Non-semi-stable loci in Hecke stacks and Fargues' conjecture

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Abstract

We show the Harris–Viehmann conjecture under some Hodge–Newton reducibility condition for a generalization of the diamond of a non-basic Rapoport–Zink space at infinite level, which appears as a cover of the non-semi-stable locus in the Hecke stack. We show also that the cohomology of the non-semi-stable locus with coefficient coming from a cuspidal Langlands parameter vanishes. As an application, we show the Hecke eigensheaf property in Fargues' conjecture for cuspidal Langlands parameters in the GL_2-case.

Introduction

In [Far16], Fargues formulated a conjecture on a geometrization of the local Langlands correspondence motivated by a formulation of the geometric Langlands conjecture in [FGV02].

Let E be a p-adic number field with residue field $\mathbb{F}_q$. Let G be a quasi-split reductive group over E. Then we can define a moduli stack Bun_G of G-bundle on the Fargues–Fontaine curve, and a moduli Div^1_X of Cartier divisors of degree 1 on the Fargues–Fontaine curve. Further, we have a diagram

$$\begin{array}{ccc} & \mathrm{Hecke}^{\leq\mu} & \\ \swarrow^{\overleftarrow{h}} & & \searrow^{\overrightarrow{h}} \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times_{\mathbb{F}_q} \mathrm{Div}^1_X, \end{array}$$

where $\mathrm{Hecke}^{\leq\mu}$ is a moduli stack of modifications of G-bundle on the Fargues–Fontaine curve with some condition determined by a cocharacter μ of G, which is called a Hecke stack. For a discrete Langlands parameter $\varphi \colon W_E \to {}^L G$, Fargues' conjecture predicts the existence of a sheaf $\mathscr{F}_\varphi$ on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ satisfying some conditions, the most intriguing one of which is the Hecke eigensheaf property

$$\overrightarrow{h}_\natural(\overleftarrow{h}^* \mathscr{F}_\varphi \otimes \mathrm{IC}'_\mu) = \mathscr{F}_\varphi \boxtimes (r_\mu \circ \varphi),$$

where r_μ is a representation of ${}^L G$ determined by μ, and IC'_μ is an object of the derived category of sheaves determined by μ via the geometric Satake correspondence. The conjecture is stated based on some conjectural objects. However, in the case φ is cuspidal and μ is minuscule, we can define every object in the conjecture assuming only the local Langlands correspondence, which is constructed in many cases.

Assume that φ is cuspidal and μ is minuscule. Then the support of the sheaf $\mathscr{F}_\varphi$ is contained in the semi-stable locus $\mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q}$ of $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$. The Hecke eigensheaf property then predicts that

$$\mathrm{supp}\, \overrightarrow{h}_\natural(\overleftarrow{h}^* \mathscr{F}_\varphi \otimes \mathrm{IC}'_\mu) \subset \mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q} \times_{\mathbb{F}_q} \mathrm{Div}^1_X .$$

This is non-trivial since the inclusion

$$\overleftarrow{h}^{-1}\big(\mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q}\big) \subset \overrightarrow{h}^{-1}\big(\mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q} \times_{\mathbb{F}_q} \mathrm{Div}^1_X\big)$$

does not hold. The vanishing of $\overrightarrow{h}_\natural(\overleftarrow{h}^* \mathscr{F}_\varphi \otimes \mathrm{IC}'_\mu)$ outside the semi-stable locus involves geometry of a non-semi-stable locus of the Hecke stack $\mathrm{Hecke}^{\leq\mu}$.

One aim of this paper is to give a partial result in this direction. Assume that φ is cuspidal, but μ can be general in the following. Let $B(G)$ be the set of σ-conjugacy classes in $G(\breve{E})$, where $\breve{E}$ is the completion of the maximal unramified extension of E. Then we have a decomposition

$$\mathrm{Bun}_{G,\overline{\mathbb{F}}_q} = \coprod_{[b]\in B(G)} \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}^{[b]}$$

into strata, where the the strata corresponding to basic elements of $B(G)$ forms the semi-stable locus. Let $[b],[b'] \in B(G)$. We define $\mathrm{Hecke}_{[b],[b']}^{\leq\mu}$ by the fiber products

$$\begin{array}{ccccc}
\mathrm{Hecke}_{[b],[b']}^{\leq\mu} & \longrightarrow & \mathrm{Hecke}_{[b]}^{\leq\mu} & \longrightarrow & \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}^{[b]} \times_{\mathbb{F}_q} \mathrm{Div}_X^1 \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & \mathrm{Hecke}_{\overline{\mathbb{F}}_q}^{\leq\mu} & \xrightarrow{\overrightarrow{h}} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_q} \times_{\mathbb{F}_q} \mathrm{Div}_X^1 \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \overleftarrow{h}} & & \\
\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}^{[b']} & \longrightarrow & \mathrm{Bun}_{G,\overline{\mathbb{F}}_q} . & &
\end{array}$$

We assume that $[b]$ is not basic, and $[b']$ is basic. Let $\mathrm{Hecke}_{[b],[b']}^{\mu}$ be an open substack of $\mathrm{Hecke}_{[b],[b']}^{\leq\mu}$, where the modifications have type μ. We find that a generalization $\mathcal{M}_{b,b'}^{\mu}$ of a diamond of a non-basic Rapoport–Zink space at infinite level covers $\mathrm{Hecke}_{[b],[b']}^{\mu}$.

We can define a Levi subgroup L^b of G such that $[b]$ is an image of a basic element $[b_{00}]$ of $B(L^b)$. Take a proper Levi subgroup L of G containing L^b. Let $[b_0]$ be the image of $[b_{00}]$ in $B(L)$. We assume that $[b']$ is in the image of an element $[b'_0] \in B(L)$. Further, we assume that $([b],[b'],\mu)$ satisfies a twisted analogue of Hodge–Newton reducibility. Our main theorem is the following:

Theorem. *The compactly supported cohomology of* $\mathcal{M}_{b,b'}^{\mu}$ *is a parabolic induction of the compactly supported cohomology of* $\mathcal{M}_{b_0,b'_0}^{\mu}$ *with some degree shift and twist.*

See Theorem 4.26 for the precise statement. This theorem is a generalization of the Harris–Viehmann conjecture on cohomology of non-basic Rapoport–Zink spaces in [RV14, Conjecture 8.5] (*cf.* [Har01, Conjecture 5.2]) up to a character twist under the Hodge–Newton reducibility condition. We also show that the compactly supported cohomology of $\mathcal{M}_{b,b'}^{\mu}$ does not contain any supercuspidal representation. These results can be viewed as generalization of results in [Man08]. Using the above theorem, we can show the following:

Theorem. *The compactly supported cohomology of* $\mathrm{Hecke}_{[b],[b']}^{\mu}$ *with coefficient in* $\overleftarrow{h}^{*}\mathscr{F}_{\varphi}$ *vanishes.*

See Theorem 4.30 for the precise statement. This result is partial, since we are assuming Hodge–Newton reducibility. On the other hand, the assumption is automatically satisfied if $\mathrm{Hecke}_{[b],[b']}^{\leq\mu}$ is not empty in the case where $G = \mathrm{GL}_2$ and $\mu(z) = \mathrm{diag}(z,1)$. As an application, we can show the following:

Theorem. *Assume that* $G = \mathrm{GL}_2$ *and* $\mu(z) = \mathrm{diag}(z,1)$*. Then the Hecke eigensheaf property for a cuspidal Langlands parameter holds.*

During the course of this work, Hansen put a related preprint [Han21] on his webpage, which shows the Harris–Viehmann conjecture for GL_n under the Hodge–Newton reducibility condition. We learned his result on canonical filtrations and some consequences of Scholze's work [Sch17] on cohomology of diamonds from [Han21]. Note that the result of [Han21] is enough for the application to Fargues' conjecture in GL_2-case. Our main points are proving the Harris–Viehmann conjecture under the Hodge–Newton reducibility condition for general reductive groups and making the relation to Fargues' conjecture clear. Note also that our main theorem on the Harris–Viehmann conjecture is independent of the work [FS21] of Fargues and Scholze on the formulation of the geometrization of the local Langlands correspondence. After this work was done, Fargues' conjecture for cuspidal Langlands parameters in the GL_n-case is proved in [ALB21] by a different method.

In Section 1, we recall a definition of the stack of G-bundle on the Fargues–Fontaine curve, and its structure. In Section 2, we recall a defintion of the Hecke stack. We explain a cohomological fromulation on the Hecke stack by Fargues, which is based on the work of Scholze. In Section 3, we construct a $\overline{\mathbb{Q}}_\ell$-Weil sheaf which satisfies properties (1), (2) and (3) of [Far16, Conjecture 4.4] and explain the Hecke eigensheaf property in Fargues' conjecture for cuspidal Langlands parameters. We also prove the character sheaf property in this case.

In Section 4, we study a non-semi-stable locus in the Hecke stack. We find that a generalization of a diamond of a non-basic Rapoport–Zink space at infinite level covers the non-semi-stable locus in the Hecke stack. We show that the cohomology of the generalizad space can be written as a parabolic induction of the cohomology of smaller space associated a Levi subgroup under the Hodge–Newton reducibility condition. In particular, we see that the cohomology does not contain any supercuspidal representation in each degree. As a result, we show that the cohomology of the non-semi-stable locus in the Hecke stack with a coefficient coming from a cuspidal Langlands parameter vanishes.

In Section 5, we see that we can recover Hecke eigensheaf property on some part of the semi-stable locus from non-abelian Lubin–Tate theory in the GL_n-case. In Section 6, we show that the Hecke eigensheaf property in the GL_2-case, using the results in the preceding sections.

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1 Stack of G-bundles

In this section we recall various results regarding the stack of G-bundles on the curve. Let p be a prime number. Fix E a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. We follow the definition of perfectoid algebra in [Fon13, 1.1] (*cf.* [Sch12, Definition 5.1]). Let $\mathrm{Perf}_{\mathbb{F}_q}$ be the category of perfectoid spaces over $\mathbb{F}_q$ equipped with v-topology (*cf.* [Sch17, Definition 8.1(iii)]). For $S \in \mathrm{Perf}_{\mathbb{F}_q}$, we have the relative Fargues–Fontaine curve $X_S = Y_S/\varphi^{\mathbb{Z}}$ as in [FS21, Definition II.1.15]. For an affinoid perfectoid $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$, we have also the schematical relative Fargues–Fontaine curve $X^{\mathrm{sch}}_{\mathrm{Spa}(R,R^+)}$ as defined just after [FS21, Remark II.2.8]. The schematic version $X^{\mathrm{sch}}_{\mathrm{Spa}(R,R^+)}$ only depends on R and so we denote it by X^{sch}_R. We have an equivalence between categories of vector bundles on $X_{\mathrm{Spa}(R,R^+)}$ and X^{sch}_R by [KL15, Theorem 8.7.7].

Let G a connected reductive group over E. Let Bun_G be the fibered category in groupoids whose fiber at $S \in \mathrm{Perf}_{\mathbb{F}_q}$ is the groupoid of G-bundles on X_S. Then Bun_G has a reasonable geometry. Let us just mention that, in particular it is a small v-stack (*cf.* [FS21, Proposition III.1.3]).

Let $\breve{E}$ be the completion of the maximal unramified extension of E. Let σ be the continuous automorphism of $\breve{E}$ lifting the q-th power Frobenius on the residue field. For $b \in G(\breve{E})$, we have an associated G-isocrystal

$$\mathcal{F}_b \colon \operatorname{Rep}_G \longrightarrow \varphi\text{-}\operatorname{Mod}_{\breve{E}};\ (V,\rho) \mapsto (V \otimes_E \breve{E}, \rho(b)\sigma).$$

Let $B(G)$ be the set of σ-conjugacy classes in $G(\breve{E})$. Then we have a bijection

$$B(G) \longrightarrow \{\text{the isomorphism classes of } G\text{-isocrystals over } \breve{E}\};\ [b] \mapsto [\mathcal{F}_b]$$

by [RR96, Remarks 3.4 (i)].

Let $S \in \operatorname{Perf}_{\mathbb{F}_q}$. We have a functor

$$\varphi\text{-}\operatorname{Mod}_{\breve{E}} \longrightarrow \operatorname{Bun}_{X_S};\ (D,\varphi) \mapsto \mathscr{E}(D,\varphi),$$

where $\mathscr{E}(D,\varphi)$ is given by

$$Y_S \times_\varphi D \longrightarrow Y_S/\varphi^{\mathbb{Z}} = X_S.$$

The composite

$$\operatorname{Rep}_G \xrightarrow{\mathcal{F}_b} \varphi\text{-}\operatorname{Mod}_{\breve{E}} \xrightarrow{\mathscr{E}(-)} \operatorname{Bun}_{X_S}$$

gives a G-bundle $\mathscr{E}_{b,X_S}$ on X_S. We simply write $\mathscr{E}_b$ for $\mathscr{E}_{b,X_S}$ sometimes. If $b' = gb\sigma(g)^{-1}$, then we have an isomophsim

$$t_g \colon \mathscr{E}_{b,X_S} \longrightarrow \mathscr{E}_{b',X_S} \tag{1.1}$$

induced by the multiplication by g. The isomorphism class of $\mathscr{E}_{b,X_S}$ depends only on the class of b in $B(G)$. Moreover by [FS21, Theorem III.2.2], this gives a complete description of the points of Bun_G.

Let $\pi_1(G)$ be an algebraic fundamental group of G defined in [Bor98, 1.4]. Let $\overline{E}$ be a separable closure of E and let $\Gamma = \operatorname{Gal}(\overline{E}/E)$ be its absolute Galois group. Let

$$\kappa \colon B(G) \longrightarrow \pi_1(G)_\Gamma$$

be the Kottwitz map in [RR96, Theorem 1.15] (*cf.* [Kot90, Lemma 6.1]). Then [FS21, Theorem III.2.7]) provides a decomposition

$$\operatorname{Bun}_{G,\overline{\mathbb{F}}_q} = \coprod_{\alpha \in \pi_1(G)_\Gamma} \operatorname{Bun}^{\alpha}_{G,\overline{\mathbb{F}}_q}$$

into open and closed substacks.

Let $\mathbb{D}$ be the split pro-algebraic torus over E such that $X_*(\mathbb{D}) = \mathbb{Q}$. For $b \in G(\breve{E})$, we have an associated homomorphism

$$\tilde{\nu}_b \colon \mathbb{D}_{\breve{E}} \longrightarrow G_{\breve{E}}$$

constructed in [Kot85, 4.2]. This gives a well-defined map

$$\nu \colon B(G) \longrightarrow \left(\operatorname{Hom}(\mathbb{D}_{\breve{E}}, G_{\breve{E}})/G(\breve{E})\right)^{\sigma};\ [b] \mapsto [\tilde{\nu}_b],$$

which is called the Newton map. We say that $b \in G(\breve{E})$ is basic, if $\tilde{\nu}_b$ factors through the center of $G_{\breve{E}}$. We say that $[b] \in B(G)$ is basic if it consists of basic elements in $G(\breve{E})$. Let $B(G)_{\text{basic}}$ denote the basic elements in $B(G)$. We recall that the Kottwitz map induces a bijection

$$\kappa \colon B(G)_{\text{basic}} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

Assume that G is quasi-split in the sequel. We fix subgroups $A \subset T \subset B$ of G, where A is a maximal split torus, T is a maximal torus and B is a Borel subgroup. We write $X_*(A)^+$ for the dominant cocharacters of A. Then we have a natural isomorphism

$$X_*(A)^+_{\mathbb{Q}} \xrightarrow{\sim} \left(\operatorname{Hom}(\mathbb{D}_{\breve{E}}, G_{\breve{E}})/G(\breve{E})\right)^{\sigma}.$$

Let $b \in G(\breve{E})$. We write $\nu_b \in X_*(A)^+_{\mathbb{Q}}$ for the representative of $[\tilde{\nu}_b]$. Let w be the maximal length element in the Weyl group of G with respect to T. Then the map

$$\mathrm{HN}\colon B(G) \to X_*(A)^+_{\mathbb{Q}};\ [b] \mapsto w \cdot (-\nu_b)$$

is called the Harder–Narasimhan map. After equipping $X_*(A)^+_{\mathbb{Q}}$ with the natural order topology, as discussed in [RR96, Section 2], the map HN is upper semicontinuous by [FS21, Theorem III.2.3].

We define an algebraic group J_b over E by

$$J_b(R) = \{g \in G(R \otimes_E \breve{E}) \mid gb\sigma(g)^{-1} = b\}$$

for any E-algebra R. Then we have $J_b(E) = \mathrm{Aut}(\mathcal{F}_b)$. We define a v-sheaf $\widetilde{J}_b$ on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ by

$$\widetilde{J}_b(S) = \mathrm{Aut}(\mathscr{E}_{b,S})$$

for an $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. We note that the isomorphism class of J_b and $\widetilde{J}_b$ depend only on $[b] \in B(G)$.

For a locally profinite group H, we write $\underline{H}$ for v-sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ associated to H. Then we have an inclusion

$$\underline{J_b(E)} \subset \widetilde{J}_b.$$

Let $\widetilde{J}_b^0$ be the connected component of the unit section of $\widetilde{J}_b$. Then we have

$$\widetilde{J}_b = \widetilde{J}_b^0 \rtimes \underline{J_b(E)}$$

and $\widetilde{J}_b^0$ is of dimension $\langle 2\rho, \nu_b \rangle$ by [FS21, Proposition III.5.1]. In particular $\underline{J_b(E)} = \widetilde{J}_b$ if and only if b is basic.

Let $\mathrm{Bun}_G^{\mathrm{ss}}$ be the semi-stable locus of Bun_G. Then $\mathrm{Bun}_G^{\mathrm{ss}}$ is an open substack of Bun_G by [FS21, Theorem III.4.5]]. Let $\alpha \in \pi_1(G)_\Gamma$. Then the upper semicontinuity of HN provides a stratification

$$\mathrm{Bun}^{\alpha}_{G,\overline{\mathbb{F}}_q} = \coprod_{\nu \in X_*(A)^+_{\mathbb{Q}}} \mathrm{Bun}^{\alpha,\mathrm{HN}=\nu}_{G,\overline{\mathbb{F}}_q}.$$

Take $\nu \in X_*(A)^+_{\mathbb{Q}}$ and assume that $\mathrm{Bun}^{\alpha,\mathrm{HN}=\nu}_{G,\overline{\mathbb{F}}_q}$ is not empty. Then we have a unique $[b] \in B(G)$ such that $\kappa([b]) = \alpha$ and $\mathrm{HN}([b]) = \nu$. Take any representative b of $[b]$. Then by [FS21, Proposition III.5.3] we have an isomorphism

$$x_b\colon [\mathrm{Spa}(\overline{\mathbb{F}}_q)/\widetilde{J}_b] \xrightarrow{\sim} \mathrm{Bun}^{\alpha,\mathrm{HN}=\nu}_{G,\overline{\mathbb{F}}_q}$$

defined by $\mathscr{E}_b$. If b is basic, then $\mathrm{Bun}^{\alpha,\mathrm{HN}=\nu}_{G,\overline{\mathbb{F}}_q}$ is equal to the semi-stable locus $\mathrm{Bun}^{\alpha,\mathrm{ss}}_{G,\overline{\mathbb{F}}_q}$ of $\mathrm{Bun}^{\alpha}_{G,\overline{\mathbb{F}}_q}$ by [FS21, Theorem III.4.5]].

The $\widetilde{J}_b$-torsor $\mathscr{T}_b$ over $\mathrm{Bun}^{\alpha,\mathrm{HN}=\nu}_{G,\overline{\mathbb{F}}_q}$ given by x_b is the torsor defined by the functor which sends $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ to

$$\left(f\colon S \longrightarrow \mathrm{Bun}^{\alpha,\mathrm{HN}=\nu}_{G,\overline{\mathbb{F}}_q}, \phi\colon \mathscr{E}_{b,S} \xrightarrow{\sim} \mathscr{E}_f\right),$$

where $\mathscr{E}_f$ is the G-bundle on X_S determined by f, and $g \in \widetilde{J}_b(S)$ acts on $\mathscr{T}_b(S)$ (on the right) by

$$(f, \phi) \mapsto (f, \phi \circ g). \tag{1.2}$$

Then we have $\mathrm{Frob}^* x_b = x_{\sigma(b)}$ and $\mathrm{Frob}^* \mathscr{T}_b = \mathscr{T}_{\sigma(b)}$. Since we have $\sigma(b) = b^{-1}b\sigma(b)$, we have a Weil descent datum

$$w_b\colon \mathrm{Frob}^* \mathscr{T}_b \longrightarrow \mathscr{T}_b \tag{1.3}$$

induced by $t_{b^{-1}}\colon \mathscr{E}_{b,S} \to \mathscr{E}_{\sigma(b),S}$ in (1.1). Explicitly at the level of S-points, (1.3) sends (f, ϕ) to $(f, \phi \circ t_{b^{-1}})$. If $b' = gb\sigma(g)^{-1}$, then t_g^{-1} induces an isomorphism $\mathscr{T}_b \to \mathscr{T}_{b'}$, which is compatible with the Weil descent data w_b and $w_{b'}$. Hence the isomorphism class of $(\mathscr{T}_b, w_b)$ depends only on $[b] \in B(G)$.

Remark 1.1. *The $\widetilde{J}_b$-torsor $\mathscr{T}_b$ is isomorphic to* $\mathrm{Spa}(\overline{\mathbb{F}}_q)$*, however it is $\mathscr{T}_b$ that allows us to define the Weil descent datum.*

2 The global Hecke stack

Let Div^1_X be the moduli space of degree 1 closed Cartier divisors defined in [FS21, Definition II.1.19], which sends $S \in \mathrm{Perf}_{\mathbb{F}_q}$ to the set of isomorphism classes of degree 1 closed Cartier divisors on X_S. By [FS21, Proposition II.1.21], $\mathrm{Div}^1_X \to *$ is representable in spatial diamonds and we have an isomorphism

$$\mathrm{Spa}(E)^\diamond/\varphi_{E^\diamond}^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1_X,$$

where $\varphi_{E^\diamond}$ is a q-th power Frobenius action on $E^\diamond$.

We write $X_*(T)^+$ for the set of dominant cocharacters of T. Let $\mu \in X_*(T)^+/\Gamma$. We define a Hecke stack $\mathrm{Hecke}^{\leq\mu}$ as the fibered category in groupoids whose fiber at an affinoid perfectoid $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$ is the groupoid of quadruples $(\mathscr{E}, \mathscr{E}', D, f)$, where

- $\mathscr{E}$ and $\mathscr{E}'$ are G-bundles on X_R^{sch},
- D is an effective Cartier divisor of degree 1 on X_R^{sch} given by some untilt of R,
- the isomorphism
$$f\colon \mathscr{E}|_{X_R^{\mathrm{sch}}\setminus D} \xrightarrow{\sim} \mathscr{E}'|_{X_R^{\mathrm{sch}}\setminus D}$$
is a modification, which is bounded by μ geometric fiberwisely.

Then we have morphisms

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}} \mathrm{Hecke}^{\leq\mu} \xrightarrow{\overrightarrow{h}} \mathrm{Bun}_G \times_{\mathbb{F}_q} \mathrm{Div}^1_X$$

defined by $\overleftarrow{h}(\mathscr{E}, \mathscr{E}', D, f) = \mathscr{E}'$ and $\overrightarrow{h}(\mathscr{E}, \mathscr{E}', D, f) = (\mathscr{E}, D)$.

In the sequel, a diamond means a diamond on $\mathrm{Perf}_{\mathbb{F}_q}$. Let ℓ be a prime number different from p. As we will need the natural functor (i.e. relative homology) constructed in [FS21], let us briefly review it. For X a small v-stack, the derived category of solid $\overline{\mathbb{Q}}_\ell$-sheaves $D_\blacksquare(X, \overline{\mathbb{Q}}_\ell)$ is constructed in [FS21, Definition VII.1.17]. For a map $f\colon X \to Y$ of small v-stacks, there is a functor

$$f_\natural\colon D_\blacksquare(X, \overline{\mathbb{Q}}_\ell) \to D_\blacksquare(Y, \overline{\mathbb{Q}}_\ell)$$

constructed in [FS21, §VII.3]. See [FS21, Proposition VII.3.1] for basic properties of this functor.

Let $\mathcal{D}_\infty$ be a diamond over $\mathbb{C}_p^\flat$ with an action of a profinite group K_0. Let $f_\infty\colon \mathcal{D}_\infty \to \mathrm{Spa}\, \mathbb{C}_p^\flat$ be the structure morphism. Assume that the action of K_0 on geometric points of $\mathcal{D}_\infty$ is free and the quotient diamond $\mathcal{D}_\infty/K_0$ is an ℓ-cohomologically smooth diamond over $\mathbb{C}_p^\flat$. For an open subgroup K of K_0, we put $\mathcal{D}_K = \mathcal{D}_\infty/K$, and let $f_K\colon \mathcal{D}_K \to \mathrm{Spa}\, \mathbb{C}_p^\flat$ be the induced morphism. Then we put

$$H^i_{\mathrm{c}}(\mathcal{D}_\infty, \overline{\mathbb{Q}}_\ell) = \varinjlim_{K \subset K_0} R^i f_{K,\natural}((f_K^! \overline{\mathbb{Q}}_\ell)^\vee)$$

for $i \geq 0$. Let $f\colon \mathcal{D} \to \mathrm{Spa}\, \mathbb{C}_p^\flat$ be an ℓ-cohomologically smooth morphism of diamonds. For $\mathscr{F} \in D_\blacksquare(\mathcal{D}, \overline{\mathbb{Q}}_\ell)$ and $i \geq 0$, we put

$$H^i_{\mathrm{c}}(\mathcal{D}, \mathscr{F}) = R^i f_\natural(\mathcal{F} \otimes (f^! \overline{\mathbb{Q}}_\ell)^\vee).$$

Let $h\colon \mathcal{M} \to \mathcal{D}$ be a G_0-torsor, where G_0 is a locally profinite group. Let π be a smooth representation of G_0 over $\overline{\mathbb{Q}}_\ell$. We define $\mathscr{F}_\pi \in D_\blacksquare(\mathcal{D}, \overline{\mathbb{Q}}_\ell)$ as the pushforward of $\mathcal{M}$ by π. Then we have a spectral sequence

$$H_i\bigl(G_0, H^j_{\mathrm{c}}(\mathcal{M}, \overline{\mathbb{Q}}_\ell) \otimes \pi\bigr) \Rightarrow H^{j-i}_{\mathrm{c}}(\mathcal{D}, \mathscr{F}_\pi). \tag{2.1}$$

This follows from [FS21, Proposition VII.3.1] as [Ima19, Lemma 1.4].

3 Fargues' conjecture

We recall the Hecke eigensheaf property in Fargues' conjecture in the case where the Langlands parameter is cuspidal and μ is minuscule. Up to some technicalities which were worked out in [FS21], we refer the reader to [Far16, Conjecture 4.4(4)] for the general case.

Let $\varphi\colon W_E \to {}^L G$ be a cuspidal Langlands parameter. We fix a Whittacker datum. For $b \in B(G)_{\mathrm{basic}}$, let $\{\pi_{\varphi,b,\rho}\}_{\rho \in \widehat{S}_\varphi}$ be the L-packet corresponding to φ by the local Langlands correspondence for the extended pure inner form J_b of G (*cf.* [Kal14, Conjecture 2.4.1]). We recall that we have a decomposition

$$\mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q} = \coprod_{\alpha \in \pi_1(G)_\Gamma} \mathrm{Bun}^{\alpha,\mathrm{ss}}_{G,\overline{\mathbb{F}}_q}$$

into open and closed substacks. Let $\mathscr{F}_\varphi$ be the $\overline{\mathbb{Q}}_\ell$-Weil sheaf with an action of S_φ on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ determined by the following conditions:

- The support of $\mathscr{F}_\varphi$ is contained in $\mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q}$.
- Let $\alpha \in \pi_1(G)_\Gamma$. Take a basic element $b \in G(\breve{E})$ such that $\alpha = \kappa([b])$. Let $\rho \in \widehat{S}_\varphi$. We put Let $\underline{\rho}$ be the constant $\overline{\mathbb{Q}}_\ell$-sheaf with action of S_φ on $\mathrm{Bun}^{\alpha,\mathrm{ss}}_{G,\overline{\mathbb{F}}_q}$ associated to ρ. Let $\underline{\pi_{\varphi,b,\rho}}$ be the $\overline{\mathbb{Q}}_\ell$-Weil sheaf on $\mathrm{Bun}^{\alpha,\mathrm{ss}}_{G,\overline{\mathbb{F}}_q}$ obtained as the pushforward of the $\underline{J_b(E)}$-torsor $\mathscr{T}_b$ under $\pi_{\varphi,b,\rho}$, where the Weil descent datum is induced by w_b in (1.3). Then we have

$$\mathscr{F}_\varphi|_{\mathrm{Bun}^{\alpha,\mathrm{ss}}_{G,\overline{\mathbb{F}}_q}} = \bigoplus_{\rho \in \widehat{S}_\varphi,\, \rho|_{Z(\widehat{G})^\Gamma} = \alpha} \underline{\rho} \otimes \underline{\pi_{\varphi,b,\rho}}, \tag{3.1}$$

  where we view α as an element of $X^*(Z(\widehat{G})^\Gamma)$ under the canonical isomorphism $\pi_1(G)_\Gamma \simeq X^*(Z(\widehat{G})^\Gamma)$. The isomorphism class of the right hand side of (3.1) as $\overline{\mathbb{Q}}_\ell$-Weil sheaves does not depend on the choice of b, since the same is true for $(\mathscr{T}_b, w_b)$.

Then properties (1), (2) and (3) of [Far16, Conjecture 4.4] are immediate. We check that $\mathscr{F}_\varphi$ satisfies the character sheaf property in [Far16, Conjecture 4.4 (5)]. This is almost tautological by the construction of $\mathscr{F}_\varphi$. Let $\delta \in G(E)$ be an elliptic element. Then $\delta \in G(\breve{E})$ is a basic element, and the morphism

$$\tilde{x}_\delta\colon\ \mathrm{Spa}(\overline{\mathbb{F}}_q) \longrightarrow [\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{J_\delta(E)}] \xrightarrow{x_\delta} \mathrm{Bun}^{\kappa([\delta]),\mathrm{ss}}_{G,\overline{\mathbb{F}}_q} \longrightarrow \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$$

is defined over $\mathbb{F}_q$ (*cf.* [Far16, 5]). In this case, the morphism $t_{\delta^{-1}}\colon \mathscr{E}_\delta \to \mathscr{E}_\delta$ in (1.1) is equal to $\delta^{-1} \in J_\delta(E)$. Hence, the morphism w_δ in (1.3) is induced from δ^{-1}. However (1.2) tell us that this is precisely the action of δ^{-1} on $\mathscr{T}_\delta$. Therefore, the Frobenius action on $\tilde{x}_\delta^* \mathscr{F}_\varphi$ is given by $\delta^{-1} \in J_\delta(E)$, which means that $\mathscr{F}_\varphi$ satisfies the character sheaf property.

Let IC_μ be the perverse sheaf on $\mathrm{Hecke}^{\leq \mu}$ constructed from μ via the geometric Satake equivalence. We put $\mathrm{IC}'_\mu = \mathbb{D}(\mathrm{IC}_\mu)^\vee$ as [FS21, IX.2].

Take a representative $\mu' \in X_*(T)^+$ of μ. Let Γ' be the stabilizer of μ' in Γ. We put

$$r_\mu = \operatorname{Ind}_{\widehat{G} \rtimes \Gamma'}^{{}^L G} r_{\mu'},$$

where $r_{\mu'}$ is the highest weight μ' irreducible representation of $\widehat{G} \rtimes \Gamma'$.

Now we can state the Hecke eigensheaf property in Fargues' conjecture:

Conjecture 3.1. *We have*

$$\overrightarrow{h}_\natural(\overleftarrow{h}^* \mathscr{F}_\varphi \otimes_{\overline{\mathbb{Q}}_\ell} \mathrm{IC}'_\mu) = \mathscr{F}_\varphi \boxtimes (r_\mu \circ \varphi)$$

as $\overline{\mathbb{Q}}_\ell$*-Weil sheaves with actions of* S_φ *on* $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q} \times_{\mathbb{F}_q} \mathrm{Div}^1_X$.

In particular, the conjecture implies

$$\operatorname{supp} H^0(\overrightarrow{h}_\natural(\overleftarrow{h}^* \mathscr{F}_\varphi \otimes \mathrm{IC}'_\mu)) \subset \mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q} \times_{\mathbb{F}_q} \mathrm{Div}^1_X,$$

since the support of $\mathscr{F}_\varphi$ is contained in $\mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q}$.

4 Non-semi-stable locus

Let $b, b' \in G(\breve{E})$. We have a natural morphism

$$y_b \colon [\mathrm{Div}^1_{X,\overline{\mathbb{F}}_q}/\widetilde{J}_b] \simeq [\mathrm{Spa}(\overline{\mathbb{F}}_q)/\widetilde{J}_b] \times_{\mathbb{F}_q} \mathrm{Div}^1_X \xrightarrow{(x_b,\mathrm{id})} \mathrm{Bun}_{G,\overline{\mathbb{F}}_q} \times_{\mathbb{F}_q} \mathrm{Div}^1_X .$$

Let

$$\tilde{y}_b \colon [\mathrm{Spa}(\breve{E})^\diamond/\widetilde{J}_b] \longrightarrow [\mathrm{Div}^1_{X,\overline{\mathbb{F}}_q}/\widetilde{J}_b] \xrightarrow{y_b} \mathrm{Bun}_{G,\overline{\mathbb{F}}_q} \times_{\mathbb{F}_q} \mathrm{Div}^1_X$$

be the composite. We consider the cartesian diagram (i.e. every sub-square is cartesian)

$$\begin{array}{ccccc}
\mathrm{Hecke}^{\leq\mu}_{b,b'} & \longrightarrow & \mathrm{Hecke}^{\leq\mu}_{b} & \longrightarrow & [\mathrm{Spa}(\breve{E})^\diamond/\widetilde{J}_b] \\
\Big\downarrow {\scriptstyle \overleftarrow{h}_{b,b'}} & & \Big\downarrow & & \Big\downarrow {\scriptstyle \tilde{y}_b} \\
 & & \mathrm{Hecke}^{\leq\mu}_{\overline{\mathbb{F}}_q} & \xrightarrow{\overrightarrow{h}} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_q} \times_{\mathbb{F}_q} \mathrm{Div}^1_X \\
 & & \Big\downarrow {\scriptstyle \overleftarrow{h}} & & \\
[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\widetilde{J}_{b'}] & \xrightarrow{x_{b'}} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_q} . & &
\end{array}$$

By the construction, for a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra (R, R^+), the groupoid $\mathrm{Hecke}^{\leq\mu}_{b,b'}(R, R^+)$ consists of quadruples $(\mathscr{E}, \mathscr{E}', D, f)$, where

- $\mathscr{E}$ and $\mathscr{E}'$ are G-bundles on X_R^{sch} which are isomorphic to $\mathscr{E}_b$ and $\mathscr{E}_{b'}$ fiberwisely over $\mathrm{Spa}(R, R^+)$.
- D is an effective Cartier divisor of degree 1 on X_R^{sch} given by some untilt of R,
- $f \colon \mathscr{E}|_{X_R^{\mathrm{sch}} \setminus D} \to \mathscr{E}'|_{X_R^{\mathrm{sch}} \setminus D}$ is a modification bounded by μ geometric fiberwisely over $\mathrm{Spa}(R, R^+)$.

Let $\mathcal{T}^{\leq\mu}_{b,b'}$ be the $\widetilde{J}_b$-torsor over $\mathrm{Hecke}^{\leq\mu}_{b,b'}$ obtained by considering an isomorphism $\phi \colon \mathscr{E}_b \xrightarrow{\sim} \mathscr{E}$. Let $\mathrm{Gr}^{\leq\mu}_{b,b'}$ and $\mathcal{M}^{\leq\mu}_{b,b'}$ be the $\widetilde{J}_{b'}$-torsors over $\mathrm{Hecke}^{\leq\mu}_{b,b'}$ and $\mathcal{T}^{\leq\mu}_{b,b'}$ obtained by considering an isomorphism $\phi'\colon \mathscr{E}_{b'} \xrightarrow{\sim} \mathscr{E}'$ respectively. Then $\mathcal{M}_{b,b'}^{\leq\mu}$ is a $\widetilde{J}_{b'}$-equivariant $\widetilde{J}_b$-torsor over $\mathrm{Gr}_{b,b'}^{\leq\mu}$. We have commutative diagrams

$$\begin{array}{ccccc} \mathcal{M}_{b,b'}^{\leq\mu} & \longrightarrow & \mathcal{T}_{b,b'}^{\leq\mu} & \longrightarrow & \mathrm{Spa}(\breve{E})^{\diamond} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Gr}_{b,b'}^{\leq\mu} & \longrightarrow & \mathrm{Hecke}_{b,b'}^{\leq\mu} & \longrightarrow & [\mathrm{Spa}(\breve{E})^{\diamond}/\widetilde{J}_b], \end{array}$$

where the sub-squares are cartesian.

By [Far16, Proposition 3.20], $\mathcal{T}_{b,b'}^{\leq\mu}$ is a diamond. Furthermore by [Sch17, Lemma 10.13, Proposition 11.5], $\mathcal{M}_{b,b'}^{\leq\mu}$ is a diamond if b' is basic.

Remark 4.1. *The maps $\mathcal{M}_{b,b'}^{\leq\mu} \to \mathrm{Gr}_{b,b'}^{\leq\mu}$ and $\mathcal{M}_{b,b'}^{\leq\mu} \to \mathcal{T}_{b,b'}^{\leq\mu}$ appearing in the above diagram are generalized versions of the Hodge–Tate period map and the Gross–Hopkins period map. Indeed if $b' = 1$ and μ is minuscule then $\mathcal{M}_{b,b'}^{\leq\mu} \to \mathrm{Gr}_{b,b'}^{\leq\mu}$ is the usual Hodge–Tate period map of a Rapoport–Zink space at infinite level associated to the isocrystal b and $\mathcal{M}_{b,b'}^{\leq\mu} \to \mathcal{T}_{b,b'}^{\leq\mu}$ is the usual Gross–Hopkins period map. On the other hand if $b = 1$ and μ is minuscule then $\mathcal{M}_{b,b'}^{\leq\mu} \to \mathrm{Gr}_{b,b'}^{\leq\mu}$ is the Gross–Hopkins map and $\mathcal{M}_{b,b'}^{\leq\mu} \to \mathcal{T}_{b,b'}^{\leq\mu}$ is the Hodge–Tate map associated to the isocrystal b'.*

For a finite dimensional algebraic representation V of G and a rational number α, we put

$$\mathrm{Fil}_b^{\alpha} V = \bigoplus_{\alpha' \leq -\alpha} V_{\alpha'},$$

where

$$V = \bigoplus_{\alpha \in \mathbb{Q}} V_{\alpha}$$

is the slope decomposition given by $\nu_b \in X_*(A)_{\mathbb{Q}}^+$. This gives a filtration Fil_b on the forgetful fiber functor $\omega\colon \mathrm{Rep}\, G \to \mathrm{Vect}_E$ (*cf.* [SR72, IV, 2.1]). The stabilizer of $\mathrm{Fil}_b\, \omega$ gives a parabolic subgroup P^b of G. Let L^b be the centralizer of $\nu_b \in X_*(A)_{\mathbb{Q}}^+$. Take a Levi subgroup L of G containing L^b. We put $P = LP^b$. Then, P is a parabolic subgroup of G and $[b] \in B(G)$ is the image of an element $b_{00} \in L^b(\breve{E})$. Let b_0 be the image of b_{00} in $L(\breve{E})$.

We take a cocharacter $\lambda \in X_*(A)$ so that P is associated to λ in the sense of [Spr98, 13.4.1]. Then we have a filtration Fil_{λ} on ω associated to λ.

We assume that $[b']$ is in the image of $B(L) \to B(G)$. Then $\mathrm{Fil}_{\lambda}\, \omega$ induces the filtrations $\mathrm{Fil}_{\lambda}\, \mathscr{E}_b$ and $\mathrm{Fil}_{\lambda}\, \mathscr{E}_{b'}$ as fiber functors by the construction, because $[b], [b']$ are in the image of $B(L) \to B(G)$ and L is the centralizer of λ in G.

We define a closed subspace $\mathcal{C}_{b,b'}^{\leq\mu}$ of $\mathrm{Gr}_{b,b'}^{\leq\mu}$ as a functor that sends a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra (R, R^+) to the isomorphism classes of $(\mathscr{E}, \mathscr{E}', D, f, \phi')$, where

- $(\mathscr{E}, \mathscr{E}', D, f)$ is as in $\mathrm{Hecke}_{b,b'}^{\leq\mu}(R, R^+)$,
- $\phi'\colon \mathscr{E}_{b'} \xrightarrow{\sim} \mathscr{E}'$ and f are compatible with $\mathrm{Fil}_{\lambda}\, \mathscr{E}_b$ and $\mathrm{Fil}_{\lambda}\, \mathscr{E}_{b'}$ geometric fiberwisely in the sense that following holds for any geometric point $\mathrm{Spa}(F, F^+)$ of $\mathrm{Spa}(R, R^+)$: Take an isomorphism $\mathscr{E}_b \xrightarrow{\sim} \mathscr{E}$ over X_F^{sch}. Let D_F be a Cartier divisor of X_F^{sch} determined by D. Then the composite

$$\mathscr{E}_b|_{X_F^{\mathrm{sch}} \setminus D_F} \xrightarrow{\sim} \mathscr{E}|_{X_F^{\mathrm{sch}} \setminus D_F} \xrightarrow{f} \mathscr{E}'|_{X_F^{\mathrm{sch}} \setminus D_F} \xrightarrow{\phi'^{-1}} \mathscr{E}_{b'}|_{X_F^{\mathrm{sch}} \setminus D_F}$$

respects the filtrations $\mathrm{Fil}_{\lambda}\, \mathscr{E}_b|_{X_F^{\mathrm{sch}} \setminus D_F}$ and $\mathrm{Fil}_{\lambda}\, \mathscr{E}_{b'}|_{X_F^{\mathrm{sch}} \setminus D_F}$.

Remark 4.2. *The condition that ϕ' and f are compatible with $\mathrm{Fil}_\lambda \mathscr{E}_b$ and $\mathrm{Fil}_\lambda \mathscr{E}_{b'}$ is independent of choice of an isomorphism $\mathscr{E}_b \xrightarrow{\sim} \mathscr{E}$, because the automorphism group $\widetilde{J}_b$ of $\mathscr{E}_b$ respects the filtration $\mathrm{Fil}_\lambda \mathscr{E}_b$.*

For $\mu \in X_*(T)$, we put

$$\overline{\mu} = \frac{1}{[\Gamma : \Gamma_\mu]} \sum_{\tau \in \Gamma/\Gamma_\mu} \tau(\mu),$$

where Γ_μ is a stabilizer of μ in Γ, and let $\mu^\natural$ denote the image of μ in $\pi_1(G)_\Gamma$.

Definition 4.3. *(cf. [RV14, Definition 2.5]) We say that $[b] \in B(G)$ is acceptable for $(\mu, [b'])$ if $\nu_b - \nu_{b'} \leq \overline{\mu}$. We say that $[b] \in B(G)$ is neutral for $(\mu, [b'])$ if $\kappa_G([b]) - \kappa_G([b']) = \mu^\natural$.*

Let $B(G, \mu, [b'])$ be the set of acceptable neutral elements in $B(G)$ for $(\mu, [b'])$.

Remark 4.4. *The set $B(G, \mu, [b'])$ is a twisted analogue of the set $B(G, \mu)$, the latter due to Kottwitz. We refer the reader to [Kot97, §6.2] for this definition.*

To state our main results we need the notion of Hodge–Newton reducibility.

Definition 4.5. *(cf. [RV14, Definition 4.28]) A triple $([b], [b'], \mu)$ such that $[b] \in B(G, \mu, [b'])$ and b' is basic is called Hodge–Newton reducible, if there is a standard proper Levi subgroup L of G and $[b_0], [b_0'] \in B(L)$ such that $[b]$ and $[b']$ are the images of $[b_0]$ and $[b_0']$ respectively, μ factors through L, $[b_0] \in B(L, \mu, [b_0'])$ and the action of ν_{b_0} on $R_{\mathrm{u}}(B)$ is non-negative.*

Lemma 4.6. *Let R be a DVR with the maximal ideal $\mathfrak{m}$, and M be an R-module such that $M \simeq \bigoplus_{1 \leq i \leq n} R/\mathfrak{m}^{k_i}$, where $k_1 \geq \cdots \geq k_n$ is a sequence of non-negative integers. Let N be a quotient of M generated by j elements, where $j \leq n$. Then we have $l(N) \leq k_1 + \cdots + k_j$. Further, if the equality holds, then N is a direct summand of M.*

Proof. This follows from [Han21, Lemma 3.2] by taking the Pontryagin dual. □

The following proposition is a slight generalization of [Han21, Theorem 3.1], where the slope of a semi-stable bundle is assumed to be zero.

Proposition 4.7. *Assume that $G = \mathrm{GL}_n$. Let $(k_1 \geq \cdots \geq k_n)$ be the sequence of integers corresponding to $\mu \in X_*(T)^+$. Let (R, R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra. Let*

$$f\colon \mathscr{E}|_{X_R^{\mathrm{sch}} \setminus D} \xrightarrow{\sim} \mathscr{E}'|_{X_R^{\mathrm{sch}} \setminus D}$$

be a modification of between G-bundles $\mathscr{E}$ and $\mathscr{E}'$ over X_R^{sch} along an effective Cartier divisor of degree 1 which is equal to μ geometric fiberwisely. We view $\mathscr{E}$ and $\mathscr{E}'$ as vector bundles of rank n. Let $\mathscr{E}^+$ be a saturated sub-vector bundle of $\mathscr{E}$ such that

$$\deg(\mathscr{E}_x^+) + \sum_{1 \leq j \leq \mathrm{rk}(\mathscr{E}^+)} k_{n+1-j} = \mathrm{rk}(\mathscr{E}^+) s \tag{4.1}$$

for every point x of $Spa(R, R^+)$.

Assume that $\mathscr{E}'$ is semi-stable of slope s geometric fiberwisely. Let $j\colon X_R^{\mathrm{sch}} \setminus D \to X_R^{\mathrm{sch}}$ be the open immersion. We put

$$\mathscr{E}'^+ = j_* f(j^* \mathscr{E}^+) \cap \mathscr{E}'.$$

Then $\mathscr{E}'^+$ is a semi-stable vector bundle of slope s such that $\mathrm{rk}(\mathscr{E}'^+) = \mathrm{rk}(\mathscr{E}^+)$.

Proof. We follow arguments in the proof of [Han21, Theorem 3.1].

Take a modification $f_1\colon \mathcal{O}|_{X_R^{\mathrm{sch}}\setminus D} \xrightarrow{\sim} \mathcal{O}(1)|_{X_R^{\mathrm{sch}}\setminus D}$ of degree 1 along D. For a large N, changing $\mathscr{E}'$, f and $(k_1,\dots,k_n)$ by $\mathscr{E}'(N)$,

$$(\mathrm{id}_{\mathscr{E}'}\otimes f_1^{\otimes N})\circ f\colon \mathscr{E}|_{X_R^{\mathrm{sch}}\setminus D} \xrightarrow{\sim} \mathscr{E}'(N)|_{X_R^{\mathrm{sch}}\setminus D}$$

and $(k_1+N,\dots,k_n+N)$ respectively, we may assume that f extends to an injective morphism $f\colon \mathscr{E}\to\mathscr{E}'$, which induces a morphism $f^+\colon \mathscr{E}^+\to\mathscr{E}'^+$. We put $\mathscr{E}^-=\mathscr{E}/\mathscr{E}^+$ and $\mathscr{E}'^-=\mathscr{E}'/\mathscr{E}'^+$. Let $f^-\colon \mathscr{E}^-\to\mathscr{E}'^-$ be the morphism induced by f.

First, we treat the case where R is a perfectoid field. In this case, $\mathscr{E}'^+$ and $\mathscr{E}'^-$ are vector bundles such that $\mathrm{rk}(\mathscr{E}'^+)=\mathrm{rk}(\mathscr{E}^+)$ and $\mathrm{rk}(\mathscr{E}'^-)=\mathrm{rk}(\mathscr{E}^-)$. Let Q^+ and Q^- be the cokernel of h^+ and h^- respectively. Then we have

$$l(Q^-)\le \sum_{1\le i\le \mathrm{rk}(\mathscr{E}^-)} k_i$$

by Lemma 4.6, since Q^- is generated by $\mathrm{rk}(\mathscr{E}^-)$-elements. Hence we have

$$l(Q^+)\ge \sum_{1\le j\le \mathrm{rk}(\mathscr{E}^+)} k_{n+1-j}.$$

By this and (4.1), we have

$$\deg(\mathscr{E}'^+)=\deg(\mathscr{E}^+)+l(Q^+)\ge \mathrm{rk}(\mathscr{E}^+)s.$$

On the other hand, we have $\deg(\mathscr{E}'^+)\le \mathrm{rk}(\mathscr{E}^+)s$, since $\mathscr{E}'$ is semi-stable. Therefore, $\mathscr{E}'^+$ is a semi-stable vector bundle of slope s.

The general case is reduced to the above case by the same argument as in [Han21, §3.2]. □

Lemma 4.8. *Let (R,R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra. For any element α of $H^1_{\mathrm{et}}(X_R^{\mathrm{sch}},\mathcal{O})$, there is a pro-etale extension (R',R'^+) of (R,R^+) such that the image of α in $H^1_{\mathrm{et}}(X_{R'}^{\mathrm{sch}},\mathcal{O})$ is zero.*

Proof. Any extension of $\mathcal{O}$ by $\mathcal{O}$ on X_R^{sch} splits after a pro-etale extension of (R,R^+) by [FF14, 6.3.1] and [Far16, Theorem 2.26] (*cf.* [KL15, Corollary 8.7.10]). This implies the claim, since $H^1_{\mathrm{et}}(X_R^{\mathrm{sch}},\mathcal{O})$ parametrize the extensions of $\mathcal{O}$ by $\mathcal{O}$ on X_R^{sch}. □

Assume that b' is basic. Let U be the unipotent radical of P. Note that we have a surjection

$$P\longrightarrow P/U\simeq L,$$

where the second isomorphism is given by $L\hookrightarrow P\to P/U$.

Lemma 4.9. *Let (R,R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra. Let $\mathscr{E}_P$ a P-bundle on X_R^{sch} such that $\mathscr{E}_P\times^P L\simeq \mathscr{E}_{b'_0}$. Then we have an isomorphism $\mathscr{E}_P\simeq\mathscr{E}_{b'_0}\times^L P$ after a pro-etale extension of (R,R^+).*

Proof. We follow arguments in the proof of [Far20, Proposition 5.16]. Let P act on U by the conjugation. We put

$$\mathscr{U}=\mathscr{E}_P\times^P U.$$

Then $H^1_{\mathrm{et}}(X_R^{\mathrm{sch}},\mathscr{U})$ parametrizes the fiber of

$$H^1_{\mathrm{et}}(X_R^{\mathrm{sch}},P)\longrightarrow H^1_{\mathrm{et}}(X_R^{\mathrm{sch}},L)$$

over the image of $\mathscr{E}_P$. Hence, it suffices to show that $H^1_{\mathrm{et}}(X_R^{\mathrm{sch}},\mathscr{U})$ is trivial after a pro-etale extension of (R,R^+). This follows from Lemma 4.8, since $\mathscr{U}$ has a filtration whose graded subquotients are semi-stable vector bundles of slope zero. □

Lemma 4.10. *Let $\mu_1, \mu_2 \in X_*(T)^+$ such that $\mu_1 \leq \mu_2$. Then* $\mathrm{Hecke}^{\leq \mu_1} \subset \mathrm{Hecke}^{\leq \mu_2}$ *is a closed substack.*

Proof. By [Far16, Proposition 3.20], it is enough to prove $\mathrm{Gr}_G^{\leq \mu_1} \subset \mathrm{Gr}_G^{\leq \mu_2}$ is closed substack. The latter follows from the semi-continuity of the map $|\mathrm{Gr}| \to X_*(T)^+/\Gamma$ in [Far16, 3.3.2] (*cf.* [SW20, Proposition 19.2.3]). □

We define a substack Hecke^{μ} of $\mathrm{Hecke}^{\leq \mu}$ by requiring the condition that modifications are equal to μ geometric fiberwisely. Then Hecke^{μ} is an open substack of $\mathrm{Hecke}^{\leq \mu}$ by Lemma 4.10. We use similar definitions and notations also for other spaces.

Let X be a scheme over E. Let $\mathrm{FilVect}_X$ be the category of filtered vector bundles on X. We consider the functor

$$\omega_\lambda \colon \mathrm{Rep}_G \longrightarrow \mathrm{FilVect}_X;\ V \mapsto (V \otimes_E \mathcal{O}_X, (\mathrm{Fil}_\lambda V) \otimes_E \mathcal{O}_X).$$

Let $\mathrm{Fil}_\lambda \mathrm{Bun}_X^G$ be the category of functors $\omega \colon \mathrm{Rep}_G \to \mathrm{FilVect}_X$ which are isomorphic to ω_λ fpqc locally on X. Let Bun_X^P be the category of P-bundles on X.

Lemma 4.11. *There is an equivalence of categories*

$$\mathrm{Fil}_\lambda \mathrm{Bun}_X^G \longrightarrow \mathrm{Bun}_X^P;\ \omega \mapsto \underline{\mathrm{Isom}}_X^{\otimes}(\omega_\lambda, \omega),$$

where $\underline{\mathrm{Isom}}_X^{\otimes}(\omega_\lambda, \omega)$ is a functor from the category of schemes over X to the category of sets which sends X' to the set of isomorphisms $\omega_\lambda|_{X'} \to \omega|_{X'}$ as filtered tensor functors.

Proof. This follows from [Zie15, Theorem 4.42 and Theorem 4.43]. □

Proposition 4.12. *Assume that $([b], [b'], \mu)$ is Hodge–Newton reducible for L. Let (R, R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra, and $(\mathscr{E}, \mathscr{E}', D, f) \in \mathrm{Hecke}_{b,b'}^{\mu}(R, R^+)$. Then, after taking a pro-etale extension of (R, R^+), there is a reduction*

$$f_P \colon \mathscr{E}_P|_{X_R^{\mathrm{sch}} \setminus D} \xrightarrow{\sim} \mathscr{E}'_P|_{X_R^{\mathrm{sch}} \setminus D}$$

of f to P such that $\mathscr{E}_P \simeq \mathscr{E}_{b_0} \times^L P$ and $\mathscr{E}'_P \simeq \mathscr{E}_{b'_0} \times^L P$.

Proof. By taking a pro-etale extension of (R, R^+), we can take an isomorphism $\mathscr{E}_b \simeq \mathscr{E}$. We put $\mathscr{E}_P = \mathscr{E}_{b_0} \times^L P$. Then $\mathscr{E}_P$ and the isomorphism

$$\mathscr{E}_P \times^P G \cong \mathscr{E}_{b_0} \times^L G \cong \mathscr{E}_b \xrightarrow{\sim} \mathscr{E}$$

give a reduction of $\mathscr{E}$ to P. We put $\phi_P = \mathrm{id}_{\mathscr{E}_{b_0} \times^L P}$. Then ϕ_P is a reduction of ϕ to P.

For any irreducible $V \in \mathrm{Rep}_G$, the vector bundle $\mathscr{E}'(V)$ is semi-stable geometric fiberwisely. By Proposition 4.7, we have a functorial construction of a filtration of $\mathscr{E}'(V)$ that is compatible under $f(V)$ with the filtration of $\mathscr{E}(V)$ coming from $\mathscr{E}_P$ by Lemma 4.11. Since the category Rep_G is semi-simple, the construction extends to all $V \in \mathrm{Rep}_G$ in a functorial way. Hence, by Lemma 4.11, we have a reduction

$$f_P \colon \mathscr{E}_P|_{X_R^{\mathrm{sch}} \setminus D} \xrightarrow{\sim} \mathscr{E}'_P|_{X_R^{\mathrm{sch}} \setminus D}$$

of f to P for some P-bundle $\mathscr{E}'_P$. By Lemma 4.9, $\mathscr{E}'_P$ is isomorphic to $\mathscr{E}_{b'_0} \times^L P$ after taking a pro-etale extension of (R, R^+). □

Let $\widetilde{P}_{b'}$ be the stabilizer of $\mathrm{Fil}_\lambda \mathscr{E}_{b'}$ in $\widetilde{J}_{b'}$. Then $\widetilde{P}_{b'} = \underline{P_{b'}(E)}$ for a parabolic subgroup $P_{b'}$ of $J_{b'}$.

Proposition 4.13. *Assume that* $([b],[b'],\mu)$ *is Hodge–Newton reducible for* L*. Then the action of* $\widetilde{P}_{b'}$ *on* $\mathrm{Gr}^{\mu}_{b,b'}$ *stabilizes* $\mathcal{C}^{\mu}_{b,b'}$*, and we have a natural* $\widetilde{J}_{b'}$*-equivariant isomorphism*

$$\mathcal{C}^{\mu}_{b,b'} \times^{\widetilde{P}_{b'}} \widetilde{J}_{b'} \xrightarrow{\sim} \mathrm{Gr}^{\mu}_{b,b'}.$$

Proof. The first claim follows from the definitions of $\widetilde{P}_{b'}$ and $\mathrm{Gr}^{\mu}_{b,b'}$. The morphism

$$\mathcal{C}^{\mu}_{b,b'} \times^{\widetilde{P}_{b'}} \widetilde{J}_{b'} \longrightarrow \mathrm{Gr}^{\mu}_{b,b'}$$

induced by the action of $\widetilde{J}_{b'}$ on $\mathrm{Gr}^{\mu}_{b,b'}$ is an epimorphism by Proposition 4.12.

We show the injectivity. Let $g \in \widetilde{J}_{b'}(R,R^+)$ for a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra (R,R^+). Assume that g sends a point of $\mathcal{C}^{\mu}_{b,b'}(R,R^+)$ to a point of $\mathcal{C}^{\mu}_{b,b'}(R,R^+)$. Then g stabilizes $\mathrm{Fil}_{\lambda}\,\mathscr{E}_{b'}$ outside the Cartier divisor corresponding to $R^{\sharp}$. This implies g stabilizes $\mathrm{Fil}_{\lambda}\,\mathscr{E}_{b'}$ on X_R^{sch}, since g stabilizes $\mathscr{E}_{b'}$ itself. Hence, we have $g \in \widetilde{P}_{b'}(R,R^+)$. □

Let $\mathcal{P}^{\mu}_{b,b'}$ be the inverse image of $\mathcal{C}^{\mu}_{b,b'}$ under $\mathcal{M}^{\mu}_{b,b'} \to \mathrm{Gr}^{\mu}_{b,b'}$.

Corollary 4.14. *Assume that* $([b],[b'],\mu)$ *is Hodge–Newton reducible for* L*. Then the action of* $\widetilde{P}_{b'}$ *on* $\mathcal{M}^{\mu}_{b,b'}$ *stabilizes* $\mathcal{P}^{\mu}_{b,b'}$*, and we have a natural* $(\widetilde{J}_b \times \widetilde{J}_{b'})$*-equivariant isomorphism*

$$\mathcal{P}^{\mu}_{b,b'} \times^{\widetilde{P}_{b'}} \widetilde{J}_{b'} \xrightarrow{\sim} \mathcal{M}^{\mu}_{b,b'}.$$

Proof. This follows from Proposition 4.13. □

We define a subsheaf $\widetilde{J}_b^U$ of $\widetilde{J}_b$ by

$$\widetilde{J}_b^U(S) = \left\{ g \in \widetilde{J}_b(S) \;\middle|\; g|_{\mathrm{Fil}^j_{\lambda}\mathscr{E}_b} \equiv \mathrm{id}_{\mathrm{Fil}^j_{\lambda}\mathscr{E}_b} \mod \mathrm{Fil}^{j+1}_{\lambda}\mathscr{E}_b \text{ for all } j \right\}$$

for $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$.

Let $U_{b'}$ be the unipotent radical of $P_{b'}$. The inner form of L determined by b' gives a Levi subgroup $L_{b'}$ of $P_{b'}$.

We use a notation that

$$\mathrm{gr}^i_{\lambda} = \mathrm{Fil}^i_{\lambda} / \mathrm{Fil}^{i+1}_{\lambda}$$

for any integer i. Let ρ_U be the half-sum of the positive roots α of T such that $-\alpha$ occurs in the adjoint action of T on $\mathrm{Lie}(U)$. We put $N_{U,b} = \langle 2\rho_U, \nu_b \rangle$.

Definition 4.15. *Let* F *be a non-archimedean field with a valuation subring* F^+*. Let* $f\colon D \to \mathrm{Spa}(F,F^+)^{\diamond}$ *be an* ℓ*-cohomologically smooth morphism of locally spatial diamonds (cf. [Sch17, Definition 23.8]). We say that* D *is* ℓ*-contractible of pure dimension* d *if* $f^!\mathbb{F}_{\ell} = \mathbb{F}_{\ell}(d)[2d]$ *and the trace morphism* $Rf_!f^!\mathbb{F}_{\ell} \to \mathbb{F}_{\ell}$ *is a quasi-isomorphism.*

Remark 4.16. *In the situation of Definition 4.15, by [FS21, Proposition VII.5.2]* $f_{\natural}\mathbb{F}_{\ell} \cong Rf_!f^!\mathbb{F}_{\ell}$*.*

Let ϖ be a uniformizer of E. Let $\mathbb{B}$ denote the v-sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ given by $\mathbb{B}(S) = \mathcal{O}(Y_S)$ (*cf.* [FS21, Proposition II.2.1]).

Lemma 4.17. *Let* d *and* h *be positive integers. Let* $f_{d,h}\colon \mathbb{B}^{\varphi^d=\varpi^h} \times \mathrm{Spa}(\breve{E})^{\diamond} \to \mathrm{Spa}(\breve{E})^{\diamond}$ *be the natural morphism.*

(1) The v-sheaf $\mathbb{B}^{\varphi^d=\varpi^h} \times \mathrm{Spa}(\breve{E})^{\diamond}$ *is an* ℓ*-cohomologically smooth* ℓ*-contractible locally spatial diamond of pure dimension* h *over* $\mathrm{Spa}(\breve{E})^{\diamond}$*.*

(2) The action of $E^{\times}$ *on* $f_{d,h,!}\mathbb{Z}_{\ell}$ *is given by* $||\cdot||^{-d}$*.*

(3) *Let F be a perfectoid field over $\breve{E}$ and $a \in \mathbb{B}^{\varphi^d=\varpi^h}(F^\flat)$. Let $f_{d,h,F^\flat}\colon \mathbb{B}^{\varphi^d=\varpi^h} \times \mathrm{Spa}(F^\flat) \to \mathrm{Spa}(F^\flat)$ denote the base change of $f_{d,h}$. Then the action of a on $f_{d,h,F^\flat,!}\mathbb{Z}_\ell$ induced by the addition on $\mathbb{B}^{\varphi^d=\varpi^h}$ is trivial.*

Proof. We may assume that $d = 1$ replacing E by the unramified extension of degree d (*cf.* [FF18, Remarque 4.2.2]). We proceed by induction on $h \geq 1$. For $h = 1$, the diamond $\mathbb{B}^{\varphi=\varpi} \times \mathrm{Spa}(\breve{E})^\diamond$ is isomorphic to $\mathrm{Spa}(\mathbb{F}_q[[x^{1/p^\infty}]]) \times \mathrm{Spa}(\breve{E})^\diamond$ by [Far16, 1.5.3]. The action of ϖ on $\mathrm{Spa}(\mathbb{F}_q[[x^{1/p^\infty}]]) \times \mathrm{Spa}(\breve{E})^\diamond$ is induced from the morphism

$$\mathrm{Spa}(\mathbb{F}_q[[x^{1/q^m}]]) \to \mathrm{Spa}(\mathbb{F}_q[[x^{1/q^m}]]);\ x^{1/q^m} \mapsto x^{1/q^{m-1}}$$

of degree q by taking limit with respect to $m \geq 0$. On the other hand, the action of $\mathcal{O}_E^\times$ on $\mathrm{Spa}(\mathbb{F}_q[[x^{1/p^\infty}]]) \times \mathrm{Spa}(\breve{E})^\diamond$ is induced from an isomorphism on $\mathrm{Spa}(\mathbb{F}_q[[x^{1/q^m}]])$ by taking limit with respect to $m \geq 0$. Further the addition of $a \in \mathrm{Spa}(\mathbb{F}_q[[x^{1/p^\infty}]])(F^\flat)$ on $\mathrm{Spa}(F^\flat[[x^{1/p^\infty}]])$ is induced from an isomorphism on $\mathrm{Spa}(\mathbb{F}_q[[x^{1/q^m}]])$ by taking limit with respect to $m \geq 0$. Hence the claims hold for $h = 1$ by [Ima19, Lemma 1.3].

Assume that the result is true for $\mathbb{B}^{\varphi=\varpi^{h-1}}$. We have an exact sequence

$$0 \longrightarrow \mathbb{B}^{\varphi=\varpi^{h-1}} \times \mathrm{Spa}(\breve{E})^\diamond \longrightarrow \mathbb{B}^{\varphi=\varpi^h} \times \mathrm{Spa}(\breve{E})^\diamond \longrightarrow \mathbb{A}^{1,\diamond}_{\breve{E}} \longrightarrow 0 \tag{4.2}$$

of diamonds which splits pro-etale locally on $\mathbb{A}^{1,\diamond}_{\breve{E}}$ as in [SW20, Example 15.2.9 (4)]. Therefore $\mathbb{B}^{\varphi=\varpi^h} \times \mathrm{Spa}(\breve{E})^\diamond$ satisfies the claims (1) and (2), since $\mathbb{A}^{1,\diamond}_{\breve{E}}$ is an ℓ-cohomologically smooth ℓ-contractible diamond of pure dimension 1 over $\mathrm{Spa}(\breve{E})^\diamond$ and the action of $c \in E^\times$ on $\mathbb{A}^{1,\diamond}_{\breve{E}}$ is induced from the isomorphism $\mathbb{A}^1_{\breve{E}} \to \mathbb{A}^1_{\breve{E}};\ x \mapsto cx$.

The action of $a \in \mathbb{B}^{\varphi=\varpi^h}(F^\flat)$ on $f_{d,h,F^\flat,!}\mathbb{Z}_\ell$ depends only on the image $\overline{a} \in \mathbb{A}^{1,\diamond}_{\breve{E}}(F^\flat)$ of a under (4.2) since the claim (3) is true for $\mathbb{B}^{\varphi=\varpi^{h-1}}$. Hence it suffices to show that the action of $\overline{a}$ on $f_{\mathbb{A},!}\mathbb{Z}_\ell$ is trivial, where $f_{\mathbb{A}}\colon \mathbb{A}^{1,\diamond}_F \to \mathrm{Spa}(F^\flat)$ is the natural morphism. This follows from that the addition by $\overline{a}$ on $\mathbb{A}^{1,\diamond}_F$ is induced from an automorphism on $\mathbb{A}^1_F$ by [SW20, Proposition 10.2.3]. □

Let $\delta_P\colon P(E) \to \overline{\mathbb{Q}}_\ell^\times$ be the modulus character of $P(E)$. Let A_b be the split center of J_b. Since J_b is an inner form of L^b, we can view A_b as an algebraic subgroup of L^b. We put $\delta_{P,A_b} = \delta_P|_{A_b(E)}$. Let $g \in J_b(E)$ act on $\widetilde{J}^U_b$ by the conjugation right action $u \mapsto g^{-1}ug$.

Lemma 4.18. *Let $f_J\colon \widetilde{J}^U_b \times \mathrm{Spa}(\breve{E})^\diamond \to \mathrm{Spa}(\breve{E})^\diamond$ be the natural morphism.*

(1) *The functor $\widetilde{J}^U_b \times \mathrm{Spa}(\breve{E})^\diamond$ is an ℓ-cohomologically smooth ℓ-contractible diamond of pure dimension $N_{U,b}$ over $\mathrm{Spa}(\breve{E})^\diamond$.*

(2) *Let $\kappa\colon J_b(E) \to \overline{\mathbb{Q}}_\ell^\times$ be the character of the action of $J_b(E)$ on $f_{J,!}\overline{\mathbb{Q}}_\ell$ induced by the conjugation right action of $J_b(E)$ on $\widetilde{J}^U_b$. Then we have $\kappa|_{A_b(E)} = \delta^{-1}_{P,A_b}$.*

(3) *Let F be a perfectoid field over $\breve{E}$. Then the action of $\widetilde{J}^U_b(F^\flat)$ on $f_{J,!}\overline{\mathbb{Q}}_\ell$ induced by the addition on $\widetilde{J}^U_b$ is trivial.*

Proof. For $i \geq 0$, we define an algebraic subgroup U_i of P by

$$U_i(R) = \left\{ g \in P(R) \;\middle|\; g|_{\mathrm{Fil}^j_\lambda V_R} \equiv \mathrm{id}_{\mathrm{Fil}^j_\lambda V_R} \mod \mathrm{Fil}^{j+i+1}_\lambda V_R \text{ for all } j \text{ and } V \in \mathrm{Rep}\, G \right\}$$

for any E-algebra R, where $V_R = V \otimes_E R$. Then $U_0 = U$, and U_i are normal in P for all i. Similarly, we define a subsheaf $\widetilde{J}^U_{b,i}$ of $\widetilde{J}_b$ for $i \geq 0$ by

$$\widetilde{J}^U_{b,i}(S) = \left\{ g \in \widetilde{J}_b(S) \;\middle|\; g|_{\mathrm{Fil}^j_\lambda \mathscr{E}_b} \equiv \mathrm{id}_{\mathrm{Fil}^j_\lambda \mathscr{E}_b} \mod \mathrm{Fil}^{j+i+1}_\lambda \mathscr{E}_b \text{ for all } j \right\}$$

for $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Then $\widetilde{J}^U_{b,0} = \widetilde{J}^U_b$. Let φ act on $G_{\breve{E}}$ and its subgroup $U_{i,\breve{E}}$ by $g \mapsto b_0\sigma(g)b_0^{-1}$. Let S be a perfectoid space over $\mathrm{Spa}(\breve{E})^\diamond$. By the internal definition of a G-torsor on the Fargues–Fontaine curve, we see that $\widetilde{J}^U_{b,i}(S)$ is equal to the sections of

$$Y_S \times_\varphi U_{i,\breve{E}} \longrightarrow X_S.$$

Hence, $(\widetilde{J}^U_{b,i}/\widetilde{J}^U_{b,i+1})(S)$ is equal to the sections of

$$Y_S \times_\varphi (U_{i,\breve{E}}/U_{i+1,\breve{E}}) \longrightarrow X_S.$$

Let L act on U_i by the conjugation. Let $\mathrm{Lie}\, G$ be the adjoint representation of G. Then the action of L on $\mathrm{Lie}\, G$ induces an action of L on $\mathrm{Lie}\, U_i/U_{i+1}$. We have an isomorphism

$$U_i/U_{i+1} \simeq \mathrm{Lie}(U_i/U_{i+1})$$

as representations of L, since U_i/U_{i+1} isomorphic to $\mathbb{G}_a^{d_i}$ for some d_i as linear algebraic groups. We have the equality

$$\mathrm{Lie}\, U_i = \mathrm{Fil}^i_\lambda \mathrm{Lie}\, G$$

by the definition of the both sides. Hence we have an isomorphism

$$\mathrm{Lie}(U_i/U_{i+1}) \simeq \mathrm{gr}^i_\lambda \mathrm{Lie}\, G$$

as representations of L. As a result we have an isomorphsim

$$U_i/U_{i+1} \simeq \mathrm{gr}^i_\lambda \mathrm{Lie}\, G \tag{4.3}$$

as representations of L. The element $b_0 \in L$ gives an L-bundle $\mathscr{E}_{b_0,S} \colon \mathrm{Rep}\, L \to \mathrm{Bun}_{X_S}$. Then we have

$$Y_S \times_\varphi (U_{i,\breve{E}}/U_{i+1,\breve{E}}) \simeq \mathscr{E}_{b_0,S}(\mathrm{gr}^i_\lambda \mathrm{Lie}\, G)$$

by (4.3). Hence, $(\widetilde{J}^U_{b,i}/\widetilde{J}^U_{b,i+1})(S)$ is equal to the sections of

$$\mathscr{E}_{b_0,S}(\mathrm{gr}^i_\lambda \mathrm{Lie}\, G) \longrightarrow X_S.$$

Then $\mathbb{D}$ acts on $\mathrm{gr}^i_\lambda \mathrm{Lie}\, G$ via ν_b and the conjugation. This action gives a slope decomposition

$$\mathrm{gr}^i_\lambda \mathrm{Lie}\, G = \bigoplus_{1 \le j \le m_i} V_{-\alpha_{i,j}}$$

where $\alpha_{i,j}$ are positive rational numbers, since L contains the centralizer L^b of ν_b. Then we have an isomorphism

$$\mathscr{E}_{b_0}(\mathrm{gr}^i_\lambda \mathrm{Lie}\, G) \simeq \bigoplus_{1 \le j \le m_i} \mathscr{O}(\alpha_{i,j}). \tag{4.4}$$

Hence $(\widetilde{J}^U_{b,i}/\widetilde{J}^U_{b,i+1}) \times \mathrm{Spa}(\breve{E})^\diamond$ is an ℓ-cohomologically smooth ℓ-contractible diamond by (4.4) and Lemma 4.17.

We show that $\widetilde{J}^U_{b,i} \times \mathrm{Spa}(\breve{E})^\diamond$ is an ℓ-cohomologically smooth ℓ-contractible diamond by a decreasing induction on i. The claim is trivial for enough large i, since $\widetilde{J}^U_{b,i} \times \mathrm{Spa}(\breve{E})^\diamond$ is one point for such i. We see that $U_{i,\breve{E}}$ is isomorphic to $U_{i+1,\breve{E}} \times (U_{i,\breve{E}}/U_{i+1,\breve{E}})$ as schemes over $U_{i,\breve{E}}/U_{i+1,\breve{E}}$ with actions of φ by [SGA70, XXVI Proposition 2.1] and its proof. Hence, $\widetilde{J}^U_{b,i} \times \mathrm{Spa}(\breve{E})^\diamond$ is isomorphic to $\widetilde{J}^U_{b,i+1} \times (\widetilde{J}^U_{b,i}/\widetilde{J}^U_{b,i+1}) \times \mathrm{Spa}(\breve{E})^\diamond$ as diamonds over $(\widetilde{J}^U_{b,i}/\widetilde{J}^U_{b,i+1}) \times \mathrm{Spa}(\breve{E})^\diamond$. Therefore, we see that $\widetilde{J}^U_{b,i} \times \mathrm{Spa}(\breve{E})^\diamond \to (\widetilde{J}^U_{b,i}/\widetilde{J}^U_{b,i+1}) \times \mathrm{Spa}(\breve{E})^\diamond$ is an ℓ-cohomologically smooth morphsm with ℓ-contractible geometric fiber, since $\widetilde{J}^U_{b,i+1} \times \mathrm{Spa}(\breve{E})^\diamond$ is an ℓ-cohomologically smooth ℓ-contractible diamond by our induction hypothesis. Then we see that $\widetilde{J}^U_{b,i} \times \mathrm{Spa}(\breve{E})^\diamond$ is an

ℓ-cohomologically smooth ℓ-contractible diamond, since we know that $(\widetilde{J}^U_{b,i}/\widetilde{J}^U_{b,i+1}) \times \mathrm{Spa}(\breve{E})^\diamond$ is an ℓ-cohomologically smooth ℓ-contractible diamond. The claim on the dimension follows from the above arguments. The claim (2) follows from the arguments above, Lemma 4.17 (2) and a calculation of δ_P (*cf.* [Ren10, V.5.4]). The claim (3) follows from Lemma 4.17 (3) by induction on i for $\widetilde{J}^U_{b,i}$ in the same way as the proof of Lemma 4.17 (3). □

Remark 4.19. *Some integral version of $\widetilde{J}_b$ is studied in [CS17, Proposition 4.2.11].*

Let $X_*(T)^{L+}$ be the set of L-dominant cocharacters in $X_*(T)$. We put

$$I_{b_0,b'_0,\mu,L} = \left\{ [\mu'] \in X_*(T)^{L+}/\Gamma \;\middle|\; \mu' \text{ is } G\text{-conjugate to } \mu \text{ and } [b_0] \in B(L,\mu',[b'_0]) \right\}.$$

We claim the set $I_{b_0,b'_0,\mu,L}$ consists of a single element. To prove this we begin with a preliminary lemma.

Lemma 4.20. *Given two cocharacters $\mu, \mu' \in X_*(T)$ which are G-conjugate, then there exists an element w of the absolute Weyl group of T in G such that $w \cdot \mu = \mu'$.*

Proof. Let L_μ be the centralizer of the cocharacter $\mathbb{G}_m \xrightarrow{\mu} T \to G$ and define similarly $L_{\mu'}$. Then, since $\mu' = g\mu g^{-1}$ for some $g \in G(\overline{E})$, it follows that $L_{\mu'} = gL_\mu g^{-1}$. Since $gTg^{-1} \subseteq L_{\mu'}$ is a maximal torus, there exists $l \in L_{\mu'}$ such that $gTg^{-1} = lTl^{-1}$. This means that $l^{-1}g$ normalizes T and gives an element w in the absolute Weyl group of T in G. Then we have $w \cdot \mu = \mu'$. □

Lemma 4.21. *$I_{b_0,b'_0,\mu,L}$ consists of a single element.*

Proof. By the definition of Hodge–Newton reducibility, we have $[\mu] \in I_{b_0,b'_0,\mu,L}$. Let $[\mu'] \in I_{b_0,b'_0,\mu,L}$ be another element. Let $\Delta(G,T)$ be the set of simple roots of G with respect to T, where the positivity of roots is given by B. Since μ is G-dominant, μ' is G-conjugate to μ and $\mu \neq \mu'$, we have that μ' is not G-dominant and

$$\mu - \mu' = \sum_{\alpha \in \Delta(G,T)} n_\alpha \alpha^\vee, \tag{4.5}$$

where $n_\alpha \geq 0$ by Lemma 4.20, [Hum78, 10.3 Lemma B] and [Bou81, VI §1 Proposition 18]. Since μ' is not G-dominant, but L-dominant, there is $\alpha_0 \in \Delta(G,T) \setminus \Delta(L,T)$ such that $\langle \mu', \alpha_0 \rangle < 0$. Then we have

$$\langle \mu - \mu', \alpha_0 \rangle > 0. \tag{4.6}$$

Substituting (4.5) to (4.6), we have

$$\sum_{\alpha \in \Delta(G,T)} n_\alpha \langle \alpha^\vee, \alpha_0 \rangle > 0.$$

This implies $n_{\alpha_0} > 0$, since we have $\langle \alpha^\vee, \alpha_0 \rangle \leq 0$ for $\alpha \neq \alpha_0$ by [Hum78, 10.1 Lemma]. Recall that

$$\pi_1(L) = X_*(T) \big/ \textstyle\sum_{\alpha \in \Delta(L,T)} \mathbb{Z}\alpha^\vee, \tag{4.7}$$

by the proof of [Bor98, Proposition 1.10] (*cf.* [RR96, §1.13]). Let $\overline{\mu}^\natural$ and $\overline{\mu'}^\natural$ be the images in $\pi_1(L)^\Gamma_\mathbb{Q}$ of $\overline{\mu}$ and $\overline{\mu'}$ in $X_*(T)^\Gamma_\mathbb{Q}$.

We show that $\overline{\mu}^\natural \neq \overline{\mu'}^\natural$. We write

$$\overline{\mu} - \overline{\mu'} = \sum_{\alpha \in \Delta(G,T)} m_\alpha \alpha^\vee,$$

where $m_\alpha \in \mathbb{Q}$. Then the equation

$$\overline{\mu} - \overline{\mu'} = [\Gamma : \Gamma_\mu \cap \Gamma_{\mu'}]^{-1} \left((\mu - \mu') + \sum_{1 \neq \tau \in \Gamma/(\Gamma_\mu \cap \Gamma_{\mu'})} \tau(\mu - \mu') \right)$$

implies $m_{\alpha_0} > 0$, since $n_{\alpha_0} > 0$ and $n_\alpha \geq 0$ for all $\alpha \in \Delta(G,T)$. Thus when passing to $\pi_1(L)^\Gamma$ the term $\alpha_0^\vee$ is not killed according to (4.7) and so $\overline{\mu}^\natural \neq \overline{\mu'}^\natural$ as claimed. This implies

$$\mu^\natural \neq \mu'^\natural \in \pi_1(L)_\Gamma,$$

since $\overline{\mu}^\natural$ and $\overline{\mu'}^\natural$ are images of $\mu^\natural$ and $\mu'^\natural$ under the map

$$\pi_1(L)_\Gamma \to \pi_1(L)^\Gamma_{\mathbb{Q}};\ [g] \mapsto \frac{1}{[\Gamma : \Gamma_g]} \sum_{\tau \in \Gamma/\Gamma_g} \tau(g),$$

where $g \in \pi_1(L)$ and Γ_g is the stabilizer of g in Γ. This contradicts that $[\mu'] \in I_{b_0, b_0', \mu, L}$, because we have

$$\mu'^\natural = \kappa_L([b_0]) - \kappa_L([b_0']) = \mu^\natural \in \pi_1(L)_\Gamma$$

by $[b_0] \in B(L, \mu', [b_0'])$ and $[b_0] \in B(L, \mu, [b_0'])$. □

Definition 4.22. *Let R be a DVR with uniformizer π, and quotient field F. Let $k_1 \geq \cdots \geq k_n$ be a sequence of integers. We say that the type of $g \in \mathrm{GL}_n(F)$ is $(k_1, \ldots, k_n)$ if we have*

$$g \in \mathrm{GL}_n(R) \begin{pmatrix} \pi^{k_1} & & \\ & \ddots & \\ & & \pi^{k_n} \end{pmatrix} \mathrm{GL}_n(R).$$

Lemma 4.23. *Let R be a DVR with uniformizer π, and quotient field F. We consider the subgroups*

$$L = \begin{pmatrix} \mathrm{GL}_{n_1} & & \\ & \ddots & \\ & & \mathrm{GL}_{n_m} \end{pmatrix} \subset P = \begin{pmatrix} \mathrm{GL}_{n_1} & & 0 \\ & \ddots & \\ * & & \mathrm{GL}_{n_m} \end{pmatrix} \subset \mathrm{GL}_n$$

of GL_n. Let $g \in P(F)$, and g_L be the image of g in the Levi quotient. We regard g_L as an element of $L(F)$. We put $N_l = n_1 + \cdots + n_l$ for $0 \leq l \leq m$.

Let $k_1 \geq \cdots \geq k_n$ be a sequence of integers. Assume that the type of

$$(g_{ij})_{N_l+1 \leq i,j \leq n} \in \mathrm{GL}_{n-N_l}(F)$$

is $(k_{N_l+1}, \ldots, k_n)$ for $0 \leq l \leq m-1$. Then we have $g_L^{-1} g \in P(R)$.

Proof. By multiplying a power of π to g, we may assume that $k_n \geq 0$. By the assumption, we see that the type of

$$(g_{ij})_{N_l+1 \leq i,j \leq N_{l+1}} \in \mathrm{GL}_{n_{l+1}}(F)$$

is $(k_{N_l+1}, \ldots, k_{N_{l+1}})$ for $0 \leq l \leq m-1$ using Lemma 4.6. Hence, we may assume that $g_L = \mathrm{diag}(\pi^{k_1}, \ldots, \pi^{k_n})$.

Let v be a normalized valuation of F. Then, it suffices to show that $v(g_{ij}) \geq k_i$ for all $1 \leq j < i \leq n$. Assume it does not hold, and take the biggest i_0 such that there is $j_0 < i_0$ satisfying $v(g_{i_0 j_0}) < k_{i_0}$. Then the type of

$$(g_{ij})_{i_0+1 \leq i,j \leq n} \in \mathrm{GL}_{n-i_0}(F)$$

is $(k_{i_0+1}, \ldots, k_n)$. Using this and Lemma 4.6, we can show that the type of

$$(g_{ij})_{1 \leq i,j \leq i_0} \in \mathrm{GL}_{i_0}(F)$$

is $(k_1, \ldots, k_{i_0})$. This implies that $v(g_{ij}) \geq k_{i_0}$ for all $1 \leq i,j \leq i_0$. This contradicts the choice of i_0. □

In the sequel, we simply write (D, f) for

$$(\mathscr{E}_b, \mathscr{E}_{b'}, D, f, \mathrm{id}_{\mathscr{E}_b}, \mathrm{id}_{\mathscr{E}_{b'}}) \in \mathcal{M}^{\mu}_{b,b'}(R, R^+).$$

Every point of $\mathcal{M}^{\mu}_{b,b'}(R, R^+)$ is represented by a datum of the above form, since we have an isomorphism of data

$$(\mathscr{E}, \mathscr{E}', D, f, \phi, \phi') \simeq (\mathscr{E}_b, \mathscr{E}_{b'}, D, \phi'^{-1} \circ f \circ \phi, \mathrm{id}_{\mathscr{E}_b}, \mathrm{id}_{\mathscr{E}_{b'}})$$

for

$$(\mathscr{E}, \mathscr{E}', D, f, \phi, \phi') \in \mathcal{M}^{\mu}_{b,b'}(R, R^+).$$

We define a morphism

$$\Phi\colon \mathcal{M}^{\mu}_{b_0,b'_0} \times \widetilde{J}^U_b \longrightarrow \mathcal{P}^{\mu}_{b,b'}$$

by sending

$$\big((D, f_L), g\big) \in \Big(\mathcal{M}^{\mu}_{b_0,b'_0} \times \widetilde{J}^U_b\Big)(R, R^+)$$

to

$$\big(D, (f_L \times^L P) \circ g\big) \in \mathcal{P}^{\mu}_{b,b'}(R, R^+)$$

for a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra (R, R^+).

Proposition 4.24. *The morphism*

$$\Phi\colon \mathcal{M}^{\mu}_{b_0,b'_0} \times \widetilde{J}^U_b \longrightarrow \mathcal{P}^{\mu}_{b,b'}$$

is an isomorphism.

Proof. Let (R, R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra, and

$$\big((D, f_L), g\big) \in \Big(\mathcal{M}^{\mu}_{b_0,b'_0} \times \widetilde{J}^U_b\Big)(R, R^+).$$

Then we have $\Phi\big((D, f_L), g\big) \times^P L = (D, f_L)$. Further, (D, f_L) and $\Phi\big((D, f_L), g\big)$ recover g. Hence, we have the injectivity of Φ.

Let

$$(D, f) \in \mathcal{P}^{\mu}_{b,b'}(R, R^+).$$

By the definition of $\mathcal{P}^{\mu}_{b,b'}$, we have a reduction

$$f_P\colon (\mathscr{E}_{b_0} \times^L P)|_{X_R^{\mathrm{sch}} \setminus D} \xrightarrow{\sim} (\mathscr{E}_{b'_0} \times^L P)|_{X_R^{\mathrm{sch}} \setminus D}$$

of f to P. We put $f_L = f_P \times^P L$.

We show that

$$(f_L \times^L P)^{-1} \circ f_P \in \widetilde{J}^U_b(R, R^+). \tag{4.8}$$

For this, it suffices to show (4.8) after taking realizations for all $V \in \mathrm{Rep}_G$. Hence, we may assume that $G = \mathrm{GL}_n$.

We view GL_n-bundles as vector bundles. We take the diagonal torus and the upper half Borel subgroup as T and B. Then we have

$$L = \begin{pmatrix} \mathrm{GL}_{n_1} & & \\ & \ddots & \\ & & \mathrm{GL}_{n_m} \end{pmatrix} \subset P = \begin{pmatrix} \mathrm{GL}_{n_1} & & 0 \\ & \ddots & \\ * & & \mathrm{GL}_{n_m} \end{pmatrix} \subset \mathrm{GL}_n .$$

We write

$$b_0 = (b_1, \dots, b_m),\ b'_0 = (b'_1, \dots, b'_m) \in \mathrm{GL}_{n_1}(\breve{E}) \times \cdots \mathrm{GL}_{n_m}(\breve{E}).$$

Then we have a decomposition

$$\mathscr{E}_b = \bigoplus_{1\leq i\leq m} \mathscr{E}_{b_i}, \quad \mathscr{E}_{b'} = \bigoplus_{1\leq i\leq m} \mathscr{E}_{b'_i}$$

as vector bundles. We put

$$\mathrm{Fil}^j \mathscr{E}_b = \bigoplus_{j\leq i\leq m} \mathscr{E}_{b_i}, \quad \mathrm{Fil}^j \mathscr{E}_{b'} = \bigoplus_{j\leq i\leq m} \mathscr{E}_{b'_i}$$

for $1 \leq j \leq m+1$. Then $f\colon \mathscr{E}_b|_{X_R^{\mathrm{sch}}\setminus D} \to \mathscr{E}_{b'}|_{X_R^{\mathrm{sch}}\setminus D}$ respects these filtrations. We can write

$$f = \bigoplus_{1\leq i\leq j\leq m} f_{ij}\colon \mathscr{E}_b|_{X_R^{\mathrm{sch}}\setminus D} \longrightarrow \mathscr{E}_{b'}|_{X_R^{\mathrm{sch}}\setminus D},$$

where $f_{ij}\colon \mathscr{E}_{b_i}|_{X_R^{\mathrm{sch}}\setminus D} \to \mathscr{E}_{b'_j}|_{X_R^{\mathrm{sch}}\setminus D}$. Then the morphism

$$f_{jj}^{-1} \circ f_{ij}\colon \mathscr{E}_{b_i}|_{X_R^{\mathrm{sch}}\setminus D} \longrightarrow \mathscr{E}_{b_j}|_{X_R^{\mathrm{sch}}\setminus D}$$

extends to a morphism $\mathscr{E}_{b_i} \to \mathscr{E}_{b_j}$ by Lemma 4.23. Hence we have (4.8) (*cf.* the proof of [Han21, Theorem 4.1]).

It remains to show that $(D, f_L) \in \mathcal{M}^{\mu}_{b_0,b'_0}(R,R^+)$. It suffices to show that the type of the modification f_L is equal to μ geometric fiberwisely. Let μ' be the type of f_L at a geometric point of $\mathrm{Spa}(R,R^+)$. The type of $f_L \times^L G$ is equal to μ by (4.8). Hence, we have $\mu' = \mu$ by Lemma 4.21. □

For a diamond $\mathcal{D}$ over $\mathrm{Spa}(\breve{E})^{\diamond}$, let $\mathcal{D}_{\mathbb{C}_p^{\flat}}$ denote $\mathcal{D} \times_{\mathrm{Spa}(\breve{E})^{\diamond}} \mathrm{Spa}\,\mathbb{C}_p^{\flat}$. Let $\kappa\colon J_b(E) \to \overline{\mathbb{Q}}_\ell^{\times}$ be the character in Lemma 4.18.

Lemma 4.25. *We have an isomorphism*

$$H^i_{\mathrm{c}}\big(\mathcal{M}^{\mu}_{b_0,b'_0,\mathbb{C}_p^{\flat}}, \overline{\mathbb{Q}}_\ell\big) \otimes \kappa \xrightarrow{\ \sim\ } H^{i+2N_{U,b}}_{\mathrm{c}}\big(\mathcal{P}^{\mu}_{b,b',\mathbb{C}_p^{\flat}}, \overline{\mathbb{Q}}_\ell\big)$$

as representations of $J_b(E) \times L_{b'}(E)$.

Proof. This follows from Lemma 4.18 and Proposition 4.24. □

Theorem 4.26. *Assume that $([b],[b'],\mu)$ is Hodge–Newton reducible for L. Then we have an isomorphism*

$$H^{i+2N_{U,b}}_{\mathrm{c}}\big(\mathcal{M}^{\mu}_{b,b',\mathbb{C}_p^{\flat}}, \overline{\mathbb{Q}}_\ell\big) \simeq \mathrm{Ind}_{P_{b'}(E)}^{J_{b'}(E)} H^i_{\mathrm{c}}\big(\mathcal{M}^{\mu}_{b_0,b'_0,\mathbb{C}_p^{\flat}}, \overline{\mathbb{Q}}_\ell\big) \otimes \kappa$$

as $J_b(E) \times J_{b'}(E)$-representations.

Proof. This follows from Corollary 4.14 and Lemma 4.25. □

Lemma 4.27. *Let (R,R^+) be a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra. Let*

$$(\mathscr{E}, \mathscr{E}', D, f, \phi, \phi') \in \mathcal{M}^{\mu}_{b,b'}(R,R^+).$$

For any $g \in \underline{U_{b'}(E)}(R,R^+)$, there exists $h \in \widetilde{J}^U_b(R,R^+)$ such that $g \circ f' = f' \circ h$, where we put

$$f' = \phi'^{-1} \circ f \circ \phi\colon \mathscr{E}_b|_{X_R^{\mathrm{sch}}\setminus D} \to \mathscr{E}_{b'}|_{X_R^{\mathrm{sch}}\setminus D}.$$

Proof. Let $j\colon X_R^{\mathrm{sch}} \setminus D \to X_R^{\mathrm{sch}}$ be the open immersion. Let $V \in \operatorname{Rep} G$. We have an embedding

$$\mathscr{E}_b(V) \hookrightarrow j_*j^*\mathscr{E}_b(V) \xrightarrow{\sim} j_*j^*\mathscr{E}_{b'}(V),$$

where the second isomorphism is induced by f'. We have an action of g on $j_*j^*\mathscr{E}_{b'}(V)$. It suffices to show that g stabilizes $\operatorname{Fil}^i_\lambda \mathscr{E}_b(V)$ and induces the identity on $\operatorname{gr}^i_\lambda \mathscr{E}_b(V)$ for all i.

We show this claim by a decreasing induction on i. For enough large i, we have $\operatorname{Fil}^i_\lambda \mathscr{E}_b(V) = 0$ and the claim is trivial for such i. Assume that the claim is true for $i+1$. We have the natural embedding

$$\operatorname{gr}^i_\lambda \mathscr{E}_b(V) \hookrightarrow j_*j^* \operatorname{gr}^i_\lambda \mathscr{E}_b(V) \xrightarrow{\sim} j_*j^* \operatorname{gr}^i_\lambda \mathscr{E}_{b'}(V)$$

where the second isomorphism is induced by f'. We have a commutative diagram

$$\begin{array}{ccc} \operatorname{gr}^i_\lambda \mathscr{E}_b(V) & \hookrightarrow & j_*j^* \operatorname{gr}^i_\lambda \mathscr{E}_{b'}(V) \\ \downarrow g & & \downarrow j_*j^* \operatorname{gr}^i_\lambda g \\ \big(g \operatorname{Fil}^i_\lambda \mathscr{E}_b(V)\big)/\operatorname{Fil}^{i+1}_\lambda \mathscr{E}_b(V) & \hookrightarrow & j_*j^* \operatorname{gr}^i_\lambda \mathscr{E}_{b'}(V), \end{array}$$

where the bottom morphism is induced by the natural inclusion

$$g \operatorname{Fil}^i_\lambda \mathscr{E}_b(V) \subset g\big(j_*j^* \operatorname{Fil}^i_\lambda \mathscr{E}_{b'}(V)\big) = j_*j^* \operatorname{Fil}^i_\lambda \mathscr{E}_{b'}(V).$$

By this diagram, we see that $g \operatorname{Fil}^i_\lambda \mathscr{E}_b(V) = \operatorname{Fil}^i_\lambda \mathscr{E}_b(V)$, since $\operatorname{gr}^i_\lambda g$ is the identity on $\operatorname{gr}^i_\lambda \mathscr{E}_{b'}(V)$. Hence, g stabilizes $\operatorname{Fil}^i_\lambda \mathscr{E}_b(V)$. Further, g induces the identity on $\operatorname{gr}^i_\lambda \mathscr{E}_b(V)$ again by the above diagram, since $\operatorname{gr}^i_\lambda g$ is the identity. □

Lemma 4.28. *The action of $U_{b'}(E)$ on $H^i_{\mathrm{c}}(\mathcal{P}^\mu_{b,b',\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell)$ is trivial.*

Proof. Let $p_{\mathcal{M}}\colon \mathcal{P}^\mu_{b,b'} \cong \mathcal{M}^\mu_{b_0,b'_0} \times \widetilde{J}^U_b \to \mathcal{M}^\mu_{b_0,b'_0}$ be the projection, where the first isomorphsim is given by Proposition 4.24. It suffices to show that the action of $U_{b'}(E)$ on $p_{\mathcal{M},!}\overline{\mathbb{Q}}_\ell$ is trivial. It suffices to show this after the pullback to each geometric point of $\mathcal{M}^\mu_{b_0,b'_0}$. It follows from Lemma 4.18 (3) and Lemma 4.27. □

Proposition 4.29. *Let π be a smooth representation of $J_{b'}(E)$. Assume that $([b],[b'],\mu)$ is Hodge–Newton reducible for L and that the Jacquet module of π with respect to $P_{b'}$ vanishes. Then we have*

$$\operatorname{Hom}_{J_{b'}(E)}\Big(\pi, H^i_{\mathrm{c}}\big(\mathcal{M}^\mu_{b,b',\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\big)\Big) = 0.$$

Proof. This follows from Theorem 4.26 and Lemma 4.28. □

We define $t_{b,b'}\colon \mathcal{T}^\mu_{b,b',\mathbb{C}_p^\flat} \to [\operatorname{Spa}(\overline{\mathbb{F}}_q)/J_{b'}(E)]$ as the composites

$$\mathcal{T}^\mu_{b,b',\mathbb{C}_p^\flat} \longrightarrow \mathcal{T}^\mu_{b,b'} \longrightarrow \mathrm{Hecke}^\mu_{b,b'} \longrightarrow [\operatorname{Spa}(\overline{\mathbb{F}}_q)/J_{b'}(E)].$$

We put $\overleftarrow{t}_{b,b'} = x_{b'} \circ t_{b,b'}$.

Theorem 4.30. *Assume that b is not basic and $([b],[b'],\mu)$ is Hodge–Newton reducible for L. Then we have*

$$H^i_{\mathrm{c}}\big(\mathcal{T}^\mu_{b,b',\mathbb{C}_p^\flat}, \overleftarrow{t}^{\,*}_{b,b'}\mathscr{F}_\varphi\big) = 0.$$

Proof. We have

$$\overleftarrow{t}^{*}_{b,b'}\mathscr{F}_{\varphi}=t^{*}_{b,b'}x^{*}_{b'}\mathscr{F}_{\varphi}=t^{*}_{b,b'}\left(\bigoplus_{\rho\in\widehat{S}_{\varphi},\,\rho|_{Z(\widehat{G})^{\Gamma}}=\kappa(b')}\underline{\rho}\otimes\underline{\pi_{\varphi,b',\rho}}\right)$$

by (3.1). We take $\rho\in\widehat{S}_{\varphi}$ such that $\rho|_{Z(\widehat{G})^{\Gamma}}=\kappa(b')$. Then it suffices to show that

$$H^{i}_{\mathrm{c}}\Big(\mathcal{T}^{\mu}_{b,b',\mathbb{C}_p^{\flat}},t^{*}_{b,b'}\underline{\pi_{\varphi,b',\rho}}\Big)=0.$$

The pullback of $\underline{\pi_{\varphi,b',\rho}}$ to $\mathcal{M}^{\mu}_{b,b'}$ is a constant sheaf, since the map $\mathcal{M}^{\mu}_{b,b'}\to[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{J_{b'}(E)}]$ factorizes via $\mathrm{Spa}(\overline{\mathbb{F}}_q)$. Hence, there is a Hochschild–Serre spectral sequence

$$H_i\Big(J_{b'}(E),H^{j}_{\mathrm{c}}\big(\mathcal{M}^{\mu}_{b,b',\mathbb{C}_p^{\flat}},\overline{\mathbb{Q}}_{\ell}\big)\otimes\pi_{\varphi,b',\rho}\Big)\Rightarrow H^{j-i}_{\mathrm{c}}\Big(\mathcal{T}^{\mu}_{b,b',\mathbb{C}_p^{\flat}},t^{*}_{b,b'}\underline{\pi_{\varphi,b',\rho}}\Big)$$

by (2.1). We show that

$$H_i\big(J_{b'}(E),H^{j}_{\mathrm{c}}\big(\mathcal{M}^{\mu}_{b,b',\mathbb{C}_p^{\flat}},\overline{\mathbb{Q}}_{\ell}\big)\otimes\pi_{\varphi,b',\rho}\big)=0$$

for all i and j. Take a projective resolution

$$\cdots\longrightarrow V_1\longrightarrow V_0\longrightarrow H^{j}_{\mathrm{c}}\big(\mathcal{P}^{\mu}_{b,b',\mathbb{C}_p^{\flat}},\overline{\mathbb{Q}}_{\ell}\big)$$

as smooth $L_{b'}(E)$-representations. By Lemma 4.25 and Theorem 4.26 we have

$$H^{j}_{\mathrm{c}}\big(\mathcal{M}^{\mu}_{b,b',\mathbb{C}_p^{\flat}},\overline{\mathbb{Q}}_{\ell}\big)\simeq\mathrm{Ind}_{P_{b'}(E)}^{J_{b'}(E)}H^{j}_{\mathrm{c}}\big(\mathcal{P}^{\mu}_{b,b',\mathbb{C}_p^{\flat}},\overline{\mathbb{Q}}_{\ell}\big)$$

as smooth $J_{b'}(E)$-representations. Moreover, the induction on the right-hand-side is parabolic by Lemma 4.28. Parabolic induction preserves projective objects, since it has a Jacquet functor as the right adjoint functor by Bernstein's second adjoint theorem (*cf.* [Bus01, Theorem 3]) and the Jacquet functor is exact. Note also that parabolic induction is exact. Thus we obtain the projective resolution

$$\cdots\longrightarrow\mathrm{Ind}_{P_{b'}(E)}^{J_{b'}(E)}V_1\longrightarrow\mathrm{Ind}_{P_{b'}(E)}^{J_{b'}(E)}V_0\longrightarrow H^{j}_{\mathrm{c}}\big(\mathcal{M}^{\mu}_{b,b',\mathbb{C}_p^{\flat}},\overline{\mathbb{Q}}_{\ell}\big)$$

as smooth $J_{b'}(E)$-representations. Finally the right adjoint of $-\otimes\pi_{\varphi,b',\rho}$ in the category of smooth $J_{b'}(E)$-representations is $-\otimes\pi^{*}_{\varphi,b',\rho}$, where $\pi^{*}_{\varphi,b',\rho}$ is the smooth dual of $\pi_{\varphi,b',\rho}$. Both functors are exact and so in particular $-\otimes\pi_{\varphi,b',\rho}$ preserves exact sequences and projective objects. Thus we obtain the projective resolution

$$\cdots\longrightarrow\mathrm{Ind}_{P_{b'}(E)}^{J_{b'}(E)}V_1\otimes\pi_{\varphi,b',\rho}\longrightarrow\mathrm{Ind}_{P_{b'}(E)}^{J_{b'}(E)}V_0\otimes\pi_{\varphi,b',\rho}\longrightarrow H^{j}_{\mathrm{c}}\big(\mathcal{M}^{\mu}_{b,b',\mathbb{C}_p^{\flat}},\overline{\mathbb{Q}}_{\ell}\big)\otimes\pi_{\varphi,b',\rho}$$

Note that $P_{b'}$ is a proper parabolic subgroup of $J_{b'}$, since b is not basic. For $i\geq0$, we have

$$\Big(\pi_{\varphi,b',\rho}\otimes\mathrm{Ind}_{P_{b'}(E)}^{J_{b'}(E)}V_i\Big)_{J_{b'}(E)}=0,$$

since $\pi_{\varphi,b',\rho}$ is cuspidal. Hence we have the claim. □

5 Non-abelian Lubin–Tate theory

Assume that $G = \mathrm{GL}_n$ and $\mu(z) = \mathrm{diag}(z, 1, \ldots, 1)$. In this case, S_φ is trivial and $\mathrm{Hecke}^{\leq\mu} = \mathrm{Hecke}^{\mu}$. We simply write $\pi_{\varphi,b}$ for $\pi_{\varphi,b,1}$ for any $[b] \in B(\mathrm{GL}_n)_{\mathrm{basic}}$. We put

$$b_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & \varpi \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathrm{GL}_n(E)$$

Then we have a bijection

$$\mathbb{Z} \xrightarrow{\sim} B(\mathrm{GL}_n)_{\mathrm{basic}};\ N \mapsto b_1^N.$$

The following proposition is a consequence of non-abelian Lubin–Tate theory.

Proposition 5.1. *We put $b = b_1^N$ for an integer N. Assume that $N \equiv 0, 1 \mod n$. Then we have*

$$y_b^* \overrightarrow{h}_\natural(\overleftarrow{h}^* \mathscr{F}_\varphi \otimes \mathrm{IC}'_\mu) = y_b^*(\mathscr{F}_\varphi \boxtimes \varphi).$$

Proof. We show the claim in the case where $N \equiv 1 \mod n$ using arguments in [MFO16, Chapter 23]. See arguments in [Far16, 8.1] for the case where $N \equiv 0 \mod n$. Since the natural morphism

$$[\mathrm{Spa}(\breve{E})^\diamond/\widetilde{J}_b] \longrightarrow [\mathrm{Div}^1_{X,\overline{\mathbb{F}}_q}/\widetilde{J}_b]$$

induces an equivalence of étale sites (*cf.* [MFO16, 22.3.2]), it suffices to show that

$$\tilde{y}_b^* \overrightarrow{h}_\natural(\overleftarrow{h}^* \mathscr{F}_\varphi \otimes \mathrm{IC}'_\mu) = \tilde{y}_b^*(\mathscr{F}_\varphi \boxtimes \varphi). \tag{5.1}$$

Suppose that $N = mn+1$ for some $m \in \mathbb{Z}$. The following lemma provides an explicit description of the stack $\mathrm{Hecke}_b^{\leq\mu}$.

Lemma 5.2. *Let $\mathrm{Spa}(F, F^+)$ be a geometric point in $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Let $\mathscr{E}$ be a vector bundle of rank n on X_F^{sch} having a degree one modification fiberwise by $\mathscr{E}_b$*

$$0 \to \mathscr{E}_b \to \mathscr{E} \to \mathscr{F} \to 0,$$

where $\mathscr{F}$ is a torsion coherent sheaf of length 1. Then $\mathscr{E}$ is isomorphic to $\mathcal{O}(-m)^n$.

Proof. This follows from [FF14, Theorem 2.94] by dualizing the modification and twisting by $\mathcal{O}(-m)$. □

We put $b' = b_1^{nm}$. Then, we have isomorphisms

$$\mathrm{Hecke}_{b,b'}^{\leq\mu} \simeq \mathrm{Hecke}_b^{\leq\mu}$$

by Lemma 5.2.

Lemma 5.3. *Let $\mathcal{M}_{\mathrm{LT}}^\infty$ be the Lubin–Tate space over $\breve{E}$ at infinite level. Then we have an isomorphism $\mathcal{M}_{b,b'}^{\leq\mu} \simeq \mathcal{M}_{\mathrm{LT}}^{\infty,\diamond}$, that is compatible with actions of $\mathrm{GL}_n(E) \times J_b(E)$ and Weil descent data.*

Proof. For a perfectoid affinoid $\overline{\mathbb{F}}_q$-algebra (R, R^+), the set $\mathcal{M}_{b,b'}^{\leq\mu}(R, R^+)$ consists of 6-tuples $(\mathscr{E}, \mathscr{E}', R^\sharp, f, \phi, \phi')$, where

- $(\mathscr{E}, \mathscr{E}', R^\sharp, f) \in \mathrm{Hecke}_{b(0)}^{\leq\mu}$

- $\phi\colon \mathscr{E}_b \xrightarrow{\sim} \mathscr{E}$ and $\phi'\colon \mathscr{E}_{b'} \xrightarrow{\sim} \mathscr{E}'$ are isomorphisms.

Hence, the claim follows from [SW13, Proposition 6.3.9] by dualizing the modification and twisting by $\mathcal{O}(-m)$. □

Let

$$p_b\colon \operatorname{Spa}\mathbb{C}_p^\flat \longrightarrow \operatorname{Spa}(\breve{E})^\diamond \longrightarrow [\operatorname{Spa}(\breve{E})^\diamond/\widetilde{J}_b] \tag{5.2}$$

be the natural projection. The equality (5.1) is equivalent to the equality

$$p_b^* \tilde{y}_b^* \overrightarrow{h}_\natural(\overleftarrow{h}^* \mathscr{F}_\varphi \otimes \mathrm{IC}'_\mu) = p_b^* \tilde{y}_b^* (\mathscr{F}_\varphi \boxtimes \varphi) \tag{5.3}$$

with action of $J_b(E) \times W_E$. Then the right hand side of (5.3) is $\pi_{\varphi,b} \otimes \varphi$ as a representation of $J_b(E) \times W_E$. Hence it suffices to show that the cohomology of the left hand side of (5.3) vanishes outside degree zero, and is equal to $\pi_{\varphi,b} \otimes \varphi$ in degree zero as representations of $J_b(E) \times W_E$.

We note that $\mathrm{IC}'_\mu = \overline{\mathbb{Q}}_\ell$ in this case. The i-th cohomology of the left hand side of (5.3) is equal to

$$H_{\mathrm{c}}^{i+n-1}\bigl(\mathcal{T}^{\leq\mu}_{b,b',\mathbb{C}_p^\flat}, \overleftarrow{t}^*_{b,b'} \mathscr{F}_\varphi\bigr)\Bigl(\frac{n-1}{2}\Bigr).$$

We have

$$\overleftarrow{t}^*_{b,b'} \mathscr{F}_\varphi = t^*_{b,b'} \underline{\pi_{\varphi,1}}$$

by (3.1), since $\pi_{\varphi,b'} = \pi_{\varphi,1}$ in our case. We have a Hochschild–Serre spectral sequence

$$H_i\Bigl(\mathrm{GL}_n(E), H_{\mathrm{c}}^j\bigl(\mathcal{M}^{\leq\mu}_{b,b',\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\bigr) \otimes \pi_{\varphi,1}\Bigr) \Rightarrow H_{\mathrm{c}}^{j-i}\Bigl(\mathcal{T}^{\leq\mu}_{b,b',\mathbb{C}_p^\flat}, t^*_{b,b'} \underline{\pi_{\varphi,1}}\Bigr)$$

by (2.1). We put

$$\mathrm{GL}_n(E)^0 = \{g \in \mathrm{GL}_n(E) \mid \det(g) \in \mathcal{O}_E^\times\}.$$

Then we have

$$H_{\mathrm{c}}^j\bigl(\mathcal{M}^{\infty,\diamond}_{\mathrm{LT},\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\bigr) = \operatorname{c-Ind}_{\mathrm{GL}_n(E)^0}^{\mathrm{GL}_n(E)} H_{\mathrm{c}}^j\bigl(\mathcal{M}^{\infty,(0),\diamond}_{\mathrm{LT},\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\bigr)$$

for a connected component $\mathcal{M}^{\infty,(0)}_{\mathrm{LT}}$ of $\mathcal{M}^\infty_{\mathrm{LT}}$ (*cf.* [Far04, 4.4.2]). By Lemma 5.3, we have

$$\begin{aligned} H_{\mathrm{c}}^j\bigl(\mathcal{M}^{\leq\mu}_{b,b',\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\bigr) \otimes \pi_{\varphi,1} &= \Bigl(\operatorname{c-Ind}_{\mathrm{GL}_n(E)^0}^{\mathrm{GL}_n(E)} H_{\mathrm{c}}^j\bigl(\mathcal{M}^{\infty,(0),\diamond}_{\mathrm{LT},\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\bigr)\Bigr) \otimes \pi_{\varphi,1} \\ &= \operatorname{c-Ind}_{\mathrm{GL}_n(E)^0}^{\mathrm{GL}_n(E)} \Bigl(H_{\mathrm{c}}^j\bigl(\mathcal{M}^{\infty,(0),\diamond}_{\mathrm{LT},\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\bigr) \otimes \pi_{\varphi,1}|_{\mathrm{GL}_n(E)^0}\Bigr). \end{aligned}$$

Therefore one has

$$H_i\Bigl(\mathrm{GL}_n(E), H_{\mathrm{c}}^j\bigl(\mathcal{M}^{\leq\mu}_{b,b',\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\bigr) \otimes \pi_{\varphi,1}\Bigr) = H_i\Bigl(\mathrm{GL}_n(E)^0, H_{\mathrm{c}}^j\bigl(\mathcal{M}^{\infty,(0),\diamond}_{\mathrm{LT},\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\bigr) \otimes \pi_{\varphi,1}|_{\mathrm{GL}_n(E)^0}\Bigr)$$

by Shapiro's Lemma. Now $\pi_{\varphi,1}|_{\mathrm{GL}_n(E)^0}$ is a compact representation and thus it is a projective object in the category of smooth $\mathrm{GL}_n(E)^0$-representations. Hence no higher homology groups appear and so

$$\Bigl(H_{\mathrm{c}}^j\bigl(\mathcal{M}^{\infty,\diamond}_{\mathrm{LT},\mathbb{C}_p^\flat}, \overline{\mathbb{Q}}_\ell\bigr) \otimes \pi_{\varphi,1}\Bigr)_{\mathrm{GL}_n(E)} = H_{\mathrm{c}}^j\Bigl(\mathcal{T}^{\leq\mu}_{b,b',\mathbb{C}_p^\flat}, t^*_{b,b'} \underline{\pi_{\varphi,1}}\Bigr).$$

Hence, the claim follows from the non-abelian Lubin–Tate theory. □

6 Hecke eigensheaf property

Assume that $G = \mathrm{GL}_2$ and $\mu(z) = \mathrm{diag}(z, 1)$ in this section.

Lemma 6.1. *Let* $\mathrm{Spa}(F, F^+)$ *be a geometric point in* $\mathrm{Perf}_{\mathbb{F}_q}$*. Let*

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}' \longrightarrow \mathscr{F} \longrightarrow 0$$

be an exact sequence of coherent sheaf over X_F^{sch}*, where* $\mathscr{E}$ *and* $\mathscr{E}'$ *are vector bundles of rank* 2 *and* $\mathscr{F}$ *is a torsion coherent sheaf of length* 1*. Assume that* $\mathscr{E}$ *is not semi-stable and* $\mathscr{E}'$ *is semi-stable. Then* $\mathscr{E} \simeq \mathcal{O}(m) \oplus \mathcal{O}(m-1)$ *and* $\mathscr{E}' \simeq \mathcal{O}(m) \oplus \mathcal{O}(m)$ *for some integer* m*.*

Proof. The vector bundle $\mathscr{E}'$ is isomorphic to $\mathcal{O}(m + \frac{1}{2})$ or $\mathcal{O}(m) \oplus \mathcal{O}(m)$ for some integer m, since it is semi-stable.

If $\mathscr{E}'$ is isomorphic to $\mathcal{O}(m + \frac{1}{2})$, then $\mathscr{E}$ is isomorphic to $\mathcal{O}(m) \oplus \mathcal{O}(m)$ by [FF14, Theorem 2.9]. This contradict to the condition that $\mathscr{E}$ is not semi-stable.

Assume $\mathscr{E}'$ is isomorphic to $\mathcal{O}(m) \oplus \mathcal{O}(m)$. Then $\mathscr{E}$ is isomorphic to $\mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$ with $m_1, m_2 \leq m$ or $\mathcal{O}(n + \frac{1}{2})$ with $n \leq m-1$ by [FF14, 6.3.1]. By considering $\deg(\mathscr{E}) + 1 = \deg(\mathscr{E}')$, the possible cases are $\mathcal{O}(m) \oplus \mathcal{O}(m-1)$ or $\mathcal{O}(m - \frac{1}{2})$. However, the latter case does not happen, since $\mathscr{E}$ is not semi-stable. □

Proposition 6.2. *Then we have*

$$\mathrm{supp}\, \overrightarrow{h}_{\natural}(\overleftarrow{h}^{*}\mathscr{F}_{\varphi} \otimes \mathrm{IC}'_{\mu}) \subset \mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q} \times \mathrm{Div}^1_X .$$

Proof. Take a non-basic element $[b] \in B(G)$. Then it suffices to show that $p_b^* \tilde{y}_b^* \overrightarrow{h}_{\natural} \overleftarrow{h}^{*} \mathscr{F}_{\varphi} = 0$, where p_b is defined at (5.2). We consider the following cartesian diagram:

$$\begin{array}{ccccc}
\mathcal{T}^{\leq\mu,\mathrm{ss}}_{b,\mathbb{C}_p^\flat} & \longrightarrow & \mathcal{T}^{\leq\mu}_{b,\mathbb{C}_p^\flat} & \longrightarrow & \mathrm{Spa}(\mathbb{C}_p^\flat) \\
 & & \downarrow & & \downarrow{\scriptstyle \tilde{y}_b \circ p_b} \\
\downarrow{\scriptstyle \overleftarrow{h}_{b,\mathrm{ss}}} & & \mathrm{Hecke}^{\leq\mu}_{\overline{\mathbb{F}}_q} & \xrightarrow{\overrightarrow{h}} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_q} \times \mathrm{Div}^1_X \\
 & & \downarrow{\scriptstyle \overleftarrow{h}} & & \\
\mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q} & \xrightarrow{j_{\mathrm{ss}}} & \mathrm{Bun}_{G,\overline{\mathbb{F}}_q} . & &
\end{array}$$

Let $\overleftarrow{h}_b \colon \mathcal{T}^{\leq\mu}_{b,\mathbb{C}_p^\flat} \to \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ be the morphism which appears in the above diagram. Then it suffices to see that

$$H^i_{\mathrm{c}}(\mathcal{T}^{\leq\mu}_{b,\mathbb{C}_p^\flat}, \overleftarrow{h}^{*}_{b}\mathscr{F}_{\varphi}) = 0.$$

On the other hand, we have

$$H^i_{\mathrm{c}}(\mathcal{T}^{\leq\mu}_{b,\mathbb{C}_p^\flat}, \overleftarrow{h}^{*}_{b}\mathscr{F}_{\varphi}) = H^i_{\mathrm{c}}(\mathcal{T}^{\leq\mu,\mathrm{ss}}_{b,\mathbb{C}_p^\flat}, \overleftarrow{h}^{*}_{b,\mathrm{ss}} j^*_{\mathrm{ss}}\mathscr{F}_{\varphi})$$

by $\mathscr{F}_{\varphi} = j_{\mathrm{ss},\natural} j^*_{\mathrm{ss}} \mathscr{F}_{\varphi}$. We have a decomposition

$$\mathcal{T}^{\leq\mu,\mathrm{ss}}_{b,\mathbb{C}_p^\flat} = \coprod_{N \in 2\mathbb{Z}} \mathcal{T}^{\leq\mu}_{b,b_1^N,\mathbb{C}_p^\flat}$$

by Lemma 6.1. Hence, we have

$$H^i_{\mathrm{c}}(\mathcal{T}^{\leq\mu,\mathrm{ss}}_{b,\mathbb{C}_p^\flat}, \overleftarrow{h}^{*}_{b,\mathrm{ss}} j^*_{\mathrm{ss}}\mathscr{F}_{\varphi}) = 0$$

by Theorem 4.30. □

Theorem 6.3. *Then we have*

$$\overrightarrow{h}_{\natural}(\overleftarrow{h}^{*}\mathscr{F}_{\varphi} \otimes \mathrm{IC}'_{\mu}) = \mathscr{F}_{\varphi} \boxtimes \varphi.$$

Proof. By Proposition 6.2, it suffices to show the equality on $\mathrm{Bun}^{\mathrm{ss}}_{G,\overline{\mathbb{F}}_q} \times \mathrm{Div}^1_X$. The equality on the semi-stable locus follows from Proposition 5.1, since we have $N \equiv 0, 1 \mod 2$ for any integer N. □

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