Geometric realization of the local Langlands correspondence for representations of conductor three

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Abstract

We prove a realization of the local Langlands correspondence for two-dimensional representations of a Weil group of conductor three in the cohomology of Lubin–Tate curves by a purely local geometric method.

Introduction

Let K be a non-archimedean local field with a residue field k of characteristic p. Let $\mathfrak{p}$ be the maximal ideal of the ring of integers of K. We take an algebraic closure K^{ac} of K. Let K^{ur} be the maximal unramified extension of K inside K^{ac}. Let $\widehat{K}^{\mathrm{ac}}$ and $\widehat{K}^{\mathrm{ur}}$ denote the completions of K^{ac} and K^{ur} respectively. For a natural number n, we write $\mathrm{LT}(\mathfrak{p}^n)$ for the Lubin–Tate curve with full level n structure over $\widehat{K}^{\mathrm{ur}}$. We write W_K for the Weil group of K. Let D be the central division algebra over K of invariant $1/2$. Let ℓ be a prime number different from p. We take an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Then the groups W_K, $GL_2(K)$ and $D^\times$ act on $\varinjlim_n H^1_{\mathrm{c}}\left(\mathrm{LT}(\mathfrak{p}^n)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell\right)$, and these actions partially realize the local Langlands correspondence and the local Jacquet–Langlands correspondence for GL_2 over K. This realization was proved by Carayol in [Car90] using global automorphic arguments. However there is no known proof using only a local geometric method.

In this paper, we give a purely local proof of a realization of the local Langlands correspondence for two-dimensional W_K-representations of conductor three, using a description of a semi-stable reduction of a Lubin–Tate curve in [IT17]. A conductor of a representation of a Weil group means the Artin conductor exponent of it. We note that three is the smallest integer which is a conductor of a primitive two-dimensional W_K-representation. The calculation in [IT17] is done by purely local methods. In [IT17], the actions of W_K and $D^\times$ on a Lubin–Tate curve are calculated in some sense. Using the calculation, we can study representations of W_K and $D^\times$ in the cohomology of Lubin–Tate curves. On the other hand, we have already known the relation between representations of $GL_2(K)$ and $D^\times$ in the cohomology by [IT17, Proposition 2.1], which is based on a realization of the local Jacquet–Langlands correspondence proved in [Mie14] by purely local methods. Therefore, we can study the relation between representations of W_K and $GL_2(K)$ in the cohomology. This enables us to show a realization of the local Langlands correspondence in the cohomology of Lubin–Tate curves.

In the study of a realization of the local Langlands correspondence for GL_2, the most difficult and interesting case is the dyadic case, which means the case where $p = 2$. A proof in the case where p is odd is similar and easier. Therefore, we have decided to write a proof only in the dyadic case.

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In the dyadic case, the irreducible two-dimensional W_K-representations of conductor three are primitive. In a construction of the local Langlands correspondence for primitive representations in [Kut80], Weil representations are used (*cf.* [BH06, §12]). On the other hand, our descriptions of representations of $GL_2(K)$ in the cohomology of Lubin–Tate curves are given by cuspidal types. Therefore, it is a non-trivial problem to check that the described representations in the cohomology actually match under the local Langlands correspondence.

We explain the contents of this paper. In Section 1, we recall definitions of Lubin–Tate curves, and give an easy consequence of a cohomological result in [IT17]. In Section 2, we recall a description of a semi-stable reduction of a Lubin–Tate curve in [IT17].

In Section 3, we study the first degree etale cohomology of an elliptic curve, which appears in the semi-stable reduction of the Lubin–Tate curve as an irreducible component. The cohomology of this elliptic curve gives a primitive W_K-representation of conductor three.

In Section 4, we show that a correspondence of explicitly described representations appear in the cohomology of Lubin–Tate curves. In Section 5, we show that the correspondence obtained in Section 4 is actually the local Langlands correspondence by calculating epsilon factors. In other word, we give a description of the local Langlands correspondence via cuspidal type for representation of conductor three. To determine the sign of an epsilon factor, we calculate the Artin map for a wildly ramified abelian extension with a non-trivial ramification filtration by reducing it to the equal characteristic case using results in [Del84].

Following a suggestion of a referee, we give comments on related progresses after this paper was written. A semi-stable reduction of a Lubin–Tate curve studied in this paper is generalized to higher dimensional cases as reductions of affinoids in Lubin–Tate perfectoid spaces in [IT13] and [IT16]. Some arguments in Section 5 of this paper is generalized to GL_n-cases in [IT15b] to study an explicit description of the local Langlands correspondence for simple supercuspidal representations of GL_n. On the other hand, we use the Deligne–Laumon product formula in [IT15b], which is not of purely local nature. In [Tsu16], a purely local proof of the non-abelian Lubin–Tate theory for GL_2 is given in the odd equal characteristic case. A purely local proof of the non-abelian Lubin–Tate theory for a primitive Galois representation is not yet known outside the case studied in this paper.

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Notation

In this paper, we use the following notation. For a field F, an algebraic closure of F is denoted by F^{ac} and a separable closure of F is denoted by F^{sep}. For a Galois extension E over F, let $\mathrm{Gal}(E/F)$ denote the Galois group of the extension. Let K be a non-archimedean local field. Let $\mathcal{O}_K$ denote the ring of integers of K and k its residue field of characteristic $p > 0$. Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}_K$. Let $q = |k|$. For $\xi \in k$, let $\hat{\xi} \in \mu_{q-1}(K) \cup \{0\}$ denote the lift of ξ. For a finite extension F of K, let W_F denote the Weil group of F and I_F denote the inertia subgroup of W_F. For a finite extension F of K and a Galois extension L of F in F^{sep}, let $W(L/F)$ denote the image of W_F in $\mathrm{Gal}(L/F)$. Let K^{ur} denote the maximal unramified extension of K in K^{ac}. The completions of K^{ac} and K^{ur} are denoted by $\widehat{K}^{\mathrm{ac}}$ and $\widehat{K}^{\mathrm{ur}}$ respectively. We write $\mathcal{O}_{\widehat{K}^{\mathrm{ac}}}$ and $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$ for the rings of integers of $\widehat{K}^{\mathrm{ac}}$ and $\widehat{K}^{\mathrm{ur}}$ respectively. For an element $a \in \mathcal{O}_{\widehat{K}^{\mathrm{ac}}}$, we write $\bar{a}$ for the image of a by the reduction map $\mathcal{O}_{\widehat{K}^{\mathrm{ac}}} \to k^{\mathrm{ac}}$. We fix a uniformizer ϖ of K. Let $v(\cdot)$ denote the valuation of K^{ac} such that $v(\varpi) = 1$. Let $|\cdot|_K$ be the absolute value of K such that $|\varpi|_K = q^{-1}$. For $a, b \in K^{\mathrm{ac}}$ and a rational number $\alpha \in \mathbb{Q}_{\geq 0}$, we write $a \equiv b \pmod{\alpha}$ if

we have $v(a-b) \geq \alpha$, and $a \equiv b \pmod{\alpha+}$ if we have $v(a-b) > \alpha$. For a local ring A, the maximal ideal of A is denoted by m_A. For an algebraic curve X over k^{ac}, we denote by X^c the smooth compactification of the normalization of X. For an affinoid $\mathbf{X}$, we write $\overline{\mathbf{X}}$ for its reduction. The category of sets is denoted by **Set**. For a representation π of a group, the dual representation of π is denoted by π^*. We take rational powers of ϖ compatibly as needed.

1 Lubin–Tate curve

Let n be a natural number. We put

$$K_1(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_K) \;\middle|\; c \equiv 0,\ d \equiv 1 \bmod \mathfrak{p}^n \right\}.$$

In the following, we define the connected Lubin–Tate curve $\mathbf{X}_1(\mathfrak{p}^n)$ with level $K_1(\mathfrak{p}^n)$.

Let Σ denote a formal $\mathcal{O}_K$-module of dimension 1 and height 2 over k^{ac}, which is unique up to isomorphism. Let $\mathcal{C}$ be the category of Noetherian complete local $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-algebras with residue field k^{ac}. For a one-dimensional formal $\mathcal{O}_K$-module $\mathcal{F} = \operatorname{Spf} A[[X]]$ over $A \in \mathcal{C}$ and an element $a \in \mathcal{O}_K$, we write $[a]_{\mathcal{F}}(X) \in A[[X]]$ for the a-multiplication on $\mathcal{F}$. For a formal one-dimensional $\mathcal{O}_K$-module $\mathcal{F} = \operatorname{Spf} A[[X]]$ over $A \in \mathcal{C}$ and an A-valued point P of $\mathcal{F}$, the corresponding element of m_A is denoted by $x(P)$. We consider the functor

$$\mathcal{A}_1(\mathfrak{p}^n) \colon \mathcal{C} \to \mathbf{Set}$$

which sends $A \in \mathcal{C}$ to the set of isomorphism classes of triples $(\mathcal{F}, \iota, P)$, where $\mathcal{F}$ is a formal $\mathcal{O}_K$-module over A with an isomorphism $\iota \colon \Sigma \simeq \mathcal{F} \otimes_A k^{\mathrm{ac}}$ and P is a ϖ^n-torsion point of $\mathcal{F}$ such that

$$\prod_{a \in \mathcal{O}_K/\mathfrak{p}^n} \bigl(X - x([a]_{\mathcal{F}}(P))\bigr) \;\Bigg|\; [\varpi^n]_{\mathcal{F}}(X)$$

in $A[[X]]$. This functor is represented by a regular local ring $\mathcal{R}_1(\mathfrak{p}^n)$. We write $\mathfrak{X}_1(\mathfrak{p}^n)$ for $\operatorname{Spf} \mathcal{R}_1(\mathfrak{p}^n)$. Its generic fiber is denoted by $\mathbf{X}_1(\mathfrak{p}^n)$, which we call the connected Lubin–Tate curve with level $K_1(\mathfrak{p}^n)$. The space $\mathbf{X}_1(\mathfrak{p}^n)$ is a rigid analytic curve over $\widehat{K}^{\mathrm{ur}}$. We can define the connected Lubin–Tate curve $\mathbf{X}(\mathfrak{p}^n)$ with full level n structure by changing P to be an $\mathcal{O}_K$-module homomorphism $\phi \colon (\mathcal{O}_K/\mathfrak{p}^n)^2 \to m_A$ such that

$$\prod_{a \in (\mathcal{O}_K/\mathfrak{p}^n)^2} (X - \phi(a)) \;\Bigg|\; [\varpi^n]_{\mathcal{F}}(X)$$

in $A[[X]]$. For $i \in \mathbb{Z}$, we can define a rigid analytic curve $\mathbf{X}(\mathfrak{p}^n)^{(i)}$ over $\widehat{K}^{\mathrm{ur}}$ by changing an isomorphism ι in the definition of $\mathbf{X}(\mathfrak{p}^n)$ to a quasi-isogeny $\iota \colon \Sigma \to \mathcal{F} \otimes_A k^{\mathrm{ac}}$ of height i. We put

$$\mathrm{LT}(\mathfrak{p}^n) = \coprod_{i \in \mathbb{Z}} \mathbf{X}(\mathfrak{p}^n)^{(i)}.$$

Let D be the central division algebra over K of invariant $1/2$. We write $\mathcal{O}_D$ for the ring of integers of D. For a positive integer m, let K_m be the unramified extension of K of degree m and k_m be the finite extension over k of degree m. Let $\kappa \in \mathrm{Gal}(K_2/K)$ be the non-trivial element. The ring $\mathcal{O}_D$ has the following description: $\mathcal{O}_D = \mathcal{O}_{K_2} \oplus \varphi \mathcal{O}_{K_2}$ with $\varphi^2 = \varpi$ and $a\varphi = \varphi a^{\kappa}$ for $a \in \mathcal{O}_{K_2}$. We define an action of $\mathcal{O}_D$ on Σ by $\zeta(X) = \bar{\zeta} X$ for $\zeta \in \mu_{q^2-1}(\mathcal{O}_{K_2})$ and $\varphi(X) = X^q$. Then this gives an isomorphism

$$\mathcal{O}_D \simeq \mathrm{End}(\Sigma) \tag{1.1}$$

by [HG94, Proposition 13.10]. Using the isomorphism (1.1), we can define a left action of $\mathcal{O}_D^\times$ on $\mathbf{X}_1(\mathfrak{p}^n)$ and $\mathbf{X}(\mathfrak{p}^n)$. We can define also a left action of $D^\times$ on $\mathrm{LT}(\mathfrak{p}^n)$ using the isomorphism $D \simeq \mathrm{End}(\Sigma)[1/\varpi]$ induced by (1.1).

Let ℓ be a prime number different from p. We take an ℓ-adic compactly supported cohomology of a rigid analytic space by regarding it as an adic space (*cf.* [Hub98]). We take an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. We put

$$H^1_{\mathrm{LT}} = \varinjlim_n H^1_{\mathrm{c}}\left(\mathrm{LT}(\mathfrak{p}^n)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell\right).$$

Then we can define an action of $GL_2(K) \times W_K \times D^\times$ on H^1_{LT} (*cf.* [Dat07, 3.2, 3.3]).

We write $\mathrm{Irr}(D^\times, \overline{\mathbb{Q}}_\ell)$ for the set of isomorphism classes of irreducible smooth representations of $D^\times$ over $\overline{\mathbb{Q}}_\ell$, and $\mathrm{Disc}(GL_2(K), \overline{\mathbb{Q}}_\ell)$ for the set of isomorphism classes of irreducible discrete series representations of $GL_2(K)$ over $\overline{\mathbb{Q}}_\ell$. Let

$$\mathrm{JL}\colon \mathrm{Irr}(D^\times, \overline{\mathbb{Q}}_\ell) \to \mathrm{Disc}(GL_2(K), \overline{\mathbb{Q}}_\ell)$$

be the local Jacquet–Langlands correspondence. We denote by LJ the inverse of JL. Let $\mathrm{Nrd}_{D/K}\colon D^\times \to K^\times$ be the reduced norm map, and $\mathrm{Trd}_{D/K}\colon D \to K$ be the reduced trace map.

Proposition 1.1. *For a supercuspidal representation π of $GL_2(K)$ over $\overline{\mathbb{Q}}_\ell$, we have*

$$\mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi) \simeq \mathrm{LJ}(\pi)^{\oplus 2}$$

as representations of $D^\times$.

Proof. Let ω_π be the central character of π. We take $c_\pi \in \overline{\mathbb{Q}}_\ell$ such that $c_\pi^2 = \omega_\pi(\varpi)$. We define a character ζ_π of $GL_2(K)$ by $\zeta_\pi(g) = c_\pi^{v(\det(g))}$, and a character ξ_π of $D^\times$ by $\xi_\pi(d) = c_\pi^{v(\mathrm{Nrd}_{D/K}(d))}$ for $d \in D^\times$. For $n \geq 0$, let $\mathrm{LT}(\mathfrak{p}^n)/\varpi^{\mathbb{Z}}$ denote the quotient of $\mathrm{LT}(\mathfrak{p}^n)$ under the action of $\varpi^{\mathbb{Z}} \subset D^\times$. We put

$$H^1_{\mathrm{LT},\varpi} = \varinjlim_n H^1_{\mathrm{c}}\left(\left(\mathrm{LT}(\mathfrak{p}^n)/\varpi^{\mathbb{Z}}\right)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell\right).$$

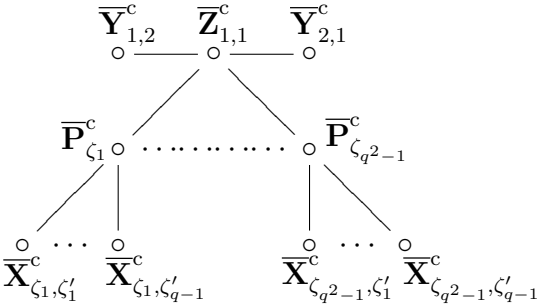
Then we have

$$\mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi) \simeq \mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT},\varpi}, \pi \otimes \zeta_\pi^{-1}) \otimes \xi_\pi$$

as representations of $D^\times$ by arguments in [Str05, 3.3]. Hence the claim follows from [IT17, Proposition 2.1], because $\mathrm{LJ}(\pi \otimes \zeta_\pi^{-1}) \otimes \xi_\pi = \mathrm{LJ}(\pi)$. □

2 Semi-stable reduction of $\mathbf{X}_1(\mathfrak{p}^3)$

From now on, we assume that $p = 2$. The dual graph of a semi-stable reduction of $\mathbf{X}_1(\mathfrak{p}^3)$ in this case is the following:



where $k_2^\times = \{\zeta_1, \ldots, \zeta_{q^2-1}\}$, $k^\times = \{\zeta'_1, \ldots, \zeta'_{q-1}\}$, $\overline{\mathbf{Y}}_{1,2}$ and $\overline{\mathbf{Y}}_{2,1}$ are defined by $x^q y - xy^q = 1$, $\overline{\mathbf{Z}}^{\mathrm{c}}_{1,1}$ and $\overline{\mathbf{P}}^{\mathrm{c}}_{\zeta}$ are isomorphic to $\mathbb{P}^1_{k^{\mathrm{ac}}}$, and $\overline{\mathbf{X}}_{\zeta,\zeta'}$ are defined by $z^2 + z = w^3$ (*cf.* [IT17, Introduction]).

For a finite extension K' of K, let $\mathrm{Art}_{K'}\colon K'^{\times} \xrightarrow{\sim} W_{K'}^{\mathrm{ab}}$ be the Artin reciprocity map normalized so that the image by $\mathrm{Art}_{K'}$ of a uniformizer is a lift of the geometric Frobenius element. We define a homomorphism $|\cdot|\colon W_K \to \mathbb{Q}_{>0}$ by the composition

$$W_K \twoheadrightarrow W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} K^{\times} \xrightarrow{|\cdot|_K} \mathbb{Q}_{>0}.$$

Definition of $\mathcal{S}$, $(W_K \times D^{\times})^0$ and r_σ

We put

$$\mathcal{S} = k_2^{\times} \times k^{\times}$$

and

$$(W_K \times D^{\times})^0 = \big\{(\sigma, d) \in W_K \times D^{\times} \;\big|\; |\mathrm{Nrd}_{D/K}(d)|_K \cdot |\sigma| = 1\big\}.$$

Then $(W_K \times D^{\times})^0$ acts on $\mathbf{X}_1(\mathfrak{p}^3)_{\widehat{K}^{\mathrm{ac}}}$. It induces an action of $(W_K \times D^{\times})^0$ on $\coprod_{(\zeta,\zeta')\in\mathcal{S}} \overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}$ by [IT17, Proposition 5.4 and Proposition 6.12]. For $\sigma \in W_K$, let r_σ be the integer such that $|\sigma| = q^{-r_\sigma}$. Let $\mathcal{O}_D^{\times} \rtimes W_K$ be the semidirect product where $\sigma \in W_K$ acts on $\mathcal{O}_D^{\times}$ by $d \mapsto \varphi^{r_\sigma} d \varphi^{-r_\sigma}$. Then we have the isomorphism

$$\mathcal{O}_D^{\times} \rtimes W_K \simeq (W_K \times D^{\times})^0;\ (d, \sigma) \mapsto (\sigma, d\varphi^{-r_\sigma}). \tag{2.1}$$

By this isomorphism, $\mathcal{O}_D^{\times} \rtimes W_K$ acts on $\coprod_{(\zeta,\zeta')\in\mathcal{S}} \overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}$. We will describe this action.

Definition of κ_1, κ_2 and f_d

For $d \in \mathcal{O}_D^{\times}$, we put

$$\kappa_1(d) = \bar{d}_1, \quad \kappa_2(d) = -\bar{d}_1^{-q}\bar{d}_2,$$

where $d = d_1 + \varphi d_2$ with $d_1 \in \mathcal{O}_{K_2}^{\times}$ and $d_2 \in \mathcal{O}_{K_2}$. We take $(\zeta, \zeta') \in \mathcal{S}$. We put

$$f_d = \mathrm{Tr}_{k_2/\mathbb{F}_2}(\zeta^{1-q}\zeta'^{-2}\kappa_2(d))$$

for $d \in \mathcal{O}_D^{\times}$.

Definition of ζ'', δ, θ, $\zeta_{3,\sigma}$, ν_σ and μ_σ

We briefly recall the definition of $\zeta_{3,\sigma}$, ν_σ and μ_σ for $\sigma \in W_K$ from [IT17, Section 6.2.2]. Consult there for detailed discussions. We choose $\zeta'' \in \mu_{3(q-1)}(K^{\mathrm{ur}})$ such that

$$\zeta''^3 = \hat{\zeta}'^4. \tag{2.2}$$

We take $\delta \in K^{\mathrm{ac}}$ such that $\delta^4 - \delta = 1/(\zeta''\varpi^{1/3})$ and $\delta\varpi^{1/12} \equiv \hat{\zeta}'\zeta''^{-1} \pmod 0$. We take $\theta \in K^{\mathrm{ac}}$ such that $\theta^2 - \theta = \delta^3$. Note that $v(\delta) = -1/12$, $v(\theta) = -1/8$ and $\delta \in K(\zeta''\varpi^{1/3}, \theta)$.

Let $\sigma \in W_K$ in this paragraph. We put

$$\zeta_{3,\sigma} = \frac{\sigma(\zeta''\varpi^{\frac{1}{3}})}{\zeta''\varpi^{\frac{1}{3}}} \in \mu_3(K^{\mathrm{ur}}).$$

We take $\nu_\sigma \in \mu_3(K^{\mathrm{ur}}) \cup \{0\}$ such that $\sigma(\delta) \equiv \zeta_{3,\sigma}^{-1}(\delta + \nu_\sigma) \pmod{5/6}$. We choose $\zeta_3 \in \mu_3(K^{\mathrm{ur}})$ such that $\zeta_3 \neq 1$. Then, we can take $\mu_\sigma \in \mu_3(K^{\mathrm{ur}}) \cup \{0\}$ such that

$$\mu_\sigma \equiv \sigma(\theta) - \theta + \nu_\sigma^2\delta + \nu_\sigma^3 + \sigma(\zeta_3) - \zeta_3 \pmod{0+}.$$

Definition of λ_σ, λ and $Q \rtimes \mathbb{Z}$

We put

$$\lambda_\sigma = \frac{\sigma\big(\varpi^{\frac{1}{2(q-1)}}\big)}{\varpi^{\frac{1}{2(q-1)}}} \in \mu_{2(q-1)}(K^{\mathrm{ac}})$$

for $\sigma \in W_K$. We define a character $\lambda\colon W_K \to k^\times$ by $\lambda(\sigma) = \bar{\lambda}_\sigma$. We put

$$Q = \left\{ g(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & \beta & \gamma \\ & \alpha^2 & \beta^2 \\ & & \alpha \end{pmatrix} \in GL_3(\mathbb{F}_4) \;\middle|\; \alpha\gamma^2 + \alpha^2\gamma = \beta^3 \right\}.$$

We note that $|Q| = 24$ (*cf.* [IT17, p. 137]). Let $Q \rtimes \mathbb{Z}$ be the semidirect product where $r \in \mathbb{Z}$ acts on Q by $g(\alpha,\beta,\gamma) \mapsto g(\alpha^{q^r}, \beta^{q^r}, \gamma^{q^r})$. To clarify the dependence on q, we sometimes write $Q \rtimes_{(q)} \mathbb{Z}$ for $Q \rtimes \mathbb{Z}$. Let $k_2^\times \rtimes \mathrm{Gal}(k_2/k)$ be the semidirect product with the natural action of $\mathrm{Gal}(k_2/k)$ on $k_2^\times$. We consider $\mathbb{F}_4$ as a subfield of $k_2 \subset k^{\mathrm{ac}}$. Let Fr_q be the q-th power Frobenius map on k^{ac}. Then $(Q \rtimes \mathbb{Z}) \times (k_2^\times \rtimes \mathrm{Gal}(k_2/k))$ acts on $\coprod_{(\zeta,\zeta')\in\mathcal{S}} \overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}$ as a scheme over k as follows: An element

$$\big((g(\alpha,\beta,\gamma), r), (a, \mathrm{Fr}_q^b)\big) \in (Q \rtimes \mathbb{Z}) \times \big(k_2^\times \rtimes \mathrm{Gal}(k_2/k)\big)$$

acts by the isomorphism

$$\overline{\mathbf{X}}_{\zeta,\zeta'} \to \overline{\mathbf{X}}_{a\zeta^{q^b},\zeta'};\ (z,w) \mapsto \big(z^{q^{-r}} + \alpha^{-1}\beta w^{q^{-r}} + \alpha^{-1}\gamma, \alpha(w^{q^{-r}} + (\alpha^{-1}\beta)^2)\big),$$

where we describe a bijection on k^{ac}-valued points. Note that the action of $(g(1,0,0), r) \in Q \rtimes \mathbb{Z}$ is induced by the action of Fr_q^r on the coefficients of $k^{\mathrm{ac}}[z,w]/(z^2 + z - w^3)$.

Proposition 2.1. *The action of $\mathcal{O}_D^\times \rtimes W_K$ on $\coprod_{(\zeta,\zeta')\in\mathcal{S}} \overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}$ is described as follows: An element $(d,1) \in \mathcal{O}_D^\times \rtimes W_K$ induces the isomorphism*

$$\overline{\mathbf{X}}_{\zeta,\zeta'} \to \overline{\mathbf{X}}_{\kappa_1(d)\zeta,\zeta'};\ (z,w) \mapsto (z + f_d, w).$$

For $\zeta' \in k^\times$, the action of $W_K \subset \mathcal{O}_D^\times \rtimes W_K$ on $\coprod_{\zeta\in k_2^\times} \overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}$ factors through

$$\begin{aligned} \Xi_{\zeta'}\colon W_K &\to (Q \rtimes \mathbb{Z}) \times (k_2^\times \rtimes \mathrm{Gal}(k_2/k)); \\ \sigma &\mapsto \Big(\big(g(\bar{\zeta}_{3,\sigma}, \bar{\zeta}_{3,\sigma}^2\bar{\nu}_\sigma^2, \bar{\zeta}_{3,\sigma}\bar{\mu}_\sigma), r_\sigma\big), (\bar{\lambda}_\sigma, \mathrm{Fr}_q^{-r_\sigma})\Big). \end{aligned}$$

Proof. This follows from [IT17, Proposition 5.4 and Proposition 6.12]. □

Definition of $\Theta_{\zeta'}$

Let $\Theta_{\zeta'}\colon W_K \to Q \rtimes \mathbb{Z}$ be the composite of $\Xi_{\zeta'}$ with the projection to $Q \rtimes \mathbb{Z}$. By [IT17, Proposition 6.13], the map $\Theta_{\zeta'}$ gives an isomorphism $W(K^{\mathrm{ur}}(\varpi^{1/3}, \theta)/K) \simeq Q \rtimes \mathbb{Z}$ and a finite extension of K inside $K^{\mathrm{ur}}(\varpi^{1/3}, \theta)$ corresponds to a finite index subgroup of $Q \rtimes \mathbb{Z}$.

3 Cohomology of elliptic curve

Let ℓ be an odd prime number. We fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. In the sequel, we consider representations of groups over $\overline{\mathbb{Q}}_\ell$.

Definition of Q_8, C_4, Z, ϕ, τ and C_3

We put

$$Q_8 = \{g(1, \beta, \gamma) \in Q\},$$

which is a normal subgroup of Q of order 8. Let $C_4 \subset Q_8$ be the cyclic subgroup of order 4 generated by $g(1, 1, \gamma)$ for $\gamma \in \mathbb{F}_4^\times \backslash \{1\}$. Let $Z \subset C_4$ be the subgroup consisting of $g(1, 0, \gamma)$ with $\gamma^2 + \gamma = 0$, which is the center of Q. We take a faithful character ϕ of C_4. By [BH06, 22.2 Lemma], there exists a unique irreducible two-dimensional representation τ of Q such that

$$\tau|_Z \simeq (\phi|_Z)^{\oplus 2}, \quad \mathrm{tr}(g(\alpha, 0, 0); \tau) = -1 \tag{3.1}$$

for $\alpha \in \mathbb{F}_4^\times \backslash \{1\}$. Let $C_3 \subset Q$ be the cyclic subgroup of order 3 consisting of $g(\alpha, 0, 0)$ with $\alpha \in \mathbb{F}_4^\times$. Then we have

$$\tau = \mathrm{Ind}_{C_4}^Q \phi - \mathrm{Ind}_{Z \times C_3}^Q (\phi|_Z \otimes 1_{C_3}) \tag{3.2}$$

by [BH06, 16.4 Lemma 2.(4)] and a proof of [BH06, 22.2 Lemma].

Definition of $\mathcal{E}$, τ_q and f

Let $\mathcal{E}$ be the elliptic curve over $\mathbb{F}_2$ defined by $z^2 + z = w^3$. Then $(g(\alpha, \beta, \gamma), r) \in Q \rtimes \mathbb{Z}$ acts on $\mathcal{E}_{k^{\mathrm{ac}}}$ by

$$(z, w) \mapsto \big(z^{q^{-r}} + \alpha^{-1}\beta w^{q^{-r}} + \alpha^{-1}\gamma, \alpha(w^{q^{-r}} + (\alpha^{-1}\beta)^2)\big).$$

The action of $Q \rtimes \mathbb{Z}$ gives a representation $H^1(\mathcal{E}_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)$ of $Q \rtimes \mathbb{Z}$ by the pullback by the inverse. For a representation V of $Q \rtimes \mathbb{Z}$ and an integer m, we write $V(m)$ for the twist of V by the character $Q \rtimes \mathbb{Z} \ni (g, n) \mapsto q^{-mn}$.

We write τ_q for the representation $H^1(\mathcal{E}_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)(1)$ of $Q \rtimes \mathbb{Z}$. Let f be the degree of the extension k over $\mathbb{F}_2$.

Definition of η_2, C and ϕ'

We are going to define a subgroup $C \subset Q_8 \rtimes \mathbb{Z}$ and a character ϕ' of C. To clarify the dependence on q, we sometimes write $C_{(q)}$ and $\phi'_{(q)}$ for C and ϕ' respectively.

First, we consider the case where $f = 1$. Let

$$C_{(2)} \subset Q_8 \rtimes_{(2)} \mathbb{Z}$$

be the subgroup which consists of $(g(1, \beta, \gamma), n)$ satisfying $g(1, \beta, \gamma) \in C_4$ if n is even, and $g(1, \beta, \gamma) \notin C_4$ if n is odd. We note that the index of $C_{(2)}$ in $Q_8 \rtimes_{(2)} \mathbb{Z}$ is two. We take $\eta_2 \in \overline{\mathbb{Q}}_\ell$ such that $\eta_2^2 - 2\eta_2 + 2 = 0$ and $\phi(g(1, 1, \bar{\zeta}_3)) = -\eta_2^2/2$. We note that $\eta_2^4 = -4$. We define a character

$$\phi'_{(2)} \colon C_{(2)} \to \overline{\mathbb{Q}}_\ell^\times$$

by sending $(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)$ to $\eta_2/2$ and $(g(1, 0, 0), 2)$ to $(-1)/2$. We note that $(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)$ and $(g(1, 0, 0), 2)$ generates $C_{(2)}$ as a group.

In general, let $C_{(q)}$ be the inverse image of $C_{(2)}$ under the group homomorphism

$$Q_8 \rtimes_{(q)} \mathbb{Z} \to Q_8 \rtimes_{(2)} \mathbb{Z}; \ (g, n) \mapsto (g, fn).$$

Let $\phi'_{(q)}$ be the character of $C_{(q)}$ induced by $\phi'_{(2)}$ and the homomorphism $C_{(q)} \to C_{(2)}$. We have $\phi'_{(q)}|_{C_4} = \phi$ by the construction. We note that $C_{(q)} = C_4 \times \mathbb{Z}$ and $\phi'_{(q)}(g, n) = \phi(g)(-2)^{(-fn)/2}$, if f is even.

Lemma 3.1. *We have an isomorphism $\tau_q|_{Q_8 \rtimes \mathbb{Z}} \simeq \mathrm{Ind}_C^{Q_8 \rtimes \mathbb{Z}} \phi'$.*

Proof. It suffices to consider the case where $f = 1$, because the claimed isomorphisms for general cases are induced from the isomorphism for this case by the group homomorphism

$$Q_8 \rtimes_{(q)} \mathbb{Z} \to Q_8 \rtimes_{(2)} \mathbb{Z};\ (g, n) \mapsto (g, fn).$$

We assume that $f = 1$. We know that $\tau_2|_{Q_8} \simeq \mathrm{Ind}_{C_4}^{Q_8} \phi$, and these representations are irreducible. Hence the $\overline{\mathbb{Q}}_\ell$-vector space

$$\mathrm{Hom}_{Q_8}\left(\tau_2|_{Q_8}, (\mathrm{Ind}_C^{Q_8 \rtimes \mathbb{Z}} \phi')|_{Q_8}\right)$$

is 1-dimensional, and $Q_8 \rtimes \mathbb{Z}$ acts on the 1-dimensional subspace by a character χ, which factors through the projection $Q_8 \rtimes \mathbb{Z} \to \mathbb{Z}$. Then we have

$$\tau_2|_{Q_8 \rtimes \mathbb{Z}} \simeq (\mathrm{Ind}_C^{Q_8 \rtimes \mathbb{Z}} \phi') \otimes \chi.$$

Therefore, it suffices to show that

$$\mathrm{tr}\big((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); \tau_2\big) = \mathrm{tr}\big((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); \mathrm{Ind}_C^{Q_8 \rtimes \mathbb{Z}} \phi'\big) \neq 0, \tag{3.3}$$

since (3.3) implies that χ is trivial. We put

$$\phi''\big((g, n)\big) = \phi'\big((g(1, 0, 0), 1)(g, n)(g(1, 0, 0), -1)\big)$$

for $(g, n) \in Q_8 \rtimes \mathbb{Z}$. Then we have

$$\begin{aligned} \mathrm{tr}\big((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); \mathrm{Ind}_C^{Q_8 \rtimes \mathbb{Z}} \phi'\big) &= \phi'\big((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)\big) + \phi''\big((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)\big) \\ &= \frac{\eta_2}{2} - \frac{\eta_2^3}{4} = 1, \end{aligned} \tag{3.4}$$

where we use

$$(g(1, 0, 0), 1)(g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)(g(1, 0, 0), -1) = (g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)^3 (g(1, 0, 0), -2)$$

at the second equality.

Let $\mathrm{Fr}_{2,\mathcal{E}}$ be the absolute 2-th power Frobenius endomorphism of $\mathcal{E}_{k^{\mathrm{ac}}}$. By the Lefschetz trace formula, we have

$$\begin{aligned} 2 + 1 - \mathrm{tr}\big((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); H^1(\mathcal{E}_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)\big) &= \big|\{P \in \mathcal{E}(k^{\mathrm{ac}}) \mid \big((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1)^{-1} \circ \mathrm{Fr}_{2,\mathcal{E}}\big)P = P\}\big| \\ &= \big|\{(z, w) \in k^{\mathrm{ac}} \times k^{\mathrm{ac}} \mid z^2 + z = w^3,\ z = z^2 + \bar{\zeta}_3^2 w^2 + \bar{\zeta}_3,\ w = w^2 + \bar{\zeta}_3\}\big| + 1 \\ &= \big|\{(z, w) \in k^{\mathrm{ac}} \times k^{\mathrm{ac}} \mid z^2 + z = w^3,\ w^3 + \bar{\zeta}_3^2 w^2 + \bar{\zeta}_3 = 0,\ w^2 + w + \bar{\zeta}_3 = 0\}\big| + 1 = 1. \end{aligned}$$

Hence, we have

$$\mathrm{tr}\big((g(1, \bar{\zeta}_3, \bar{\zeta}_3), 1); \tau_2\big) = 1. \tag{3.5}$$

The claim (3.3) follows from (3.4) and (3.5). □

For any positive integer m, let

$$\mathrm{fr}_{2^m} \colon \mathcal{E}_{k^{\mathrm{ac}}} \to \mathcal{E}_{k^{\mathrm{ac}}}$$

be the base change to k^{ac} of the 2^m-th power absolute Frobenius endomorphism of $\mathcal{E}$.

Lemma 3.2. *We assume that $f = 1$. Then we have*

$$\mathrm{tr}\big((g(1, 0, 0), n); \tau_2\big) = \begin{cases} 0 & \text{if } n = 1, \\ -1 & \text{if } n = 2 \end{cases}$$

for $(g(1, 0, 0), n) \in Q \rtimes \mathbb{Z}$.

Proof. We have

$$\mathrm{tr}\big((g(1,0,0),n);H^1(\mathcal{E}_{k^{\mathrm{ac}}},\overline{\mathbb{Q}}_\ell)\big)=\mathrm{tr}\big(\mathrm{fr}_{2^n}^*;H^1(\mathcal{E}_{k^{\mathrm{ac}}},\overline{\mathbb{Q}}_\ell)\big)=\begin{cases}0 & \text{if } n=1,\\ -4 & \text{if } n=2,\end{cases}$$

where the last equality follows from $|\mathcal{E}(\mathbb{F}_2)|=3$, $|\mathcal{E}(\mathbb{F}_4)|=9$ and the Lefschetz trace formula. The claim follows from this. □

Lemma 3.3. *We have* $\det\big((g,n);\tau_q\big)=q^{-n}$ *for* $(g,n)\in Q\rtimes\mathbb{Z}$.

Proof. We have an isomorphism $\tau_q|_Q\simeq\tau$ as Q-representations by [IT17, Lemma 7.7].

First we are going to show that $\det\tau=1$. We see that $\det\tau$ factors through Q/Q_8, because Q/Q_8 is the maximal abelian quotient of Q. By (3.2), we know that τ is self-dual. Hence, the character of Q/Q_8 induced from $\det\tau$ is trivial, since $|Q/Q_8|=3$. Therefore, we have $\det\tau=1$.

Since τ_q is induced from τ_2 by the group homomorphism

$$Q\rtimes_{(q)}\mathbb{Z}\to Q\rtimes_{(2)}\mathbb{Z};\ (g,n)\mapsto(g,fn),$$

it suffices to show the claim in the case where $f=1$.

We assume that $f=1$. Let ω_1 and ω_2 be the non-trivial characters of C_3. Then, we have a direct decomposition

$$\tau_2|_{C_3}\cong\omega_1\oplus\omega_2$$

by (3.1). We fix a basis in the above decomposition. Then the action of $(g(1,0,0),1)\in Q\rtimes\mathbb{Z}$ can be written as

$$\begin{pmatrix}0 & a\\ b & 0\end{pmatrix}$$

for some $a,b\in\overline{\mathbb{Q}}_\ell^\times$, since we have $(g(1,0,0),1)c=c^2(g(1,0,0),1)$ in $Q\rtimes\mathbb{Z}$ for $c\in C_3$. By Lemma 3.2, we have $2ab=-1$. Hence, we have

$$\det\big((g(1,0,0),1);\tau_2\big)=-ab=2^{-1}.$$

The claim follows from this and $\det\tau=1$. □

Definition of $\tau_{\zeta'}$

Let $\tau_{\zeta'}$ be the representation of W_K induced from the $(Q\rtimes\mathbb{Z})$-representation τ_q by $\Theta_{\zeta'}$. We say that a continuous two-dimensional representation V of W_K over $\overline{\mathbb{Q}}_\ell$ is primitive, if there is no pair of a quadratic extension K' and a continuous character χ of $W_{K'}$ such that $V\simeq\mathrm{Ind}_{W_{K'}}^{W_K}\chi$. The representation $\tau_{\zeta'}$ is primitive of conductor 3 by [IT17, Lemma 7.8].

4 Realization of correspondence

Definition of χ_2, L and ψ_K

Let $\chi_2\colon\mathbb{F}_2\to\overline{\mathbb{Q}}_\ell^\times$ be the non-trivial character. We put

$$\mathfrak{J}=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in M_2(\mathcal{O}_K)\ \middle|\ c\equiv 0 \bmod\mathfrak{p}\right\},\quad \mathfrak{P}=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathfrak{J}\ \middle|\ a\equiv d\equiv 0 \bmod\mathfrak{p}\right\},\quad (4.1)$$

and $U_{\mathfrak{J}}^1=1+\mathfrak{P}\subset M_2(\mathcal{O}_K)$. We put $L=K(\varphi)\subset D$, and consider L as a K-subalgebra of $M_2(K)$ by the embedding

$$L\hookrightarrow M_2(K);\ \varphi\mapsto\begin{pmatrix}0 & 1\\ \varpi & 0\end{pmatrix}.\quad (4.2)$$

We put $U_D^1 = 1 + \varphi\mathcal{O}_D$. We take an additive character $\psi_K \colon K \to \overline{\mathbb{Q}}_\ell^\times$ such that $\psi_K(a) = (\chi_2 \circ \mathrm{Tr}_{k/\mathbb{F}_2})(\bar{a})$ for $a \in \mathcal{O}_K$.

For a finite abelian group A, the character group $\mathrm{Hom}_{\mathbb{Z}}(A, \overline{\mathbb{Q}}_\ell^\times)$ is denoted by $A^\vee$.

Definition of $\Lambda_{\zeta',\chi,c}$ and $\pi_{\zeta',\chi,c}$

Let $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$. We define a character $\Lambda_{\zeta',\chi,c} \colon L^\times U_{\mathfrak{J}}^1 \to \overline{\mathbb{Q}}_\ell^\times$ by

$$\begin{aligned}
&\Lambda_{\zeta',\chi,c}(\varphi) = -c, \\
&\Lambda_{\zeta',\chi,c}(a) = \chi(\bar{a}) \text{ for } a \in \mathcal{O}_L^\times, \\
&\Lambda_{\zeta',\chi,c}(x) = (\psi_K \circ \mathrm{tr}) \left(\hat{\zeta}'^{-2} \varphi^{-1}(x-1) \right) \text{ for } x \in U_{\mathfrak{J}}^1.
\end{aligned}$$

We put

$$\pi_{\zeta',\chi,c} = \text{c-Ind}_{L^\times U_{\mathfrak{J}}^1}^{GL_2(K)} \Lambda_{\zeta',\chi,c}.$$

By [IT18, Proposition 1.3], $\pi_{\zeta',\chi,c}$ is a supercuspidal representation of conductor 3, and any supercuspidal representation of conductor 3 is isomorphic to $\pi_{\zeta',\chi,c}$ for some $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$.

Definition of $\theta_{\zeta',\chi,c}$ and $\rho_{\zeta',\chi,c}$

Next, we define a character $\theta_{\zeta',\chi,c} \colon L^\times U_D^1 \to \overline{\mathbb{Q}}_\ell^\times$ by

$$\begin{aligned}
&\theta_{\zeta',\chi,c}(\varphi) = c, \\
&\theta_{\zeta',\chi,c}(a) = \chi(\bar{a}) \text{ for } a \in \mathcal{O}_L^\times, \\
&\theta_{\zeta',\chi,c}(d) = (\chi_2 \circ \mathrm{Tr}_{k_2/\mathbb{F}_2}) \left(\zeta'^{-2} \kappa_2(d) \right) \text{ for } d \in U_D^1.
\end{aligned}$$

We put

$$\rho_{\zeta',\chi,c} = \mathrm{Ind}_{L^\times U_D^1}^{D^\times} \theta_{\zeta',\chi,c}.$$

The representation $\rho_{\zeta',\chi,c}$ is irreducible by [BH06, 54.4 Proposition (1)].

Proposition 4.1. *For $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$, we have $\mathrm{JL}(\rho_{\zeta',\chi,c}) = \pi_{\zeta',\chi,c}$.*

Proof. This follows from [BH06, 56.5], because

$$(\psi_K \circ \mathrm{Trd}_{D/K}) \left(\hat{\zeta}'^{-2} \varphi^{-1}(d-1) \right) = (\chi_2 \circ \mathrm{Tr}_{k_2/\mathbb{F}_2}) \left(\zeta'^{-2} \kappa_2(d) \right)$$

for $d \in U_D^1$. □

Remark 4.2. *In [BH06, 56.1], the local Jacquet–Langlands correspondence for $D^\times$ is characterized by coincidences of L-functions and ϵ-factors. However, we can check that the correspondence between $\rho_{\zeta',\chi,c}$ and $\pi_{\zeta',\chi,c}$ satisfies the characterization by trace identities (cf. [IT18]). We note that the existence of the local Jacquet–Langlands correspondence for $D^\times$ satisfying the characterization by trace identities is proved in [Mie14] by purely local methods.*

Definition of ϕ_c and $\tau_{\zeta',\chi,c}$

For $c \in \overline{\mathbb{Q}}_\ell^\times$, let $\phi_c \colon W_K \to \overline{\mathbb{Q}}_\ell^\times$ be the character defined by $\phi_c(\sigma) = c^{r_\sigma}$. For $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$, we put

$$\tau_{\zeta',\chi,c} = \tau_{\zeta'} \otimes (\chi \circ \lambda) \otimes \phi_c.$$

For a representation V of a Weil group and an integer m, we write $V(m)$ for the m-times Tate twist of V. We choose $(-2)^{1/2} \in \overline{\mathbb{Q}}_\ell$.

Theorem 4.3. *For $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$, we have*

$$\mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi_{\zeta',\chi,c}) \simeq \tau_{\zeta',\chi,c} \otimes \rho_{\zeta',\chi,c}$$

as representations of $W_K \times D^\times$.

Proof. Let $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$. By Proposition 1.1 and Proposition 4.1, we know that

$$\mathrm{Hom}_{D^\times}\left(\rho_{\zeta',\chi,c}, \mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi_{\zeta',\chi,c})\right) \simeq \tau'$$

for some two-dimensional W_K-representation τ'. First, we will show that $\tau' = \tau_{\zeta',\chi,c}$.

We put

$$H^1_{\mathbf{X}} = \varinjlim_n H^1_{\mathrm{c}}\left(\mathbf{X}(\mathfrak{p}^n)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell\right)$$

and

$$(GL_2(K) \times W_K \times D^\times)^0 = \left\{(g,\sigma,d) \in GL_2(K) \times W_K \times D^\times \;\middle|\; |\det(g)^{-1}\,\mathrm{Nrd}_{D/K}(d)|_K \cdot |\sigma| = 1\right\}.$$

Then we have

$$\begin{aligned}
\mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi_{\zeta',\chi,c}) &\simeq \mathrm{Hom}_{GL_2(K)}\left(\text{c-Ind}^{GL_2(K)\times W_K\times D^\times}_{(GL_2(K)\times W_K\times D^\times)^0} H^1_{\mathbf{X}}, \pi_{\zeta',\chi,c}\right) \\
&\subset \mathrm{Hom}_{K_1(\mathfrak{p}^3)}\left(\text{c-Ind}^{K_1(\mathfrak{p}^3)\times W_K\times D^\times}_{K_1(\mathfrak{p}^3)\times(W_K\times D^\times)^0} H^1_{\mathbf{X}}, \pi_{\zeta',\chi,c}\right) \\
&\subset \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}\left(\text{c-Ind}^{W_K\times D^\times}_{(W_K\times D^\times)^0} H^1_{\mathrm{c}}\left(\mathbf{X}_1(\mathfrak{p}^3)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell\right), \overline{\mathbb{Q}}_\ell\right),
\end{aligned}$$

where the last inclusion follows by taking the $K_1(\mathfrak{p}^3)$-invariant part and using [JPSS81, 5 Théorème], because the conductor of $\pi_{\zeta',\chi,c}$ is three. Hence, we obtain

$$\begin{aligned}
\tau' &\simeq \mathrm{Hom}\left(\rho_{\zeta',\chi,c}, \mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi_{\zeta',\chi,c})\right) \\
&\subset \mathrm{Hom}_{D^\times}\left(\rho_{\zeta',\chi,c}, \left(\text{c-Ind}^{W_K\times D^\times}_{(W_K\times D^\times)^0} H^1_{\mathrm{c}}\left(\mathbf{X}_1(\mathfrak{p}^3)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell\right)\right)^*\right) \qquad (4.3) \\
&\simeq \left(\rho^*_{\zeta',\chi,c} \otimes \left(\text{c-Ind}^{W_K\times D^\times}_{(W_K\times D^\times)^0} H^1_{\mathrm{c}}\left(\mathbf{X}_1(\mathfrak{p}^3)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell\right)\right)^*\right)^{D^\times} \qquad (4.4) \\
&\simeq \mathrm{Hom}_{D^\times}\left(\text{c-Ind}^{W_K\times D^\times}_{(W_K\times D^\times)^0} H^1_{\mathrm{c}}\left(\mathbf{X}_1(\mathfrak{p}^3)_{\widehat{K}^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell\right), \rho^*_{\zeta',\chi,c}\right) \\
&\simeq \mathrm{Hom}_{D^\times}\left(\text{c-Ind}^{W_K\times D^\times}_{(W_K\times D^\times)^0}\left(\bigoplus_{\zeta\in k_2^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}, \overline{\mathbb{Q}}_\ell)^*(-1)\right), \rho^*_{\zeta',\chi,c}\right),
\end{aligned}$$

where the last isomorphism follows from [IT17, Proposition 7.3, Proposition 7.9 and Theorem 7.16] by studying only $\mathcal{O}_D^\times$-actions. We remark that [IT17, Theorem 7.16] is based on [IT15a, Theorem 5.3] where we use Berkovich spaces, but it does not matter for Lubin–Tate spaces by [Far04, Lemma 4.4.6]. As vector spaces, the last space is isomorphic to

$$\mathrm{Hom}_{D^\times}\left(\text{c-Ind}^{D^\times}_{\mathcal{O}_D^\times}\left(\bigoplus_{\zeta\in k_2^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}, \overline{\mathbb{Q}}_\ell)^*\right), \rho^*_{\zeta',\chi,c}\right) \simeq \mathrm{Hom}_{\mathcal{O}_D^\times}\left(\bigoplus_{\zeta\in k_2^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}, \overline{\mathbb{Q}}_\ell)^*, \rho^*_{\zeta',\chi,c}|_{\mathcal{O}_D^\times}\right)$$

which is two-dimensional by [IT17, Proposition 7.9]. Hence, the inclusion in (4.3) is an equality. Therefore it suffices to show that there is a non-trivial homomorphism

$$\text{c-Ind}_{(W_K\times D^\times)^0}^{W_K\times D^\times}\left(\bigoplus_{\zeta\in k_2^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'},\overline{\mathbb{Q}}_\ell)^*(-1)\right)\longrightarrow \tau^*_{\zeta',\chi,c}\otimes\rho^*_{\zeta',\chi,c}$$

as representations of $W_K\times D^\times$. By the Frobenius reciprocity, this is equivalent to give a non-trivial homomorphism

$$(\tau_{\zeta',\chi,c}\otimes\rho_{\zeta',\chi,c})|_{(W_K\times D^\times)^0}\longrightarrow\bigoplus_{\zeta\in k_2^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'},\overline{\mathbb{Q}}_\ell)(1)$$

as representations of $(W_K\times D^\times)^0$. We put

$$(\mathcal{O}_D^\times\rtimes W_K)^0=\{(d,\sigma)\in\mathcal{O}_D^\times\rtimes W_K\mid \kappa_1(d)\bar{\lambda}_\sigma=1\}$$

and consider this group as a subgroup of $(W_K\times D^\times)^0$ by the isomorphism (2.1). Then we have

$$\bigoplus_{\zeta\in k_2^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'},\overline{\mathbb{Q}}_\ell)(1)\simeq \mathrm{Ind}_{(\mathcal{O}_D^\times\rtimes W_K)^0}^{(W_K\times D^\times)^0} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{1,\zeta'},\overline{\mathbb{Q}}_\ell)(1),$$

because the action of $\mathcal{O}_D^\times\rtimes W_K$ on $\coprod_{(\zeta,\zeta')\in\mathcal{S}}\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'}$ permutes the connected components transitively and $(\mathcal{O}_D^\times\rtimes W_K)^0$ is the stabilizer of the connected component $\overline{\mathbf{X}}^{\mathrm{c}}_{1,\zeta'}$ by Proposition 2.1. Hence, we have

$$\begin{aligned}\mathrm{Hom}_{(W_K\times D^\times)^0}&\left((\tau_{\zeta',\chi,c}\otimes\rho_{\zeta',\chi,c})|_{(W_K\times D^\times)^0},\bigoplus_{\zeta\in k_2^\times} H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{\zeta,\zeta'},\overline{\mathbb{Q}}_\ell)(1)\right)\\ &\simeq \mathrm{Hom}_{(\mathcal{O}_D^\times\rtimes W_K)^0}\left((\tau_{\zeta',\chi,c}\otimes\rho_{\zeta',\chi,c})|_{(\mathcal{O}_D^\times\rtimes W_K)^0},H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{1,\zeta'},\overline{\mathbb{Q}}_\ell)(1)\right).\end{aligned}$$

Since $\tau_{\zeta',\chi,c}\otimes\rho_{\zeta',\chi,c}\simeq\mathrm{Ind}_{W_K\times L^\times U_D^1}^{W_K\times D^\times}(\tau_{\zeta',\chi,c}\otimes\theta_{\zeta',\chi,c})$ and $(\mathcal{O}_D^\times\rtimes W_K)^0\subset W_K\times L^\times U_D^1$, we have a non-trivial homomorphism

$$(\tau_{\zeta',\chi,c}\otimes\rho_{\zeta',\chi,c})|_{(\mathcal{O}_D^\times\rtimes W_K)^0}\longrightarrow(\tau_{\zeta',\chi,c}\otimes\theta_{\zeta',\chi,c})|_{(\mathcal{O}_D^\times\rtimes W_K)^0}$$

by the Frobenius reciprocity. Hence, it suffices to show there is a non-trivial homomorphism

$$(\tau_{\zeta',\chi,c}\otimes\theta_{\zeta',\chi,c})|_{(\mathcal{O}_D^\times\rtimes W_K)^0}\longrightarrow H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{1,\zeta'},\overline{\mathbb{Q}}_\ell)(1)$$

as representations of $(\mathcal{O}_D^\times\rtimes W_K)^0$. We put

$$W_K'=\{(\lambda_\sigma^{-1},\sigma)\in(\mathcal{O}_D^\times\rtimes W_K)^0\mid\sigma\in W_K\}.$$

We consider U_D^1 as a subgroup of $(\mathcal{O}_D^\times\rtimes W_K)^0$ by identifying $d\in U_D^1$ with $(d,1)\in(\mathcal{O}_D^\times\rtimes W_K)^0$. Then we have an isomorphism

$$(\tau_{\zeta',\chi,c}\otimes\theta_{\zeta',\chi,c})|_{W_K'}\longrightarrow H^1(\overline{\mathbf{X}}^{\mathrm{c}}_{1,\zeta'},\overline{\mathbb{Q}}_\ell)(1)|_{W_K'}$$

as representations of W_K' by Proposition 2.1 and the definition of $\tau_{\zeta',\chi,c}$ and $\theta_{\zeta',\chi,c}$. This isomorphism is compatible with the action of U_D^1 by Proposition 2.1. Then this is an isomorphism

as representations of $(\mathcal{O}_D^\times \rtimes W_K)^0$, because $(\mathcal{O}_D^\times \rtimes W_K)^0$ is generated by W_K' and U_D^1. Thus we have proved that

$$\mathrm{Hom}_{D^\times}\left(\rho_{\zeta',\chi,c}, \mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi_{\zeta',\chi,c})\right) \simeq \tau_{\zeta',\chi,c}. \tag{4.5}$$

By (4.5), we see that $\mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi_{\zeta',\chi,c})$ is an irreducible representation of $W_K \times D^\times$. The group $Q \rtimes (\mathbb{Z}/2\mathbb{Z})$ is regarded as a quotient of W_K via $\Theta_{\zeta'}$. Let $\xi_c\colon D^\times \to \overline{\mathbb{Q}}_\ell^\times$ be the character defined by $\xi_c(d) = c^{v(\mathrm{Nrd}_{D/K}(d))}$. By (4.5), we have

$$\mathrm{Hom}_{D^\times}\left(\rho_{\zeta',\chi,c} \otimes \xi_c^{-1}, \mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi_{\zeta',\chi,c}) \otimes \phi^{-1}_{c(-2)^{-f/2}} \otimes (\chi\circ\lambda)^{-1} \otimes \xi_c^{-1}\right) \simeq \tau_{\zeta',\chi,c} \otimes \phi^{-1}_{c(-2)^{-f/2}} \otimes (\chi\circ\lambda)^{-1}.$$

Then we see that $\tau_{\zeta',\chi,c} \otimes \phi^{-1}_{c(-2)^{-f/2}} \otimes (\chi\circ\lambda)^{-1}$ and $\rho_{\zeta',\chi,c} \otimes \xi_c^{-1}$ factor through representations of $Q \rtimes (\mathbb{Z}/2\mathbb{Z})$ and $D^\times/(\varpi^{\mathbb{Z}}(1+\varpi\mathcal{O}_D))$ respectively, where we use Lemma 3.1 for the first factorization. Hence, the $(W_K \times D^\times)$-representation

$$\mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi_{\zeta',\chi,c}) \otimes \phi^{-1}_{c(-2)^{-f/2}} \otimes (\chi\circ\lambda)^{-1} \otimes \xi_c^{-1}$$

factors through a representation of the finite group $(Q \rtimes (\mathbb{Z}/2\mathbb{Z})) \times \left(D^\times/(\varpi^{\mathbb{Z}}(1+\varpi\mathcal{O}_D))\right)$. Then we have

$$\mathrm{Hom}_{GL_2(K)}(H^1_{\mathrm{LT}}, \pi_{\zeta',\chi,c}) \otimes \phi^{-1}_{c(-2)^{-f/2}} \otimes (\chi\circ\lambda)^{-1} \otimes \xi_c^{-1} \simeq \left(\tau_{\zeta',\chi,c} \otimes \phi^{-1}_{c(-2)^{-f/2}} \otimes (\chi\circ\lambda)^{-1}\right) \otimes (\rho_{\zeta',\chi,c} \otimes \xi_c^{-1}),$$

because an irreducible representation of a product of two finite groups is isomorphic to a tensor product of irreducible representations of the two groups. Therefore, we have the claim. □

5 Local Langlands correspondence

In this section, we prove that the correspondence in Theorem 4.3 actually gives the local Langlands correspondence. After we introduce some notations, we give an explicit description of the Artin map in Subsection 5.1. This enables us to calculate an epsilon factor explicitly. In Subsection 5.2, we give an explicit description of the local Langlands correspondence for $\tau_{\zeta',\chi,c}$ using a result in Subsection 5.1.

We write $\mathcal{G}_2(K, \overline{\mathbb{Q}}_\ell)$ for the set of equivalent classes of two-dimensional Frobenius semisimple Weil–Deligne representations of W_K over $\overline{\mathbb{Q}}_\ell$, and $\mathrm{Irr}(GL_2(K), \overline{\mathbb{Q}}_\ell)$ for the set of equivalent classes of irreducible smooth representations of $GL_2(K)$. For $\pi \in \mathrm{Irr}(GL_2(K), \overline{\mathbb{Q}}_\ell)$, let ω_π denote the central character of π. Let

$$\mathrm{LL}_\ell\colon \mathcal{G}_2(K, \overline{\mathbb{Q}}_\ell) \to \mathrm{Irr}(GL_2(K), \overline{\mathbb{Q}}_\ell)$$

be the ℓ-adic Langlands correspondence. We follow the normalization in [BH06, 35.1]. If we take an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, then ${}^\iota\tau$ and ${}^\iota\pi$ denote the representations over $\mathbb{C}$ associated to τ and π by ι respectively for $\tau \in \mathcal{G}_2(K, \overline{\mathbb{Q}}_\ell)$ and $\pi \in \mathrm{Irr}(GL_2(K), \overline{\mathbb{Q}}_\ell)$. We use similar notations also over a finite extension of K.

Remark 5.1. *The ℓ-adic Langlands correspondence LL_ℓ satisfies that*

$$\omega_{\mathrm{LL}_\ell(\tau)} \circ \mathrm{Art}_K^{-1} = (\det\tau) \otimes |\cdot|^{-1}$$

for $\tau \in \mathcal{G}_2(K, \overline{\mathbb{Q}}_\ell)$. If we take an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, then we have

$$\varepsilon\left({}^\iota\tau, s, \psi\right) = \varepsilon\left({}^\iota\mathrm{LL}_\ell(\tau), s + \frac{1}{2}, \psi\right)$$

for any non-trivial additive character $\psi\colon K \to \mathbb{C}^\times$.

For a finite extension K' of K, we define an additive character $\psi_{K'}\colon K' \to \overline{\mathbb{Q}}_\ell^\times$ by $\psi_{K'} = \psi_K \circ \mathrm{Tr}_{K'/K}$, and let $v_{K'}$ be the normalized discrete valuation of K' that sends a uniformizer to 1.

Definition of F, L', $\epsilon_{F/K}$, $\Lambda_{F,\zeta'}$ and $\pi_{F,\zeta'}$

We take $\zeta' \in k^\times$. We simply write $\pi_{\zeta'}$, $\Lambda_{\zeta'}$ and $\tau_{\zeta'}$ for $\pi_{\zeta',1,1}$, $\Lambda_{\zeta',1,1}$ and $\tau_{\zeta',1,1}$ respectively. We put

$$F = K(\zeta''\varpi^{1/3}) \quad \text{and} \quad L' = F(\varphi).$$

We define $\mathfrak{J}_F, \mathfrak{P}_F \subset M_2(\mathcal{O}_F)$ similarly to $\mathfrak{J}$ and $\mathfrak{P}$ as in (4.1). We put $U^i_{\mathfrak{J}_F} = 1 + \mathfrak{P}_F^i$ for any positive integer i. We consider L' as an F-subalgebra of $M_2(F)$ similarly as (4.2). We put

$$\epsilon_{F/K} = (-1)^f.$$

Let $\pi_{F,\zeta'}$ be the tame lifting of $\pi_{\zeta'}$ to F. See [BH06, 46.5 Definition] for the tame lifting. We define a character $\Lambda_{F,\zeta'}\colon L'^\times U^2_{\mathfrak{J}_F} \to \overline{\mathbb{Q}}_\ell^\times$ by

$$\begin{aligned}
\Lambda_{F,\zeta'}(x) &= \epsilon_{F/K}^{v_{L'}(x)} \Lambda_{\zeta'}(\mathrm{Nr}_{L'/L}(x)) \quad \text{for } x \in L'^\times, \\
\Lambda_{F,\zeta'}(x) &= (\psi_F \circ \mathrm{tr}) \left(\hat{\zeta}'^{-2} \varphi^{-1}(x-1) \right) \quad \text{for } x \in U^2_{\mathfrak{J}_F}.
\end{aligned}$$

Then we have $\pi_{F,\zeta'} = \text{c-Ind}_{L'^\times U^2_{\mathfrak{J}_F}}^{GL_2(F)} \Lambda_{F,\zeta'}$ by [BH06, 46.3 Proposition] and the construction of the tame lifting.

We will describe the restriction of $\tau_{\zeta'}$ to W_F. The field F corresponds to the subgroup $Q_8 \rtimes \mathbb{Z}$ of $Q \rtimes \mathbb{Z}$.

Definition of δ_2, δ_4 and θ_2

First, we consider the case where f is even. We put $h_0(x) = x^2 - x$. Then we have

$$h_0(\delta^2 - \delta) \equiv 1/(\zeta''\varpi^{1/3}) \pmod{3/4}.$$

Hence we can take $\delta_2 \in F(\delta)$ such that $h_0(\delta_2) = 1/(\zeta''\varpi^{1/3})$ and $\delta_2 \equiv \delta^2 - \delta \pmod{3/4}$ by Newton's method. Similarly, we can take $\delta_4 \in F(\delta)$ such that $\delta_4^2 - \delta_4 = \delta_2$ and $\delta_4 \equiv \delta \pmod{3/4}$. Then we have $F(\delta_4) = F(\delta)$. Further, we can take $\theta_2 \in F(\theta)$ such that $\theta_2^2 - \theta_2 = \delta_4^3$ and $\theta_2 \equiv \theta \pmod{7/12}$. We have $F(\theta_2) = F(\theta)$. Then $F(\delta_2)$ corresponds to the subgroup $C_4 \times \mathbb{Z}$ of $Q \times \mathbb{Z}$.

Next, we consider the case where f is odd. We put $h_1(x) = x^2 - x + 1$. Then we have

$$h_1(\delta^2 - \delta + \zeta_3) \equiv 1/(\zeta''\varpi^{1/3}) \pmod{3/4}.$$

Hence we can take $\delta_2 \in F(\zeta_3, \delta)$ such that $h_1(\delta_2) = 1/(\zeta''\varpi^{1/3})$ and $\delta_2 \equiv \delta^2 - \delta + \zeta_3 \pmod{3/4}$ by Newton's method. Similarly, we can take $\delta_4 \in F(\zeta_3, \delta)$ such that $\delta_4^2 - \delta_4 + \zeta_3 = \delta_2$ and $\delta_4 \equiv \delta \pmod{3/4}$. Then we have $F(\zeta_3, \delta_4) = F(\zeta_3, \delta)$. Further, we can take $\theta_2 \in F(\zeta_3, \theta)$ such that $\theta_2^2 - \theta_2 = \delta_4^3$ and $\theta_2 \equiv \theta \pmod{7/12}$. We have $F(\zeta_3, \theta_2) = F(\zeta_3, \theta)$. Then $F(\delta_2)$ corresponds to the subgroup C of $Q \rtimes \mathbb{Z}$.

We note that $v(\delta_2) = -1/6$ for any f.

Definition of E, $\phi_{\zeta'}$ and $\varkappa_{E/F}$

We put $E = F(\delta_2)$. The image of W_E under $\Theta_{\zeta'}$ equals C. Let $\phi_{\zeta'}$ be the character of W_E induced from ϕ' by $\Theta_{\zeta'}$. Then we have

$$\tau_{\zeta'}|_{W_F} \simeq \mathrm{Ind}_{W_E}^{W_F} \phi_{\zeta'}.$$

We consider $\phi_{\zeta'}$ as a character of $E^\times$ by the Artin reciprocity map Art_E. For a finite extension K' of K and integer i, we write $\mathfrak{p}_{K'}$ for the maximal ideal of $\mathcal{O}_{K'}$, and put $U_{K'}^i = 1 + \mathfrak{p}_{K'}^i$. Let E_m be the unramified extension over E of degree m for a positive integer m. Let $\varkappa_{E/F}$ be the character of $F^\times$ with kernel $\mathrm{Nr}_{E/F}(E^\times)$.

5.1 Explicit Artin reciprocity law

The results in this subsection will be used in the proof of Proposition 5.6.

For a Galois group G of a finite Galois extension of a non-Archimedean field, let G_s and G^t be the ramification subgroups of G with lower numbering and upper numbering respectively. Note that

$$\mathrm{Ker}\,\mathrm{Tr}_{k/\mathbb{F}_2} = \{\xi + \xi^2 \mid \xi \in k\}. \tag{5.1}$$

Lemma 5.2. *We have*

$$\phi_{\zeta'}(1+x) = \psi_E(\delta_2^3 x)$$

for $x \in \mathfrak{p}_E^2$ *and*

$$\varkappa_{E/F}(1+y) = \psi_F\left((\zeta''\varpi^{1/3})^{-1}y\right)$$

for $y \in \mathfrak{p}_F$.

Proof. We prove the first statement only in the case where f is odd. It is easier to prove the first statement in the case where f is even.

We put $G = \mathrm{Gal}(E_2(\theta)/E)$. For $\sigma \in I_E$, we can show that

$$v\left(\sigma\left(\frac{\delta}{\theta}\right) - \frac{\delta}{\theta}\right) = \begin{cases} \frac{1}{12} & \text{if } \zeta_{3,\sigma} = 1,\ \nu_\sigma = 1, \\ \frac{1}{6} & \text{if } \zeta_{3,\sigma} = 1,\ \nu_\sigma = 0,\ \mu_\sigma = 1 \end{cases}$$

by the definition of ζ_σ, ν_σ and μ_σ. Then we have

$$\mathrm{Gal}(E_2(\theta)/E_2) = G_0 = G_1 \supset \mathrm{Gal}(E_2(\theta)/E_2(\delta)) = G_2 = G_3 \supset \{1\} = G_4$$

and

$$G^t = \begin{cases} \mathrm{Gal}(E_2(\theta)/E_2) & \text{if } 0 \le t \le 1, \\ \mathrm{Gal}(E_2(\theta)/E_2(\delta)) & \text{if } 1 < t \le 2, \\ \{1\} & \text{if } 2 < t. \end{cases}$$

Then the restriction of $\phi_{\zeta'}$ to U_E^2 equals the composite

$$U_E^2 \twoheadrightarrow U_E^2/(U_E^3\,\mathrm{Nr}_{E_2(\theta)/E}(U_{E(\theta)}^3)) \xrightarrow{\sim} \mathrm{Gal}(E_2(\theta)/E_2(\delta)) \simeq Z \xrightarrow{\phi|_Z} \overline{\mathbb{Q}}_\ell^\times$$

by [Ser68, XV §2 Corollaire 3 au Théorème 1]. We define $N_2 \colon k_2 \to k$ by $N_2(x) = \mathrm{Tr}_{k_2/k}(x)^2 + \mathrm{Tr}_{k_2/k}(x)$. Then we can check that

$$\mathrm{Nr}_{E_2(\theta)/E} \colon U_{E_2(\theta)}^3/U_{E_2(\theta)}^4 \to U_E^2/U_E^3$$

becomes $N_2 \colon k_2 \to k$ under the identifications

$$U_{E_2(\theta)}^3/U_{E_2(\theta)}^4 \simeq k_2;\ 1 + \theta^{-1}x \mapsto \bar{x} \quad \text{and} \quad U_E^2/U_E^3 \simeq k;\ 1 + \delta_2^{-2}x \mapsto \bar{x}.$$

Therefore we have

$$\phi_{\zeta'}(1+x) = (\chi_2 \circ \mathrm{Tr}_{k/\mathbb{F}_2})(\overline{\delta_2^2 x}) \tag{5.2}$$

for $x \in \mathfrak{p}_E^2$, because $\mathrm{Im}\, N_2 = \mathrm{Ker}\, \mathrm{Tr}_{k/\mathbb{F}_2}$ by (5.1). Since we have $(\chi_2 \circ \mathrm{Tr}_{k/\mathbb{F}_2})(\bar{x}) = \psi_E(\delta_2 x)$ for $x \in \mathcal{O}_E$, the first statement follows.

We can prove the second statement similarly. □

Lemma 5.3. *We consider $\hat{\zeta}'^{-2}\varphi^{-1}$ as an element of $GL_2(F)$ by the embedding* (4.2)*. Then we have*

$$\begin{aligned} \det(\hat{\zeta}'^{-2}\varphi^{-1}) &\equiv \mathrm{Nr}_{E/F}(\delta_2^3) && \bmod U_F^1, \\ \mathrm{tr}(\hat{\zeta}'^{-2}\varphi^{-1}) &\equiv (\zeta''\varpi^{\frac{1}{3}})^{-1} + \mathrm{Tr}_{E/F}(\delta_2^3) && \bmod \mathcal{O}_F. \end{aligned}$$

Proof. We have

$$\det(\hat{\zeta}'^{-2}\varphi^{-1}) = -\hat{\zeta}'^{-4}\varpi^{-1} = -\zeta''^{-3}\varpi^{-1}$$

by (2.2). On the other hand, we have

$$\mathrm{Nr}_{E/F}(\delta_2^3) = \mathrm{Nr}_{E/F}(\delta_2)^3 = \begin{cases} -(\zeta''^3\varpi)^{-1} & \text{if } f \text{ is even,} \\ (1-(\zeta''\varpi^{1/3})^{-1})^3 & \text{if } f \text{ is odd.} \end{cases}$$

Hence, we have the first congruence. We have $\mathrm{tr}(\hat{\zeta}'^{-2}\varphi^{-1}) = 0$. On the other hand, we have

$$\begin{aligned} \mathrm{Tr}_{E/F}(\delta_2^3) &= \mathrm{Tr}_{E/F}(\delta_2)^3 - 3\,\mathrm{Nr}_{E/F}(\delta_2)\,\mathrm{Tr}_{E/F}(\delta_2) \\ &= 1 - 3\,\mathrm{Nr}_{E/F}(\delta_2) = \begin{cases} 1+3(\zeta''\varpi^{1/3})^{-1} & \text{if } f \text{ is even,} \\ -2+3(\zeta''\varpi^{1/3})^{-1} & \text{if } f \text{ is odd.} \end{cases} \end{aligned}$$

Hence, we have the second congruence. □

Let $\mathcal{E}'$ be the elliptic curve over $\mathbb{F}_2$ defined by $z^2+z = w^3+w$. Let $\alpha_1, \alpha_2 \in \overline{\mathbb{Q}}_\ell$ be the roots of $x^2+2x+2=0$.

Lemma 5.4. *We have*

$$\begin{aligned} |\mathcal{E}(\mathbb{F}_q)| &= q+1-\left((-2)^{1/2}\right)^f - \left(-(-2)^{1/2}\right)^f, \\ |\mathcal{E}'(\mathbb{F}_q)| &= q+1-\alpha_1^f-\alpha_2^f. \end{aligned}$$

Proof. We have $\mathrm{tr}(\mathrm{fr}_2^*; H^1(\mathcal{E}_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)) = 0$ and $\mathrm{tr}(\mathrm{fr}_4^*; H^1(\mathcal{E}_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)) = -4$ as in the proof of Lemma 3.2. Hence we obtain

$$\mathrm{tr}(\mathrm{fr}_q^*; H^1(\mathcal{E}_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)) = \left((-2)^{1/2}\right)^f + \left(-(-2)^{1/2}\right)^f.$$

The first claim follows from this and the Lefschetz trace formula. The second claim is proved similarly by $\mathrm{tr}(\mathrm{fr}_2^*; H^1(\mathcal{E}'_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)) = -2$ and $\mathrm{tr}(\mathrm{fr}_4^*; H^1(\mathcal{E}'_{k^{\mathrm{ac}}}, \overline{\mathbb{Q}}_\ell)) = 0$. □

If f is odd, then the map $\Theta_{\zeta'}$ induces an isomorphism $W(E^{\mathrm{ur}}(\theta)/E) \simeq C$, and we write $\mathfrak{a}_E$ for the composite

$$E^\times \xrightarrow{\mathrm{Art}_E} W_E^{\mathrm{ab}} \twoheadrightarrow W(E^{\mathrm{ur}}(\theta)/E) \simeq C.$$

Lemma 5.5. *We assume that f is odd. Let n_f and m_f be the integers such that $1 \le n_f, m_f \le 2$, $n_f \equiv (f+1)/2 \bmod 2$ and $m_f \equiv (f^2+7)/8 \bmod 2$. Then we have $\mathfrak{a}_E(\delta_2) = (g(1, \bar{\zeta}_3^{2n_f}, \bar{\zeta}_3^{m_f}), -1)$.*

Proof. Let $\overline{C}$ be the image of C in $Q_8 \rtimes (\mathbb{Z}/2\mathbb{Z})$. Then $\overline{C}$ is a cyclic group of order 8. Let $\overline{\mathfrak{a}}_E$ be the composite of $\mathfrak{a}_E$ with the natural projection $C \to \overline{C}$. It suffices to show that

$$\overline{\mathfrak{a}}_E(\delta_2) = (g(1, \bar{\zeta}_3^{2n_f}, \bar{\zeta}_3^{m_f}), 1),$$

because we know that the second component of $\mathfrak{a}_E(\delta_2)$ is -1. We note that the isomorphism $W(E^{\mathrm{ur}}(\theta)/E) \simeq C$ induces $\mathrm{Gal}(E_2(\theta_2)/E) \simeq \overline{C}$. By this isomorphism, we consider $(g(1, \bar{\zeta}_3^{2n_f}, \bar{\zeta}_3^{m_f}), 1)$ as an element of $\mathrm{Gal}(E_2(\theta_2)/E)$.

We write $E_{(0)}$ for E in the mixed characteristic case, and $E_{(p)}$ for E in the equal characteristic case. We use similar notations for other fields and elements of the fields. Then we have the isomorphism

$$E_{(0)}^\times/U_{E_{(0)}}^3 \simeq E_{(p)}^\times/U_{E_{(p)}}^3;\ \hat{\xi}_0 + \hat{\xi}_1\delta_{2,(0)}^{-1} + \hat{\xi}_2\delta_{2,(0)}^{-2} \mapsto \xi_0 + \xi_1\delta_{2,(p)}^{-1} + \xi_2\delta_{2,(p)}^{-2}$$

where $\xi_0, \xi_1, \xi_2 \in k \subset E_{(p)}$. This isomorphism induces an isomorphism

$$\big(\mathrm{Gal}(E_{(0)}^{\mathrm{sep}}/E_{(0)})/\mathrm{Gal}(E_{(0)}^{\mathrm{sep}}/E_{(0)})^3\big)^{\mathrm{ab}} \simeq \big(\mathrm{Gal}(E_{(p)}^{\mathrm{sep}}/E_{(p)})/\mathrm{Gal}(E_{(p)}^{\mathrm{sep}}/E_{(p)})^3\big)^{\mathrm{ab}}$$

by [Del84, (3.5.2)]. It further induces an isomorphism

$$\mathrm{Gal}\big(E_{2,(0)}(\theta_{2,(0)})/E_{(0)}\big) \simeq \mathrm{Gal}\big(E_{2,(p)}(\theta_{2,(p)})/E_{(p)}\big).$$

Then we have a commutative diagram

$$\begin{array}{ccccc} E_{(0)}^\times/U_{E_{(0)}}^3 & \xrightarrow{\mathrm{Art}_{E_{(0)}}} & \mathrm{Gal}\big(E_{2,(0)}(\theta_{2,(0)})/E_{(0)}\big) & \xrightarrow{\sim} & \overline{C} \\ \big\downarrow\wr & & \big\downarrow\wr & & \big\| \\ E_{(p)}^\times/U_{E_{(p)}}^3 & \xrightarrow{\mathrm{Art}_{E_{(p)}}} & \mathrm{Gal}\big(E_{2,(p)}(\theta_{2,(p)})/E_{(p)}\big) & \xrightarrow{\sim} & \overline{C} \end{array}$$

by [Del84, (3.6.1)] and the construction of the isomorphisms. Therefore, it suffices to show that $\overline{\mathfrak{a}}_E(\delta_2) = (g(1, \bar{\zeta}_3^{2n_f}, \bar{\zeta}_3^{m_f}), 1)$ in the equal characteristic case.

We assume that the characteristic of E is p. We define the central division algebra D_g over E of degree 64 by

$$D_g = \bigoplus_{i=0}^{7} E_2(\theta_2)s^i$$

where $s^8 = \delta_2$ and $sas^{-1} = (g(1, \bar{\zeta}_3^{2n_f}, \bar{\zeta}_3^{m_f}), 1)(a)$ for $a \in E_2(\theta_2)$. Let $\sigma_q \in \mathrm{Gal}(E_8/E)$ be the lift of Fr_q. We define the central division algebra D_σ over E of degree 64 by

$$D_\sigma = \bigoplus_{i=0}^{7} E_8 t^i$$

where $t^8 = \delta_2$ and $tat^{-1} = \sigma_q(a)$ for $a \in E_8$. By the construction of the Artin reciprocity map, it suffices to show $D_g \simeq D_\sigma$ to prove the claim. To show this isomorphism, it suffices to find $s', \delta_4', \theta_2' \in D_\sigma$ such that

$$\begin{aligned} &s'^8 = \delta_2, \quad \delta_4'^2 - \delta_4' + \zeta_3 = \delta_2, \quad \theta_2'^2 - \theta_2' = \delta_4'^3, \quad \delta_4'\theta_2' = \theta_2'\delta_4', \\ &s'\zeta_3 s'^{-1} = \zeta_3^2, \quad s'\delta_4' s'^{-1} = \delta_4' + \zeta_3^{n_f}, \quad s'\theta_2' s'^{-1} = \theta_2' + \zeta_3^{2n_f}\delta_4' + \zeta_3^{m_f}. \end{aligned}$$

We put $s' = t$. Then we have $s'^8 = \delta_2$ and $s'\zeta_3 s'^{-1} = \zeta_3^2$. We take $a_0 \in \mu_{q^4-1}(E_4)$ such that $a_0^2 - a_0 = \zeta_3$. We put

$$\delta_4' = a_0 + t^2 + t^4.$$

Then we can check that ${\delta'_4}^2 - \delta'_4 + \zeta_3 = \delta_2$ using $t^2 a_0 t^{-2} = a_0 + 1$. We can check also that $t\delta'_4 t^{-1} = \delta'_4 + \zeta_3^{n_f}$ using $t a_0 t^{-1} = a_0 + \zeta_3^{n_f}$.

We take $b_0 \in \mu_{q^8-1}(E_8)$ and $b_4 \in \mu_{q^4-1}(E_4)$ such that $b_0^2 - b_0 = a_0\zeta_3^2 + \zeta_3$ and $b_4^2 = a_0$. We put

$$\theta'_2 = b_0 + (a_0 + \zeta_3)t^2 + b_4 t^4 + t^6.$$

Then we can check that ${\theta'_2}^2 - \theta'_2 = {\delta'_4}^3$, $\delta'_4\theta'_2 = \theta'_2\delta'_4$ and $t\theta'_2 t^{-1} = \theta'_2 + \zeta_3^{2n_f}\delta'_4 + \zeta_3^{m_f}$ using $t b_0 t^{-1} = b_0 + a_0\zeta_3^{2n_f} + \zeta_3^{m_f}$ and $t b_4 t^{-1} = b_4 + \zeta_3^{2n_f}$. Therefore, we have proved the claim. □

5.2 Explicit local Langlands correspondence

In the next proposition, we show that $\tau_{\zeta'}|_{W_F}$ corresponds to $\pi_{F,\zeta'}$ under the local Langlands correspondence by calculating their epsilon factors. We will show a correspondence over K in Theorem 5.7 using the correspondence over F and a construction of the local Langlands correspondence for primitive representations in [BH06, 50.3].

Proposition 5.6. *We have* $\mathrm{LL}_\ell(\tau_{\zeta'}|_{W_F}) = \pi_{F,\zeta'}$.

Proof. We put $\mathrm{LL}_\ell(\tau_{\zeta'}) = \pi'_{\zeta'}$ and $\mathrm{LL}_\ell(\tau_{\zeta'}|_{W_F}) = \pi'_{F,\zeta'}$. We want to show that $\pi'_{F,\zeta'} = \pi_{F,\zeta'}$.

By Lemma 5.2, Lemma 5.3 and [BH06, 44.7 Proposition], the representation $\pi'_{F,\zeta'}$ contains the ramified simple stratum $(\mathfrak{J}_F, 3, \hat{\zeta}'^{-2}\varphi^{-1})$. Then the representation $\pi'_{\zeta'}$ contains the ramified simple stratum $(\mathfrak{J}, 1, \hat{\zeta}'^{-2}\varphi^{-1})$ by the construction of $\pi'_{\zeta'}$ in [BH06, 50.3]. Therefore we have

$$\pi'_{\zeta'} = \text{c-Ind}_{L^\times U^1_{\mathfrak{J}}}^{GL_2(K)} \Lambda'_{\zeta'}$$

for a character $\Lambda'_{\zeta'} \colon L^\times U^1_{\mathfrak{J}} \to \overline{\mathbb{Q}}_\ell^\times$ such that $\Lambda'_{\zeta'} = \Lambda_{\zeta'}$ on $U^1_{\mathfrak{J}}$.

Let 1_F denote the trivial character of W_F. We put

$$\varkappa_{F/K} = \det \mathrm{Ind}_{W_F}^{W_K} 1_F.$$

Then $\varkappa_{F/K}|_{W_F} = 1_F$ if f is even, and $\varkappa_{F/K}|_{W_F}$ is the unramified character of order two if f is odd. Hence, the definition of $\epsilon_{F/K}$ in [BH06, 46.3] coincides with that in this paper. By [BH06, 46.3 Proposition], we have

$$\pi'_{F,\zeta'} = \text{c-Ind}_{L'^\times U^2_{\mathfrak{J}_F}}^{GL_2(F)} \Lambda'_{F,\zeta'}$$

for a character $\Lambda'_{F,\zeta'} \colon L'^\times U^2_{\mathfrak{J}_F} \to \overline{\mathbb{Q}}_\ell^\times$ such that $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ on $U^2_{\mathfrak{J}_F}$ and

$$\Lambda'_{F,\zeta'}(x) = \epsilon_{F/K}^{v_{L'}(x)} \Lambda'_{\zeta'}(\mathrm{Nr}_{L'/L}(x))$$

for $x \in L'^\times$. Hence we have $\Lambda'_{F,\zeta'}(x) = 1$ for $x \in U^1_{L'}$, because $\Lambda'_{\zeta'}(x) = 1$ for $x \in U^1_L$. Then we see that $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ on $U^1_{L'} F^\times U^2_{\mathfrak{J}_F}$, because $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ on $F^\times$ by Remark 5.1 and Lemma 3.3.

We define $\kappa_F \colon F^\times \to \overline{\mathbb{Q}}_\ell^\times$ by $\kappa_F(x) = (-1)^{v_F(x)}$. Since $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ on $U^1_{L'} F^\times U^2_{\mathfrak{J}_F}$, we know that $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$ or $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'} \otimes (\kappa_F \circ \det)$. We take an isomorphism $\iota \colon \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Then, to show $\Lambda'_{F,\zeta'} = \Lambda_{F,\zeta'}$, it suffices to show that

$$\varepsilon({}^\iota\pi'_{F,\zeta'}, 1/2, \iota \circ \psi_F) = \varepsilon({}^\iota\pi_{F,\zeta'}, 1/2, \iota \circ \psi_F).$$

We note that we have already known this equality up to sign.

In the sequel of this proof, we identify $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$ by ι, and omit to write ι. By [BH06, 25.5 Corollary], we obtain $\varepsilon(\pi_{F,\zeta'}, 1/2, \psi_F) = -\epsilon_{F/K}$ using that 5 is the least integer $m \geq 0$ such that $U^{m+1}_{\mathfrak{J}_F} \subset \mathrm{Ker}\, \Lambda_{F,\zeta'}$. On the other hand, we have

$$\varepsilon(\pi'_{F,\zeta'}, 1/2, \psi_F) = \varepsilon(\tau_{\zeta'}|_{W_F}, 0, \psi_F) = q^{\frac{3}{2}} \varepsilon(\tau_{\zeta'}|_{W_F}, 1/2, \psi_F),$$

because the conductor of $\tau_{\zeta'}|_{W_F}$ is three. Hence, it suffices to show that

$$\varepsilon(\tau_{\zeta'}|_{W_F}, 1/2, \psi_F) = -\epsilon_{F/K} q^{-3/2}.$$

Let $\lambda_{E/F}(\psi_F)$ be the Langlands constant of E over F with respect to ψ_F (*cf.* [BH06, 30.4]). Let 1_E denote the trivial character of W_E. Then we have

$$\lambda_{E/F}(\psi_F) = \varepsilon(\mathrm{Ind}_{W_E}^{W_F} 1_E, 1/2, \psi_F)\varepsilon(1_E, 1/2, \psi_E)^{-1} = \varepsilon(\varkappa_{E/F}, 1/2, \psi_F) = \varkappa_{E/F}(\zeta''\varpi^{\frac{1}{3}}) = \epsilon_{F/K},$$

where we use $\mathrm{Ind}_{W_E}^{W_F} 1_E \simeq 1_F \oplus \varkappa_{E/F}$ and [BH06, 23.5 Lemma 1 and Proposition] at the second equality, Lemma 5.2 and [BH06, (23.6.2) and 23.6 Proposition] at the third equality, and Lemma 5.2 and the equality

$$\mathrm{Nr}_{E/F}(\delta_2) = \begin{cases} -1/(\zeta''\varpi^{1/3}) & \text{if } f \text{ is even,} \\ -1/(\zeta''\varpi^{1/3}) + 1 & \text{if } f \text{ is odd} \end{cases}$$

at the last equality. Therefore, it suffices to show that

$$\varepsilon(\phi_{\zeta'}, 1/2, \psi_E) = -q^{-3/2},$$

because we have $\varepsilon(\tau_{\zeta'}|_{W_F}, 1/2, \psi_F) = \varepsilon(\phi_{\zeta'}, 1/2, \psi_E)\lambda_{E/F}(\psi_F)$.

We define ψ'_E by $\psi'_E(x) = \psi_E(\delta_2 x)$ for $x \in E^\times$. Then ψ'_E has level one (*cf.* [BH06, 1.7 Definition]), and we have

$$\begin{aligned}
\varepsilon(\phi_{\zeta'}, 1/2, \psi_E) = \phi_{\zeta'}(\delta_2)^{-1}\varepsilon(\phi_{\zeta'}, 1/2, \psi'_E) &= q^{-\frac{1}{2}}\phi_{\zeta'}(\delta_2)^{-1} \sum_{y \in U_E^1/U_E^2} \phi_{\zeta'}(\delta_2^2 y)^{-1}\psi'_E(\delta_2^2 y) \\
&= q^{-\frac{1}{2}}\phi_{\zeta'}(\delta_2^3)^{-1} \sum_{x \in \mathfrak{p}_E/\mathfrak{p}_E^2} \phi_{\zeta'}(1+x)^{-1}\psi_E\left(\delta_2^3(1+x)\right) \\
&= q^{-\frac{1}{2}}\phi_{\zeta'}(\delta_2^3)^{-1} \sum_{x \in \mathfrak{p}_E/\mathfrak{p}_E^2} \phi_{\zeta'}(1+x)^{-1}\psi_E(\delta_2^3 x)
\end{aligned}$$

by [BH06, 23.5 Lemma 1, (23.6.2) and (23.6.4)] and $\psi_E(\delta_2^3) = 1$. Therefore it suffices to show that

$$\phi_{\zeta'}(\delta_2^3)^{-1} \sum_{x \in \mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1+\delta_2^{-1}x)^{-1}\psi_E(\delta_2^2 x) = -q^{-1}. \tag{5.3}$$

Note that we have already known this equality up to sign.

First, we consider the case where f is even. Then we have

$$\phi_{\zeta'}(\delta_2^3) = (-2)^{3f/2}\phi_{\zeta'}(-1) = (-2)^{3f/2}$$

by

$$\mathrm{Nr}_{E(\theta_2)/E}(\theta_2) = \mathrm{Nr}_{E(\delta_4)/E}(-\delta_4^3) = \mathrm{Nr}_{E(\delta_4)/E}(\delta_4)^3 = -\delta_2^3$$

and Lemma 5.2 with $x = -2$. Hence, it suffices to show

$$\sum_{x \in \mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1+\delta_2^{-1}x)^{-1}\psi_E(\delta_2^2 x) = -(-2)^{\frac{f}{2}}.$$

We simply write $\hat{\xi}^{2,4}$ for $\hat{\xi}^2 + \hat{\xi}^4$, and use similar notations also for other sums. We have

$$\phi_{\zeta'}(1 + \hat{\xi}^{2,4}\delta_2^{-1} + \hat{\xi}^{1,2,3}\delta_2^{-2}) = 1 \tag{5.4}$$

for $\xi \in k$, since

$$\begin{aligned}\mathrm{Nr}_{E(\theta_2)/E}(1+\hat{\xi}\delta_4\theta_2^{-1}) &= \mathrm{Nr}_{E(\theta_2)/E}\left(\theta_2^{-1}(\theta_2+\hat{\xi}\delta_4)\right) = \mathrm{Nr}_{E(\delta_4)/E}\left(\delta_4^{-2}(\delta_4^2-\hat{\xi}-\hat{\xi}^2\delta_4)\right)\\ &= \mathrm{Nr}_{E(\delta_4)/E}\left(\delta_4^{-2}\left((1-\hat{\xi}^2)\delta_4+\delta_2-\hat{\xi}\right)\right)\\ &\equiv 1+\hat{\xi}^{2,4}\delta_2^{-1}+\hat{\xi}^{1,2,3}\delta_2^{-2} \pmod{1/2}.\end{aligned}$$

Therefore we see that $\phi_{\zeta'}(1+\delta_2^{-1}x) = \pm\sqrt{-1}$ for $x \in \mathcal{O}_E$ if $\bar{x} \neq \xi^2+\xi^4$ for any $\xi \in k$. Then we have

$$\sum_{x\in\mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1+\delta_2^{-1}x)^{-1}\psi_E(\delta_2^2 x) = \frac{1}{2}\sum_{\xi\in k} \phi_{\zeta'}\big(1+\hat{\xi}^{2,4}\delta_2^{-1}\big)^{-1}\psi_E\big(\delta_2^2\hat{\xi}^{2,4}\big),$$

since we have already known that

$$\sum_{x\in\mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1+\delta_2^{-1}x)^{-1}\psi_E(\delta_2^2 x) = \pm(-2)^{f/2} \in \mathbb{Q}.$$

We have

$$\sum_{\xi\in k} \phi_{\zeta'}\big(1+\hat{\xi}^{2,4}\delta_2^{-1}\big)^{-1}\psi_E\big(\delta_2^2\hat{\xi}^{2,4}\big) = \sum_{\xi\in k} \phi_{\zeta'}\big(1+\hat{\xi}^{3}\delta_2^{-2}\big)\psi_E\big(\delta_2^2\hat{\xi}^{2,4}\big),$$

since

$$\phi_{\zeta'}(1+\hat{\xi}^{2,4}\delta_2^{-1})^{-1} = \phi_{\zeta'}(1+\hat{\xi}^{1,2,3}\delta_2^{-2}) = \phi_{\zeta'}(1+\hat{\xi}^{3}\delta_2^{-2})$$

by (5.1), (5.2) and (5.4). Further we have

$$\begin{aligned}\sum_{\xi\in k} \phi_{\zeta'}(1+\hat{\xi}^{3}\delta_2^{-2})\psi_E(\delta_2^2\hat{\xi}^{2,4}) &= \sum_{\xi\in k} \chi_2\left(\mathrm{Tr}_{k/\mathbb{F}_2}(\xi^3+\xi^2+\xi^4)\right)\\ &= \sum_{\xi\in k} \chi_2\left(\mathrm{Tr}_{k/\mathbb{F}_2}(\xi^3)\right) = -2(-2)^{\frac{f}{2}}\end{aligned}$$

by Lemma 5.2 and (5.1), because

$$|\{(x,y)\in k^2 \mid x^2+x=y^3\}| = |\mathcal{E}(\mathbb{F}_q)| - 1 = q - 2(-2)^{\frac{f}{2}}$$

by Lemma 5.4. Thus we have the claim in the case where f is even.

Next we consider the case where f is odd. We define n_f and m_f as in Lemma 5.5. We treat only the case where $n_f = m_f = 1$. The other cases are proved similarly.

We assume that $n_f = m_f = 1$, which is equivalent to that $f \equiv 1 \bmod 8$. We put $\eta = \eta_2^f$. By Lemma 5.5 and the definition of $\phi_{\zeta'}$, we have $\phi_{\zeta'}(\delta_2) = q/\eta$. Hence, to prove (5.3), it suffices to show

$$\sum_{x\in\mathcal{O}_E/\mathfrak{p}_E} \phi_{\zeta'}(1+\delta_2^{-1}x)^{-1}\psi_E(\delta_2^2 x) = -q^2\eta^{-3}.$$

By (5.2), we have

$$\phi_{\zeta'}(1+\hat{\xi}^{1,2}\delta_2^{-2}) = 1 \tag{5.5}$$

for $\xi \in k$. We have

$$\phi_{\zeta'}\Big(1+\mathrm{Tr}_{E_2/E}(\hat{\xi}^{2,4})\delta_2^{-1}+\big(\mathrm{Tr}_{E_2/E}(\hat{\xi}^{1,3})+\mathrm{Nr}_{E_2/E}(\hat{\xi}^{2,4})\big)\delta_2^{-2}\Big) = 1 \tag{5.6}$$

for $\xi \in k_2$ by (5.5) and

$$\begin{aligned}
\mathrm{Nr}_{E_2(\theta_2)/E}(1+\hat{\xi}\delta_4\theta_2^{-1}) &= \mathrm{Nr}_{E_2(\delta_4)/E}\left(\delta_4^{-2}(\delta_4^2-\hat{\xi}-\hat{\xi}^2\delta_4)\right)\\
&= \mathrm{Nr}_{E_2(\delta_4)/E}\left(\delta_4^{-2}\left((1-\hat{\xi}^2)\delta_4+\delta_2-\hat{\xi}-\zeta_3\right)\right)\\
&= \mathrm{Nr}_{E_2/E}\left((\zeta_3-\delta_2)^{-2}\left((1-\hat{\xi}^2)^2(\zeta_3-\delta_2)+(1-\hat{\xi}^2)(\delta_2-\hat{\xi}-\zeta_3)+(\delta_2-\hat{\xi}-\zeta_3)^2\right)\right)\\
&\equiv 1+\mathrm{Tr}_{E_2/E}(\hat{\xi}^{2,4})\delta_2^{-1}+\left(\mathrm{Tr}_{E_2/E}(\hat{\xi}^{1,3}+\hat{\xi}^2\zeta_3^2+\hat{\xi}^4\zeta_3)+\mathrm{Nr}_{E_2/E}(\hat{\xi}^{2,4})\right)\delta_2^{-2} \pmod{1/2}.
\end{aligned}$$

Then we have

$$\phi_{\zeta'}(1+\hat{\xi}^{2,4}\delta_2^{-1}+\hat{\xi}^{1,3}\delta_2^{-2}) = 1 \tag{5.7}$$

for $\xi \in k$ by (5.5) and (5.6), because

$$\begin{aligned}
\mathrm{Tr}_{k_2/k}(\xi^3)+\mathrm{Nr}_{k_2/k}(\xi^2+\xi^4) = \mathrm{Tr}_{k_2/k}(\xi)^3 &+ \left(\mathrm{Nr}_{k_2/k}(\xi^2)+\xi^{q+1}\,\mathrm{Tr}_{k_2/k}(\xi)\right)\\
&+ \left(\mathrm{Nr}_{k_2/k}(\xi^2)+\xi^{q+1}\,\mathrm{Tr}_{k_2/k}(\xi)\right)^2
\end{aligned}$$

for $\xi \in k_2$. Since

$$\mathrm{Nr}_{E_2(\theta_2)/E}(\theta_2/\delta_4) = \mathrm{Nr}_{E_2(\delta_4)/E}(-\delta_4) = \mathrm{Nr}_{E_2/E}(\zeta_3-\delta_2) = \delta_2^2+\delta_2+1,$$

we have

$$\phi_{\zeta'}(\delta_2^2+\delta_2+1) = \phi'(g(1,0,0),-2) = (-2^{-1})^{-f} = -q.$$

Hence, we obtain $\phi_{\zeta'}(1+\delta_2^{-1}+\delta_2^{-2}) = -\eta^2 q^{-1}$. Therefore we have

$$\phi_{\zeta'}\left(1+(\hat{\xi}^{2,4}+1)\delta_2^{-1}\right)^{-1}\psi_E\left(\delta_2^2(\hat{\xi}^{2,4}+1)\right) = -\frac{q}{\eta^2}\phi_{\zeta'}(1+\hat{\xi}^{2,4}\delta_2^{-1})^{-1}\psi_E(\delta_2^2\hat{\xi}^{2,4})$$

by (5.5) and $\phi_{\zeta'}(1+\delta_2^{-2}) = \psi_E(\delta_2) = \psi_E(\delta_2^2)$, which follows from Lemma 5.2 and

$$\mathrm{Tr}_{E/F}(\delta_2) = 1 \equiv -1+2(\zeta''\varpi^{1/3})^{-1} = \mathrm{Tr}_{E/F}(\delta_2^2) \pmod{2/3}.$$

Then we have

$$\sum_{x\in\mathcal{O}_E/\mathfrak{p}_E}\phi_{\zeta'}(1+\delta_2^{-1}x)^{-1}\psi_E(\delta_2^2x) = \frac{1}{2}\left(1-\frac{q}{\eta^2}\right)\sum_{\xi\in k}\phi_{\zeta'}(1+\hat{\xi}^{2,4}\delta_2^{-1})^{-1}\psi_E(\delta_2^2\hat{\xi}^{2,4}),$$

because $\xi \in \mathrm{Ker}\,\mathrm{Tr}_{k/\mathbb{F}_2} = \{\xi'^2+\xi'^4 \mid \xi' \in k\}$ if and only if $\xi+1 \notin \mathrm{Ker}\,\mathrm{Tr}_{k/\mathbb{F}_2}$. Therefore, it suffices to show that

$$\sum_{\xi\in k}\phi_{\zeta'}(1+\hat{\xi}^{2,4}\delta_2^{-1})^{-1}\psi_E(\delta_2^2\hat{\xi}^{2,4}) = -(-2)^{\frac{f+1}{2}}.$$

On the other hand, we have

$$\begin{aligned}
\sum_{\xi\in k}\phi_{\zeta'}(1+\hat{\xi}^{2,4}\delta_2^{-1})^{-1}\psi_E(\delta_2^2\hat{\xi}^{2,4}) &= \sum_{\xi\in k}\phi_{\zeta'}(1+\hat{\xi}^{1,3}\delta_2^{-2})\psi_E(\delta_2^2\hat{\xi}^{2,4})\\
&= \sum_{\xi\in k}\chi_2\left(\mathrm{Tr}_{k/\mathbb{F}_2}(\xi+\xi^3)\right) = -(-2)^{\frac{f+1}{2}}
\end{aligned}$$

by (5.1), (5.7) and Lemma 5.2, because

$$|\{(x,y)\in k^2 \mid x^2+x = y^3+y\}| = |\mathcal{E}'(\mathbb{F}_q)|-1 = q-(-2)^{\frac{f+1}{2}}$$

by Lemma 5.4 under the assumption $f \equiv 1 \bmod 8$. Thus we have proved the claim. □

Theorem 5.7. *For $\zeta' \in k^\times$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$, we have* $\mathrm{LL}_\ell(\tau_{\zeta',\chi,c}) = \pi_{\zeta',\chi,c}$.

Proof. We may assume that $\chi = 1$ and $c = 1$ by character twists. By [BH06, 50.3 Lemma 1 and (50.3.2)] and Proposition 5.6, it suffices to show that

$$\begin{aligned}
&\Lambda_{\zeta'}|_{K^\times} \circ \mathrm{Art}_K^{-1} = (\det \tau_{\zeta'}) \otimes |\cdot|^{-1}, \\
&\Lambda^3_{\zeta'}|_{U^1_{\mathfrak{J}}} = \Lambda_{F,\zeta'}|_{U^1_{\mathfrak{J}}}, \\
&\Lambda^3_{\zeta'}(x) = \epsilon_{F/K}^{v_{L'}(x)} \Lambda_{F,\zeta'}(x) \text{ for } x \in L^\times.
\end{aligned}$$

The first equality follows from Lemma 3.3, and the third equality follows from the definition of $\Lambda_{F,\zeta'}$.

We are going to show the second equality. We put

$$U'_{\mathfrak{J}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{J} \;\middle|\; a \equiv d \equiv 1,\ b \equiv 0 \bmod \mathfrak{p} \right\}, \quad U''_{\mathfrak{J}} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathfrak{J} \right\}.$$

Then we have $\Lambda^3_{\zeta'}|_{U'_{\mathfrak{J}}} = \Lambda_{F,\zeta'}|_{U'_{\mathfrak{J}}}$ by the definition of $\Lambda_{\zeta'}$ and $\Lambda_{F,\zeta'}$, because $U'_{\mathfrak{J}} \subset U^2_{\mathfrak{J}_F}$. On the other hand, we have

$$\begin{aligned}
\Lambda_{F,\zeta'}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) &= \Lambda_{F,\zeta'}\left(\varphi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \varphi^{-1}\right) = \Lambda_{F,\zeta'}\left(\begin{pmatrix} 1 & 0 \\ b\varpi & 1 \end{pmatrix}\right) \\
&= \psi_F(\hat{\zeta}'^{-2} b) = \psi_K(\hat{\zeta}'^{-2} b)^3 = \Lambda_{\zeta'}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)^3
\end{aligned}$$

for $b \in \mathcal{O}_K$. Therefore we have the claim, because $U^1_{\mathfrak{J}}$ is generated by $U'_{\mathfrak{J}}$ and $U''_{\mathfrak{J}}$. □

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