The syntomic realization functor for Shimura varieties of abelian type

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Abstract

This is a survey article on prismatic F-gauges and their application to Shimura varieties.

1 Introduction

Let (G, X) be a Shimura datum, and K a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$. Let E be the reflex field of (G, X). We write $\mathrm{Sh}_K(G, X)$ for the Shimura variety over E attached to (G, X) and K. This should be "a moduli of G-motives of type X with K-level structure".

We define G^{c} as in [KSZ21, 1.5.8], which is a quotient of G by a subtorus of the center of G. We have $G^{\mathrm{c}} = G$ if (G, X) is of Hodge type (*cf.* [IKY23, Remark 2.6]).

Let p be a prime number. Assume that $G_{\mathbb{Q}_p}$ has a reductive model $\mathcal{G}$ over $\mathbb{Z}_p$. We assume that $K_p = \mathcal{G}(\mathbb{Z}_p)$. Let $\mathcal{G}^{\mathrm{c}}$ be an integral analogue of G^{c} define in [IKY23, §2.3]. Let v be a prime of E above p. We can construct a tensor functor

$$\omega_{\mathrm{et}} \colon \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}}) \to \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{K,E_v})$$

using a tower of Shimura varieties, where $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}})$ is the category of finite rank algebraic representations of $\mathcal{G}^{\mathrm{c}}$ over $\mathbb{Z}_p$ and $\mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{K,E_v})$ is the category of etale $\mathbb{Z}_p$-local systems on Sh_{K,E_v}.

Assume (G, X) is of abelian type. Then there is a canonical integral model $\mathcal{S}_K$ of Sh_{K,E_v} constructed by Kisin in [Kis10]. In [Lov17], Lovering constructed a tensor functor

$$\omega_{\mathrm{crys}} \colon \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}}) \to \mathrm{FilF\text{-}Crys}(\widehat{\mathcal{S}}_K)$$

compatible with ω_{et}, where $\mathrm{FilF\text{-}Crys}(\widehat{\mathcal{S}}_K)$ is a category of filtered F-crystals on the p-adic completion $\widehat{\mathcal{S}}_K$ of $\mathcal{S}_K$.

If (G, X) is of Hodge type, a shtuka realization compatible with ω_{et} is constructed in [PR24].

In [IKY23], we give a refinement of these realizations in prismatic F-gauges.

2 Prismatic F-crystal

We recall the notion of prism and prismatic F-crystal from [BS22] and [BS23].

Definition 2.1. *A δ-ring is a commutative $\mathbb{Z}_{(p)}$-algebra A with a map $\delta\colon A \to A$ such that $\delta(1) = 0$ and*

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p},$$
$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

for any $x, y \in A$.

For a δ-ring A with δ, we define a Frobenius lift $\phi\colon A \to A$ by $\phi(x) = x^p + p\delta(x)$. A morphism of δ-rings is a ring homomorphism that intertwines the δ-structures.

Example 2.2 ([Joy85]). *For a commutative ring R, let $W(R)$ be the ring of p-typical Witt vectors in R, and let*

$$\delta_{W(R)}\colon W(R) \to W(R);\ (x_0, x_1, \ldots) \mapsto (x_1, x_2, \ldots).$$

Then $W(R)$ with $\delta_{W(R)}$ is a δ-ring. The functor $R \mapsto (W(R), \delta_{W(R)})$ gives the right adjoint of the forgetful functor from the category of δ-rings to the category of commutative rings.

Definition 2.3. *A prism is a pair (A, I) where A is a δ-ring and $I \subset A$ is an invertible ideal such that A is derived (p, I)-adically complete, and $I + \phi(I)$ contains p.*

For a prism (A, I), we equip A with the (p, I)-adic topology.

Definition 2.4. *We say that a commutative ring R has bounded p^∞-torsion if there is some positive integer n such that $R[p^\infty] = R[p^n]$. We say that a prism (A, I) is bounded if A/I has bounded p^∞-torsion.*

Let $\mathfrak{X}$ be a p-adic formal scheme.

Definition 2.5. *The absolute prismatic site $\mathfrak{X}_{\Delta}$ of $\mathfrak{X}$ is the opposite category of the category of bounded prisms (A, I) with $\mathrm{Spf}(A/I) \to \mathfrak{X}$, with topology given by flat covers of prisms.*

We define presheaves $\mathcal{O}_{\Delta}$ and $\mathcal{I}_{\Delta}$ on $\mathfrak{X}_{\Delta}$ by $\mathcal{O}_{\Delta}(A, I) = A$ and $\mathcal{I}_{\Delta}(A, I) = I$. These are sheaves by [BS22, Corollary 3.12].

Definition 2.6. *A prismatic F-crystal on $\mathfrak{X}$ is a vector bundle $\mathcal{E}$ on $\mathfrak{X}_{\Delta}$ with an isomorphism $\varphi_{\mathcal{E}}\colon (\phi^*\mathcal{E})[\frac{1}{\mathcal{I}_{\Delta}}] \xrightarrow{\sim} \mathcal{E}[\frac{1}{\mathcal{I}_{\Delta}}]$.*

Let $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ be the category of prismatic F-crystals on $\mathfrak{X}$.

Example 2.7. *If $\mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over $\mathbb{F}_p$ with an untilt $\mathrm{Spa}(R^\sharp, R^{+\sharp})$ and a morphism $\mathrm{Spf}(R^{+\sharp}) \to \mathfrak{X}$, then $(W(R^+), \mathrm{Ker}(W(R^+) \to R^{+\sharp}))$ with $\mathrm{Spf}(R^{+\sharp}) \to \mathfrak{X}$ gives an object of $\mathfrak{X}_{\Delta}$. Using the value of $(\mathcal{E}, \varphi_{\mathcal{E}})$ at this object, we can obtain a shtuka with one leg at $\mathrm{Spa}(R^\sharp, R^{+\sharp})$.*

The construction in Example 2.7 gives a connection between the prismatic theory and shtukas. See [IKY24, Theorem 3] for details.

3 Prismatic F-gauge

The notion of prismatic F-gauge is introduced by Bhatt–Lurie (*cf.* [Bha23]) based on previous works [Dri24], [BL22a], [BL22b] on the stacky approach to the integral p-adic Hodge theory.

Let W be the ring scheme of p-typical Witt vectors over $\mathbb{Z}_{(p)}$. We have the Frobenius morphism $F\colon W \to W$.

Definition 3.1. *Let R be a p-nilpotent ring. A Cartier–Witt divisor over R is a morphism $\alpha\colon I \to W(R)$ of $W(R)$-modules, where I is an invertible $W(R)$-module, such that the image of*

$$I \xrightarrow{\alpha} W(R) \xrightarrow{\gamma_0} R$$

is a nilpotent ideal, where γ_0 is the projection to the 0-th component, and the image of

$$I \xrightarrow{\alpha} W(R) \xrightarrow{\delta_{W(R)}} W(R)$$

generates the unit ideal.

We assume that $\mathfrak{X}$ is bounded in the sense that for any affine open U of $\mathfrak{X}$ the ring $\mathcal{O}_{\mathfrak{X}}(U)$ has bounded p^∞-torsion.

Definition 3.2. *The prismatization of $\mathfrak{X}$ is a formal stack $\mathfrak{X}^{\Delta}$ over $\mathbb{Z}_p$ defined as follows: For a p-nilpotent ring R, the groupoid $\mathfrak{X}^{\Delta}(R)$ consists of Cartier–Witt divisors $I \to W(R)$ with a morphism $\mathrm{Spec}(\mathrm{Cone}(I \to W(R))) \to \mathfrak{X}$ of derived formal schemes.*

For a p-nilpotent ring R and Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ over R, the induced map $F^*I \xrightarrow{F^*\alpha} W(R)$ is also a Cartier–Witt divisor over R and we have the map

$$\mathrm{Cone}(I \xrightarrow{\alpha} W(R)) \to \mathrm{Cone}(F^*I \xrightarrow{F^*\alpha} W(R))$$

of animated rings induced from F. This gives a morphism $\phi_{\mathfrak{X}^{\Delta}}\colon \mathfrak{X}^{\Delta} \to \mathfrak{X}^{\Delta}$.

We define a group scheme $\mathbb{G}_a^\sharp$ over $\mathbb{Z}_{(p)}$ by

$$\mathbb{G}_a^\sharp = \mathrm{Ker}(F\colon W \to F_*W).$$

Definition 3.3. *Let R be a p-nilpotent ring.*

(1) A $\sharp$-invertible W-module over R is an affine W-module scheme over R that is fpqc locally isomorphic to $\mathbb{G}_a^\sharp$ as W-modules.

(2) An admissible W-module over R is an affine W-module scheme over R that can be realized as an extension of an invertible F_*W-module over R by a $\sharp$-invertible W-module over R.

Proposition 3.4 ([Bha23, Remark 5.2.5])**.** *Let R be a p-nilpotent ring. Let M and N be admissible W-modules over R which sit in short exact sequences*

$$0 \to L_M \to M \to F_*I_M \to 0, \qquad (*_M)$$

$$0 \to L_N \to N \to F_*I_N \to 0, \qquad (*_N)$$

where L_M, L_N are $\sharp$-invertible W-modules over R, and I_M and I_N are invertible W-module over R. Then any morphism $M \to N$ of W-modules induces a morphism of short exact sequences from $(*_M)$ to $(*_N)$.

In particular, the short exact sequence $(*_M)$ is unique up to unique isomorphism.

Definition 3.5. *Let R be a p-nilpotent ring. A filtered Cartier–Witt divisor over R is a morphism $d\colon M \to W$ of W-modules over R, where M is an admissible W-module over R, such that the induced morphism*

$$F_* I_M \to F_* W \tag{3.1}$$

comes from a Cartier–Witt divisor over R via F_*, where (3.1) is induced by d, $(*_M)$ and the sequence

$$0 \to \mathbb{G}_{\mathrm{a}}^{\sharp} \to W \xrightarrow{F} F_* W \to 0$$

using Proposition 3.4.

Definition 3.6. *The filtered prismatization of $\mathfrak{X}$ is a formal stack $\mathfrak{X}^{\mathcal{N}}$ over $\mathbb{Z}_p$ defined as follows: For a p-nilpotent ring R, the groupoid $\mathfrak{X}^{\mathcal{N}}(R)$ consists of filtered Cartier–Witt divisors $M \to W$ with a morphism $\operatorname{Spec}((W/M)(R)) \to \mathfrak{X}$ of derived formal schemes, where*

$$(W/M)(R) = \mathrm{R}\Gamma(\operatorname{Spec}(R), \operatorname{Cone}(M \to W)).$$

Definition 3.7. *Let R be a p-nilpotent ring.*

Associating a filtered Cartier–Witt divisor $d\colon M \to W$ over R with the Cartier–Witt divisor over R giving (3.1), we obtain a morphism $\pi\colon \mathfrak{X}^{\mathcal{N}} \to \mathfrak{X}^{\Delta}$.

Associating a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ with the filtered Cartier–Witt divisor

$$I \otimes_{W(R)} W \xrightarrow{\alpha \otimes \mathrm{id}_W} W(R) \otimes_{W(R)} W \cong W,$$

we obtain a morphism $j_{\mathrm{HT}}\colon \mathfrak{X}^{\Delta} \to \mathfrak{X}^{\mathcal{N}}$.

Associating a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ with the filtered Cartier–Witt divisor M given by the following pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_{\mathrm{a}}^{\sharp} & \longrightarrow & M & \longrightarrow & F_*(I \otimes_{W(R)} W) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow {\scriptstyle F_*(\alpha \otimes \mathrm{id}_W)} & & \\ 0 & \longrightarrow & \mathbb{G}_{\mathrm{a}}^{\sharp} & \longrightarrow & W & \xrightarrow{F} & F_* W & \longrightarrow & 0, \end{array}$$

we obtain a morphism $j_{\mathrm{dR}}\colon \mathfrak{X}^{\Delta} \to \mathfrak{X}^{\mathcal{N}}$.

By the construction, we have $\pi \circ j_{\mathrm{HT}} = \phi_{\mathfrak{X}^{\Delta}}$ and $\pi \circ j_{\mathrm{dR}} = \mathrm{id}_{\mathfrak{X}^{\Delta}}$. Further, j_{HT} and j_{dR} are open immersion with the disjoint images (*cf.* [Bha23, Remark 5.3.6]).

Example 3.8 ([Bha23, §5.5.1])**.** *Assume that $\mathfrak{X} = \operatorname{Spf} R$ where R is a perfectoid ring in the sense of [BMS18, Definition 3.5]. Then the natural homomorphism*

$$\theta\colon W(R^{\flat}) \to R$$

defined in [BMS18, §3.1] is surjective by [BMS18, Lemma 3.9]. We put $(\Delta_R, I) = (W(R^{\flat}), \ker\theta)$. Let ϕ act on $\Delta_R = W(R^{\flat})$ via the p-th power map on $R^{\flat}$. We define a filtration $\operatorname{Fil}^{\bullet}_{\mathcal{N}} \Delta_R$ of Δ_R by

$$\operatorname{Fil}^i_{\mathcal{N}} \Delta_R = \begin{cases} \phi^{-1}(I^i) & \textit{if } i \geq 0, \\ \Delta_R & \textit{if } i < 0. \end{cases}$$

We put

$$\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} {\mathbb{\Delta}}_R) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}^{i}_{\mathcal{N}} {\mathbb{\Delta}}_R t^{-i} \subset {\mathbb{\Delta}}_R[t, t^{-1}].$$

Let $\mathbb{G}_{\mathrm{m}}$ *act on this ring by* i*-th power on* $\mathrm{Fil}^{i}_{\mathcal{N}} {\mathbb{\Delta}}_R t^{-i}$*. Then*

$$\mathfrak{X}^{\mathbb{\Delta}} = \mathrm{Spf}\, {\mathbb{\Delta}}_R, \quad \mathfrak{X}^{\mathcal{N}} = [\mathrm{Spec}(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} {\mathbb{\Delta}}_R))/\mathbb{G}_{\mathrm{m}}] \times_{\mathrm{Spec}\, {\mathbb{\Delta}}_R} \mathrm{Spf}\, {\mathbb{\Delta}}_R,$$

and $\pi \colon \mathfrak{X}^{\mathcal{N}} \to \mathfrak{X}^{\mathbb{\Delta}}$ *is the natural projection. Further,* j_{HT} *and* j_{dR} *are induced from*

$$\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} {\mathbb{\Delta}}_R) \xrightarrow{\phi} \bigoplus_{i \in \mathbb{Z}} I^i t^{-i} \quad \textit{and} \quad \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} {\mathbb{\Delta}}_R) \hookrightarrow {\mathbb{\Delta}}_R[t^{\pm 1}]$$

respectively.

Definition 3.9. *We define the syntomification* $\mathfrak{X}^{\mathrm{syn}}$ *of* $\mathfrak{X}$ *by the cocartesian diagram*

$$\begin{array}{ccc} \mathfrak{X}^{\mathbb{\Delta}} \coprod \mathfrak{X}^{\mathbb{\Delta}} & \xhookrightarrow{j_{\mathrm{HT}} \coprod j_{\mathrm{dR}}} & \mathfrak{X}^{\mathcal{N}} \\ \downarrow & & \downarrow \\ \mathfrak{X}^{\mathbb{\Delta}} & \longrightarrow & \mathfrak{X}^{\mathrm{syn}}. \end{array}$$

The category of prismatic F*-gauges in vector bundles is defined as the category of vector bundles on* $\mathfrak{X}^{\mathrm{syn}}$*, for which we write* $\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}})$*.*

4 Tannakian framework

Let k be a perfect field of characteristic p. Assume that $\mathfrak{X}$ is a smooth p-adic formal scheme over $W(k)$.

For an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\mathbb{\Delta}})$, we define the filtration $\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi^* \mathcal{E}) \subset \phi^* \mathcal{E}$ by $\mathcal{O}_{\mathbb{\Delta}}$-submodules, which we call the Nygaard filtration, so that

$$\mathrm{Fil}^{r}_{\mathrm{Nyg}}(\phi^* \mathcal{E})(A, I) = \{x \in \phi^* \mathcal{E}(A, I) \mid \varphi_{\mathcal{E}}(x) \in I^r \mathcal{E}(A, I)\}$$

for an object (A, I) of $\mathfrak{X}_{\mathbb{\Delta}}$.

Let $\mathrm{Vect}^{\varphi, \mathrm{lff}}(\mathfrak{X}_{\mathbb{\Delta}})$ be the category of prismatic F-crystal on $\mathfrak{X}$ with locally filtered free Nygaard filtration.

Let $\mathscr{G}$ be a smooth group scheme over $\mathbb{Z}_p$. For a $\mathbb{Z}_p$-linear tensor category $\mathcal{C}$, let $\mathscr{G}$-$\mathcal{C}$ denote the category of $\mathbb{Z}_p$-linear exact tensor functors $\mathrm{Rep}_{\mathbb{Z}_p}(\mathscr{G}) \to \mathcal{C}$. Using [GL23, Theorem 2.31], we can show the following:

Proposition 4.1 ([IKY23, §1.3])**.** *There is a bi-exact* $\mathbb{Z}_p$*-linear tensor-equivalence*

$$\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}}) \cong \mathrm{Vect}^{\varphi, \mathrm{lff}}(\mathfrak{X}_{\mathbb{\Delta}}).$$

In particular, we have an equivalence

$$\mathscr{G}\text{-}\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}}) \cong \mathscr{G}\text{-}\mathrm{Vect}^{\varphi, \mathrm{lff}}(\mathfrak{X}_{\mathbb{\Delta}}).$$

Definition 4.2. *An analytic prismatic F-crystal on $\mathfrak{X}$ is a compatible system of pairs $(\mathcal{E}_{(A,I)}, \varphi_{(A,I)})$ of a vector bundle $\mathcal{E}_{(A,I)}$ on* $\mathrm{Spec}(A) \setminus V(p,I)$ *and* $\varphi_{(A,I)} \colon (\phi^*\mathcal{E}_{(A,I)})[\frac{1}{I}] \xrightarrow{\sim} \mathcal{E}_{(A,I)}[\frac{1}{I}]$ *for* $(A,I) \in \mathfrak{X}_{\Delta}$*, where $V(p,I)$ is the closed subscheme of* $\mathrm{Spec}\, A$ *defined by the ideal (p,I).*

Let $\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ be the category of analytic prismatic F-crystals on $\mathfrak{X}$. We write $\mathfrak{X}_\eta$ for the generic fiber of $\mathfrak{X}$. Let $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$ be the category of crystalline $\mathbb{Z}_p$-local systems on $\mathfrak{X}_\eta$.

Theorem 4.3 ([DLMS24, Theorem 3.46], [GR24, Theorem A])**.** *There is an equivalence*

$$\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}) \xrightarrow{\sim} \mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$$

of categories.

The restriction functor $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ is fully faithful by [GR24, Proposition 3.7].

We say that an object of $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$ is prismatically good reduction if it comes from $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ under the equivalence in Theorem 4.3. Let $\mathrm{Loc}^{\Delta\text{-gr}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta) \subset \mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$ be the exact full subcategory of prismatically good reduction objects. By the definition, we have an equivalence

$$\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \xrightarrow{\sim} \mathrm{Loc}^{\Delta\text{-gr}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta) \tag{4.1}$$

of categories.

Remark 4.4. *The equivalence in Theorem 4.3 is bi-exact, but the equivalence* (4.1) *may not be bi-exact. More precisely, a quasi-inverse of* (4.1) *may not be exact. The problem is that there is a non-exact sequence in* $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ *whose image in* $\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ *is exact.*

In spite of Remark 4.4, we still can show the following:

Theorem 4.5 ([IKY24, §2.4])**.** *Assume that $\mathscr{G}$ is a reductive group scheme over $\mathbb{Z}_p$. Then there is an equivalence*

$$\mathscr{G}\text{-}\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \xrightarrow{\sim} \mathscr{G}\text{-}\mathrm{Loc}^{\Delta\text{-}gr}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$$

of categories.

5 Integral crystalline functor

In this section, we construct the integral crystalline functor

$$\mathbb{D}_{\mathrm{crys}} \colon \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathrm{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}),$$

where $\mathrm{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ is the category of naive filtered F-crystals on $\mathfrak{X}$ in the sense of [IKY25, §2.1.1]. In [BS23, Construction 4.12], Bhatt–Scholze constructed

$$\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}});\ \mathcal{E} \mapsto \mathcal{E}^{\mathrm{crys}}.$$

Therefore it is enough to define a filtration on $\mathcal{E}^{\mathrm{crys}}$. Let

$$\mathbb{D}^+_{\mathrm{dR}} \colon \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathrm{FilVect}(\mathfrak{X})$$

be the de Rham realization functor defined in [IKY25, §1.2]. As explained there, the filtration of $\mathbb{D}^+_{\mathrm{dR}}(\mathcal{E})$ can be described using the Nygaard filtration of $\phi^*\mathcal{E}$.

Theorem 5.1 ([IKY25, §1.3])**.** *For $\mathcal{E} \in \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$, there is a canonical crystalline-de Rham comparison isomorphism $\mathcal{E}^{\mathrm{crys}} \cong \mathbb{D}^{+}_{\mathrm{dR}}(\mathcal{E})$.*

For $\mathcal{E} \in \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$, we put $\mathbb{D}_{\mathrm{crys}}(\mathcal{E}) = \mathcal{E}^{\mathrm{crys}}$ and define the filtration on $\mathbb{D}_{\mathrm{crys}}(\mathcal{E})$ by the filtration on $\mathbb{D}^{+}_{\mathrm{dR}}(\mathcal{E})$ using the crystalline-de Rham comparison isomorphism in Theorem 5.1.

The functor $\mathbb{D}_{\mathrm{crys}}$ induces an equivalence between the category of prismatic F-crystals on $\mathfrak{X}$ with Hodge–Tate weights in $[0, p-2]$ and the category of Fontaine–Laffaille modules on $\mathfrak{X}$. See [IKY25, §3] for details.

6 Syntomic realization

Let $\mu_h \colon \mathbb{G}_{\mathrm{m},\mathcal{O}_{E_v}} \to \mathcal{G}_{\mathcal{O}_{E_v}}$ be the extension of a Hodge cocharacter of (G, X). Let $\mu_h^{\mathrm{c}} \colon \mathbb{G}_{\mathrm{m},\mathcal{O}_{E_v}} \to \mathcal{G}^{\mathrm{c}}_{\mathcal{O}_{E_v}}$ be the cocharacter induced by μ_h. We assume that p is odd.

Theorem 6.1 ([IKY23, §2.4 and §2.6])**.** *There is a tensor functor*

$$\omega_{\mathrm{syn}} \colon \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}}) \to \mathrm{Vect}(\widehat{\mathcal{S}}_K^{\mathrm{syn}})$$

of type μ_h^{c} that is compatible with ω_{crys} and ω_{et}.

For construction of ω_{syn}, first we show that ω_{et} belongs to $\mathcal{G}^{\mathrm{c}}\text{-}\mathrm{Loc}_{\mathbb{Z}_p}^{\Delta\text{-}\mathrm{gr}}(\widehat{\mathcal{S}}_{K,\eta})$. Then we obtain an object of $\mathcal{G}^{\mathrm{c}}\text{-}\mathrm{Vect}^{\varphi}(\widehat{\mathcal{S}}_{K,\Delta})$ by Theorem 4.5. Next we show that the obtained object belongs to $\mathcal{G}^{\mathrm{c}}\text{-}\mathrm{Vect}^{\varphi,\mathrm{lff}}(\widehat{\mathcal{S}}_{K,\Delta})$. Then we can obtain ω_{syn} using Proposition 4.1.

We note that to make sense of the compatibility with ω_{crys} in Theorem 6.1, we need the functor $\mathbb{D}_{\mathrm{crys}}$ constructed in §5.

See [IKY23, §2.6.2] for cohomological consequences of Theorem 6.1.

7 Characterization of the integral model

Definition 7.1. *A potentially crystalline locus of $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$ is a quasi-compact open subspace U of $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$ uniquely determined by the following condition:*

> *For any classical point x of $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$, the point x is in U if and only if $\omega_{\mathrm{et}}(V)_x$ is potentially crystalline for any $V \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}})$.*

Let $\mathrm{Sh}^{\mathrm{pc}}_{K,E_v} \subset \mathrm{Sh}^{\mathrm{an}}_{K,E_v}$ be the potentially crystalline locus constructed in [IM20, Theorem 5.17] (*cf.* [IM20, Remark 2.12] and [LZ17, Theorem 1.2]). We write ω_{pc} for the restriction of $\omega^{\mathrm{an}}_{\mathrm{et}}$ to $\mathrm{Sh}^{\mathrm{pc}}_{K,E_v}$.

We write $\mathrm{BT}^{\mathcal{G}^{\mathrm{c}},-\mu_h^{\mathrm{c}}}$ for the moduli stack of prismatic F-gauges with $\mathcal{G}^{\mathrm{c}}$-structure of type $-\mu_h^{\mathrm{c}}$ constructed by [GM24], confirming a conjecture in [Dri23]. Then we have $\widehat{\mathcal{S}}_K \to \mathrm{BT}^{\mathcal{G}^{\mathrm{c}},-\mu_h^{\mathrm{c}}}$ given by ω_{syn}.

Definition 7.2. *A syntomic integral canonical model of Sh_{K,E_v} is a smooth, separated integral model $\mathcal{X}$ of Sh_{K,E_v} over $\mathcal{O}_{E_v}$ satisfying the following conditions:*

(1) $\widehat{\mathcal{X}}_{\eta} = \mathrm{Sh}^{\mathrm{pc}}_{K,E_v}$.

(2) There is a tensor functor

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}}) \to \mathrm{Vect}(\widehat{\mathscr{X}}^{\mathrm{syn}})$$

of type $-\mu_h^{\mathrm{c}}$ whose etale realization is equal to ω_{pc} such that the resulting morphism $\widehat{\mathscr{X}} \to \mathrm{BT}^{\mathcal{G}^{\mathrm{c}},-\mu_h^{\mathrm{c}}}$ is formally etale.

Remark 7.3. *To make sense of a condition like Definition 7.2 (2), the moduli of shtukas is not enough because the theory of v-sheaves can not distinguish nilpotent thickenings.*

Theorem 7.4 ([IKY23, §3.4])**.** *The scheme $\mathcal{S}_K$ over $\mathcal{O}_{E_v}$ is the unique syntomic integral canonical model.*

In the proof of Theorem 7.4, confirming the condition (1) in Definition 7.2 generalizes a result in [IM13, §7] for the PEL type case, and confirming the condition (2) in Definition 7.2 amounts to proving a Serre–Tate theorem for Shimura varieties of abelian type.

Remark 7.5. *The characterization in Theorem 7.4 works for each level K contrary to previously known characterization in [Kis10].*

We write $\overline{\mathcal{S}}_K$ and $\overline{\mathcal{G}}^{\mathrm{c}}$ for the special fibers of $\mathcal{S}_K$ and $\mathcal{G}^{\mathrm{c}}$ respectively. Let $\overline{\mathcal{G}}^{\mathrm{c}}\text{-Zip}^{-\mu_h^{\mathrm{c}}}$ be the moduli stack of $\overline{\mathcal{G}}^{\mathrm{c}}$-zips of type $-\mu_h^{\mathrm{c}}$. As an application of the syntomic realization functor, we can obtain the following theorem:

Theorem 7.6 ([IKY23, §3.5])**.** *There is a natural smooth morphism*

$$\overline{\mathcal{S}}_K \to \overline{\mathcal{G}}^{\mathrm{c}}\text{-Zip}^{-\mu_h^{\mathrm{c}}}.$$

Theorem 7.6 generalizes [Zha18, Theorem 4.12] in the Hodge type case to the abelian type case. This allows us to apply results in [IK24] to Shimura varieties of abelian type as well.

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References

[Bha23] B. Bhatt, Prismatic F-gauges, 2023, course notes.

[BL22a] B. Bhatt and J. Lurie, Absolute prismatic cohomology, 2022, arXiv:2201.06120.

[BL22b] B. Bhatt and J. Lurie, The prismatization of p-adic formal schemes, 2022, arXiv:2201.06124.

[BMS18] B. Bhatt, M. Morrow and P. Scholze, Integral p-adic Hodge theory, Publ. Math. Inst. Hautes Études Sci. 128 (2018), 219–397.

[BS22] B. Bhatt and P. Scholze, Prisms and prismatic cohomology, Ann. of Math. (2) 196 (2022), no. 3, 1135–1275.

[BS23] B. Bhatt and P. Scholze, Prismatic F-crystals and crystalline Galois representations, Camb. J. Math. 11 (2023), no. 2, 507–562.

[DLMS24] H. Du, T. Liu, Y. S. Moon and K. Shimizu, Completed prismatic F-crystals and crystalline $\mathbf{Z}_p$-local systems, Compos. Math. 160 (2024), no. 5, 1101–1166.

[Dri23] V. Drinfeld, On Shimurian generalizations of the stack $\mathrm{BT}_1 \otimes \mathbb{F}_p$, 2023, arXiv:2304.11709.

[Dri24] V. Drinfeld, Prismatization, Selecta Math. (N.S.) 30 (2024), no. 3, Paper No. 49, 150.

[GL23] H. Guo and S. Li, Frobenius height of prismatic cohomology with coefficients, 2023, arXiv:2309.06663.

[GM24] Z. Gardner and K. Madapusi, An algebraicity conjecture of Drinfeld and the moduli of p-divisible groups, 2024, arXiv:2412.10226.

[GR24] H. Guo and E. Reinecke, A prismatic approach to crystalline local systems, Invent. Math. 236 (2024), no. 1, 17–164.

[IK24] N. Imai and J.-S. Koskivirta, Partial Hasse invariants for Shimura varieties of Hodge-type, Adv. Math. 440 (2024), Paper No. 109518, 47.

[IKY23] N. Imai, H. Kato and A. Youcis, The prismatic realization functor for Shimura varieties of abelian type, 2023, arXiv:2310.08472.

[IKY24] N. Imai, H. Kato and A. Youcis, A Tannakian framework for prismatic F-crystals, 2024, arXiv:2406.08259.

[IKY25] N. Imai, H. Kato and A. Youcis, An integral analogue of Fontaine's crystalline functor, 2025, arXiv:2504.16282.

[IM13] N. Imai and Y. Mieda, Toroidal compactifications of Shimura varieties of PEL type and its applications, in Algebraic number theory and related topics 2011, vol. B44 of RIMS Kôkyûroku Bessatsu, pp. 3–24, Res. Inst. Math. Sci. (RIMS), Kyoto, 2013.

[IM20] N. Imai and Y. Mieda, Potentially good reduction loci of Shimura varieties, Tunis. J. Math. 2 (2020), no. 2, 399–454.

[Joy85] A. Joyal, δ-anneaux et vecteurs de Witt, C. R. Math. Rep. Acad. Sci. Canada 7 (1985), no. 3, 177–182.

[Kis10] M. Kisin, Integral models for Shimura varieties of abelian type, J. Amer. Math. Soc. 23 (2010), no. 4, 967–1012.

[KSZ21] M. Kisin, S. W. Shin and Y. Zhu, The stable trace formula for Shimura varieties of abelian type, 2021, arXiv:2110.05381.

[Lov17] T. Lovering, Filtered F-crystals on Shimura varieties of abelian type, 2017, arXiv:1702.06611.

[LZ17] R. Liu and X. Zhu, Rigidity and a Riemann-Hilbert correspondence for p-adic local systems, Invent. Math. 207 (2017), no. 1, 291–343.

[PR24] G. Pappas and M. Rapoport, p-adic shtukas and the theory of global and local Shimura varieties, Camb. J. Math. 12 (2024), no. 1, 1–164.

[Zha18] C. Zhang, Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type, Canad. J. Math. 70 (2018), no. 2, 451–480.

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