphism $(B\times {}^zB)/S\to U_{\max}$, where S is as in (1). Hence $\mathcal{U}_{\max}$ is isomorphic to $[E'\backslash B\times {}^zB/S]$. We have an isomorphism

$$E'\backslash(B\times {}^zB)\to B_M,\quad E'\cdot(x,y)\mapsto \varphi(\theta^P_L(x))^{-1}\theta^Q_M(y) \tag{4.2.2}$$

whose inverse is $B_M\to E'\backslash B\times {}^zB$; $b\mapsto E'\cdot(1,b)$. Identify T and S via the isomorphism $T\to S$; $t\mapsto (t,\sigma(w_{0,I})t\sigma(w_{0,I})^{-1})$. The action of S on $E'\backslash B\times {}^zB$ by multiplication on the right transforms via the isomorphism (4.2.2) to the right action of T defined by $B_M\times T\to B_M$; $(b,t)\mapsto\varphi(t)^{-1}b\sigma(w_{0,I})t\sigma(w_{0,I})^{-1}$. This proves (2). Finally, we show (3). Assume that P is defined over $\mathbb{F}_q$. Then U_μ coincides with the unique open $P\times Q$-orbit by [Wed14, Corollary 2.15]. Since $B\times {}^zB\subset P\times Q$, the set U_μ is a union of $B\times {}^zB$-orbits, hence contains $U_{\max}$. Since $\mathcal{U}'_\mu=[E'\backslash U_\mu]$, we have $\mathcal{U}_{\max}\subset\mathcal{U}'_\mu$. □

For $\lambda\in X^*(T)$, let $S(\lambda)$ denote the space of functions $h\colon B_M\to\mathbb{A}^1$ satisfying

$$h(\varphi(t)^{-1}b\sigma(w_{0,I})t\sigma(w_{0,I})^{-1})=\lambda(t)^{-1}h(b),\quad \forall t\in T,\ \forall b\in B_M.$$

Corollary 4.2.2. *The isomorphism from Lemma 4.2.1(2) induces an isomorphism*

$$\vartheta\colon H^0(\mathcal{U}_{\max},\mathcal{V}_{\mathsf{flag}}(\lambda))\to S(\lambda).$$

We describe explicitly this isomorphism. Let $f \in H^0(\mathcal{U}_{\max}, \mathcal{V}_{\mathsf{flag}}(\lambda))$, viewed as a function $f\colon U_{\max} \to \mathbb{A}^1$ satisfying (4.2.1). The corresponding element $\vartheta(f) \in S(\lambda)$ is the function $B_M \to \mathbb{A}^1$; $b \mapsto f(\sigma(w_{0,I})b^{-1})$. Conversely, if $h\colon B_M \to \mathbb{A}^1$ is an element of $S(\lambda)$, the function $f = \vartheta^{-1}(h)$ is given by

$$f(b_1\sigma(w_{0,I})b_2^{-1}) = \lambda(b_1)h(\varphi(\theta_L^P(b_1))^{-1}\theta_M^Q(b_2)), \quad (b_1, b_2) \in B \times {}^zB. \tag{4.2.3}$$

By the property of h, the function f is well-defined.

In particular, for a section of $\mathcal{V}_{\mathsf{flag}}(\lambda)$ over $G\text{-}\mathtt{ZipFlag}^\mu$, we can restrict it to the open substack $\mathcal{U}_{\max}$, and then apply ϑ to obtain an element of $S(\lambda)$. Assume now that P is defined over $\mathbb{F}_q$. In particular, we have $\sigma(w_{0,I}) = w_{0,I}$ and $z = w_{0,I}w_0$. We also have $\mathcal{U}_{\max} \subset \mathcal{U}'_\mu$ (*cf.* Lemma 4.2.1(3)) and inclusions

$$H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{V}_{\mathsf{flag}}(\lambda)) \subset H^0(\mathcal{U}'_\mu, \mathcal{V}_{\mathsf{flag}}(\lambda)) \subset H^0(\mathcal{U}_{\max}, \mathcal{V}_{\mathsf{flag}}(\lambda)).$$

Write $S_{\mathsf{flag}}(\lambda) \subset S_\mu(\lambda) \subset S(\lambda)$ respectively for the images under ϑ of these three spaces. Choose a realization $(u_\alpha)_{\alpha\in\Phi}$ (see §2.1). For $\alpha \in \Delta$, define a map $\Gamma_\alpha\colon B_L \times \mathbb{A}^1 \to G$ by

$$\Gamma_\alpha\colon (b,t) \mapsto b\phi_\alpha(A(t))w_{0,I}, \quad \text{where } A(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2$$

and $\phi_\alpha\colon \mathrm{SL}_2 \to G$ is the map attached to α. For $\alpha \in \Delta$, define an open subset

$$G_\alpha := G \setminus \bigcup_{\substack{\beta\in\Delta \\ \beta\neq\alpha}} \overline{F}_{s_\beta w_0} = U_{\max} \cup F_{s_\alpha w_0}.$$

Since U_μ coincides with the open $P \times Q$-orbit, one sees that $G_\alpha \subset U_\mu$ if and only if $\alpha \in I$. In this setting, one has an analogue of [IK21, Proposition 3.1.4]:

Proposition 4.2.3. *The following properties hold:*
(1) *The image of Γ_α is contained in G_α.*
(2) *For all $b \in B_L$ and $t \in \mathbb{A}^1$, one has $\Gamma_\alpha(b,t) \in U_{\max} \Longleftrightarrow t \neq 0$.*

Proof. We have $U_{\max} = Bw_0Bz^{-1} = BB^+w_{0,I}$. As in [IK21, (3.1.3)], one has a decomposition

$$A(t) = \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}.$$

Thus for $t \neq 0$, we have $\phi_\alpha(A(t)) \in BB^+$, hence $\Gamma(b,t) \in U_{\max}$. For $t = 0$, we have $\phi_\alpha(A(0)) = s_\alpha$ and $\Gamma_\alpha(b,t) \in Bs_\alpha w_{0,I} \subset Bs_\alpha w_{0,I}{}^zB = F_{s_\alpha w_0}$. This shows (1) and (2). $\square$

Let $f \in H^0(\mathcal{U}_{\max}, \mathcal{V}_{\mathsf{flag}}(\lambda))$, viewed as a function $f\colon U_{\max} \to \mathbb{A}^1$ satisfying (4.2.1). Let $h := \vartheta(f)$ be the corresponding element of $S(\lambda)$. Using (4.2.3), we have for $\alpha \in \Delta^P$ and $(b,t) \in B_L \times \mathbb{G}_{\mathrm{m}}$:

$$\begin{aligned} f \circ \Gamma_\alpha(b,t) &= f\left(b\phi_\alpha\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} w_{0,I}\left(w_{0,I}\phi_\alpha\begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix} w_{0,I}\right)\right) \\ &= \lambda(b)\ h\left(\varphi(b)^{-1}w_{0,I}\alpha^\vee(t)^{-1}w_{0,I}\right). \end{aligned}$$

Similarly, for $\alpha \in I$ and $(b,t) \in B_L \times \mathbb{G}_{\mathrm{m}}$, one can show the following (we leave out the computation, since we will only need the case $\alpha \in \Delta^P$ in §4.3):

$$f \circ \Gamma_\alpha(b,t) = \lambda(b)\ h\left(\phi_{\sigma(\alpha)}\begin{pmatrix} 1 & 0 \\ t^{-q} & 1 \end{pmatrix}\varphi(b)^{-1}\phi_{-w_{0,I}\alpha}\begin{pmatrix} t & 0 \\ -1 & t^{-1} \end{pmatrix}\right).$$

For $\alpha \in \Delta$, define a function $F_{h,\alpha}\colon B_L \times \mathbb{G}_\mathrm{m} \to \mathbb{A}^1$ by

$$F_{h,\alpha}(b,t) := h\left(\phi_{\sigma(\alpha)}\begin{pmatrix} 1 & 0 \\ t^{-q} & 1\end{pmatrix} b\phi_{-w_{0,I}\alpha}\begin{pmatrix} t & 0 \\ -1 & t^{-1}\end{pmatrix}\right) \qquad \text{if } \alpha \in I,$$
$$F_{h,\alpha}(b,t) := h\left(bw_{0,I}\alpha^\vee(t)^{-1}w_{0,I}\right) \qquad \text{if } \alpha \in \Delta^P.$$

The function $F_{h,\alpha}(b,t)$ lies in $k[B_L][t,\frac{1}{t}]$, where $k[B_L]$ denotes the ring of functions of B_L. Moreover, $F_{h,\alpha}(b,t) \in k[B_L][t]$ if and only if $f \circ \Gamma_\alpha(b,t)$ extends to a map $B_L \times \mathbb{A}^1 \to G$.

Proposition 4.2.4. *Let $h \in S(\lambda)$.*
(1) $h \in S_{\mathsf{flag}}(\lambda)$ if and only if $F_{h,\alpha} \in k[B_L][t]$ for all $\alpha \in \Delta$.
(2) $h \in S_\mu(\lambda)$ if and only if $F_{h,\alpha} \in k[B_L][t]$ for all $\alpha \in I$.

Proof. Let $f = \vartheta^{-1}(h) \in H^0(\mathcal{U}_{\max}, \mathcal{V}_{\mathsf{flag}}(\lambda))$. In the terminology of [Kos19, Definition 3.2.1], the map Γ_α is adapted to f by [Kos19, Lemma 3.2.4], because f is an eigenfunction for the action of E' and we have $E' \cdot \Gamma_\alpha(B_L \times \{0\}) = F_{s_\alpha w_0}$ using $B \times {}^zB = E'(B_L \times \{1\})$. By [Kos19, Lemma 3.2.2], f extends to G if and only if $f \circ \Gamma_\alpha$ extends to $B_L \times \mathbb{A}^1$ for all $\alpha \in \Delta$, which shows (1). Assertion (2) is proved similarly. $\square$

4.3 Main result

We state the main result of this section, which is the reciprocal of Corollary 4.1.7.

Theorem 4.3.1. *Let (G,μ) be a cocharacter datum of Hasse-type. Then G-$\texttt{ZipFlag}^\mu$ satisfies the Hasse property. Combining with Corollary 4.1.7, we have:*

$$(G,\mu) \textit{ is of Hasse-type} \iff \mathcal{C}_{\mathsf{zip}} = \mathcal{C}_{\mathsf{pHa}}.$$

We prove Theorem 4.3.1 in the rest of this section. Fix a cocharacter datum (G,μ), with zip datum $\mathcal{Z} = \mathcal{Z}_\mu = (G,P,L,Q,M,\varphi)$. For now, we only assume that P is defined over $\mathbb{F}_q$ (hence $L = M$). Fix also a frame (B,T,z) with $z = w_{0,I}w_0$.

Proposition 4.3.2 ([ABD[+]66, XXII, Proposition 5.5.1])**.** *Let G be a reductive group over k and let (B,T) be a Borel pair. Choose a total order on Φ^-. The k-morphism*

$$\gamma\colon T \times \prod_{\alpha\in\Phi^-} U_\alpha \to G \tag{4.3.1}$$

defined by taking the product with respect to the chosen order is a closed immersion with image B.

We apply this proposition to (L, B_L). Choose an order on Φ_L^- and consider the corresponding map γ as in (4.3.1), with image B_L. For a function $h\colon B_L \to \mathbb{A}^1$, put $P_h := h\circ\gamma$. Via the isomorphism $u_\alpha\colon \mathbb{G}_a \to U_\alpha$, we can view P_h as a polynomial $P_h \in k[T][(x_\alpha)_{\alpha\in\Phi_L^-}]$, where the x_α are indeterminates indexed by Φ_L^-. For $m = (m_\alpha)_\alpha \in \mathbb{N}^{\Phi_L^-}$ and $\lambda \in X^*(T)$, denote by $P_{m,\lambda}$ the monomial

$$P_{m,\lambda} = \lambda(t)\prod_{\alpha\in\Phi_L^-} x_\alpha^{m_\alpha} \;\in\; k[T][(x_\alpha)_{\alpha\in\Phi_L^-}].$$

We can write any element P of $k[T][(x_\alpha)_{\alpha\in\Phi_L^-}]$ as a sum of monomials

$$P = \sum_{i=1}^N c_i P_{m_i,\lambda_i} \tag{4.3.2}$$

where for all $1 \leq i \leq N$, we have $m_i \in \mathbb{N}^{\Phi_L^-}$, $\lambda_i \in X^*(T)$ and $c_i \in k$. Furthermore, we may assume that the (m_i, λ_i) are pairwise distinct. Under this assumption, the expression (4.3.2) is uniquely determined (up to permutation of the indices). For $P \in k[T][(x_\alpha)_{\alpha \in \Phi_L^-}]$, write $h_P \colon B_L \to \mathbb{A}^1$ for the function $P \circ \gamma^{-1}$. For $m = (m_\alpha)_\alpha \in \mathbb{N}^{\Phi_L^-}$ and $\lambda \in X^*(T)$, write $h_{m,\lambda} := h_{P_{m,\lambda}}$.

Lemma 4.3.3. *Let $(m, \lambda) \in \mathbb{N}^{\Phi_L^-} \times X^*(T)$. For all $a \in T$ and $b \in B_L$, we have*

$$\begin{aligned} h_{m,\lambda}(ab) &= \lambda(a) h_{m,\lambda}(b), \quad \textit{and} \\ h_{m,\lambda}(ba) &= \Big(\lambda(a) \prod_{\alpha \in \Phi_L^-} \alpha(a)^{-m_\alpha}\Big) h_{m,\lambda}(b). \end{aligned}$$

Proof. The first formula is an immediate computation. For the second, let $b = \gamma(t, (u_\alpha(x_\alpha))_\alpha)$ with $t \in T$ and $(x_\alpha)_\alpha \in \mathbb{G}_a^{\Phi_L^-}$. Then

$$\begin{aligned} h_{m,\lambda}(ba) &= h_{m,\lambda}\Big(ta \prod_{\alpha \in \Phi_L^-} a^{-1} u_\alpha(x_\alpha) a\Big) = h_{m,\lambda}\Big(ta \prod_{\alpha \in \Phi_L^-} u_\alpha(\alpha(a)^{-1} x_\alpha)\Big) \\ &= \lambda(ta) \prod_{\alpha \in \Phi_L^-} (\alpha(a)^{-1} x_\alpha)^{m_\alpha} = \Big(\lambda(a) \prod_{\alpha \in \Phi_L^-} \alpha(a)^{-m_\alpha}\Big) h_{m,\lambda}(b), \end{aligned}$$

where we used the formula $a^{-1} u_\alpha(x) a = u_\alpha(\alpha(a)^{-1} x)$ for all $x \in \mathbb{A}^1$ and all $a \in T$. □

For $(m, \lambda) \in \mathbb{N}^{\Phi_L^-} \times X^*(T)$ as above, define the weight $\omega(m, \lambda)$ as

$$\omega(m, \lambda) := q\sigma^{-1}(\lambda) - w_{0,I}\lambda + \sum_{\beta \in \Phi_L^-} m_\alpha(w_{0,I}\beta) \in X^*(T). \tag{4.3.3}$$

It follows immediately from Lemma 4.3.3 that $h_{m,\lambda} \colon B_L \to \mathbb{A}^1$ lies in $S(\omega(m, \lambda))$.

Lemma 4.3.4. *Let $\lambda \in X^*(T)$ and $h \in k[B_L]$ be nonzero. Write $P_h = \sum_{i=1}^N c_i P_{m_i,\lambda_i}$ as in (4.3.2), with (m_i, λ_i) pairwise disjoint and $c_i \neq 0$ for all $1 \leq i \leq N$. Then we have*

$$h \in S(\lambda) \Longleftrightarrow \omega(m_i, \lambda_i) = \lambda \textit{ for all } i = 1, \ldots, N.$$

Proof. The implication "$\Longleftarrow$" is obvious. Conversely, if $h \in S(\lambda)$, then for all $t \in T$, $b \in B$, we have $\lambda(t)h(b) = h(\varphi(t) b w_{0,I} t^{-1} w_{0,I}) = \sum_{i=1}^N \omega(m_i, \lambda_i)(t) c_i h_{m_i,\lambda_i}(b)$. The result follows by linear independence of characters. □

For $m \in \mathbb{N}^{\Phi_L^-}$, $\lambda \in X^*(T)$ and $\alpha \in \Delta^P$, we write $F_{m,\lambda,\alpha} := F_{h_{m,\lambda},\alpha}$ (see §4.2). For all $\alpha \in \Delta^P$, and all $(b, t) \in B_L \times \mathbb{G}_m$, we find:

$$F_{m,\lambda,\alpha}(b, t) = t^{-q\langle \lambda, \sigma\alpha^\vee \rangle + \langle \omega(m,\lambda), \alpha^\vee \rangle} h_{m,\lambda}(b). \tag{4.3.4}$$

In particular, $F_{m,\lambda,\alpha}$ is in $k[B_L][t]$ if and only if $-q\langle \lambda, \sigma\alpha^\vee \rangle + \langle \omega(m, \lambda), \alpha^\vee \rangle \geq 0$. Using (4.3.3), this inequality can also be written as

$$\langle w_{0,I}\lambda, \alpha^\vee \rangle \leq \sum_{\beta \in \Phi_L^-} m_\beta \langle w_{0,I}\beta, \alpha^\vee \rangle. \tag{4.3.5}$$

Corollary 4.3.5. *Let $\lambda \in X^*(T)$ and $h \in S(\lambda)$ be nonzero. Write $P_h = \sum_{i=1}^N c_i P_{m_i,\lambda_i}$ as in (4.3.2), with the (m_i, λ_i) pairwise distinct and $c_i \neq 0$ for $1 \leq i \leq N$. Let $\alpha \in \Delta^P$.*

(1) *We have*

$$\begin{aligned} F_{h,\alpha} \in k[B_L][t] &\iff \forall i = 1, \dots, N,\ F_{m_i,\lambda_i,\alpha} \in k[B_L][t]. \\ &\iff \forall i = 1, \dots, N,\ -q\langle \lambda_i, \sigma\alpha^\vee \rangle + \langle \lambda, \alpha^\vee \rangle \geq 0. \end{aligned}$$

(2) *Moreover, if $F_{h,\alpha} \in k[B_L][t]$ then $\langle w_{0,I}\lambda_i, \alpha^\vee \rangle \leq 0$ for all $1 \leq i \leq N$.*

Proof. By (4.3.4), we have $F_{m_i,\lambda_i,\alpha}(b,t) = t^{d_i} h_{m_i,\lambda_i}(b)$ for some integer $d_i \in \mathbb{Z}$. Hence, the first equivalence of (1) follows from the assumption that (m_i, λ_i) for $1 \leq i \leq N$ are pairwise distinct. The second equivalence follows from the previous discussion, using $\omega(m_i, \lambda_i) = \lambda$ (Lemma 4.3.4). Assertion (2) follows from the inequality (4.3.5) and the fact that $\langle w_{0,I}\beta, \alpha^\vee \rangle \leq 0$ for all $\beta \in \Phi_L^-$. Indeed, recall that $\langle \beta, \alpha^\vee \rangle \leq 0$ for any two distinct simple roots $\alpha, \beta \in \Delta$. Since $\beta \in \Phi_L^-$, we have $w_{0,I}\beta \in \Phi_L^+$, hence $w_{0,I}\beta$ is a sum of simple roots in I. Since $\alpha \in \Delta^P$, the result follows. □

We now study the partial Hasse invariant cone C_{pHa} (Definition 3.6.1). Fix a positive integer n such that G is split over $\mathbb{F}_{q^n}$. By inverting the map $h_{\mathcal{Z}} \colon \lambda \mapsto \lambda - qw_{0,I}(\sigma^{-1}\lambda)$, we can write $\mathcal{C}_{\mathsf{pHa}}$ as the set of $\lambda \in X^*(T)$ such that

$$\sum_{i=0}^{2n-1} q^i \langle (w_{0,I})^i \sigma^{-i} \lambda, \alpha^\vee \rangle \leq 0, \quad \forall \alpha \in \Delta.$$

For $\alpha \in \Delta$ and $\lambda \in X^*(T)$, define $K_\alpha(\lambda) := \sum_{i=0}^{2n-1} q^i \langle (w_{0,I})^i \sigma^{-i} \lambda, \alpha^\vee \rangle$.

Lemma 4.3.6. *Assume that (G, μ) is of Hasse-type. For all $\lambda \in X^*_{+,I}(T)$ and $\alpha \in I$, we have $K_\alpha(\lambda) \leq 0$. In particular, we have*

$$\mathcal{C}_{\mathsf{pHa}} = \{\lambda \in X^*_{+,I}(T) |\ \forall \alpha \in \Delta^P,\ K_\alpha(\lambda) \leq 0\}.$$

Proof. For all $\alpha \in I$, we have

$$\begin{aligned} \sum_{i=0}^{2n-1} q^i \langle (w_{0,I})^i \sigma^{-i} \lambda, \alpha^\vee \rangle &= \sum_{i=0}^{2n-1} q^i \langle \lambda, \sigma^i((w_{0,I})^i \alpha^\vee) \rangle = \sum_{i=0}^{2n-1} (-1)^i q^i \langle \lambda, \alpha^\vee \rangle \\ &= -\langle \lambda, \alpha^\vee \rangle \left(\frac{q^{2n} - 1}{q + 1} \right) \leq 0, \end{aligned}$$

where we used that (G, μ) is of Hasse-type in the second equality and the fact that λ is I-dominant in the last inequality. This shows the result. □

For example, if P is a maximal parabolic, we have $|\Delta^P| = 1$, hence $\mathcal{C}_{\mathsf{pHa}}$ is given inside $X^*_{+,I}(T)$ by a single inequality. This is in contrast to cases which are not of Hasse-type. For example, if $G = \mathrm{Sp}(6)$ as explained in [Kos19, §5.5], the cone $\mathcal{C}_{\mathsf{pHa}}$ is defined by $|\Delta| = 3$ inequalities inside $X^*_{+,I}(T)$.

From now on, assume that (G, μ) is of Hasse-type. We prove Theorem 4.3.1 by showing that if $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0$, then $\lambda \in \mathcal{C}_{\mathsf{pHa}}$. First, recall that $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda))$ identifies with $H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{V}_{\mathsf{flag}}(\lambda))$, and also with $S_{\mathsf{flag}}(\lambda) \subset S(\lambda)$. Let $h \in S_{\mathsf{flag}}(\lambda)$ be nonzero. By Proposition 4.2.4 (2), $F_{h,\alpha} \in k[B_L][t]$ for all $\alpha \in \Delta$. We will only need this information for $\alpha \in \Delta^P$. Write again $P_h = \sum_{i=1}^N c_i P_{m_i,\lambda_i}$ as in (4.3.2), with the (m_i, λ_i) pairwise distinct and $c_i \neq 0$ for $1 \leq i \leq N$. By Lemma 4.3.4 and formula (4.3.3), we have in particular

$$\lambda = q\sigma^{-1}(\lambda_1) - w_{0,I}\lambda_1 + \sum_{\beta \in \Phi_L^-} m_{1,\beta}(w_{0,I}\beta).$$

We want to show $\lambda \in \mathcal{C}_{\mathsf{pHa}}$, which amounts to $K_\alpha(\lambda) \leq 0$ for all $\alpha \in \Delta^P$ by Lemma 4.3.6. We first compute $K_\alpha(\beta)$ for any $\beta \in \Phi_L$. We find:

$$K_\alpha(\beta) = \sum_{i=0}^{2n-1} q^i \langle (w_{0,I})^i \sigma^{-i}\beta, \alpha^\vee \rangle = \sum_{i=0}^{2n-1} (-1)^i q^i \langle \beta, \alpha^\vee \rangle = -\langle \beta, \alpha^\vee \rangle \left(\frac{q^{2n}-1}{q+1} \right).$$

On the other hand, we have

$$\begin{aligned} K_\alpha(q\sigma^{-1}(\lambda_1) - w_{0,I}\lambda_1) &= \sum_{i=0}^{2n-1} q^i \langle (w_{0,I})^i \sigma^{-i}(q\sigma^{-1}(\lambda_1) - w_{0,I}\lambda_1), \alpha^\vee \rangle \\ &= \sum_{i=0}^{2n-1} q^{i+1} \langle (w_{0,I})^i \sigma^{-(i+1)}(\lambda_1), \alpha^\vee \rangle - \sum_{i=0}^{2n-1} q^i \langle (w_{0,I})^{i+1} \sigma^{-i}(\lambda_1), \alpha^\vee \rangle \\ &= (q^{2n}-1) \langle w_{0,I}\lambda_1, \alpha^\vee \rangle, \end{aligned}$$

where we used that $\sigma^{2n}\lambda_1 = \lambda_1$. Hence, we find for all $\alpha \in \Delta^P$:

$$\begin{aligned} K_\alpha(\lambda) &= K_\alpha(q\sigma^{-1}(\lambda_1) - w_{0,I}\lambda_1) + \sum_{\beta \in \Phi_L^-} m_{1,\beta} K_\alpha(w_{0,I}\beta) \\ &= \frac{q^{2n}-1}{q+1} \left((q+1)\langle w_{0,I}\lambda_1, \alpha^\vee \rangle - \sum_{\beta \in \Phi_L^-} m_{1,\beta} \langle w_{0,I}\beta, \alpha^\vee \rangle \right). \end{aligned}$$

One has $K_\alpha(\lambda) \leq 0$ using the fact that $\langle w_{0,I}\lambda_1, \alpha^\vee \rangle \leq 0$ (Corollary 4.3.5(2)) and equation (4.3.5) applied to F_{m_1,λ_1}. This terminates the proof of Theorem 4.3.1.

5 $R_{\mathrm{u}}(P_0)$-invariant subspace

Let (G, μ) be an arbitrary cocharacter datum, and let $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ the attached zip datum. Fix a frame (B, T, z) with $z = \sigma(w_{0,I})w_0$. Let (V, ρ) be an L-representation. For $f \in V^{L_\varphi}$, we can view f as a section of $\mathcal{V}_I(\lambda)$ over $\mathcal{U}_\mu$, by Lemma 2.4.2. When P is defined over $\mathbb{F}_q$, this section extends to $G\text{-}\mathtt{Zip}^\mu$ if and only if $f \in V(\lambda)_{\geq 0}^{\Delta^P}$, by Corollary 2.4.4. For general P, the condition on f is given by the Brylinski–Kostant filtration on $V_I(\lambda)$ (see [IK21, Theorem 3.4.1]). Unfortunately, this condition is too complex to understand explicitly. However, let P_0 be the parabolic L_0B with L_0 as in (2.4.3), and assume further that $f \in V_I(\lambda)^{R_{\mathrm{u}}(P_0)}$. In this case, we can give a more explicit condition for when f extends. In particular, the lowest weight vector of $V_I(\lambda)$ satisfies this condition. This makes it possible to define a "lowest weight cone" $\mathcal{C}_{\mathsf{lw}}$ (see §5.2 below) similar to the highest weight cone $\mathcal{C}_{\mathsf{hw}}$. When P is not defined over $\mathbb{F}_q$, one sees on examples that $\mathcal{C}_{\mathsf{hw}}$ is usually very small. On the other hand, the lowest weight cone will be quite large.

5.1 Statement

As in the proof of Proposition 3.5.1, define for $\alpha \in \Delta^P$:

$$m_\alpha = \min\{m \geq 1 \mid \sigma^{-m}(\alpha) \notin I\}$$

and $t_\alpha = t^{-1}\alpha(\varphi(\delta_\alpha(t)))^{-1} = t\alpha(\delta_\alpha(t))^{-1} \in t^{\mathbb{Q}}$, where t is an indeterminate. Set also

$$u_{t,\alpha} = \prod_{i=1}^{m_\alpha - 1} \phi_{\sigma^{-i}(\alpha)} \left(\begin{pmatrix} 1 & -t_\alpha^{\frac{1}{q^i}} \\ 0 & 1 \end{pmatrix} \right). \tag{5.1.1}$$

For $\alpha \in \Phi$, write $G_\alpha \subset G$ for the image of the map $\phi_\alpha\colon \mathrm{SL}_2 \to G$. For simplicity, we consider the following condition:

Condition 5.1.1. *For all $1 \le i, j \le m_\alpha - 1$ with $i \ne j$ we have $\langle \sigma^{-i}(\alpha), \sigma^{-j}(\alpha^\vee)\rangle = 0$ and the subgroups $G_{\sigma^{-i}(\alpha)}$ and $G_{\sigma^{-j}(\alpha)}$ commute with each other.*

Remark 5.1.2. Condition 5.1.1 is satisfied in many cases. For example, if G splits over $\mathbb{F}_{q^2}$, then $m_\alpha \in \{1, 2\}$ and the condition is trivially satisfied. In particular, all absolutely simple unitary groups satisfy it. The condition also holds for $G = \mathrm{Res}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(G_{0,\mathbb{F}_{q^n}})$ where G_0 is a split reductive over $\mathbb{F}_q$.

Let (V, ρ) be an L-representation and let $V = \bigoplus_{\nu \in X^*(T)} V_\nu$ be its T-weight decomposition. For $\alpha \in \Delta$, set $\delta_\alpha = \wp_*^{-1}(\alpha^\vee)$ (where $\wp_*$ was defined in (2.4.4)). Put $P_1 := \sigma^{-(m_\alpha - 1)}(P)$. We have $\Delta^{P_1} = \sigma^{-(m_\alpha-1)}(\Delta^P)$. Since $P_0 \subset P_1$, we have $\Delta^{P_1} \subset \Delta^{P_0}$. Define $V_{\ge 0}^{\Delta^{P_1}}$ similarly to (2.4.5) by

$$V_{\ge 0}^{\Delta^{P_1}} = \bigoplus_{\langle \nu, \delta_\beta\rangle \ge 0,\ \forall \beta \in \Delta^{P_1}} V_\nu.$$

Proposition 5.1.3. *Assume that Condition 5.1.1 holds. Then we have*

$$V^{R_u(P_0)} \cap V^{L_\varphi} \cap V_{\ge 0}^{\Delta^{P_1}} \subset H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)).$$

Proof. Let $f \in V^{R_\mathrm{u}(P_0)} \cap V^{L_\varphi}$ and let $\widetilde{f}\colon U_\mu \to V$ be the function corresponding to f by Lemma 2.4.2. It suffices to check that $\widetilde{f}$ extends to G. By the proof of [IK21, Theorem 3.4.1], it is enough to show that for all $\alpha \in \Delta^P$, the function

$$F_\alpha\colon t \mapsto \rho\left(\phi_\alpha\left(\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1\end{pmatrix}\right)\delta_\alpha(t) u_{t,\alpha}\right) f$$

lies in $k[t] \otimes V$. Since it lies in $k[t, t^{-1}] \otimes V$ by the proof of [IK21, Theorem 3.4.1], it suffices to show that it also lies in $k[(t^r)_{r \in \mathbb{Q}_{\ge 0}}] \otimes V$. Since ρ is trivial on $R_\mathrm{u}(P)$ and $\alpha \in \Delta^P$, one has simply $F_\alpha(t) = \rho(\delta_\alpha(t) u_{t,\alpha}) f$. Using (5.1.1), we can write

$$F_\alpha(t) = \rho\left(\delta_\alpha(t) \prod_{i=1}^{m_\alpha - 1} \phi_{\sigma^{-i}(\alpha)}\left(\begin{pmatrix} 1 & -t_\alpha^{\frac{1}{q^i}} \\ 0 & 1\end{pmatrix}\right)\right) f = \rho\left(\prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{-i}(\alpha)}\left(\begin{pmatrix} 1 & \gamma_i \\ 0 & 1\end{pmatrix}\right)\delta_\alpha(t)\right) f$$

where $\gamma_i = -t^{\langle \sigma^{-i}(\alpha), \delta_\alpha\rangle} t_\alpha^{\frac{1}{q^i}}$. We have $q^{-1}\sigma^{-1}(\delta_\alpha) = \delta_\alpha + q^{-1}\sigma^{-1}\alpha^\vee$ and hence by induction $q^{-i}\sigma^{-i}(\delta_\alpha) = \delta_\alpha + (q^{-1}\sigma^{-1}\alpha^\vee + \cdots + q^{-i}\sigma^{-i}\alpha^\vee)$. Let $1 \le i \le m_\alpha - 1$. By Condition 5.1.1, we deduce $\langle \sigma^{-i}(\alpha), \delta_\alpha\rangle = q^{-i}(\langle \alpha, \delta_\alpha\rangle - 2)$. Thus

$$\gamma_i = -t^{\langle \sigma^{-i}(\alpha), \delta_\alpha\rangle + q^{-i}(1 - \langle \alpha, \delta_\alpha\rangle)} = -t^{-1/q^i}.$$

Let $f = \sum_\nu f_\nu$ be the T-weight decomposition of f. By assumption, we have:

$$\begin{aligned}
F_\alpha(t) &= \sum_\nu \rho\left(\prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{-i}(\alpha)}\left(\begin{pmatrix} 1 & -t^{-1/q^i} \\ 0 & 1\end{pmatrix}\right)\delta_\alpha(t)\right) f_\nu \\
&= \sum_\nu t^{\langle \nu, \delta_\alpha\rangle} \rho\left(\prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{-i}(\alpha)}\left(\begin{pmatrix} 1 & -t^{-1/q^i} \\ 0 & 1\end{pmatrix}\right)\right) f_\nu \\
&= \sum_\nu t^{\langle \nu, \delta_\alpha\rangle} \rho\left(\prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{-i}(\alpha)}\left(\begin{pmatrix} t^{-1/q^i} & -1 \\ 0 & t^{1/q^i}\end{pmatrix}\right)\sigma^{-i}(\alpha)^\vee(t^{1/q^i})\right) f_\nu \\
&= \sum_\nu t^{\langle \nu, \delta_\alpha + \sum_{i=1}^{m_\alpha-1} q^{-i}\sigma^{-i}(\alpha)^\vee\rangle} \rho\left(\prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{-i}(\alpha)}\left(\begin{pmatrix} t^{-1/q^i} & -1 \\ 0 & t^{1/q^i}\end{pmatrix}\right)\right) f_\nu.
\end{aligned}$$

As before, we have $\delta_\alpha + \sum_{i=1}^{m_\alpha-1} q^{-i}\sigma^{-i}(\alpha)^\vee = q^{-(m_\alpha-1)}\sigma^{-(m_\alpha-1)}(\delta_\alpha)$. Furthermore, we have

$$\begin{pmatrix} t^{-1/q^i} & -1 \\ 0 & t^{1/q^i} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & t^{1/q^i} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1/q^i} & 1 \end{pmatrix}.$$

Since P_0 is defined over $\mathbb{F}_q$, we have $\sigma^{-i}(\alpha) \notin I_{P_0}$ for all $i \in \mathbb{Z}$. By invariance of f under $R_{\mathrm{u}}(P_0)$, we deduce

$$F_\alpha(t) = \sum_\nu t^{\langle \nu, \sigma^{-(m_\alpha-1)}(\delta_\alpha)\rangle / q^{m_\alpha-1}} \rho \left(\prod_{i=1}^{m_\alpha-1} \phi_{\sigma^{-i}(\alpha)} \left(\begin{pmatrix} 0 & -1 \\ 1 & t^{1/q^i} \end{pmatrix} \right) \right) f_\nu.$$

Since $f \in V_{\geq 0}^{\Delta^{P_1}}$, we have $\langle \nu, \sigma^{-(m_\alpha-1)}(\delta_\alpha)\rangle = \langle \nu, \delta_{\sigma^{-(m_\alpha-1)}(\alpha)}\rangle \geq 0$. Hence, the t-valuation of $F_\alpha(t)$ is ≥ 0. The result follows. □

5.2 Lowest weight cone

We examine the case $V = V_I(\lambda)$ for $\lambda \in X^*_{+,I}(T)$. The L_0-representation $V_I(\lambda)^{R_{\mathrm{u}}(P_0)}$ is isomorphic to $V_{I_0}(w_{0,I_0}w_{0,I}\lambda)$ by [IK24, Proposition 6.3.1]. Put $\lambda_0 = w_{0,I_0}w_{0,I}\lambda$.

Let $f_{\mathsf{low},\lambda} \in V_I(\lambda)$ be a nonzero element in the lowest weight line of $V_I(\lambda)$. Consider the element $\mathrm{Norm}_{L_\varphi}(f_{\mathsf{low},\lambda}) \in V_I(N_\varphi\lambda)$, defined in (3.5.1), where $N_\varphi = |L_0(\mathbb{F}_q)|q^m$. By construction, this element lies in $V_I(N_\varphi\lambda)^{L_\varphi}$. For $\alpha \in \Delta$, write r_α for the smallest integer $r \geq 1$ such that $\sigma^r(\alpha) = \alpha$.

Theorem 5.2.1. *Assume Condition 5.1.1. Suppose that for all $\alpha \in \Delta^{P_0}$, one has*

$$\sum_{w \in W_{L_0}(\mathbb{F}_q)} \sum_{i=0}^{r_\alpha-1} q^{i+\ell(w)} \ \langle w\lambda_0, \sigma^i(\alpha^\vee)\rangle \leq 0. \tag{5.2.1}$$

Then $\mathrm{Norm}_{L_\varphi}(f_{\mathsf{low},\lambda})$ *extends to G-$\mathtt{Zip}^\mu$.*

Remark 5.2.2. Formulas (5.2.1) and (3.5.2) (in the case of $f_{\mathsf{high},\lambda}$) differ in two aspects: λ changes to $\lambda_0 = w_{0,I_0}w_{0,I}\lambda$ and "for all $\alpha \in \Delta^P$" changes to "for all $\alpha \in \Delta^{P_0}$".

Proof. The lowest weight vector $f_{\mathsf{low},\lambda}$ is contained in the L_0-subrepresentation $V_I(\lambda)^{R_{\mathrm{u}}(P_0)} \cong V_{I_0}(\lambda_0)$, which has highest weight λ_0, lowest weight vector $f_{\mathsf{low},\lambda}$ and highest weight vector $f_{\mathsf{high},\lambda_0} := w_{0,I_0}(f_{\mathsf{low},\lambda})$. Since $w_{0,I_0} \in W_{L_0}(\mathbb{F}_q)$, we have

$$\mathrm{Norm}_{L_\varphi}(f_{\mathsf{low},\lambda}) = \mathrm{Norm}_{L_\varphi}(f_{\mathsf{high},\lambda_0}) = \mathrm{Norm}_{L_0(\mathbb{F}_q)}(f_{\mathsf{high},\lambda_0})^{q^m}. \tag{5.2.2}$$

Consider the zip datum $\mathcal{Z}_0 = (G, P_0, L_0, Q_0, L_0, \varphi)$, where Q_0 is the opposite parabolic to P_0 with Levi subgroup L_0. By Remark 2.2.1, we have $\mathcal{Z}_0 = \mathcal{Z}_{\mu_0}$ for some cocharacter $\mu_0 \colon \mathbb{G}_{\mathrm{m},k} \to G_k$. Since P_0 is defined over $\mathbb{F}_q$, we have by Corollary 2.4.4:

$$H^0(G\text{-}\mathtt{Zip}^{\mu_0}, \mathcal{V}_{I_0}(\lambda_0)) = V_{I_0}(\lambda_0)^{L_0(\mathbb{F}_q)} \cap V_{I_0}(\lambda_0)^{\Delta^{P_0}}_{\geq 0}.$$

Applying Proposition 3.5.1 to G-$\mathtt{Zip}^{\mu_0}$ and the L_0-representation $V_{I_0}(\lambda_0)$, we deduce

$$\mathrm{Norm}_{L_0(\mathbb{F}_q)}(f_{\mathsf{high},\lambda_0}) \in V_{I_0}(N_0\lambda_0)^{\Delta^{P_0}}_{\geq 0}$$

where $N_0 = |L_0(\mathbb{F}_q)|$. Combining this with (5.2.2), and using that $\Delta^{P_1} \subset \Delta^{P_0}$, we find

$$\mathrm{Norm}_{L_\varphi}(f_{\mathsf{low},\lambda}) \in V_I(N_\varphi\lambda)^{\Delta^{P_0}}_{\geq 0} \subset V_I(N_\varphi\lambda)^{\Delta^{P_1}}_{\geq 0}.$$

The result follows from Proposition 5.1.3 applied to $V_I(N_\varphi\lambda)$. □

Definition 5.2.3. *Define* $\mathcal{C}_{\mathsf{lw}}$ *as the set of* $\lambda \in X^*_{+,I}(T)$ *satisfying the inequalities* (5.2.1).

We call $\mathcal{C}_{\mathsf{lw}}$ the lowest weight cone. Under Condition 5.1.1, one has $\mathcal{C}_{\mathsf{lw}} \subset \mathcal{C}_{\mathsf{zip}}$ by Theorem 5.2.1. We do not know if this inclusion holds in general. When P is defined over $\mathbb{F}_q$, one has $P_0 = P$ and hence $\mathcal{C}_{\mathsf{lw}} = \mathcal{C}_{\mathsf{hw}}$.

Lemma 5.2.4. *One has* $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{lw}}$.

Proof. For $\lambda \in \mathcal{C}_{\mathsf{GS}}$, the character $w_{0,I}\lambda$ is anti-dominant. For all $w \in W_{L_0}(\mathbb{F}_q)$, we have $\langle w\lambda_0, \sigma^i(\alpha^\vee)\rangle = \langle w_{0,I}\lambda, w_{0,I_0}w^{-1}\sigma^i(\alpha^\vee)\rangle$. Since $w_{0,I_0}w^{-1} \in W_{L_0}$ and $\alpha \in \Delta^{P_0}$, the root $w_{0,I_0}w^{-1}\sigma^i(\alpha)$ is positive. Hence $\langle w\lambda_0, \sigma^i(\alpha^\vee)\rangle \leq 0$ for all $w \in W_{L_0}(\mathbb{F}_q)$, and the result follows. $\square$

In particular, if Condition 5.1.1 holds, we deduce $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{zip}}$ from Lemma 5.2.4. We will prove this inclusion in the next section in the general case.

6 Weil restriction

When Condition 5.1.1 does not hold, we cannot use Proposition 5.1.3 to show $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{zip}}$. We show here that a version of Proposition 5.1.3 holds in general (see Theorem 6.3.1 below). To eliminate the need for Condition 5.1.1, we first study the case of a Weil restriction. More generally, we will prove a useful result that makes it possible to reduce certain questions pertaining to the cone $\mathcal{C}_{\mathsf{zip}}$ to the case of a split group.

6.1 Zip strata of a Weil restriction

We recall some results from [KW18, §4]. Note that *loc. cit.* uses the convention $B \subset Q$, whereas we assume $B \subset P$. We make the appropriate changes in this section. Let $r \geq 1$ and let G_1 be a connected, reductive group over $\mathbb{F}_{q^r}$. Put $G = \mathrm{Res}_{\mathbb{F}_{q^r}/\mathbb{F}_q} G_1$. Over k, we can decompose

$$G_k = G_{1,k} \times G_{2,k} \times \cdots \times G_{r,k}$$

where $G_i = \sigma^{i-1}(G_1)$. The Frobenius homomorphim $\varphi \colon G \to G$ maps $(x_1, \ldots, x_r) \in G_k$ to $(\varphi(x_r), \varphi(x_1), \ldots, \varphi(x_{r-1}))$. We choose a cocharacter $\mu \colon \mathbb{G}_{\mathrm{m},k} \to G_k$ written as $(\mu_1, \ldots, \mu_r)$ with $\mu_i \colon \mathbb{G}_{\mathrm{m},k} \to G_{i,k}$. Consider the attached zip datum (G, P, L, Q, M, φ). Assume that there is a Borel pair (B, T) defined over $\mathbb{F}_q$ and $B \subset P$. For all $\square = P, L, Q, M, B, T$, one can decompose $\square = \prod_{i=1}^r \square_i$. Note that $\sigma(B_i) = B_{i+1}$ and $\sigma(T_i) = T_{i+1}$ and $\sigma(L_i) = M_{i+1}$, where indices are taken modulo r. Moreover, $\sigma(P_i)$ and Q_{i+1} are opposite in $G_{i+1,k}$. Write Δ_i for the set of simple roots of G_i. The Weyl group $W := W(G_k, T)$ decomposes also as $W = W_1 \times \cdots \times W_r$ where $W_i := W(G_{i,k}, T_i)$. Let $w_{0,i}$ be the longest element in W_i. The Frobenius induces an automorphism of W again denoted by σ, and we have $\sigma(W_i) = W_{i+1}$. Similarly, we have ${}^IW = {}^{I_1}W_1 \times \cdots \times {}^{I_r}W_r$ and $W^J = W_1^{J_1} \times \cdots \times W_r^{J_r}$, where $I_i, J_i \subset \Delta_i$ are the types of the parabolic subgroups P_i and Q_i respectively.

We obtain a frame (B, T, z) by setting $z := \sigma(w_{0,I})w_0 = w_0 w_{0,J}$ (Lemma 2.2.3). Thus $z = (z_1, \ldots, z_r)$ with $z_i = w_{0,i}w_{0,J_i}$ for all $i = 1, \ldots, r$. By the dual parametrization (2.2.2) and the dimension formula for E-orbits in Theorem 2.2.4, the E-orbits of codimension one in G are

$$C_{i,\alpha} := E \cdot (1, \ldots, 1, w_{0,i}s_\alpha w_{0,i}, 1, \ldots, 1)\,, \quad 1 \leq i \leq r,\ \alpha \in \Delta_i \setminus J_i. \tag{6.1.1}$$

For each $1 \leq j \leq r$, define parabolic subgroups in $G_{j,k}$ by

$$P'_j = \bigcap_{i=0}^{r-1} \sigma^{-i}(P_{i+j}) \quad \text{and} \quad Q'_j = \bigcap_{i=0}^{r-1} \sigma^{i}(Q_{j-i})$$

where the indices are taken modulo r. The unique Levi subgroups of P'_j and Q'_j containing T_j are respectively

$$L'_j = \bigcap_{i=0}^{r-1} \sigma^{-i}(L_{i+j}) \quad \text{and} \quad M'_j = \bigcap_{i=0}^{r-1} \sigma^{i}(M_{j-i}).$$

By [KW18, Lemma 4.2.1], the tuple $\mathcal{Z}_j := (G_j, P'_j, L'_j, Q'_j, M'_j, \varphi^r)$ is a zip datum over $\mathbb{F}_{q^r}$. Clearly $B_j \subset P'_j$ and $B_j^+ \subset Q'_j$, since B is defined over $\mathbb{F}_q$. It follows that $\sigma^r(P'_j)$ and Q'_j are opposite parabolics of $G_{j,k}$. By Remark 2.2.1, $\mathcal{Z}_j$ is of cocharacter-type. We denote the zip group of $\mathcal{Z}_j$ by $E_j \subset P'_j \times Q'_j$ (in [KW18], this group is denoted by E'_j, but we want to avoid confusion with the group E' defined in §2.5).

Write $\iota_j \colon G_{j,k} \to G_k$ for the natural embedding $x \mapsto (1, \dots, x, \dots, 1)$. Denote by $\mathcal{X}$ the set of E-orbits in G_k, and by $\mathcal{X}_j \subset \mathcal{X}$ the set of E-orbits which intersect $G_{j,k}$ (viewed as a subset of G_k via ι_j). We have the following result ([KW18, Theorem 4.3.1]):

Theorem 6.1.1. *The map $C \mapsto C \cap G_{j,k}$ defines a bijection between $\mathcal{X}_j$ and the set of E_j-orbits in $G_{j,k}$. Furthermore one has $\operatorname{codim}_{G_k}(C) = \operatorname{codim}_{G_{j,k}}(C \cap G_{j,k})$ for all $C \in \mathcal{X}_j$.*

Note that $\mathcal{X}_j$ always contains the open E-orbit, since this orbit contains $1 \in G_k$. Furthermore, by equation (6.1.1), any E-orbit of codimension 1 lies in at least one of the $\mathcal{X}_j$. There is a natural group homomorphism $\gamma_j \colon E_j \to E$, defined as follows. For $(x, y) \in E_j$ write $\overline{x} := \theta^{P'_j}_{L'_j}(x)$ and set

$$\begin{aligned}
u_j(x,y) &:= (\varphi^{r-j+1}(\overline{x}), \dots, \varphi^{r-1}(\overline{x}), x, \varphi(\overline{x}), \dots, \varphi^{r-j}(\overline{x})) \in P \\
v_j(x,y) &:= (\varphi^{r-j+1}(\overline{x}), \dots, \varphi^{r-1}(\overline{x}), y, \varphi(\overline{x}), \dots, \varphi^{r-j}(\overline{x})) \in Q \\
\gamma_j(x,y) &:= (u_j(x,y), v_j(x,y)) \in E.
\end{aligned}$$

The pair (ι_j, γ_j) induces a morphism of stacks

$$\theta_j \colon [E_j \backslash G_{j,k}] \to [E \backslash G_k].$$

By the previous discussion, the image of θ_j contains a nonempty open subset, and each codimension 1 stratum in $G\text{-}\mathtt{Zip}^\mu$ is contained in the image of at least one θ_j. Note that $u_j(x,y)$ only depends on x. By abuse of notation, we denote again by γ_j the map

$$\gamma_j \colon P'_j \to P, \quad x \mapsto (\varphi^{r-j+1}(\overline{x}), \dots, \varphi^{r-1}(\overline{x}), x, \varphi(\overline{x}), \dots, \varphi^{r-j}(\overline{x})).$$

We have a commutative diagram

$$\begin{tikzcd}
E_j \arrow[r, "\gamma_j"] \arrow[d, "\mathrm{pr}_1"'] & E \arrow[d, "\mathrm{pr}_1"] \\
P'_j \arrow[r, "\gamma_j"'] & P
\end{tikzcd}$$

For $x \in L'_j$, we have $\gamma_j(x) \in L$. Hence, we also have a map $\gamma_j \colon L'_j \to L$.

6.2 Space of global sections

For each $1 \leq i \leq r$, let (V_i, ρ_i) be an L_i-representation and let (V, ρ) be the L-representation $\boxtimes_{i=1}^r \rho_i$. For example, if $\lambda = (\lambda_1, \dots, \lambda_r)$ is in $X^*(T) = X^*(T_1) \times \cdots \times X^*(T_r)$, then we have $V_I(\lambda) = \boxtimes_{i=1}^r V_{I_i}(\lambda_i)$. View ρ_i as a map $P_i \to \mathrm{GL}(V_i)$ trivial on $R_{\mathrm{u}}(P_i)$. Using the maps $\gamma_j \colon P'_j \to P$, we have

$$\theta_j^* \left(\mathcal{V}(\rho)\right) = \bigotimes_{i=1}^r \mathcal{V}(\rho_{j+i}^{[i]})$$

where $\rho_{j+i}^{[i]}$ denotes the P'_j-representation $P'_j \xrightarrow{\varphi^i} P_{j+i} \xrightarrow{\rho_{j+i}} \mathrm{GL}(V_{j+i})$ (indices modulo r). By definition of P'_j, this composition is well-defined. Note that $\rho_{j+i}^{[i]}$ may not be trivial on the unipotent radical of P'_j. Let L_φ be the stabilizer of $1 \in G$ in E, as defined in §2.4 and fix $f \in V^{L_\varphi}$. By Lemma 2.4.2, we may view f as a section of $\mathcal{V}(\rho)$ over the open substack $\mathcal{U}_\mu \subset G\text{-}\mathtt{Zip}^\mu$. Similarly, since θ_j maps $\mathcal{U}_{\mu_j}$ into $\mathcal{U}_\mu$ (Theorem 6.1.1), we have $\theta_j^*(f) \in H^0(\mathcal{U}_{\mu_j}, \theta_j^*(\mathcal{V}(\rho)))$.

Lemma 6.2.1. *The section f extends to $G\text{-}\mathtt{Zip}^\mu$ if and only if $\theta_j^*(f)$ extends to $G_j\text{-}\mathtt{Zip}^{\mathcal{Z}_j}$ for all $1 \leq j \leq r$.*

Proof. The only if implication is clear. Conversely, assume that $\theta_j^*(f) \in H^0(G\text{-}\mathtt{Zip}^{\mathcal{Z}_j}, \theta_j^*(\mathcal{V}(\rho)))$ for all $1 \leq j \leq r$. Viewing f as a section over $\mathcal{U}_\mu$, consider the unique regular map $\widetilde{f} \colon U_\mu \to V$ satisfying $\widetilde{f}(1) = f$ and $\widetilde{f}(axb^{-1}) = \rho(a)\widetilde{f}(x)$ for all $x \in U_\mu$ and all $(a, b) \in E$. It suffices to show that $\widetilde{f}$ extends to a regular map $\widetilde{f} \colon G \to V$ (by density, this regular map will automatically satisfy the E-equivariance condition).

Consider a codimension one E-orbit $C_{i,\alpha}$ for some $1 \leq i \leq r$ and $\alpha \in \Delta_i \setminus J_i$ (where $C_{i,\alpha}$ was defined in equation (6.1.1)). Set $Y := U_\mu \cup C_{i,\alpha}$. It is the complement in G of the union of the Zariski closures of all other codimension one E-orbits. In particular Y is open in G. Define $X := \iota_i^{-1}(Y)$ and consider the map $\iota_i \colon X \to Y$. This map satisfies conditions (1) and (2) of Lemma 6.2.2 below (for the group $H = E$). By assumption, the function $\iota_i^*(\widetilde{f}) = \widetilde{f} \circ \iota_i \colon U_{\mu_i} \to V$ extends to a function $G_i \to V$ (in particular to a map $X \to V$). Therefore, we can apply Lemma 6.2.2 to deduce that $\widetilde{f}$ extends to a regular map $Y \to V$. To show that $\widetilde{f}$ extends to G, let $\widetilde{f}_0 \colon U_\mu \to \mathbb{A}^1$ be a coordinate function of f in some basis of V. By the above discussion, $\widetilde{f}_0$ cannot have a pole along any codimension one E-orbit of G, hence extends to G by normality. Hence $\widetilde{f}$ itself extends to G and the result follows. $\square$

Lemma 6.2.2. *Let Y, X be irreducible normal k-varieties, and assume that Y is endowed with an action of an algebraic group H. Suppose that Y has an open subset $U_Y \subset Y$ stable by H. Set $Z_Y := Y \setminus U_Y$. Let (V, ρ) be an H-representation and let $f \colon U_Y \to V$ be an H-equivariant regular map on U_Y. Let $\iota \colon X \to Y$ be a regular map satisfying the following:*
(1) $\iota(X) \cap U_Y \neq \emptyset$,
(2) $H \cdot (\iota(X) \cap Z_Y)$ is Zariski dense in Z_Y.
Define $U_X := \iota^{-1}(U_Y)$. Then the following holds: The morphism f extends to an H-equivariant regular map $Y \to V$ if and only if $\iota^*(f) \colon U_X \to V$ extends to a regular map $X \to V$.

Proof. The only if direction is obvious. Conversely, assume that $\iota^*(f) \colon U_X \to V$ extends to a regular map $X \to V$. Consider the map

$$\phi \colon H \times X \to Y, \quad (h, x) \mapsto h \cdot \iota(x).$$

We have $\phi^{-1}(U_Y) = H \times U_X$. Then f extends to a regular map $Y \to V$ if and only if $\phi^*(f)\colon H \times U_X \to V$ extends to a regular map $H \times X \to V$. Indeed, choose a basis of V. Let $f_i\colon U_Y \to \mathbb{A}^1_k$ for $1 \leq i \leq \dim V$ be coordinate maps of f with respect to that basis. Since the image of ϕ is dense in Z_Y by assumption, f_i cannot have a pole along Z_Y, hence extends to Y by normality. Thus, it suffices to show that if $\iota^*(f)$ extends, then so does $\phi^*(f)$. But since f is H-equivariant, we have for all $h \in H$, $x \in U_X$:

$$\phi^*(f)(h,x) = f(h \cdot \iota(x)) = h \cdot (\iota^*(f)(x)) .$$

Hence if $\iota^*(f)$ extends to X, we can define a function $H \times X \to V$ using the above formula, and it must coincide with $\phi^*(f)$ on the open subset $H \times U_X$. The result follows. $\square$

Now, assume that for all $1 \leq j \leq r$, P_j is defined over $\mathbb{F}_{q^r}$ (for example, this is the case if T_1 is split over $\mathbb{F}_{q^r}$). It is clear that P'_j is then also defined over $\mathbb{F}_{q^r}$. We apply Corollary 2.4.4 to the $\mathbb{F}_{q^r}$-zip datum $\mathcal{Z}_j$. We deduce that for any L'_j-representation (W, ρ_W), we have

$$H^0(G_j\text{-}\mathtt{Zip}^{\mathcal{Z}_j}, \mathcal{V}(\rho_W)) = W^{L'_j(\mathbb{F}_{q^r})} \cap W^{\Delta^{P'_j}}_{\geq 0} . \tag{6.2.1}$$

However, since $\gamma_j^*(\rho) = \rho \circ \gamma_j \in \operatorname{Rep}(P'_j)$ may be non-trivial on $R_{\mathrm{u}}(P'_j)$, we cannot apply this formula directly to $\gamma_j^*(\rho)$. Denote by $V^{\#} \subset V$ the subspace of $f \in V$ which are invariant under $\gamma_j(R_{\mathrm{u}}(P'_j))$ for all $1 \leq j \leq r$. We deduce from (6.2.1) and Lemma 6.2.1:

Corollary 6.2.3. *Let $f \in V^{L_\varphi} \cap V^{\#}$. Then f extends to G-$\mathtt{Zip}^\mu$ if and only if $f \in \left(V|_{L'_j}\right)^{\Delta^{P'_j}}_{\geq 0}$ for all $1 \leq j \leq r$, where $V|_{L'_j}$ denotes the L'_j-representation $\gamma_j^*(\rho)\colon L'_j \xrightarrow{\gamma_j} L \xrightarrow{\rho} \mathrm{GL}(V)$.*

Write $V = \bigoplus_{\chi \in X^*(T)} V_\chi$ for the T-weight space decomposition of V, and write $\chi = (\chi_1, \ldots, \chi_r)$ where $\chi_i \in X^*(T_i)$. Similarly, let $f = \sum_\chi f_\chi$ be the decomposition of f. We determine the T_j-weight decomposition of $V|_{L'_j}$. For $\chi \in X^*(T)$, define

$$S_j(\chi) := \sum_{i=0}^{r-1} q^i \sigma^{-i}(\chi_{j+i}) \in X^*(T_j)$$

(indices taken modulo r). Then, the T_j-weight decomposition of $V|_{L'_j}$ is given by

$$V|_{L'_j} = \bigoplus_{\eta \in X^*(T_j)} V_\eta, \qquad \text{where} \qquad V_\eta = \bigoplus_{\substack{\chi \in X^*(T) \\ S_j(\chi) = \eta}} V_\chi .$$

Define $V^{\cap}_{\geq 0} \subset V$ as the intersection of all $\left(V|_{L'_j}\right)^{\Delta^{P'_j}}_{\geq 0}$ for $1 \leq j \leq r$ inside V. Put

$$\wp^{(r)}_{j,*}\colon X_*(T_j)_{\mathbb{R}} \to X_*(T_j)_{\mathbb{R}}, \quad \delta \mapsto \delta - q^r \sigma^r(\delta)$$

as in (2.4.4) (but changing φ to φ^r). For $\alpha \in \Delta_j$, define $\delta^{(r)}_{j,\alpha} := (\wp^{(r)}_{j,*})^{-1}(\alpha^\vee) \in X_*(T_j)_{\mathbb{R}}$. By definition, $(V|_{L'_j})^{\Delta^{P'_j}}_{\geq 0}$ is the direct sum of V_η for $\eta \in X^*(T_j)$ satisfying $\langle \eta, \delta^{(r)}_{j,\alpha} \rangle \geq 0$ for all $\alpha \in \Delta^{P'_j}$. Hence $V^{\cap}_{\geq 0} \subset V$ is the direct sum of weight spaces V_χ satisfying $\langle S_j(\chi), \delta^{(r)}_{j,\alpha} \rangle \geq 0$ for all $\alpha \in \Delta^{P'_j}$ and all $1 \leq j \leq r$. We have shown that f extends to G-$\mathtt{Zip}^\mu$ if and only if $f \in V^{\cap}_{\geq 0}$. In other words:

Proposition 6.2.4. *Let $\Gamma(\rho)$ be the set of all $\chi \in X^*(T)$ such that $\langle S_j(\chi), \delta_{j,\alpha}^{(r)} \rangle \geq 0$ for all $1 \leq j \leq r$ and all $\alpha \in \Delta^{P'_j}$. For $f \in V^{L_\varphi} \cap V^{\#}$, f extends to G-$\mathtt{Zip}^\mu$ if and only if $f \in V_{\geq 0}^{\cap} = \bigoplus_{\chi \in \Gamma(\rho)} V_\chi$.*

Now, assume that T_1 is split over $\mathbb{F}_{q^r}$. Then for all $1 \leq j \leq r$, T_j is split over $\mathbb{F}_{q^r}$, hence $\delta_{j,\alpha}^{(r)} = -\frac{1}{q^r-1}\alpha^\vee$ for all $\alpha \in \Delta_j$. Therefore, in this case $\Gamma(\rho)$ is the set of $\chi \in X^*(T)$ satisfying $\langle S_j(\chi), \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Delta^{P'_j}$ and all $1 \leq j \leq r$.

6.3 Consequence for $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$

We derive consequences from the above considerations. Let G be a connected, reductive group over $\mathbb{F}_q$, $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ a cocharacter, and $\mathcal{Z} = (G, P, L, Q, M, \varphi)$ the associated zip datum over $\mathbb{F}_q$. Choose a frame (B, T, z) as in §2.2.3. For $r \geq 1$, consider the diagonal embedding

$$\Delta\colon G \to \widetilde{G} := \mathrm{Res}_{\mathbb{F}_{q^r}/\mathbb{F}_q}(G_{\mathbb{F}_{q^r}}).$$

The cocharacter $\widetilde{\mu} := \Delta \circ \mu$ induces a zip datum $\widetilde{\mathcal{Z}} = (\widetilde{G}, \widetilde{P}, \widetilde{L}, \widetilde{Q}, \widetilde{M}, \widetilde{\varphi})$, where for each $\square = G, P, L, Q, M$ we have $\widetilde{\square}_k = \square_k \times \cdots \times \square_k$. Write $\widetilde{E}$ for the zip group of $\widetilde{\mathcal{Z}}$. We obtain a morphism of stacks

$$\Delta\colon\ G\text{-}\mathtt{Zip}^\mu \to \widetilde{G}\text{-}\mathtt{Zip}^{\widetilde{\mu}}.$$

For all $1 \leq i \leq r$, let (V_i, ρ_i) be an L-representation, and write $\widetilde{\rho} := \boxtimes_{i=1}^r \rho_i$, viewed as an $\widetilde{L}$-representation. We have

$$\Delta^*(\mathcal{V}(\widetilde{\rho})) = \bigotimes_{i=1}^r \mathcal{V}(\rho_i).$$

Since $\Delta\colon G \to \widetilde{G}$ is a group homomorphism, it satisfies $\Delta(1) = 1$, hence the induced map $\Delta\colon\ G\text{-}\mathtt{Zip}^\mu \to \widetilde{G}\text{-}\mathtt{Zip}^{\widetilde{\mu}}$ is dominant (1 lies in the open zip stratum). Therefore, pullback via Δ induces an injection on the spaces of global sections:

$$\Delta^*\colon H^0(\widetilde{G}\text{-}\mathtt{Zip}^{\widetilde{\mu}}, \mathcal{V}(\widetilde{\rho})) \to H^0(G\text{-}\mathtt{Zip}^\mu, \bigotimes_{i=1}^r \mathcal{V}(\rho_i)).$$

In particular, let (V, ρ) be an L-representation and let $\rho_0\colon L \to \{1\}$ be the trivial character of L. Put $\rho_1 = \rho$ and $\rho_i = \rho_0$ for all $2 \leq i \leq r$. We obtain an injection

$$\Delta^*\colon H^0(\widetilde{G}\text{-}\mathtt{Zip}^{\widetilde{\mu}}, \mathcal{V}(\mathrm{pr}_1^* \rho)) \to H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) \tag{6.3.1}$$

where $\mathrm{pr}_1\colon \widetilde{L} \to L$ is the first projection and $\mathrm{pr}_1^* \rho$ is the $\widetilde{L}$-representation $\rho \circ \mathrm{pr}_1$. Fix $r \geq 1$ such that P is defined over $\mathbb{F}_{q^r}$. We apply Proposition 6.2.4 to $\mathrm{pr}_1^* \rho$. In this case, for each $1 \leq j \leq r$, the parabolic subgroup P'_j is equal to $P_0 = \bigcap_{i \in \mathbb{Z}} \sigma^i(P)$, the largest parabolic subgroup defined over $\mathbb{F}_q$ contained in P. Let $L_0 \subset P_0$ be the Levi subgroup containing T, as in (2.4.3). The space $V^{\#}$ is clearly $V^{R_{\mathrm{u}}(P_0)}$. Any weight of the $\widetilde{T}$-representation $\mathrm{pr}_1^* \rho$ is of the form $\widetilde{\chi} = (\chi, 0, \ldots, 0)$ where χ is a T-weight of V. Hence, for each $1 \leq j \leq r$, we have $S_j(\widetilde{\chi}) = q^{r-j+1} \sigma^{-(r-j+1)} \chi$. Thus, $V_{\geq 0}^{\cap}$ is the direct sum of T-weight spaces V_χ satisfying $\langle \sigma^{-(r-j+1)} \chi, \delta_\alpha^{(r)} \rangle \leq 0$ for all $\alpha \in \Delta^{P_0}$ and all $1 \leq j \leq r$ (here $\delta_{j,\alpha}^{(r)}$ is independent of j, so we denote it simply by $\delta_\alpha^{(r)}$). But since P_0 is defined over $\mathbb{F}_q$, this condition is also equivalent to $\langle \chi, \delta_\alpha^{(r)} \rangle \leq 0$ for all $\alpha \in \Delta^{P_0}$. Note that $V_{\geq 0}^{\cap}$ is very close to the space $V_{\geq 0}^{\Delta^{P_0}}$, the only difference being that δ_α is replaced by $\delta_\alpha^{(r)}$ in the definition. In other words, we could say that $V^{\cap}_{\geq 0} = V_{\geq 0}^{\Delta^{P_0 \otimes \mathbb{F}_{q^r}}}$, where we changed P_0 to $P_0 \otimes \mathbb{F}_{q^r}$. To simplify notation, for any L-representation (V, ρ) define

$$V_{\geq 0}^{\Delta^{P_0},(r)} := \bigoplus_{\langle \nu, \delta_\alpha^{(r)} \rangle \geq 0, \ \forall \alpha \in \Delta^{P_0}} V_\nu.$$

We showed that $V^{\cap}_{\geq 0} = V_{\geq 0}^{\Delta^{P_0},(r)}$. Denote by $L_\varphi^{(r)}$ the image of $\mathrm{Stab}_{\widetilde{E}}(1)$ via the composition of the projection $\widetilde{E} \to \widetilde{P}$ and the first projection $\mathrm{pr}_1 \colon \widetilde{P} \to P$. By Lemma 2.4.1, we have $L_\varphi^{(r)} \subset L$. We deduce from Proposition 6.2.4:

$$V^{L_\varphi^{(r)}} \cap V_{\geq 0}^{\Delta^{P_0},(r)} \cap V^{R_{\mathrm{u}}(P_0)} \subset H^0(\widetilde{G}\text{-}\mathtt{Zip}^{\widetilde{\mu}}, \mathcal{V}(\mathrm{pr}_1^*(\rho))). \tag{6.3.2}$$

The largest Levi subgroup of $\widetilde{G}$ defined over $\mathbb{F}_q$ contained in $\widetilde{L}$ is $\widetilde{L}_0 := \mathrm{Res}_{\mathbb{F}_{q^r}/\mathbb{F}_q} L_0$. Since $\widetilde{L}_0(\mathbb{F}_q) = L_0(\mathbb{F}_{q^r})$, we have $L_\varphi^{(r)} = L_\varphi^{(r),\circ} \rtimes L_0(\mathbb{F}_{q^r})$ by Lemma 2.4.1. Furthermore, Δ induces an injection $\Delta \colon L_\varphi \to L_\varphi^{(r)}$. Combining (6.3.2) with (6.3.1), we deduce:

Theorem 6.3.1. *Let $r \geq 1$ such that P is defined over $\mathbb{F}_{q^r}$. One has*

$$V^{L_\varphi^{(r)}} \cap V_{\geq 0}^{\Delta^{P_0},(r)} \cap V^{R_u(P_0)} \subset H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\rho)). \tag{6.3.3}$$

This theorem is slightly weaker than Proposition 5.1.3, but holds in general, independently of Condition 5.1.1. Put $V_{\mathrm{Weil}}^{(r)} := V^{L_\varphi^{(r)}} \cap V_{\geq 0}^{\Delta^{P_0},(r)} \cap V^{R_{\mathrm{u}}(P_0)}$.

6.4 Applications to C_{zip}

Consider the L-representation $V = V_I(\lambda)$ for $\lambda \in X^*_{+,I}(T)$. Let $r \geq 1$ such that P is defined over $\mathbb{F}_{q^r}$. Consider the sub-L_0-representation $V_{I_0}(\lambda_0) \subset V_I(\lambda)$ with $\lambda_0 := w_{0,I_0} w_{0,I} \lambda$. Then, we have $V^{R_{\mathrm{u}}(P_0)} = V_{I_0}(\lambda_0)$. Let Q_0 be the opposite parabolic to P_0 with Levi subgroup L_0. Let $\mu_0 \colon \mathbb{G}_{\mathrm{m},k} \to G_k$ be any dominant cocharacter with centralizer L_0 (hence μ_0 defines the parabolics P_0, Q_0). If we base-change G to $\mathbb{F}_{q^r}$, we have by Corollary 2.4.4:

$$\begin{aligned} H^0(G_{\mathbb{F}_{q^r}}\text{-}\mathtt{Zip}^{\mu_0}, \mathcal{V}_{I_0}(\lambda_0)) &= V_{I_0}(\lambda_0)^{L_0(\mathbb{F}_{q^r})} \cap V_{I_0}(\lambda_0)_{\geq 0}^{\Delta^{P_0},(r)} \\ &= V^{L_0(\mathbb{F}_{q^r})} \cap V_{\geq 0}^{\Delta^{P_0},(r)} \cap V^{R_{\mathrm{u}}(P_0)} \end{aligned} \tag{6.4.1}$$

Hence, the space $V_{\mathrm{Weil}}^{(r)}$ given in (6.3.3) is very close to the space (6.4.1). The only difference is that we take invariants under $L_\varphi^{(r)} = L_\varphi^{(r),\circ} \rtimes L_0(\mathbb{F}_{q^r})$ instead of $L_0(\mathbb{F}_{q^r})$.

Fix $m \geq 1$ such that the finite unipotent group $L_\varphi^{(r),\circ}$ is annihilated by φ^m. If $f \in H^0(G_{\mathbb{F}_{q^r}}\text{-}\mathtt{Zip}^{\mu_0}, \mathcal{V}_{I_0}(\lambda_0))$, then f^{q^m} is stable by $L_\varphi^{(r)}$, and hence lies in $V_I(q^m\lambda)_{\mathrm{Weil}}^{(r)}$. We deduce the following: Assume that $\lambda \in X^*_{+,I}(T)$ satisfies $\lambda_0 \in C_{\mathsf{zip}}(G_{\mathbb{F}_{q^r}}, \mu_0)$, where $C_{\mathsf{zip}}(G_{\mathbb{F}_{q^r}}, \mu_0)$ is the zip cone of the zip datum $(G_{\mathbb{F}_{q^r}}, \mu_0)$. Then $\lambda \in \mathcal{C}_{\mathsf{zip}}$. We have shown

Theorem 6.4.1. *Assume that P is defined over $\mathbb{F}_{q^r}$. Then*

$$X^*_{+,I}(T) \cap \Big(w_{0,I} w_{0,I_0} \mathcal{C}_{\mathsf{zip}}(G_{\mathbb{F}_{q^r}}, \mu_0) \Big) \subset \mathcal{C}_{\mathsf{zip}}.$$

Remark 6.4.2. We can apply all results and constructions about the zip cone to $(G_{\mathbb{F}_{q^r}}, \mu_0)$. For example, consider the highest weight cone of $(G_{\mathbb{F}_{q^r}}, \mu_0)$. We deduce from Theorem 6.4.1 and Proposition 3.5.1 that if $\lambda \in X^*_{+,I}(T)$ satisfies

$$\sum_{w \in W_{L_0}(\mathbb{F}_q)} q^{r\ell(w)} \ \langle w\lambda_0, \alpha^\vee \rangle \leq 0, \quad \forall \alpha \in \Delta^{P^0},$$

then $\lambda \in \mathcal{C}_{\mathsf{zip}}$. This is slightly weaker than Theorem 5.2.1, but holds without any assumption on (G, μ).

We can finally prove in general:

Theorem 6.4.3. *One has* $\mathcal{C}_{\mathsf{GS}} \subset \mathcal{C}_{\mathsf{zip}}$.

Proof. Write $\mathcal{C}_{\mathsf{GS},I} = \mathcal{C}_{\mathsf{GS}}$ and $\mathcal{C}_{\mathsf{GS},I_0}$ for the Griffiths–Schmid cones of I and I_0 respectively. By Lemma 3.7.2, we have $\mathcal{C}_{\mathsf{GS},I_0} \subset \mathcal{C}_{\mathsf{zip}}(G_{\mathbb{F}_{q^r}}, \mu_0)$. Since $w_{0,I}w_{0,I_0}\mathcal{C}_{\mathsf{GS},I_0} = \mathcal{C}_{\mathsf{GS},I}$, the result follows from Theorem 6.4.1. $\square$

7 Examples

7.1 The case $G = U(2,1)$ with p inert

We consider the example of Picard modular surfaces. More precisely, let $\mathbf{E}/\mathbb{Q}$ be a quadratic totally imaginary extension and $(\mathbf{V}, \psi)$ a hermitian space over $\mathbf{E}$ of dimension 3 such that $\psi_{\mathbb{R}}$ has signature $(2,1)$. There is a Shimura variety of dimension 2 of PEL-type attached to $\mathbf{G} = \mathrm{GU}(\mathbf{V}, \psi)$. It parametrizes abelian varieties of dimension 3 with a polarization, an action of $\mathcal{O}_{\mathbf{E}}$ and a level structure. Let p be a prime of good reduction, and let X be the special fiber of the Kisin–Vasiu (canonical) integral model of the Shimura variety. By (2.6.1), we have a smooth, surjective morphism $\zeta\colon X \to G\text{-}\mathtt{Zip}^{\mu}$, where G is the special fiber of a reductive $\mathbb{Z}_p$-model of $\mathbf{G}_{\mathbb{Q}_p}$. In this section, we study the cones attached to $G\text{-}\mathtt{Zip}^{\mu}$ when p is inert in $\mathbf{E}$. To simplify, we consider the case of a unitary group $G = \mathrm{U}(V, \psi)$ (the case of $G = \mathrm{GU}(V, \psi)$ is very similar).

Let (V, ψ) be a 3-dimensional vector space over $\mathbb{F}_{q^2}$ endowed with a non-degenerate hermitian form $\psi\colon V \times V \to \mathbb{F}_{q^2}$ (in the context of Shimura varieties, take $q = p$). Write $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) = \{\mathrm{Id}, \sigma\}$. Choose a basis $\mathcal{B} = (v_1, v_2, v_3)$ of V where ψ is given by the matrix

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

We define a reductive group G by

$$G(R) = \{f \in \mathrm{GL}_{\mathbb{F}_{q^2}}(V \otimes_{\mathbb{F}_q} R) \mid \psi_R(f(x), f(y)) = \psi_R(x, y),\ \forall x, y \in V \otimes_{\mathbb{F}_q} R\}$$

for any $\mathbb{F}_q$-alegebra R. There is an isomorphism $G_{\mathbb{F}_{q^2}} \simeq \mathrm{GL}(V) \simeq \mathrm{GL}_{3,\mathbb{F}_{q^2}}$. It is induced by the $\mathbb{F}_{q^2}$-algebra isomorphism $\mathbb{F}_{q^2} \otimes_{\mathbb{F}_q} R \to R \times R$, $a \otimes x \mapsto (ax, \sigma(a)x)$ (where $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) = \{\mathrm{Id}, \sigma\}$). The corresponding action of σ on $\mathrm{GL}_3(k)$ is given by $\sigma \cdot A = J\sigma({}^tA)^{-1}J$. Let T denote the diagonal torus and B the lower-triangular Borel subgroup of G_k (note that B and T are defined over $\mathbb{F}_q$). Identify $X^*(T) = \mathbb{Z}^3$ such that $(a_1, a_2, a_3) \in \mathbb{Z}^3$ corresponds to the character $\mathrm{diag}(x_1, x_2, x_3) \mapsto \prod_{i=1}^{3} x_i^{a_i}$. The simple roots are $\Delta = \{e_1 - e_2, e_2 - e_3\}$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{Z}^3$. Define a cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},k} \to G_k$ by $x \mapsto \mathrm{diag}(x, x, 1)$ via the identification $G_k \simeq \mathrm{GL}_{3,k}$. Let $\mathcal{Z}_{\mu} = (G, P, L, Q, M, \varphi)$ be the associated zip datum. We have $\Delta^P = \{e_2 - e_3\}$. Note that the determinant $\det\colon \mathrm{GL}_{3,k} \to \mathbb{G}_{\mathrm{m},k}$ is an invertible section of the line bundle $\mathcal{V}_I(p+1, p+1, p+1)$ on $G\text{-}\mathtt{Zip}^{\mu}$. Set $D := \mathbb{Z}(1,1,1) = X^*(G)$. We have $D \subset \mathcal{C}_{\mathsf{zip}}$. Identify

$$\mathbb{Z}^3/D \simeq \mathbb{Z}^2, \quad (a_1, a_2, a_3) \mapsto (a_1 - a_3, a_2 - a_3). \tag{7.1.1}$$

Hence, subcones of $\mathbb{Z}^3$ containing D correspond bijectively to subcones of $\mathbb{Z}^2$ via (7.1.1). For a subcone C of $\mathbb{Z}^3$ containing D and a subcone $C' \subset \mathbb{Z}^2$, we write $C \leftrightarrow C'$ if they correspond via the bijection (7.1.1).

Proposition 7.1.1. *Via this identification, we have*

$$\begin{aligned}
X^*_{+,I}(T) &\leftrightarrow \{(a_1,a_2)\in\mathbb{Z}^2 \mid \ a_1\geq a_2\}\\
X^*_-(L) &\leftrightarrow \mathbb{N}(-1,-1)\\
\mathcal{C}_{\mathsf{GS}} &\leftrightarrow \{(a_1,a_2)\in X^*_{+,I}(T) \mid \ 0\geq a_1\}\\
\mathcal{C}_{\mathsf{zip}} &\leftrightarrow \{(a_1,a_2)\in X^*_{+,I}(T) \mid \ (q-1)a_1+a_2\leq 0\}\\
\mathcal{C}_{\mathsf{pHa}} &\leftrightarrow \{(a_1,a_2)\in X^*_{+,I}(T) \mid \ qa_1-(q-1)a_2\geq 0 \textit{ and } (q-1)a_1+a_2\leq 0\}\\
\mathcal{C}_{\mathsf{hw}} &\leftrightarrow \{(a_1,a_2)\in X^*_{+,I}(T) \mid \ qa_1-(q-1)a_2\leq 0\}\\
\mathcal{C}_{\mathsf{lw}} &= \mathcal{C}_{\mathsf{zip}}.
\end{aligned}$$

Proof. The cone $\mathcal{C}_{\mathsf{zip}}$ was determined in [IK21, Corollary 6.3.3]. The rest is a straightforward computation. □

This example is not of Hasse-type since P is not defined over $\mathbb{F}_q$. As predicted by Proposition 4.1.5, $C_{\mathsf{pHa},\mathbb{R}_{\geq 0}}$ is not a neighborhood of $X^*_-(L)_{\mathrm{reg}}$ in $X^*_{+,I}(T)_{\mathbb{R}_{\geq 0}}$. Condition 5.1.1 is satisfied, and we have indeed $\mathcal{C}_{\mathsf{GS}}\subset\mathcal{C}_{\mathsf{lw}}$ (Lemma 5.2.4). However, $\mathcal{C}_{\mathsf{GS}}\subset\mathcal{C}_{\mathsf{hw}}$ does not hold. For this group, Conjecture 3.3.3 holds by [GK22a, Theorem 4.3.3], i.e. we have $\mathcal{C}(\overline{\mathbb{F}}_p)=\mathcal{C}_{\mathsf{zip}}$.

7.2 The orthogonal group $\mathrm{SO}(2n+1)$

We consider the case of odd orthogonal groups. This example arises in the theory of Shimura varieties of Hodge-type attached to general spin groups $\mathrm{GSpin}(2n-1,2)$ $(n\geq 1)$. This furnishes an interesting infinite family of examples of zip data of Hasse-type (Definition 4.1.6). To simplify, we only consider the case of odd special orthogonal groups $\mathrm{SO}(2n+1)$, which is completely similar. Assume $p>2$. Let J be the symmetric square matrix of size $2n+1$ defined by

$$J := \begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Let $n\geq 1$ and let G be the reductive, connected, algebraic group over $\mathbb{F}_q$ defined by

$$G(R) := \{A\in \mathrm{SL}_{2n+1}(R) \mid {}^tAJA=J\}$$

for all $\mathbb{F}_q$-algebra R. Let T be the maximal diagonal torus, given by matrices of the form $t=\mathrm{diag}(t_1,\dots,t_n,1,t_n^{-1},\dots,t_1^{-1})$. Identify $X^*(T)\simeq\mathbb{Z}^n$ such that $(a_1,\dots,a_n)\in\mathbb{Z}^n$ corresponds to $t\mapsto t_1^{a_1}\dots t_n^{a_n}$. Let $e_1,\dots,e_n$ be the canonical basis of $\mathbb{Z}^n$. Fix the Borel subgroup of lower-triangular matrices in G. The positive roots Φ^+ and the simple roots Δ are respectively

$$\begin{aligned}
\Phi^+ &:= \{e_i\pm e_j,\ 1\leq i<j\leq n\}\cup\{e_i,\ 1\leq i\leq n\},\\
\Delta &:= \{e_1-e_2,\dots,e_{n-1}-e_n,e_n\}.
\end{aligned}$$

The Weyl group identifies as the group of permutations σ of $\{1,\dots,2n+1\}$ satisfying $\sigma(i)+\sigma(2n+2-i)=2n+2$. In particular, we have $\sigma(n+1)=n+1$. Moreover, σ is entirely determined by $\sigma(1),\dots,\sigma(n)$. For $\sigma\in W$ such that $\sigma(i)=a_i$ for $i=1,\dots,n$, write $\sigma=[a_1\ \dots\ a_n]$. Hence, the identity element is $[1\ 2\dots\ n]$ and the longest element is $w_0=[2n+1\ 2n\dots\ n+2]$. The action of w_0 on $X^*(T)$ is given by $w_0\lambda=-\lambda$. Consider the cocharacter

$$\mu\colon z\mapsto \mathrm{diag}(z,1,\dots,1,z^{-1}).$$

Let $\mathcal{Z}_\mu := (G, P, L, Q, M, \varphi)$ be the zip datum attached to μ (since μ is defined over $\mathbb{F}_q$ we have $M = L$). For $n \geq 2$, one has:

$$I = \Delta \setminus \{e_1 - e_2\}, \quad \Delta^P = \{e_1 - e_2\}$$

(for $n = 1$, one has $I = \emptyset$, $\Delta^P = \Delta = \{e_1\}$). The Levi L is isomorphic to $\mathrm{SO}(2n-1) \times \mathbb{G}_{\mathrm{m}}$. In particular, $w_{0,I}$ acts on I by $w_{0,I}\alpha = -\alpha$. Since T is $\mathbb{F}_q$-split, one has $\sigma(\alpha) = \alpha = -w_{0,I}\alpha$ for all $\alpha \in I$. This shows that (G, μ) is of Hasse-type. Put $z := w_{0,I} w_0 = [2n+1 \ 2 \ \dots \ n]$. Then (B, T, z) is a frame for $\mathcal{Z}_\mu$ (Lemma 2.2.3). We determine the cones appearing in Diagram (3.7.1).

Proposition 7.2.1. *For $n \geq 2$, we have*

$$\begin{aligned}
X^*_{+,I}(T) &= \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \ a_2 \geq \dots \geq a_n \geq 0.\} \\
X^*(L)_- &= \mathbb{Z}_{\leq 0}(1, 0, \dots, 0) \\
\mathcal{C}_{\mathsf{GS}} &= \{(a_1, \dots, a_n) \in \mathbb{Z}^n \in X^*_{+,I}(T) \mid \ a_1 + a_2 \leq 0\} \\
\mathcal{C}_{\mathsf{pHa}} &= \{(a_1, \dots, a_n) \in X^*_{+,I}(T) \mid \ (q+1)a_1 + (q-1)a_2 \leq 0\} \\
\mathcal{C}_{\mathsf{zip}} &= \mathcal{C}_{\mathsf{pHa}} \\
\mathcal{C}_{\mathsf{hw}} = \mathcal{C}_{\mathsf{lw}} &= \{(a_1, \dots, a_n) \in X^*_{+,I}(T) \mid \ (q^{2n-2} - 1)a_1 \leq (q-1)\sum_{i=2}^{n}(q^{i-2} - q^{2n-1-i})a_i\}.
\end{aligned}$$

Proof. The equality $\mathcal{C}_{\mathsf{zip}} = \mathcal{C}_{\mathsf{pHa}}$ follows from Theorem 4.3.1. Since P is defined over $\mathbb{F}_q$, we have $\mathcal{C}_{\mathsf{hw}} = \mathcal{C}_{\mathsf{lw}}$. The only nontrivial computation is $\mathcal{C}_{\mathsf{hw}}$. Since T is split over $\mathbb{F}_q$, we can use [Kos19, §3.6] (changing p to q). Put $\alpha = e_1 - e_2$. Denote by $L_\alpha \subset L$ the centralizer in L of $\alpha^\vee$, and $I_\alpha \subset I$ its type. Then $\mathcal{C}_{\mathsf{hw}}$ is the set of $\lambda \in X^*_{+,I}(T)$ satisfying

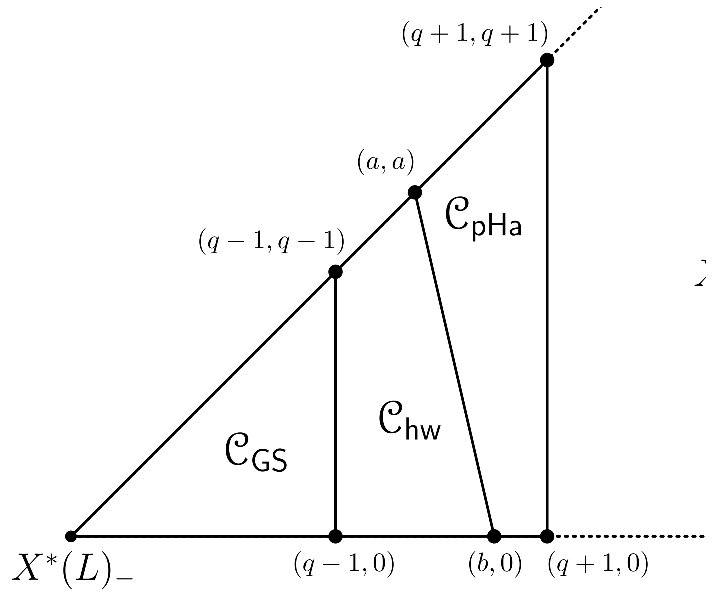
$$\sum_{w \in {}^{I_\alpha}W_I} q^{\ell(w)} \langle w\lambda, \alpha^\vee \rangle \leq 0. \tag{7.2.1}$$

We only carry out the case $n \geq 3$. The set ${}^{I_\alpha}W_I$ has cardinality $2(n-1)$. Any permutation $w \in {}^{I_\alpha}W_I$ is entirely determined by $w^{-1}(2)$, and it can be any integer $2 \leq w^{-1}(2) \leq 2n$ different from $n+1$. Writing $w^{-1}(2) = i$, there are two cases to consider: $2 \leq i \leq n$ and $n+2 \leq i \leq 2n$. In the first case, the length of w is $i-2$ and one has $\langle \lambda, w^{-1}\alpha^\vee \rangle = a_1 - a_i$ (where $\lambda = (a_1, \dots, a_n)$). In the second case, the length of w is $i-3$ and $\langle \lambda, w^{-1}\alpha^\vee \rangle = a_1 + a_{2n+2-i}$. Hence we find that the sum in (7.2.1) is equal to

$$\sum_{i=2}^{n} q^{i-2}(a_1 - a_i) + \sum_{i=n+2}^{2n} q^{i-3}(a_1 + a_{2n+2-i}) = \frac{q^{2n-2} - 1}{q-1} a_1 - \sum_{i=2}^{n}(q^{i-2} - q^{2n-1-i})a_i.$$

The result follows. $\square$

As predicted by Theorem 4.3.1, one sees that $\mathcal{C}_{\mathsf{pHa}}$ contains all cones of Proposition 7.2.1 (except of course $X^*_{+,I}(T)$). For example, assume that $\lambda \in \mathcal{C}_{\mathsf{hw}}$. We find $\frac{q^{2n-2}-1}{q-1}a_1 \leq \sum_{i=2}^{n}(q^{i-2} - q^{2n-1-i})a_i \leq (1 - q^{2n-3})a_2$, and hence $\frac{q^{2n-2}-1}{q^{2n-3}-1}a_1 + (q-1)a_2 \leq 0$. In particular, this implies $a_1 \leq 0$. Since $q + 1 \geq \frac{q^{2n-2}-1}{q^{2n-3}-1}$, we have $(q+1)a_1 + (q-1)a_2 \leq 0$. This shows $\mathcal{C}_{\mathsf{hw}} \subset \mathcal{C}_{\mathsf{pHa}}$ (for $n = 2$ one has actually $\mathcal{C}_{\mathsf{hw}} = \mathcal{C}_{\mathsf{pHa}}$). Here is a representation of the cones for $n = 3$. We represent the intersections with the affine hyperplane $a_1 = -(q-1)$. In other words, the weight $(-(q-1), x, y)$ appears as the point (x, y). Set $a := \frac{q^4-1}{q^3+q^2-q-1}$ and $b := \frac{q^4-1}{q^3-1}$ (hence we have $q - 1 < a < b < q + 1$).



A Appendix: Classification of Hasse-type zip data (by Wushi Goldring)

In this appendix, we classify the pairs (G,μ) which are of Hasse-type (Definition 4.1.6).

A.1 Hasse-type Dynkin triples

Let $\mathfrak{D}$ be a Dynkin diagram, let $\sigma \in \mathrm{Aut}(\mathfrak{D})$ be a diagram automorphism and let $\mathfrak{I} \subset \mathfrak{D}$ be a σ-stable sub-diagram. In this appendix, we classify such Dynkin triples $(\mathfrak{D},\mathfrak{I},\sigma)$ satisfying the following:

Condition A.1.1. *The actions of σ and the opposition involution $-w_{0,\mathfrak{I}}$ of $\mathfrak{I}$ on $\mathfrak{I}$ coincide.*

The case $\mathfrak{I}=\mathfrak{D}$ is allowed; if $\mathfrak{I}=\mathfrak{D}$, then the conditions A.1.1 hold precisely when the opposition involution of $\mathfrak{D}$ is trivial: $-w_0 := -w_{0,\mathfrak{D}} = 1$.

A.2 Translation

In the setting of 4.1.5, let $\mathfrak{D}$ denote the Dynkin diagram of the simple roots Δ associated to (G,B,T) and let $\mathfrak{I}$ denote the Dynkin sub-diagram of the type $I \subset \Delta$ of the parabolic $P \supset B$. Then the triples $(\mathfrak{D},\mathfrak{I},\sigma)$ satisfying A.1.1 are precisely those which arise from Hasse-type zip data, as characterized by the root-data-theoretic condition 4.1.5(4.1.5).

A.3 Classification

The classification is broken up into two cases, according to whether or not σ is trivial, see A.3.1 and A.3.4. The special cases where $(\mathfrak{D},\mathfrak{I},\sigma)$ is maximal and of Hodge-type are singled out in §A.4 and §A.5 respectively. Proofs are given in §A.6.

Isolated vertices of $\mathfrak{I}$ We call a vertex $v \in \mathfrak{I}$ isolated if its connected component in $\mathfrak{I}$ is $\{v\}$. Assume that $(\mathfrak{D},\mathfrak{I},\sigma)$ satisfies A.1.1. If σ fixes an isolated vertex v of $\mathfrak{I}$, then it is clear that $(\mathfrak{D},\mathfrak{I}\setminus\{v\},\sigma)$ also satisfies A.1.1. More generally, if $v \in \mathfrak{I}$ is isolated but not necessarily fixed by σ, then the orbit $\langle\sigma\rangle v$ consists of isolated vertices of $\mathfrak{I}$ and $(\mathfrak{D},\mathfrak{I},\sigma)$

satisfies A.1.1 if and only if $(\mathfrak{D}, \mathfrak{I} \setminus \langle\sigma\rangle v, \sigma)$ does. For this reason, it suffices to consider triples $(\mathfrak{D}, \mathfrak{I}, \sigma)$ where $\mathfrak{I}$ contains no isolated vertices.

Let $\mathfrak{I}^{\geq 2} \subset \mathfrak{I}$ denote the sub-diagram consisting of all connected components with at least two vertices (i.e., the sub-diagram obtained by throwing out all the isolated vertices). When $\mathfrak{I}$ consists entirely of isolated vertices, write $\mathfrak{I}^{\geq 2} = \emptyset$.

Theorem A.3.1. *Assume that $\mathfrak{D}$ is connected, that $\sigma = 1$ and that $\mathfrak{I}^{\geq 2} \neq \emptyset$. Then A.1.1 holds precisely in the following cases:*

(1) *$\mathfrak{D}$ is of type B_n or C_n ($n \geq 2$) and $\mathfrak{I}^{\geq 2}$ is of type B_m or C_m respectively, for some $2 \leq m \leq n$; in other words $\mathfrak{I}^{\geq 2}$ is connected and contains the multi-laced vertices.*

(2) *$\mathfrak{D}$ is of type D_4 or G_2 and $\mathfrak{D} = \mathfrak{I}^{\geq 2}$.*

(3) *$\mathfrak{D}$ is of type D_n ($n \geq 5$) and $\mathfrak{I}^{\geq 2}$ is the unique sub-diagram of type D_{2m} for some $2 \leq m \leq n/2$; in other words $\mathfrak{I}^{\geq 2}$ is connected of even size and contains the two extremal vertices which are permuted by* $\mathrm{Aut}(\mathfrak{D})$.

(4) *$\mathfrak{D}$ is of type F_4 and $\mathfrak{I}^{\geq 2}$ is the unique sub-diagram of type $B_2 \cong C_2$ B_3, C_3 or F_4.*

(5) *$\mathfrak{D}$ is of type E_6 and $\mathfrak{I}^{\geq 2} \cong D_4$.*

(6) *$\mathfrak{D}$ is of type E_7 and the type of $\mathfrak{I}^{\geq 2}$ is either D_4, D_6 or E_7.*

(7) *$\mathfrak{D}$ is of type E_8 and the type of $\mathfrak{I}^{\geq 2}$ is either D_4, D_6, E_7 or E_8.*

The following two remarks explain how A.1.1 behaves when $\mathfrak{D}$ is disconnected:

Remark A.3.2. If $\mathfrak{D} = \bigsqcup \mathfrak{D}_{i=1}^{m}$ has multiple connected components $\mathfrak{D}_i$ and $\sigma = 1$, then $(\mathfrak{D}, \mathfrak{I}, \sigma = 1)$ satisfies A.1.1 if and only if every component $(\mathfrak{D}_i, \mathfrak{I} \cap \mathfrak{D}_i, \sigma = 1)$ does.

Remark A.3.3. Assume that σ permutes the connected components of $\mathfrak{D}$ non-trivially, i.e., that there exists distinct components $\mathfrak{D}_i$ and $\mathfrak{D}_j$ of $\mathfrak{D}$ with $\sigma\mathfrak{D}_i = \mathfrak{D}_j$. Then a triple $(\mathfrak{D}, \mathfrak{I}, \sigma)$ satisfies A.1.1 if and only if the type $\mathfrak{I} = \emptyset$ (i.e., if and only if the parabolic $P = B$ is a Borel). Indeed, the opposition involution $-w_{0,\mathfrak{I}}$ leaves stable each component $\mathfrak{D}_i$ (and acts on $\mathfrak{D}_i$ by the ith component $-w_{0,\mathfrak{D}_i \cap \mathfrak{I}}$ of $-w_{0,\mathfrak{I}}$).

Theorem A.3.4. *Assume that $\sigma \neq 1$, that $\mathfrak{I}^{\geq 2} \neq \emptyset$ and that A.1.1 holds. Then $\mathfrak{D}$ is necessarily connected, of type A_n ($n \geq 2$), D_n ($n \geq 4$) or E_6. More precisely, the conditions $\sigma \neq 1$, $\mathfrak{I}^{\geq 2} \neq \emptyset$ and A.1.1 hold jointly precisely in the following cases:*

(1) *$\mathfrak{D}$ is of type A_n for some $n \geq 2$, the automorphism $\sigma = -w_0$ is the opposition involution of $\mathfrak{D}$ and there is some m satisfying $2 \leq m \leq n$ and $m \equiv n \pmod 2$ such that $\mathfrak{I}^{\geq 2}$ is the unique σ-stable sub-diagram of type A_m.*

(2) *$\mathfrak{D}$ is of type D_4, the automorphism σ is one of the three transpositions in $\mathrm{Aut}(\mathfrak{D}) \cong S_3$ and $\mathfrak{I}^{\geq 2} = \mathfrak{I} \cong D_3 \cong A_3$ is obtained by removing any one of the three extremal vertices.*

(3) *$\mathfrak{D}$ is of type D_n for some $n \geq 5$, $\sigma \in \mathrm{Aut}(\mathfrak{D})$ is the unique nontrivial element and $\mathfrak{I}^{\geq 2}$ is the unique sub-diagram of type D_{2m+1}, for some $1 \leq m \leq (n-1)/2$.*

(4) *$\mathfrak{D}$ is of type E_6, $\sigma = -w_0$ is the opposition involution and $\mathfrak{I}^{\geq 2}$ is the unique $-w_0$-stable sub-diagram of type A_3, A_5 or E_6.*

A.4 Special cases I: Maximal Dynkin triples

Definition A.4.1. *We say that a Dynkin pair $(\mathfrak{D}, \mathfrak{I})$ is <u>maximal</u> if, for every connected component $\mathfrak{D}_i$ of $\mathfrak{D}$, either $\mathfrak{D}_i \cap \mathfrak{I} = \mathfrak{D}_i$ or $\mathfrak{D}_i \cap \mathfrak{I}$ is a proper, maximal sub-diagram of $\mathfrak{D}$, i.e.,* $\mathrm{Card}(\mathfrak{D}_i \cap \mathfrak{I}) = \mathrm{Card}(\mathfrak{D}_i) - 1$. *We say that a Dynkin triple $(\mathfrak{D}, \mathfrak{I}, \sigma)$ is maximal if the underlying pair $(\mathfrak{D}, \mathfrak{I})$ is.*

Remark A.4.2. By definition, a Dynkin pair $(\mathfrak{D}, \mathfrak{I})$ is maximal if and only if this is true of every component $(\mathfrak{D}_i, \mathfrak{D}_i \cap \mathfrak{I})$.

Remark A.4.3. If the Dynkin triple $(\mathfrak{D}, \mathfrak{I}, \sigma)$ arises from (G, μ, P, B, T) as in A.2, then $(\mathfrak{D}, \mathfrak{I})$ is maximal if and only if, for every nontrivial, minimal, normal, connected k-subgroup G_i of G, the Levi subgroup $L = \mathrm{Cent}(\mu) \cap G_i$ of G_i is either all of G_i or a proper, maximal Levi of G_i.

A σ-orbit of a Dynkin triple $(\mathfrak{D}, \mathfrak{I}, \sigma)$ is a Dynkin triple $(\mathfrak{D}', \mathfrak{I}', \sigma')$ such that $\mathfrak{D}'$ is a σ-orbit of connected components of $\mathfrak{D}$, $\mathfrak{I}' := \mathfrak{D}' \cap \mathfrak{I}$ and $\sigma' = \sigma_{|\mathfrak{D}'} \in \mathrm{Aut}(\mathfrak{D}')$.

Corollary A.4.4. *A maximal Dynkin triple $(\mathfrak{D}, \mathfrak{I}, \sigma)$ satisfies A.1.1 if and only if every σ-orbit $(\mathfrak{D}', \mathfrak{I}', \sigma')$ is one of the following:*
(1) *$\mathfrak{D}'$ is of type A_1^m for some $m \geq 1$, $\mathfrak{I}' = \emptyset$ and necessarily σ' permutes the m components of $\mathfrak{D}'$ cyclically.*
(2) *$\mathfrak{D}'$ is of type A_2, $\mathfrak{I}'$ is of type A_1 and necessarily $\sigma' = 1$.*
(3) *$\mathfrak{D}'$ is of type B_n (resp. C_n) for some $n \geq 2$, necessarily $\sigma' = 1$ and $\mathfrak{I}'$ is the unique sub-diagram of type B_{n-1} (resp. C_{n-1}).*
(4) *$\mathfrak{D}'$ is of type D_4, σ' has order 2 and $\mathfrak{I}' \cong D_3 \cong A_3$ is any one of the three sub-diagrams obtained by removing an extremal vertex.*
(5) *$\mathfrak{D}'$ is of type D_{2m} for some $m \geq 3$, $\sigma' \neq 1$ is the unique nontrivial element and $\mathfrak{I}'$ is the unique sub-diagram of type D_{2m-1}.*
(6) *$\mathfrak{D}'$ is of type D_{2m+1} for some $m \geq 2$, $\sigma' = 1$ and $\mathfrak{I}'$ is the unique sub-diagram of type D_{2m}.*
(7) *$\mathfrak{D}'$ is of type G_2 (resp. F_4), necessarily $\sigma' = 1$ and $\mathfrak{I}'$ is obtained by removing an extremal vertex, so $\mathfrak{I}'$ is of type A_1 (resp. B_3 or C_3).*
(8) *$\mathfrak{D}'$ is of type E_6, $\sigma' = -w_0$ and $\mathfrak{I}'$ is the unique sub-diagram of type A_5.*
(9) *$\mathfrak{D}'$ is of type E_7, necessarily $\sigma' = 1$ and $\mathfrak{I}'$ is the unique sub-diagram of type D_6.*
(10) *$\mathfrak{D}'$ is of type E_8, necessarily $\sigma' = 1$ and $\mathfrak{I}'$ is the unique sub-diagram of type E_7.*

A.5 Special cases II: Dynkin triples of Hodge-type

Let $(\mathbf{G}, \mathbf{X})$ be a Hodge-type Shimura datum, i.e., a Shimura datum which admits a symplectic embedding $(\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathrm{GSp}(2g), \mathbf{X}_g)$ into a Siegel-type Shimura datum for some $g \geq 1$. For every prime p such that $\mathbf{G}_{\mathbb{Q}_p}$ is unramified, the process recalled in §2.6 produces a connected, reductive $\mathbb{F}_p$-group G from $\mathbf{G}$ and a cocharacter $\mu \in X_*(G)$ from $\mathbf{X}$. Then A.2 associates a Dynkin triple $(\mathfrak{D}, \mathfrak{I}, \sigma)$ to (G, μ) (which may depend on p).

Definition A.5.1. *We say that a Dynkin triple $(\mathfrak{D}, \mathfrak{I}, \sigma)$ is of Hodge-type if it arises from some Hodge-type Shimura datum $(\mathbf{G}, \mathbf{X})$ and some prime p by the process described above.*

Combining Deligne's classification of symplectic embeddings [Del79, 1.3.9, 2.3.4-2.3.10] with A.4.4 gives:

Corollary A.5.2. *Assume $(\mathfrak{D}, \mathfrak{I}, \sigma)$ is of Hodge-type. Then $(\mathfrak{D}, \mathfrak{I}, \sigma)$ is maximal and A.1.1 holds precisely in cases* (1), (2), (6) *and the sub-case B_n of* (3) *of A.4.4.*

In other words the only maximal triples $(\mathfrak{D}, \mathfrak{I}, \sigma)$ with $\mathfrak{D}$ classical and connected which satisfy A.1.1 but are not of Hodge-type are those where $\mathfrak{D}$ is of type C_n for some $n \geq 3$ or of type D_{2m} for some $m \geq 2$.

A.6 Proofs

The proofs are exercises in reading the *Planches* of Bourbaki [Bou68, Chap. 6, Planches I-IX]. Most importantly, consulting *loc. cit.*, one finds:

Lemma A.6.1. *The connected Dynkin diagrams $\mathfrak{D}$ which have trivial opposition involution $-w_0 = 1$ in* $\mathrm{Aut}(\mathfrak{D})$ *are precisely those of type A_1, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_{2n} $(n \geq 2)$, G_2, F_4, E_7 and E_8.*

Proof of A.3.1. Assume $\sigma = 1$. Then the problem is to identify the sub-diagrams $\mathfrak{I}$ which satisfy $-w_{0,\mathfrak{I}} = 1$. Using A.6.1, one sees that $\mathfrak{I}$ should contain none of the following
(1) A connected component of type A_m with $m \geq 2$,
(2) a connected component of type D_{2k+1} with $k \geq 2$,
(3) a sub-diagram of type E_6.
The first two restrictions (1)–(2) establish A.3.1 when $\mathfrak{D}$ is not of type E. Type E is handled the same way, except that in addition one disqualifies the unique sub-diagram of type E_6. Note that the sub-diagrams of type D_5 in E_6 are excluded by (2). □

Proof of A.3.4. Since $\sigma \neq 1$ and $\mathfrak{I} \neq \emptyset$, $\mathfrak{D}$ is connected by A.3.3. Thus $\mathfrak{D}$ must be of type A_n $(n \geq 2)$, D_n $(n \geq 3)$ or E_6.

Consider type E_6. Since $\sigma \neq 1$, it must be the opposition involution $\sigma = -w_0$. There are precisely six $-w_0$-stable sub-diagrams without isolated points, of types $A_2, A_3, A_2 \times A_2$, D_4, A_5 and E_6. The action of σ on the unique sub-diagram of type D_4 is nontrivial, while $-w_0 = 1$ for D_4. On the other hand, the action of σ on the σ-stable A_2 is trivial, while $-w_0 \neq 1$ for A_2. Thus both of these sub-diagrams fail to satisfy A.1.1. The sub-diagram of type $A_2 \times A_2$ also fails to satisfy A.1.1 because the Weyl group preserves connected components, hence so does the opposition involution $-w_0$. The remaining three sub-diagrams A_3, A_5 and E_6 do satisfy A.1.1. This proves A.3.4(4).

In type A, if $\sigma \neq 1$ then again $\sigma = -w_0$ is the opposition involution. We then conclude the same way as was argued for $A_2 \times A_2$ in E_6, that the opposition involution of a diagram preserves its connected components.

In type D, σ will act trivially on σ-stable sub-diagrams of type A with more than one point, while these have $-w_0 \neq 1$. On the other hand, $\sigma \neq 1$ will act non-trivially on a sub-diagram of type D, so the latter must be of type D_{2k+1} rather than D_{2k}. □

Proof of A.4.4. As in A.4.2, $(\mathfrak{D}, \mathfrak{I}, \sigma)$ is maximal if and only if all its σ-orbits are maximal. The cases where $(\mathfrak{I}')^{\geq 2} \neq \emptyset$ follow directly from A.3.1 and A.3.4. Assume that $(\mathfrak{D}', \mathfrak{I}', \sigma')$ is maximal and $(\mathfrak{I}')^{\geq 2} = \emptyset$. If $\mathfrak{D}'$ is disconnected, then $\mathfrak{I}' = \emptyset$ and $\mathfrak{D}'$ is of type A_1^m by A.3.3. If $\mathfrak{D}'$ is connected and and $(\mathfrak{I}')^{\geq 2} = \emptyset$, then $\mathfrak{D}'$ must have size one or two. □

Recall from [Del79, 1.2.5] that a vertex $v \in \mathfrak{D}$ is special if the corresponding simple root $\alpha_v \in \Delta$ has multiplicity one in the decomposition of the highest root of the connected component $\mathfrak{D}_i$ of $\mathfrak{D}$ containing v. Equivalently, v is special if and only if the corresponding fundamental coweight is minuscule.

Proof of A.5.2. Assume that $(\mathfrak{D}, \mathfrak{I}, \sigma)$ is of Hodge-type. By [Del79, 1.3.9], every connected component $\mathfrak{D}_i$ of $\mathfrak{D}$ is of classical type (A_n, B_n, C_n or D_n) and for every component satisfying $\mathfrak{I} \cap \mathfrak{D}_i \subsetneq \mathfrak{D}_i$, the complement $\mathfrak{D}_i \setminus (\mathfrak{I} \cap \mathfrak{D}_i) = \{v_i\}$ for some special vertex $v_i \in \mathfrak{D}_i$. Let $(\mathfrak{D}', \mathfrak{I}', \sigma)$ be a σ-orbit and assume that $\mathfrak{I}' \neq \emptyset$. Then $\mathfrak{D}'$ is connected. The cases (4) and (5) of type D_{2m} with nontrivial σ in A.4.4 are excluded as follows: Since $(\mathbf{G}, \mathbf{X})$ is a Shimura datum, the adjoint group $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$ over $\mathbb{R}$ is an inner form of its compact form. The compact form of the adjoint split group of type D_{2m} is inner to the split form.[1] Therefore, if $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple of type D_{2m}, then the $\mathbb{F}_p$-group G^{ad} is $\mathbb{F}_p$-split, because every

[1]This follows from the facts that $\mathrm{SO}(2m, 2m)$ is $\mathbb{R}$-split of type D_{2m}, that $\mathrm{SO}(4m) = \mathrm{SO}(4m, 0)$ is its compact form, and that, for $a + b = c + d$ even, $\mathrm{SO}(a, b)$ and $\mathrm{SO}(c, d)$ are inner forms of one another if and only if a, b, c, d all have the same parity.

connected, reductive group over a finite field is quasi-split (Lang-Steinberg). So $\sigma = 1$ if $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple of type D_{2m}.

We have excluded all the cases of A.4.4 which don't appear in A.5.2. Conversely, it follows from Deligne's classification [Del79, 2.3.4-2.3.10] that the cases listed in A.4.4 all arise from Shimura data of Hodge-type. Concretely, A.4.4(1) is realized by Hilbert modular varieties, (2) is realized by Picard modular surfaces associated to unitary similitude groups of signature $(2, 1)$ at infinity and p split in the reflex field, and the remaining cases are realized with $\mathbf{G}$ a spin similitude group of signature $(2, j)$ at infinity. □

Remark A.6.2. In [GK18] it was shown that the cone conjecture (Conjecture 2 of the introduction, 2.1.6 in [GK18]) holds when $G = \mathrm{GSp}(4)$ (or equivalently when $G = \mathrm{Sp}(4)$). Note that $\mathrm{GSp}(4) \cong \mathrm{GSpin}(2, 3)$ and $\mathrm{Sp}(4)/\{\pm 1\} \cong \mathrm{SO}(2, 3)$, so this example is part of the B_n sub-case of A.4.4(3) listed in the Hodge cases A.5.2.

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