Shintani lifts for Weil representations of unitary groups over finite fields

Naoki Imai and Takahiro Tsushima

Abstract

We construct extended Weil representations of unitary groups over finite fields geometrically, and show that they are Shintani lifts for Weil representations.

1 Introduction

Let q be a power of a prime number p. Let n be a positive integer. We define

$$\mathrm{U}_n(q) = \{g \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \mid g^* g = \mathrm{I}_n\},$$

where $g^* = (a_{j,i}^q)$ for $g = (a_{i,j})$. Let $\ell \neq p$ be a prime number. Let ψ be a non-trivial character of $\mathbb{F}_{q,+} = \{x \in \mathbb{F}_{q^2} \mid x^q + x = 0\}$ over $\overline{\mathbb{Q}}_\ell$. A Weil representation of a unitary group $\mathrm{U}_n(q)$ associated to ψ is constructed in [2], which we denote by $\rho_{\mathrm{U}_n(q),\psi}$. Let m be a positive odd integer.

In this paper, we investigate the behavior of $\rho_{\mathrm{U}_n(q),\psi}$ via Shintani lifting from $\mathbb{F}_q$ to $\mathbb{F}_{q^m}$. Let Γ be a cyclic group of order m with generator σ. Let $F\colon \mathrm{U}_n(q^m) \to \mathrm{U}_n(q^m);\ g \mapsto ({}^t g^\tau)^{-1}$, where $\tau((a_{i,j})) = (a_{i,j}^q)$. Let σ act on $\mathrm{U}_n(q^m)$ as F. We consider the semidirect group $\mathrm{U}_n(q^m) \rtimes \Gamma$. In [4], Gyoja defines a norm map from the set of $\mathrm{U}_n(q^m)$-conjugacy classes in $\mathrm{U}_n(q^m)\sigma$ to the set of conjugacy classes in $\mathrm{U}_n(q)$, which is denoted by N. We set $\psi_m = \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^2}} \colon \mathbb{F}_{q^m,+} \to \overline{\mathbb{Q}}_\ell^\times$.

Theorem. *There is a unique extension $\widetilde{\rho}_{\mathrm{U}_n(q^m),\psi_m}$ of $\rho_{\mathrm{U}_n(q^m),\psi_m}$ to $\mathrm{U}_n(q^m) \rtimes \Gamma$ such that*

$$\mathrm{tr}\, \widetilde{\rho}_{\mathrm{U}_n(q^m),\psi_m}(g, \sigma) = \mathrm{tr}\, \rho_{\mathrm{U}_n(q),\psi}(\mathrm{N}(g, \sigma))$$

for any $g \in \mathrm{U}_n(q^m)$.

This is a version of [5, Theorem in §1] in unitary cases. Similarly to [5, Theorem 4.1], we actually show a stronger equality. Namely, we show a similar equality for Heisenberg–Weil representations and more general norm maps. See Theorem 7.1 for the precise statement. As mentioned in [5, §1], the arguments in [5] work also in unitary cases. In this paper, we use an inductive argument similar to that in [5], but replace some ingredients in the proof with geometric inputs: In [5], an extended Weil representation is constructed based on the Schrödinger model of a Weil representation. On the other hand, we construct $\widetilde{\rho}_{\mathrm{U}_n(q^m),\psi_m}$ in a geometric way and study the representation by geometric tools.

2010 *Mathematics Subject Classification.* Primary: 20C33; Secondary: 11F27.

In [6, Theorem 2.5], it is known that the representation $\rho_{\mathrm{U}_n(q^m),\psi_m}$ is realized in the ψ_m-isotypic part of the middle ℓ-adic cohomology of the smooth affine variety over $\mathbb{F}_{q^2}$ defined by

$$z^{q^m}+z=\sum_{i=1}^{n} x_i^{q^m+1}$$

in $\mathbb{A}^{n+1}_{\mathbb{F}_{q^2}}$. In order to extend this representation to $\mathrm{U}_n(q^m)\rtimes\Gamma$, we use the Frobenius action over $\mathbb{F}_{q^2}$. A merit in our geometric construction is that the the action of the cyclic group Γ in the Shintani lift appears very naturally from the rationality of the above variety. We believe that this gives a new insight on the relation between liftings of representations and geometry.

We briefly explain the content of each section. In §2, we collect several known facts on Gyoja's norm map. In §3, we recall unitary Heisenberg–Weil representations. In §4, we construct an extension $\widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}$ of a Heisenberg–Weil representation geometrically and show Theorem 7.1 when $n=1$ in Proposition 4.5. In §5, we study the behavior of $\widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}$ restricted to several subgroups. In §6, we study the support of the trace of $\widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}$. In §7, we state our main theorem. Under the knowledges in §5–§6, we reduce Theorem 7.1 to a special case in Lemma 8.3 in §8. By using the character formula of a tensor induction in [3], we show Theorem 7.1 in the special case in §9.

Notation

- Let q be a power of a prime number p.
- Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. For a scheme X over $\mathbb{F}_q$ and an endomorphism F of $X_{\overline{\mathbb{F}}_q}$, let
$$X^F=\{x\in X(\overline{\mathbb{F}}_q)\mid F(x)=x\}.$$
- For a positive integer m, let Fr_{q^m} denote the geometric Frobenius element of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^m})$ given by $\overline{\mathbb{F}}_q\to\overline{\mathbb{F}}_q;\ x\mapsto x^{q^{-m}}$.
- Let ℓ be a prime number different from p.
- For an abelian group A, let $A^\vee=\mathrm{Hom}_{\mathbb{Z}}(A,\overline{\mathbb{Q}}_\ell^\times)$.
- For a representation W over $\overline{\mathbb{Q}}_\ell$ of an abelian group A and $\chi\in A^\vee$, let $W[\chi]$ denote the χ-isotypic part of W.

2 Norm map

We follow [5, §3] and [4, §3]. Let $\mathbf{G}$ be a connected algebraic group over $\mathbb{F}_q$ with Frobenius endomorphism F. Let m be a positive integer. We put $G=\mathbf{G}^{F^m}$.

Let Γ be a cyclic group of order m with a generator σ. Let σ act on G as F. We consider the semidirect group $G\rtimes\Gamma$. Let $1\le i\le m$ be an integer. We set $d=(m,i)$. We choose an integer t such that $ti\equiv d \pmod{m}$. We set $\mu=m/d$. For $g\in G$, we can choose

$$\alpha=\alpha(g)\in\mathbf{G}(\overline{\mathbb{F}}_q)$$

such that

$$\alpha^{-1}F^d(\alpha)=(g,\sigma^i)^t(1,\sigma^{-it})$$

by Lang's theorem. We put

$$\mathrm{N}_{i,t}(g,\sigma^i)=\alpha g F^i(g)\cdots F^{i(\mu-1)}(g)\alpha^{-1}.$$

The element $\mathrm{N}_{i,t}(g,\sigma^i)$ does belong to G^{F^i}. Its conjugacy class does not depend on the choice of α. By [4, Lemma 3.2(1)], $\mathrm{N}_{i,t}$ induces a bijection from the set of G-conjugacy classes in $G\sigma^i$ to the set of conjugacy classes in G^{F^i}. The norm map is originally defined in [10] in the case where $i=t=1$ and generalized in [8].

Let C be an algebraically closed field of characteristic 0. Let $\mathcal{C}(G\sigma^i)$ denote the vector space of C-valued functions on $G\sigma^i$ which are invariant under conjugation by G. For a finite group H, let $\mathcal{C}(H)$ denote the set of C-valued class functions on H. Then we have the bijection

$$\mathcal{N}_{i,t}\colon \mathcal{C}(G^{F^i})\to\mathcal{C}(G\sigma^i);\ \chi\mapsto\chi\circ\mathrm{N}_{i,t}.$$

This induces the bijection

$$\mathcal{C}(G^{F^i})^F\xrightarrow{\sim}\mathcal{C}(G\sigma^i)^\sigma$$

by [4, Lemma 3.2(2)], where σ acts on $G\sigma^i$ by the conjugation. Note that $G^{F^i}=\mathbf{G}^{F^d}$.

Lemma 2.1. *Let $\mathbf{H}$ be a connected algebraic subgroup of $\mathbf{G}$ defined over $\mathbb{F}_q$, and $\mathbf{G}_1,\mathbf{G}_2$ two connected algebraic groups over $\mathbb{F}_q$ with Frobenius endomorphisms F. Let $H=\mathbf{H}^{F^m}$ and $G_j=\mathbf{G}_j^{F^m}$ for $j=1,2$.*

(1) Let $\widetilde{\chi}\in\mathcal{C}(H\rtimes\Gamma)$, and take $\chi\in\mathcal{C}(H^{F^i})^F$ such that $\widetilde{\chi}|_{H\sigma^i}=\mathcal{N}_{i,t}(\chi)$. Then we have

$$(\mathrm{Ind}_{H\rtimes\Gamma}^{G\rtimes\Gamma}\widetilde{\chi})|_{G\sigma^i}=\mathcal{N}_{i,t}(\mathrm{Ind}_{H^{F^i}}^{G^{F^i}}\chi).$$

(2) Let $\widetilde{\chi}\in\mathcal{C}(G\rtimes\Gamma)$, and take $\chi\in\mathcal{C}(G^{F^i})^F$ such that $\widetilde{\chi}|_{G\sigma^i}=\mathcal{N}_{i,t}(\chi)$. Then we have

$$(\widetilde{\chi}|_{H\rtimes\Gamma})|_{H\sigma^i}=\mathcal{N}_{i,t}(\chi|_{H^{F^i}}).$$

(3) Assume that $\mathbf{H}$ is a normal algebraic subgroup of $\mathbf{G}$. We identify G/H with $(\mathbf{G}/\mathbf{H})^{F^m}$. Let $\widetilde{\chi}\in\mathcal{C}((G/H)\rtimes\Gamma)$, and take $\chi\in\mathcal{C}((G/H)^{F^i})^F$ such that $\widetilde{\chi}|_{(G/H)\sigma^i}=\mathcal{N}_{i,t}(\chi)$. Let

$$p'\colon G\rtimes\Gamma\to(G/H)\rtimes\Gamma,\quad p\colon G^{F^i}\to(G/H)^{F^i}$$

be the canonical projections. Then we have

$$(\widetilde{\chi}\circ p')|_{G\sigma^i}=\mathcal{N}_{i,t}(\chi\circ p).$$

(4) Let $\widetilde{\chi}_j\in\mathcal{C}(G_j\rtimes\Gamma)$ for $j=1,2$, and take $\chi_j\in\mathcal{C}(G_j^{F^i})^F$ such that $\widetilde{\chi}_j|_{G_j\sigma^i}=\mathcal{N}_{i,t}(\chi_j)$. Then we have

$$(\widetilde{\chi}_1\times\widetilde{\chi}_2)|_{(G_1\times G_2)\sigma^i}=\mathcal{N}_{i,t}(\chi_1\times\chi_2),$$

where $(G_1\times G_2)\sigma^i$ is identified with

$$\bigl\{\bigl((g_1,\sigma^i),(g_2,\sigma^i)\bigr)\in(G_1\rtimes\Gamma)\times(G_2\rtimes\Gamma)\bigm| g_1\in G_1,\ g_2\in G_2\bigr\}.$$

Proof. This is proved in [4, Lemma 3.6] if $C=\mathbb{C}$. The same proof works also for a general C. □

3 Heisenberg–Weil representation

Let V be a vector space over $\mathbb{F}_{q^2}$ with a hermitian form h. In this paper, a hermitian form is supposed to be sesquilinear on the first coordinate. For the hermitian space (V,h), we define $\mathbf{H}_{(V,h)}$ over $\mathbb{F}_q$ as

$$\mathbf{H}_{(V,h)}(R) = \{(v,a) \in (V \otimes_{\mathbb{F}_q} R) \times (\mathbb{F}_{q^2} \otimes_{\mathbb{F}_q} R) \mid (\mathrm{Fr}_q \otimes \mathrm{id}_R)(a) + a = h_R(v,v)\}$$

for an $\mathbb{F}_q$-algebra R, where h_R is the R-linear extension of h, with the group operation

$$(v,a)(v',a') = (v+v', a+a'+h_R(v,v')).$$

We put $\mathrm{H}(V,h) = \mathbf{H}_{(V,h)}(\mathbb{F}_q)$ and call it the unitary Heisenberg group associated to (V,h).

Assume that h is nondegenerate in the sequel. Let $\mathbf{U}_{(V,h)}$ be the unitary algebraic group over $\mathbb{F}_q$ defined as

$$\begin{aligned}&\mathbf{U}_{(V,h)}(R)\\ &\quad = \{g \in \mathrm{Aut}_{\mathbb{F}_{q^2}\otimes_{\mathbb{F}_q} R}(V \otimes_{\mathbb{F}_q} R) \mid h_R(gv_1, gv_2) = h_R(v_1,v_2) \text{ for } v_1, v_2 \in V \otimes_{\mathbb{F}_q} R\}\end{aligned}$$

for an $\mathbb{F}_q$-algebra R, with an action on $\mathbf{H}_{(V,h)}$ defined by $(v,a) \mapsto (gv,a)$ for $(v,a) \in \mathbf{H}_{(V,h)}(R)$ and $g \in \mathbf{U}_{(V,h)}(R)$. We put

$$\mathbf{HU}_{(V,h)} = \mathbf{H}_{(V,h)} \rtimes \mathbf{U}_{(V,h)}.$$

Further we put

$$\mathrm{U}(V,h) = \mathbf{U}_{(V,h)}(\mathbb{F}_q), \quad \mathrm{HU}(V,h) = \mathbf{HU}_{(V,h)}(\mathbb{F}_q).$$

We simply write $\mathbf{H}_V$, $\mathbf{U}_V$ and $\mathbf{HU}_V$ for $\mathbf{H}_{(V,h)}$, $\mathbf{U}_{(V,h)}$ and $\mathbf{HU}_{(V,h)}$ if it is clear which hermitian form is involved.

For a positive integer m, we put

$$\mathbb{F}_{q^m,+} = \{x \in \overline{\mathbb{F}}_q \mid x^{q^m} + x = 0\}.$$

Let $\psi \in \mathbb{F}_{q,+}^\vee \setminus \{1\}$. Let $\rho_{\mathrm{H}(V,h),\psi}$ denote the unique irreducible representation of $\mathrm{H}(V,h)$ whose restriction to the center $Z(V,h)$ contains ψ. We put $n = \dim_{\mathbb{F}_{q^2}} V$. Then $\rho_{\mathrm{H}(V,h),\psi}|_{Z(V,h)} \simeq \psi^{\oplus q^n}$. The following is shown in [2] and stated in [6, Lemma 2.2].

Lemma 3.1. *There exists a unique representation $\rho_{\mathrm{HU}(V,h),\psi}$ of $\mathrm{HU}(V,h)$ characterized by*

- *an isomorphism $\rho_{\mathrm{HU}(V,h),\psi}|_{\mathrm{H}(V,h)} \simeq \rho_{\mathrm{H}(V,h),\psi}$, and*
- *the equality $\mathrm{tr}\,\rho_{\mathrm{HU}(V,h),\psi}(g) = (-1)^n(-q)^{N(g)}$ for $g \in \mathrm{U}(V,h)$, where*

$$N(g) = \dim_{\mathbb{F}_{q^2}} \ker(g - \mathrm{id}_V).$$

We call $\rho_{\mathrm{HU}(V,h),\psi}$ the Heisenberg–Weil representation of $\mathrm{HU}(V,h)$ associated to ψ. The restriction of $\rho_{\mathrm{HU}(V,h),\psi}$ to the subgroup $\mathrm{U}(V,h)$ is called the Weil representation of $\mathrm{U}(V,h)$ associated to ψ.

Let

$$h_n \colon \mathbb{F}_{q^2}^n \times \mathbb{F}_{q^2}^n \to \mathbb{F}_{q^2};\ ((x_1,\ldots,x_n),(x_1',\ldots,x_n')) \mapsto \sum_{i=1}^n x_i^q x_i'.$$

Any nondegenerate hermitian space (V,h) of dimension n is isometric to $(\mathbb{F}_{q^2}^n, h_n)$. We simply write $\mathrm{H}_n(q)$, $\mathrm{U}_n(q)$ and $\mathrm{HU}_n(q)$ for $\mathrm{H}(\mathbb{F}_{q^2}^n, h_n)$, $\mathrm{U}(\mathbb{F}_{q^2}^n, h_n)$ and $\mathrm{HU}(\mathbb{F}_{q^2}^n, h_n)$, respectively.

4 Geometric construction

We give a geometric construction of an extended Heisenberg–Weil representation. Let m be an odd positive integer. We consider the smooth affine variety over $\mathbb{F}_{q^2}$ defined by

$$z^{q^m} + z = \sum_{i=1}^{n} x_i^{q^m+1}$$

in $\mathbb{A}^{n+1}_{\mathbb{F}_{q^2}}$, which is denoted by $X_{m,n}$. The group $\mathrm{HU}_n(q^m)$ acts on $X_{m,n,\mathbb{F}_{q^{2m}}}$ by

$$\begin{aligned}
&((x_i), z) \mapsto (g(x_i), z) \quad \text{for } g \in \mathrm{U}_n(q^m), \\
&((x_i), z) \mapsto ((x_i) + v, z + a + h_n(v, (x_i))) \quad \text{for } (v, a) \in \mathrm{H}_n(q^m).
\end{aligned}$$

Let $\mathrm{HU}_n(q^m)$ act on $H^n_{\mathrm{c}}(X_{m,n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ by letting $g \in \mathrm{HU}_n(q^m)$ act as $(g^{-1})^*$.

Let $\nu \colon \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^2}) \to \overline{\mathbb{Q}}_\ell^\times$ be the character which sends the geometric Frobenius element Fr_{q^2} to $-q^{-1}$. Let $\psi \in \mathbb{F}_{q,+}^\vee \setminus \{1\}$. The restriction of the trace map $\mathrm{Tr}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^2}}$ to $\mathbb{F}_{q^m,+}$ induces $\mathrm{Tr}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^2}} \colon \mathbb{F}_{q^m,+} \to \mathbb{F}_{q,+}$. We set

$$\psi_m = \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^2}} \in \mathbb{F}_{q^m,+}^\vee.$$

The Frobenius endomorphism F of the algebraic group $\mathbf{HU}_{(\mathbb{F}^n_{q^2}, h_n)}$ over $\mathbb{F}_q$ induces an automorphism of $\mathrm{HU}_n(q^m)$, for which we use the same symbol F. Viewing $\mathrm{HU}_n(q^m)$ as a subset of $\mathbb{F}^n_{q^{2m}} \times \mathbb{F}_{q^{2m}} \times \mathrm{GL}_n(\mathbb{F}_{q^{2m}})$ naturally, the coordinatewise q-th power map on $\mathbb{F}^n_{q^{2m}} \times \mathbb{F}_{q^{2m}} \times \mathrm{GL}_n(\mathbb{F}_{q^{2m}})$ induces an automorphism τ of $\mathrm{HU}_n(q^m)$. Then we have $F^2 = \tau^2$ by the definitions. Hence we have $F = F^{m+1} = \tau^{m+1}$ as an automorphism of $\mathrm{HU}_n(q^m)$ because $\mathrm{HU}_n(q^m) = \mathbf{HU}^{F^m}_{(\mathbb{F}^n_{q^2}, h_n)}$ and m is odd.

We regard $H^n_{\mathrm{c}}(X_{m,n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ as a $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^2})$-representation as usual. By [1, Rapport (1.8.1)], the action of Fr_{q^2} on $H^n_{\mathrm{c}}(X_{m,n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ is equal to the pullback under the q^2-nd power Frobenius endomorphism F_X of $X_{m,n}$ over $\mathbb{F}_{q^2}$. Let $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^2})$ act on $\mathrm{HU}_n(q^m)$ by letting Fr_{q^2} act as F^{-2}, since we have

$$F_X^*(g^{-1})^* = (g^{-1}F_X)^* = (F_X F^{-2}(g)^{-1})^* = (F^{-2}(g)^{-1})^* F_X^*.$$

We regard

$$H^n_{\mathrm{c}}(X_{m,n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi_m] \otimes \nu^n$$

as an $\mathrm{HU}_n(q^m) \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^2})$-representation, which we denote by $\widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}$.

Lemma 4.1 ([6, Theorem 2.5]). *We have $H^i_{\mathrm{c}}(X_{m,n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi_m] = 0$ for $i \neq n$. The restriction of $\widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}$ to $\mathrm{HU}_n(q^m)$ is isomorphic to $\rho_{\mathrm{HU}_n(q^m),\psi_m}$.*

Lemma 4.2. *(1) The geometric Frobenius element Fr_{q^2} acts on*

$$H^1_{\mathrm{c}}(X_{1,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]$$

as scalar multiplication by $-q$.

(2) We have $\widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}(\mathrm{Fr}_{q^{2m}}) = 1$.

Proof. By Lemma 4.1, we have $H^i_{\mathrm{c}}(X_{1,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi] = 0$ for $i \neq 1$. Hence we have

$$\mathrm{Tr}(\mathrm{Fr}_{q^2}; H^1_{\mathrm{c}}(X_{1,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]) = -q^{-1} \sum_{\eta \in \mathbb{F}_{q,+}} \psi(\eta)\, \mathrm{Tr}((-\eta)\mathrm{Fr}_{q^2}; H^*_{\mathrm{c}}(X_{1,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)).$$

By the Lefschetz fixed point formula, we have

$$\begin{aligned}\mathrm{Tr}((-\eta)\mathrm{Fr}_{q^2}; H^*_{\mathrm{c}}(X_{1,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)) &= \sharp\{(x,z) \in X_{1,1}(\overline{\mathbb{F}}_q) \mid x^{q^2} = x,\ z^{q^2} - \eta = z\} \\ &= \begin{cases} q^3 & \text{if } \eta = 0, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Thus we obtain

$$\mathrm{Tr}(\mathrm{Fr}_{q^2}; H^1_{\mathrm{c}}(X_{1,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]) = -q^2. \tag{4.1}$$

The action of Fr_{q^2} commutes with the one of $\mathrm{H}_1(q)$. Since $H^1_{\mathrm{c}}(X_{1,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]$ is an irreducible $\mathrm{H}_1(q)$-representation by Lemma 4.1, Fr_{q^2} acts on it as scalar multiplication by Schur's lemma. Hence the first claim follows from (4.1), because $\dim H^1_{\mathrm{c}}(X_{1,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi] = q$ by Lemma 4.1.

We show the second claim. By the first assertion and the Künneth formula, $\mathrm{Fr}_{q^{2m}}$ acts on $H^n_{\mathrm{c}}(X_{m,n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]$ as scalar multiplication by $(-1)^n q^{mn}$. We have $\nu^n(\mathrm{Fr}_{q^{2m}}) = (-1)^{mn} q^{-mn}$. Since m is odd, the claim follows. □

We set $\Gamma = \mathrm{Gal}(\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^2})$. By Lemma 4.2 (2), we can regard $\widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}$ as an $\mathrm{HU}_n(q^m) \rtimes \Gamma$-representation. This is an extension of $\rho_{\mathrm{HU}_n(q^m),\psi_m}$ to $\mathrm{HU}_n(q^m) \rtimes \Gamma$ by Lemma 4.1. The restriction of $\widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}$ to the subgroup $\mathrm{U}_n(q^m) \rtimes \Gamma$ is denoted by $\widetilde{\rho}_{\mathrm{U}_n(q^m),\psi_m}$ as in §1. We put

$$\sigma = \mathrm{Fr}_{q^2}^{\frac{m-1}{2}}.$$

Then σ is the generator of Γ acting on $\mathrm{HU}_n(q^m)$ as F.

Definition 4.3. *Let (V,h) be a nondegenerate hermitian space of dimension n over $\mathbb{F}_{q^2}$. We take an isometry $(V,h) \simeq (\mathbb{F}^n_{q^2}, h_n)$. This induces $\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma \simeq \mathrm{HU}_n(q^m) \rtimes \Gamma$. Via this isomorphism and $\widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}$, we define a representation of $\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma$, which is denoted by $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$.*

Remark 4.4. *The isomorphism class of $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ is independent of the choice of the isometry $(V,h) \simeq (\mathbb{F}^n_{q^2}, h_n)$, since the difference is a conjugation by an element of $\mathbf{U}_V(\mathbb{F}_q)$.*

For a positive integer r, let $\mu_r = \{x \in \overline{\mathbb{F}}_q \mid x^r = 1\}$.

Proposition 4.5. *Let $1 \leq j \leq m$. We set $(j,m) = d$. Let $\zeta \in \mu_{q^m+1}$.*

(1) We have

$$\mathrm{tr}\, \widetilde{\rho}_{\mathrm{HU}_1(q^m),\psi_m}(\zeta, \mathrm{Fr}^j_{q^2}) = \begin{cases} q^d & \text{if } \zeta \in \mu_{\frac{q^m+1}{q^d+1}}, \\ -1 & \text{otherwise.} \end{cases}$$

(2) We have

$$\mathrm{tr}\, \widetilde{\rho}_{\mathrm{HU}_1(q^m),\psi_m}(\zeta, \mathrm{Fr}^j_{q^2}) = \mathrm{tr}\, \widetilde{\rho}_{\mathrm{HU}_1(q^d),\psi_d}(\zeta^{\frac{q^m+1}{q^d+1}}).$$

(3) We have $\mathrm{tr}\, \widetilde{\rho}_{\mathrm{HU}_n(q^m),\psi_m}(\mathrm{Fr}^j_{q^2}) = q^{nd}$.

Proof. The third claim follows from the Künneth formula and the first one. The second claim follows from the first one and Lemma 3.1.

We show the first claim. We consider the curve X_m defined by $z^q + z = x^{q^m+1}$ over $\mathbb{F}_{q^2}$. We have the finite étale morphism

$$X_{m,1} \to X_m;\ (x,z) \mapsto \left(x, \sum_{i=0}^{m-1} (-1)^i z^{q^i}\right).$$

The pull-back of this induces

$$H^1_{\mathrm{c}}(X_{m,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi] \xrightarrow{\sim} H^1_{\mathrm{c}}(X_{m,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi_m]. \tag{4.2}$$

Thus

$$\operatorname{tr} \widetilde{\rho}_{\mathrm{HU}_1(q^m),\psi_m}(\zeta, \mathrm{Fr}^j_{q^2}) = (-q^{-1})^j \operatorname{Tr}(\zeta \mathrm{Fr}^j_{q^2}; H^1_{\mathrm{c}}(X_{m,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]). \tag{4.3}$$

In the sequel, we identify $\mathrm{U}_1(q^i)$ with μ_{q^i+1} for any positive integer i. We have an isomorphism

$$H^1_{\mathrm{c}}(X_{m,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi] \simeq \bigoplus_{\chi \in \mu^\vee_{q^m+1} \setminus \{1\}} \chi \tag{4.4}$$

as μ_{q^m+1}-representations as in [7, Lemma 2.2]. Let $\chi \in \mu^\vee_{q^m+1}$. We have

$$\chi^{q^{2j}} = \chi \iff \chi|_{\mu_{\frac{q^m+1}{q^d+1}}} = 1, \tag{4.5}$$

because of $(q^m+1, q^{2j}-1) = q^d+1$. We regard $\mu^\vee_{q^d+1}$ as a subset of $\mu^\vee_{q^m+1}$ by the dual of

$$\mu_{q^m+1} \to \mu_{q^d+1};\ x \mapsto x^{\frac{q^m+1}{q^d+1}}.$$

We have the finite morphism

$$X_m \to X_d;\ (x,z) \mapsto \left(x^{\frac{q^m+1}{q^d+1}}, z\right).$$

Then the image of its pull-back map

$$H^1_{\mathrm{c}}(X_{d,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi] \to H^1_{\mathrm{c}}(X_{m,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]$$

equals $\bigoplus_{\chi \in \mu^\vee_{q^d+1} \setminus \{1\}} \chi$ under (4.4). We write as $j = dk$. We compute

$$\begin{aligned} \operatorname{Tr}(\zeta \mathrm{Fr}^j_{q^2}; H^1_{\mathrm{c}}(X_{m,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]) &= \operatorname{Tr}(\zeta^{\frac{q^m+1}{q^d+1}} \mathrm{Fr}^k_{q^{2d}}; H^1_{\mathrm{c}}(X_{d,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]) \\ &= (-q^d)^k \operatorname{Tr}(\zeta^{\frac{q^m+1}{q^d+1}}; H^1_{\mathrm{c}}(X_{d,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]) \\ &= (-q^d)^k \sum_{\chi \in \mu^\vee_{q^d+1} \setminus \{1\}} \chi(\zeta^{\frac{q^m+1}{q^d+1}}) \\ &= \begin{cases} (-1)^k q^{d(k+1)} & \text{if } \zeta \in \mu_{\frac{q^m+1}{q^d+1}}, \\ (-1)^{k+1} q^{dk} & \text{otherwise,} \end{cases} \end{aligned}$$

where the first equality follows from (4.5) and the second one follows from Lemma 4.2 (1) since

$$H^1_{\mathrm{c}}(X_{d,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi] \xrightarrow{\sim} H^1_{\mathrm{c}}(X_{d,1,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi_d]$$

similarly to (4.2). Hence the claim follows from (4.3), since d is odd. □

5 Compatibility

Let (V, h) be a nondegenerate hermitian space over $\mathbb{F}_{q^2}$ of dimension n.

5.1 Orthogonal decomposition

Let $(V, h) = (V_1, h_1) \oplus (V_2, h_2)$ be a decomposition of V into the orthogonal direct sum of two hermitian spaces. Then we have the natural homomorphism

$$\mathbf{HU}_{V_1} \times \mathbf{HU}_{V_2} \to \mathbf{HU}_V.$$

This morphism and the projections induce

$$\begin{array}{ccc} \big(\mathbf{HU}_{V_1}(\mathbb{F}_{q^m}) \times \mathbf{HU}_{V_2}(\mathbb{F}_{q^m})\big) \rtimes \Gamma & \xrightarrow{\delta} & \mathbf{HU}_V(\mathbb{F}_{q^m}) \rtimes \Gamma \\ \downarrow i & & \\ \big(\mathbf{HU}_{V_1}(\mathbb{F}_{q^m}) \rtimes \Gamma\big) \times \big(\mathbf{HU}_{V_2}(\mathbb{F}_{q^m}) \rtimes \Gamma\big), & & \end{array} \tag{5.1}$$

where Γ acts on $\mathbf{HU}_{V_1}(\mathbb{F}_{q^m}) \times \mathbf{HU}_{V_2}(\mathbb{F}_{q^m})$ by $(h_1, h_2) \mapsto (\sigma' h_1, \sigma' h_2)$ for $\sigma' \in \Gamma$.

Proposition 5.1. *The inflation of $\widetilde{\rho}_{\mathbf{HU}_V(\mathbb{F}_{q^m}),\psi_m}$ by δ is isomorphic to the restriction of $\widetilde{\rho}_{\mathbf{HU}_{V_1}(\mathbb{F}_{q^m}),\psi_m} \boxtimes \widetilde{\rho}_{\mathbf{HU}_{V_2}(\mathbb{F}_{q^m}),\psi_m}$ by i.*

Proof. The claim follows from Definition 4.3 and the Künneth formula. □

5.2 Parabolic subgroup

Let W be an isotropic subspace of V and $W^\perp$ be its orthogonal. Let (W_0, h_0) be the hermitian space $W^\perp/W$ with induced hermitian form by h. Let $\mathbf{P}_W$ be the stabilizer of W in $\mathbf{U}_V$. Then $\mathbf{P}_W$ naturally acts on $\mathbf{H}_{W^\perp}$. We have the natural homomorphism $\mathbf{H}_{W^\perp} \rtimes \mathbf{P}_W \to \mathbf{HU}_{W_0}$. This induces the homomorphism

$$(\mathbf{H}_{W^\perp} \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma \to \mathbf{HU}_{W_0}(\mathbb{F}_{q^m}) \rtimes \Gamma. \tag{5.2}$$

Proposition 5.2. *The restriction of $\widetilde{\rho}_{\mathbf{HU}_V(\mathbb{F}_{q^m}),\psi_m}$ to $(\mathbf{H}_V \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma$ is isomorphic to*

$$\mathrm{Ind}_{(\mathbf{H}_{W^\perp} \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma}^{(\mathbf{H}_V \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma} \widetilde{\rho}_{\mathbf{HU}_{W_0}(\mathbb{F}_{q^m}),\psi_m},$$

where the inflation of $\widetilde{\rho}_{\mathbf{HU}_{W_0}(\mathbb{F}_{q^m}),\psi_m}$ via (5.2) is denoted by the same symbol.

Proof. We have $\mathrm{tr}\, \widetilde{\rho}_{\mathbf{HU}_V(\mathbb{F}_{q^m}),\psi_m}(\sigma) = q^n$ by Proposition 4.5 (3). Hence it suffices to check that the trace of the induced representation at σ is q^n by [2, Theorem 3.3(b)]. We see that the trace equals

$$|\mathbf{H}_V(\mathbb{F}_q)/\mathbf{H}_{W^\perp}(\mathbb{F}_q)| q^{n_0} = q^n,$$

where $n_0 = \dim W_0$, by the character formula of an induced representation and Proposition 4.5 (3) (*cf.* [5, the proof of Proposition 6.2]). Hence we obtain the claim. □

6 Support of trace of extended Weil representation

We have the isomorphism

$$\iota\colon \mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma \xrightarrow{\sim} \mathbf{HU}_{(V,-h)}(\mathbb{F}_{q^m}) \rtimes \Gamma;\ (v,a,g) \mapsto (v,-a,g).$$

Lemma 6.1. *The $\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma$-representation $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m^{-1}}$ is isomorphic to the inflation of $\widetilde{\rho}_{\mathbf{HU}_{(V,-h)}(\mathbb{F}_{q^m}),\psi_m}$ by ι.*

Proof. This follows from Lemma 3.1 and Proposition 4.5 (3). □

Let $(2V,\pm h) = (V,h) \oplus (V,-h)$ be the orthogonal direct sum. By (5.1), we have the group homomorphism

$$\delta'\colon (\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \times \mathbf{HU}_{(V,-h)}(\mathbb{F}_{q^m})) \rtimes \Gamma \to \mathbf{HU}_{(2V,\pm h)}(\mathbb{F}_{q^m}) \rtimes \Gamma.$$

Let

$$\Delta\colon \mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma \to \mathbf{HU}_{(2V,\pm h)}(\mathbb{F}_{q^m}) \rtimes \Gamma;\ (g,\sigma) \mapsto \delta'(g,\iota(g,\sigma)).$$

Lemma 6.2. *The inflation of $\widetilde{\rho}_{\mathbf{HU}_{(2V,\pm h)}(\mathbb{F}_{q^m}),\psi_m}$ by Δ is isomorphic to the restriction of $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m} \otimes \widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m^{-1}}$.*

Proof. The inflation of $\widetilde{\rho}_{\mathbf{HU}_{(2V,\pm h)}(\mathbb{F}_{q^m}),\psi_m}$ by δ' is isomorphic to the restriction of the representation

$$\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m} \boxtimes \widetilde{\rho}_{\mathbf{HU}_{(V,-h)}(\mathbb{F}_{q^m}),\psi_m}$$

to

$$(\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \times \mathbf{HU}_{(V,-h)}(\mathbb{F}_{q^m})) \rtimes \Gamma$$

by Proposition 5.1. Hence the claim follows from Lemma 6.1. □

Let $\mathbf{Z}_{(V,h)}$ be the center of $\mathbf{H}_{(V,h)}$. We put $\mathbf{ZU}_{(V,h)} = \mathbf{Z}_{(V,h)} \times \mathbf{U}_{(V,h)}$.

Lemma 6.3. *The tensor product $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m} \otimes \widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m^{-1}}$ is isomorphic to the induction of the trivial character of $\mathbf{ZU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma$.*

Proof. Take $\zeta \in \mathbb{F}_{q^2} \setminus \{1\}$ such that $\mathrm{Nr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\zeta) = 1$. Let W be the Lagrangian subspace $\{(v,\zeta v) \in V \oplus V\}$ of $2V$, which is isotropic via $\pm h$. Note that $W^\perp = W$. Applying Proposition 5.2 in this situation, we have $W_0 = \{0\}$, $\mathbf{HU}_{W_0}(\mathbb{F}_{q^m}) \simeq \mathbb{F}_{q^m,+}$ and $\widetilde{\rho}_{\mathbf{HU}_{W_0}(\mathbb{F}_{q^m}),\psi_m}$ is the character

$$\mathbb{F}_{q^m,+} \rtimes \Gamma \to \overline{\mathbb{Q}}_\ell^\times;\ (x,\sigma^i) \mapsto \psi_m(x).$$

Let

$$\widetilde{\psi}_m\colon (\mathbf{H}_{(W,\pm h)} \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma \to \overline{\mathbb{Q}}_\ell^\times;\ \big((v,\zeta v),a,g,\sigma^i\big) \mapsto \psi_m(a).$$

Then $\widetilde{\psi}_m$ is the inflation of $\widetilde{\rho}_{\mathbf{HU}_{W_0}(\mathbb{F}_{q^m}),\psi_m}$ via (5.2). Thus Proposition 5.2 gives an isomorphism

$$\widetilde{\rho}_{\mathbf{HU}_{(2V,\pm h)}(\mathbb{F}_{q^m}),\psi_m}\big|_{(\mathbf{H}_{(2V,\pm h)} \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma} \simeq \mathrm{Ind}_{(\mathbf{H}_{(W,\pm h)} \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma}^{(\mathbf{H}_{(2V,\pm h)} \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma} \widetilde{\psi}_m. \tag{6.1}$$

The image of Δ is contained in $(\mathbf{H}_{(2V,\pm h)} \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma$. The map Δ induces a bijection

$$(\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma)/(\mathbf{ZU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma) \xrightarrow{\sim} ((\mathbf{H}_{(2V,\pm h)} \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma)/((\mathbf{H}_{(W,\pm h)} \rtimes \mathbf{P}_W)(\mathbb{F}_{q^m}) \rtimes \Gamma).$$

Inflating (6.1) under Δ, we obtain the claim by Lemma 6.2. □

Remark 6.4. *The proof of Lemma 6.3 is slightly different from that of [5, Lemma 7.3] in the sense that we use* $\{(v, \zeta v) \in V \oplus V\}$ *instead of* $\{(v, v) \in V \oplus V\}$ *as* W. *By this choice, the bijection at the end of the proof holds and the proof becomes clearer.*

Corollary 6.5. *The trace of the representation* $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ *is zero outside the conjugates of* $\mathbf{ZU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma$.

Proof. We take an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Using this isomorphism, we consider $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ and $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m^{-1}}$ as representations over $\mathbb{C}$. First, we show that the representation $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m^{-1}}$ is isomorphic to the complex conjugate of $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$. Clearly ψ_m^{-1} is equal to the complex conjugate of ψ_m. Hence $\rho_{\mathbf{H}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m^{-1}}$ is isomorphic to the complex conjugate of $\rho_{\mathbf{H}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$. Therefore, we obtain the claim by Lemma 3.1 and Proposition 4.5 (3), since $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m^{-1}}$ is irreducible as an $\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m})$-representation.

The required claim follows from this and Lemma 6.3. □

7 Main theorem

Let the notation be as in §2.

Theorem 7.1. *Let* $1 \leq i \leq m$ *be an integer. We set* $d = (m, i)$. *We take an integer* t *such that* $ti \equiv d \pmod{m}$. *Then there is a unique extension* $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ *of* $\rho_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ *to* $\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma$ *such that*

$$\operatorname{tr} \widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}(g, \sigma^i) = \operatorname{tr} \widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^d}),\psi_d}(\mathrm{N}_{i,t}(g, \sigma^i)) \tag{7.1}$$

for any $g \in \mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m})$.

In §4, we already know that $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ in Definition 4.3 is an extension of $\rho_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$. Since $\rho_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ is irreducible, only one extension can satisfy (7.1). The aim in the rest of this paper is to prove that the extension $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ actually satisfies (7.1).

8 Reduction steps

To show Theorem 7.1, we imitate the arguments in [5, §8]. We recall a known fact.

Lemma 8.1. *Let* $g_0 \in \mathrm{U}(V, h)$. *Assume that* g_0 *stabilizes no non-trivial isotropic subspace. Then* g_0 *is semisimple.*

Proof. The claim is stated in [2, proof of Theorem 4.9.2]. We recall a proof here (*cf.* [5, §8]). Let $g_0 = su$ be the Jordan decomposition in $\mathrm{U}(V, h)$. If $u \neq \mathrm{id}_V$, then $\mathrm{Im}(u - \mathrm{id}_V) \cap \mathrm{Ker}(u - \mathrm{id}_V)$ is a non-trivial isotropic subspace of V stable under g_0. Hence the claim follows. □

We fix i and t. Changing the base field from $\mathbb{F}_q$ to $\mathbb{F}_{q^d}$, we may assume that $d = (m, i) = 1$ (*cf.* [5, Remark 3.1 (ii)]). Let $g \in \mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m})$. We set

$$g_0 = \mathrm{N}_{i,t}(g, \sigma^i).$$

If g_0 does not belong to $\mathbf{ZU}_{(V,h)}(\mathbb{F}_q)$, the both sides of (7.1) are 0 by Corollary 6.5. Assume $g \in \mathbf{Z}_{(V,h)}(\mathbb{F}_{q^m}) \simeq \mathbb{F}_{q^m,+}$. The restriction of $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ to $\mathbf{Z}_{(V,h)}(\mathbb{F}_{q^m})$ is a multiple of ψ_m.

Clearly $((m-1)i/2, m) = 1$ by $(m, i) = 1$. Hence the left hand side of (7.1) equals $\psi_m(g)q^n$ by Proposition 4.5 (3) for $j = (m-1)i/2$, because $\sigma = \mathrm{Fr}_{q^2}^{\frac{m-1}{2}}$. By the definition of $\mathrm{N}_{i,t}$ and $(m, i) = 1$, we have

$$g_0 = g\sigma^i(g)\cdots\sigma^{i(m-1)}(g) = \mathrm{Tr}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^2}}(g).$$

Thus the right hand side of (7.1) is $\psi(g_0)\dim\widetilde{\rho}_{\mathrm{HU}(V,h),\psi} = \psi_m(g)q^n$. Hence we have (7.1) on $\mathbf{Z}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma$. By applying Lemma 2.1 (4) to the product $\mathbf{ZU}_{(V,h)} = \mathbf{Z}_{(V,h)} \times \mathbf{U}_{(V,h)}$, we may assume that $g_0 \in \mathrm{U}(V, h)$.

We show Theorem 7.1 in the case where $g_0 \in \mathrm{U}(V, h)$ by induction on n. The claim for $n = 1$ is shown in Proposition 4.5 (2). Assume $n > 1$. If g_0 stabilizes a non-trivial orthogonal decomposition of (V, h), the claim follows from the induction hypothesis by Lemma 2.1 (4), and Proposition 5.1. If g_0 stabilizes a non-trivial isotropic subspace of V, the claim follows from the induction hypothesis by Lemma 2.1 (1) and Proposition 5.2.

Now, we may assume that g_0 *stabilizes no non-trivial isotropic subspace and stabilizes no non-trivial orthogonal decomposition.* By Lemma 8.1, g_0 is semisimple. We write s for g_0.

We set $A = \mathrm{End}_{\mathbb{F}_{q^2}}(V)$. Let

$$\dagger_h \colon A \to A;\ f \mapsto f^{\dagger_h}$$

be the adjoint involution associated to h. Namely, we have $h(f(x), y) = h(x, f^{\dagger_h}(y))$ for any $x, y \in V$. Then we have $s^{\dagger_h} = s^{-1}$ by definition. Hence the involution stabilizes $\mathbb{F}_{q^2}[s]$.

Lemma 8.2. *The subalgebra $\mathbb{F}_{q^2}[s] \subset \mathrm{End}_{\mathbb{F}_{q^2}}(V)$ is a field.*

Proof. Since s is semisimple, we can write as

$$\mathbb{F}_{q^2}[s] \simeq \prod_{\alpha\in I} F_\alpha,$$

where F_α is a field. Let e_α be the idempotent in $\mathbb{F}_{q^2}[s]$ associated to F_α. We have a direct sum

$$V = \bigoplus_{\alpha\in I} V_\alpha,$$

where $V_\alpha = \{v \in V \mid e_\alpha v = v\}$. The subspaces V_α are s-stable. The involution $\dagger_h$ gives a permutation $\alpha \mapsto \overline{\alpha}$ on I, with an isomorphism $F_\alpha \simeq F_{\overline{\alpha}}$.

Assume that I has at least two elements. We take $\alpha \in I$. If $\alpha = \overline{\alpha}$, then V is the orthogonal sum of V_α and $\bigoplus_{\beta\neq\alpha} V_\beta$ as hermitian spaces. Actually, we have

$$h(x_\alpha, x_\beta) = h(e_\alpha x_\alpha, x_\beta) = h(x_\alpha, e_{\overline{\alpha}} x_\beta) = h(x_\alpha, e_\alpha x_\beta) = 0$$

for any $\alpha \neq \beta$, $x_\alpha \in V_\alpha$ and $x_\beta \in V_\beta$. Hence V has a non-trivial s-stable orthogonal decomposition. This contradicts to the assumption.

Assume $\alpha \neq \overline{\alpha}$. Then V_α is a non-trivial s-stable isotropic subspace of V. Actually, we have

$$h(x_\alpha, x'_\alpha) = h(e_\alpha x_\alpha, x'_\alpha) = h(x_\alpha, e_{\overline{\alpha}} x'_\alpha) = 0$$

for any $x_\alpha, x'_\alpha \in V_\alpha$. Again this contradicts to the assumption. Hence I consists of one element. The claim follows. □

We put

$$E = \mathbb{F}_{q^2}[s] \subset A, \quad E_+ = E^{\dagger_h}.$$

We regard V as an E-vector space.

Lemma 8.3. *(1) The extension E/E_+ is a quadratic extension and $[E_+ : \mathbb{F}_q]$ is odd.*

(2) There exists a nondegenerate hermitian form $\widetilde{h}\colon V \times V \to E$ such that $h = \mathrm{Tr}_{E/\mathbb{F}_{q^2}} \circ \widetilde{h}$.

(3) We have $\dim_E V = 1$.

Proof. The first claim follows from that $\dagger_h$ on $\mathbb{F}_{q^2}$ is the q-th power map.

Since $\mathrm{Tr}_{E/\mathbb{F}_{q^2}} \colon E \times E \to \mathbb{F}_{q^2}$ is nondegenerate, we can define a nondegenerate hermitian form $\widetilde{h}\colon V \times V \to E$ by the condition that

$$\mathrm{Tr}_{E/\mathbb{F}_{q^2}}(a\widetilde{h}(v, v')) = h(v, av')$$

for $a \in E, v, v' \in V$. Hence we obtain the second claim.

The element $s \in E^\times$ acts on V as scalar multiplication. Since s stabilizes no non-trivial orthogonal decomposition of $(V, \widetilde{h})$ as hermitian spaces over E by the assumption, we obtain the third claim. □

9 Proof in the reduced case

We will show Theorem 7.1 in the situation of Lemma 8.3. Note that n is odd and $E = \mathbb{F}_{q^{2n}}$ by Lemma 8.3. By the natural homomorphism $\mathrm{U}(V, \widetilde{h}) \hookrightarrow \mathrm{U}(V, h)$, we obtain

$$r\colon\ \mathrm{HU}(V, \widetilde{h}) \to \mathrm{HU}(V, h);\ (v, a, g) \mapsto (v, \mathrm{Tr}_{E/\mathbb{F}_{q^2}}(a), g).$$

We put $\psi_E = \psi \circ \mathrm{Tr}_{E/\mathbb{F}_{q^2}} \in \mathbb{F}_{q^n,+}^\vee$.

Lemma 9.1. *(1) The inflation of $\rho_{\mathrm{HU}(V,h),\psi}$ by r is isomorphic to $\rho_{\mathrm{HU}(V,\widetilde{h}),\psi_E}$.*

(2) Let $s \in \mathrm{U}(V, \widetilde{h})$. Then we have

$$\mathrm{tr}\, \rho_{\mathrm{HU}(V,h),\psi}(s) = \mathrm{tr}\, \rho_{\mathrm{HU}(V,\widetilde{h}),\psi_E}(s) = \begin{cases} q^n & \text{if } s = 1, \\ -1 & \text{otherwise.} \end{cases}$$

Proof. Let $r^*\rho_{\mathrm{HU}(V,h),\psi}$ denote the inflation of $\rho_{\mathrm{HU}(V,h),\psi}$ by r. The restriction of $r^*\rho_{\mathrm{HU}(V,h),\psi}$ to the center of $\mathrm{H}(V, \widetilde{h})$ is a multiple of ψ_E with multiplicity q^n. Hence we have

$$(r^*\rho_{\mathrm{HU}(V,h),\psi})|_{\mathrm{H}(V,\widetilde{h})} \simeq \rho_{\mathrm{H}(V,\widetilde{h}),\psi_E}.$$

Therefore the second claim implies the first one. The second claim follows from Lemma 3.1 (*cf.* [2, Corollary 3.5]). □

Let $e = (m, n)$. We put $\Gamma_e = \mathrm{Gal}(\mathbb{F}_{q^e}/\mathbb{F}_q)$. For $\alpha \in \Gamma_e$, we put

$$E_\alpha = E \otimes_{\mathbb{F}_{q^e},\alpha} \mathbb{F}_{q^m}, \quad V_\alpha = V \otimes_{\mathbb{F}_{q^e},\alpha} \mathbb{F}_{q^m},$$

where $\mathbb{F}_{q^m}$ is regarded as an $\mathbb{F}_{q^e}$-algebra via the composite $\mathbb{F}_{q^e} \xrightarrow{\alpha} \mathbb{F}_{q^e} \hookrightarrow \mathbb{F}_{q^m}$. Note that E_α is isomorphic to $\mathbb{F}_{q^{2mn/e}}$. We have

$$\begin{aligned} E \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m} &\xrightarrow{\sim} \prod_{\alpha \in \Gamma_e} E_\alpha;\ x \otimes a \mapsto (x \otimes a)_\alpha, \\ V \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m} &\xrightarrow{\sim} \prod_{\alpha \in \Gamma_e} V_\alpha;\ v \otimes a \mapsto (v \otimes a)_\alpha. \end{aligned} \tag{9.1}$$

The base change of $\widetilde{h}$ for $\mathbb{F}_{q^m}/\mathbb{F}_q$ induces a hermitian form $\widetilde{h}_\alpha \colon V_\alpha \times V_\alpha \to E_\alpha$. We put

$$h_\alpha = \mathrm{Tr}_{E_\alpha/\mathbb{F}_{q^{2m}}} \circ \widetilde{h}_\alpha,$$

which is a hermitian form on V_α over $\mathbb{F}_{q^{2m}}$.

Let (V_m, h_m) be the base change of (V, h) for $\mathbb{F}_{q^m}/\mathbb{F}_q$. Then (V_m, h_m) is isomorphic to the orthogonal sum $\bigoplus_\alpha (V_\alpha, h_\alpha)$. Hence we have the homomorphisms

$$\prod_{\alpha\in\Gamma_e} \mathrm{HU}(V_\alpha, \widetilde{h}_\alpha) \hookrightarrow \prod_{\alpha\in\Gamma_e} \mathrm{HU}(V_\alpha, h_\alpha) \hookrightarrow \mathrm{HU}(V_m, h_m) = \mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}). \tag{9.2}$$

We put $\alpha_0 = \mathrm{id}_{\mathbb{F}_{q^e}} \in \Gamma_e$. We omit the index set Γ_e in the notation. Let

$$\mathrm{pr}_{\alpha_0} \colon \prod_\alpha \mathrm{HU}(V_\alpha, \widetilde{h}_\alpha) \to \mathrm{HU}(V_{\alpha_0}, \widetilde{h}_{\alpha_0}).$$

be the projection. We consider the homomorphism

$$\left(\prod_\alpha \mathrm{HU}(V_\alpha, \widetilde{h}_\alpha)\right) \rtimes \langle\sigma^e\rangle \xrightarrow{\mathrm{pr}_{\alpha_0}\times\mathrm{id}} \mathrm{HU}(V_{\alpha_0}, \widetilde{h}_{\alpha_0}) \rtimes \langle\sigma^e\rangle, \tag{9.3}$$

where the left hand side is regarded as a subgroup of $\left(\prod_\alpha \mathrm{HU}(V_\alpha, \widetilde{h}_\alpha)\right) \rtimes \Gamma$. We put

$$\psi_{E_\alpha} = \psi \circ \mathrm{Tr}_{E_\alpha/\mathbb{F}_{q^2}} \in \mathbb{F}^\vee_{q^{mn/e},+}.$$

Let R denote the tensor induction to

$$\left(\prod_\alpha \mathrm{HU}(V_\alpha, \widetilde{h}_\alpha)\right) \rtimes \Gamma$$

of the inflation of $\widetilde{\rho}_{\mathrm{HU}(V_{\alpha_0},\widetilde{h}_{\alpha_0}),\psi_{E_{\alpha_0}}}$ under (9.3). Here and in the sequel, we identify $\langle\sigma^e\rangle = \mathrm{Gal}(\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^{2e}})$ with $\mathrm{Gal}(\mathbb{F}_{q^{2mn/e}}/\mathbb{F}_{q^{2n}})$ via the natural isomorphism $\mathbb{F}_{q^{2m}} \otimes_{\mathbb{F}_{q^{2e}}} \mathbb{F}_{q^{2n}} \cong \mathbb{F}_{q^{2mn/e}}$. We take $s_0' \in \mathrm{U}(V_{\alpha_0}, \widetilde{h}_{\alpha_0})$ with norm $s \in \mathrm{U}(V, \widetilde{h})$. We set

$$s' = (s_0', 1, \ldots, 1) \in \prod_\alpha \mathrm{U}(V_\alpha, \widetilde{h}_\alpha).$$

Recall that we assume $(m, i) = 1$.

Lemma 9.2. *We have*

$$\mathrm{tr}\, R(s', \sigma^i) = \begin{cases} q^n & \textit{if } s = 1, \\ -1 & \textit{otherwise.} \end{cases}$$

Proof. The group Γ permutes the factors $\{V_\alpha\}_{\alpha\in\Gamma_e}$ in (9.1) transitively and the stabilizer of each factor is $\langle\sigma^e\rangle$. Hence by [3, §2],

$$\begin{aligned} \mathrm{tr}\, R(s', \sigma^i) &= \mathrm{tr}\, \widetilde{\rho}_{\mathrm{HU}(V_{\alpha_0},\widetilde{h}_{\alpha_0}),\psi_{E_{\alpha_0}}}((\mathrm{pr}_{\alpha_0} \times \mathrm{id})(s', \sigma^i)^e) \\ &= \mathrm{tr}\, \widetilde{\rho}_{\mathrm{HU}(V_{\alpha_0},\widetilde{h}_{\alpha_0}),\psi_{E_{\alpha_0}}}(s_0', \sigma^{ie}) \end{aligned}$$

(*cf.* [9, Definitions 10 and 11]). We note $s_0'^{\frac{q^{mn/e}+1}{q^n+1}} = s$. Thus the claim follows from $(m, i) = 1$ and Proposition 4.5 (1) with taking $(q^n, m/e)$ as (q, m). □

Corollary 9.3. *(1) The* $(\prod_\alpha \mathrm{HU}(V_\alpha, \widetilde{h}_\alpha)) \rtimes \Gamma$*-representation* R *is isomorphic to the inflation of* $\widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ *by the natural homomorphism*

$$\left(\prod_\alpha \mathrm{HU}(V_\alpha, \widetilde{h}_\alpha)\right) \rtimes \Gamma \to \mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}) \rtimes \Gamma.$$

(2) We have

$$\mathrm{tr}\, \widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}(s', \sigma^i) = \begin{cases} q^n & \text{if } s = 1, \\ -1 & \text{otherwise.} \end{cases}$$

Proof. By [3, §2], the restriction of R to the subgroup $\prod_\alpha \mathrm{HU}(V_\alpha, \widetilde{h}_\alpha)$ is isomorphic to $\boxtimes_\alpha \rho_{\mathrm{HU}(V_\alpha,\widetilde{h}_\alpha),\psi_{E_\alpha}}$. The inflation of $\rho_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}$ by (9.2) is isomorphic to $\boxtimes_\alpha \rho_{\mathrm{HU}(V_\alpha,\widetilde{h}_\alpha),\psi_{E_\alpha}}$ by Proposition 5.1 and Lemma 9.1 (1). By Proposition 4.5 (3), we have

$$\mathrm{tr}\, \widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}(\sigma) = q^n.$$

Since σ is a generator of Γ, the first claim follows from Lemma 9.2. The second claim follows from the first one and Lemma 9.2. □

Note that (g, σ^i) is $\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m})$-conjugate to (s', σ^i) for any $g \in \mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m})$ satisfying $s = \mathrm{N}_{i,t}(g, \sigma^i)$. Hence, it suffices to show

$$\mathrm{tr}\, \widetilde{\rho}_{\mathbf{HU}_{(V,h)}(\mathbb{F}_{q^m}),\psi_m}(s', \sigma^i) = \mathrm{tr}\, \rho_{\mathrm{HU}(V,h),\psi}(s).$$

This follows from Lemma 9.1 (2) and Corollary 9.3 (2).

Acknowledgements

The authors would like to sincerely thank the reviewer for reading the paper and giving helpful comments. This work was supported by JSPS KAKENHI Grant Numbers 20K03529, 21H00973.

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Naoki Imai
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
naoki@ms.u-tokyo.ac.jp

Takahiro Tsushima
Department of Mathematics and Informatics, Faculty of Science, Chiba University 1-33 Yayoi-cho, Inage, Chiba, 263-8522, Japan
tsushima@math.s.chiba-u.ac.jp