Mod ℓ Weil representations and Deligne–Lusztig inductions for unitary groups

Naoki Imai and Takahiro Tsushima

Abstract

We study the mod ℓ Weil representation of a finite unitary group and related Deligne–Lusztig inductions. In particular, we study their decomposition as representations of a symplectic group, and give a construction of a mod ℓ Howe correspondence for $(\mathrm{Sp}_{2n}, \mathrm{O}_2^-)$ including the case where $p = 2$.

1 Introduction

Let q be a power of a prime number p. Weil representations of symplectic groups over $\mathbb{F}_q$ are studied in [Sai72] and [How73] after [Wei64] if q is odd. Weil representations of general linear groups and unitary groups over $\mathbb{F}_q$ are constructed in [Gér77] for any q. The Howe correspondence is constructed using the Weil representations.

In [IT23], we construct Weil representations of unitary groups by using cohomology of varieties over finite fields. More concretely, we consider the affine smooth variety X_n defined by $z^q + z = \sum_{i=1}^{n} x_i^{q+1}$ in $\mathbb{A}^{n+1}_{\mathbb{F}_{q^2}}$, where $n \geq 2$. Let $\mathbb{F}_{q,+} = \{x \in \mathbb{F}_{q^2} \mid x^q + x = 0\}$. This variety admits an action of a finite unitary group $\mathrm{U}_n(\mathbb{F}_q)$ and a natural action of $\mathbb{F}_{q,+}$. Let $\ell \neq p$ be a prime number and $\psi \in \mathrm{Hom}(\mathbb{F}_{q,+}, \overline{\mathbb{Q}}_\ell^\times) \setminus \{1\}$. Then the ψ-isotypic part $V_n = H^n_{\mathrm{c}}(X_{n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]$ realizes the Weil representation of $\mathrm{U}_n(\mathbb{F}_q)$ with a natural action of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^2})$. We can use this Galois action to construct Shintani lifts for Weil representations as in [IT19]. Further V_n is isomorphic to middle cohomology of a μ_{q+1}-torsor over the complement of a Fermat hypersurface in a projective space as $\mathrm{U}_n(\mathbb{F}_q)$-representations. Let S_n be the Fermat hypersurface defined by the homogenous polynomial $\sum_{i=1}^{n} x_i^{q+1} = 0$ in $\mathbb{P}^{n-1}_{\mathbb{F}_{q^2}}$. Let $Y_n = \mathbb{P}^{n-1}_{\mathbb{F}_{q^2}} \setminus S_n$. Here let $\mathrm{U}_n(\mathbb{F}_q)$ act on $\mathbb{P}^{n-1}_{\mathbb{F}_{q^2}}$ by multiplication. Let $\widetilde{Y}_n$ be the affine smooth variety defined by $\sum_{i=1}^{n} x_i^{q+1} = 1$ in $\mathbb{A}^n_{\mathbb{F}_{q^2}}$. The natural morphism $f \colon \widetilde{Y}_n \to Y_n;\ (x_1, \ldots, x_n) \mapsto [x_1 : \cdots : x_n]$ is a μ_{q+1}-torsor. These varieties appear as Deligne–Lusztig varieties.

Let $\Lambda \in \{\overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell\}$. Let $\mathscr{K}_\chi$ denote the sheaf of Λ-modules on Y_n associated to a character $\chi^{-1} \colon \mu_{q+1} \to \Lambda^\times$ and the μ_{q+1}-torsor f. We identify μ_{q+1} with the center of $\mathrm{U}_n(\mathbb{F}_q)$. For $\chi \in \mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{Q}}_\ell^\times)$, we have an isomorphism $V_n[\chi] \simeq H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)$ as $\mathrm{U}_n(\mathbb{F}_q)$-representations (*cf.* (2.3)), which we can write using Deligne–Lusztig induction ([IT23, Proposition 6.1]). In this paper, we study a modular coefficients case of this cohomology. Namely, we analyze

$$H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_\xi)$$

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as $\mathrm{U}_n(\mathbb{F}_q)$-representations over $\overline{\mathbb{F}}_\ell$ for $\xi \in \mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{F}}_\ell^\times)$. We show that these representations are irreducible in the most cases, but it can be an extension of the trivial representation by an irreducible representations when $\ell \mid q+1$. This is contrary to the $\overline{\mathbb{Q}}_\ell$-coefficients case, where they are all irreducible (*cf.* Proposition 2.1). See Proposition 3.6 for a more precise result.

In [IT23], we consider a rational form of X_{2n} over $\mathbb{F}_q$, which is denoted by X'_{2n}. Using the Frobenius action coming from the rationality of X'_{2n} over $\mathbb{F}_q$, we can obtain a representation of $\mathrm{Sp}_{2n}(\mathbb{F}_q) \times \mathrm{O}_2^-(\mathbb{F}_q)$ on $H_c^{2n}(X'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)[\psi]$ (*cf.* [IT23, §7.1.1]), for which we simply write $W_{n,\psi}$. We also consider a rational form of Y_{2n}, which we denote by Y'_{2n}. For $\chi \in \mathrm{Hom}(\mu_{q+1}, \Lambda^\times)$, we can define a sheaf of Λ-modules $\mathscr{K}_\chi$ on Y'_{2n} similarly, and then the cohomology $H_c^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)$ is regarded as an $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation. Similarly as above, we have an isomorphism $W_{n,\psi}[\chi] \simeq H_c^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)$ as $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representations for $\chi \in \mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{Q}}_\ell^\times)$. If $\chi^2 = 1$, we can define the plus part $H_c^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)^+$ and the minus part $H_c^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)^-$ as $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representations using the Frobenius action. Any irreducible representation σ of $\mathrm{O}_2^-(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$ is attached to

$$[\xi] \in \{\xi \in \mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{F}}_\ell^\times) \mid \xi^2 \neq 1\}/(\mathbb{Z}/2\mathbb{Z}) \quad \text{or} \quad (\xi, \kappa) \in \mathrm{Hom}(\mu_{q+1}, \mu_2(\overline{\mathbb{F}}_\ell)) \times \{\pm\}$$

as [IT23, §7.2], where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{F}}_\ell^\times)$ by $\xi \mapsto \xi^{-1}$. Then $W_{n,\psi}[\sigma]$ is isomorphic to

$$H_c^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi) \quad \text{or} \quad H_c^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)^\kappa$$

accordingly. This realizes the Howe correspondence for the dual pair $\mathrm{Sp}_{2n}(\mathbb{F}_q) \times \mathrm{O}_2^-(\mathbb{F}_q)$.

The aim of this paper is to propose the modular coefficients version of this correspondence as a mod ℓ Howe correspondence for $\mathrm{Sp}_{2n}(\mathbb{F}_q) \times \mathrm{O}_2^-(\mathbb{F}_q)$ (*cf.* §6.2) and study the mod ℓ correspondence. Our main result is the following:

Theorem (Theorem 6.1)**.** *Assume that $\ell \neq 2$. The $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representations*

$$\begin{aligned} &H_c^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\xi) \quad \textit{for } [\xi] \in \{\xi \in \mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{F}}_\ell^\times) \mid \xi^2 \neq 1\}/(\mathbb{Z}/2\mathbb{Z}), \\ &H_c^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\xi)^\kappa \quad \textit{for } \xi \in \mathrm{Hom}(\mu_{q+1}, \mu_2(\overline{\mathbb{F}}_\ell)),\ \kappa \in \{\pm\} \end{aligned}$$

are irreducible except the case where $\ell \mid q+1$ and $(\xi, \kappa) = (1, +)$, in which case $H_c^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\xi)^\kappa$ is a non-trivial extension of the trivial representation by an irreducible representation. Furthermore, the above representations have no irreducible constituent in common.

In the following, we briefly introduce a content of each section. In §2.1, we recall several fundamental facts proved in [IT23]. In §2.2, we recall general facts on étale cohomology. In §2.3, we recall results on Lusztig series and ℓ-blocks.

In §3, we investigate the mod ℓ cohomology of Y_n as a $\mathrm{U}_n(\mathbb{F}_q)$-representation. Our fundamental result is Proposition 3.3. To show this proposition, we need a transcendental result in [Dim92] (*cf.* the proof of Lemma 3.2). In §4, we prepare some geometric results on cohomology of Y'_{2n}. In §5, we study mod ℓ cohomology of Y'_{2n} as an $\mathrm{Sp}_n(\mathbb{F}_q)$-representation for $\ell \neq 2$. In a modular representation theoretic view point, Brauer characters associated to V_n and $W_{n,\psi}$ have been studied in [GMST02], [GT04] and [HM01] etc. Using these results, we study the cohomology of Y_n and Y'_{2n} as mentioned above.

In §6, we formulate a mod ℓ Howe correspondence for $(\mathrm{Sp}_{2n}, \mathrm{O}_2^-)$ using mod ℓ cohomology of Y'_{2n}, and state our result in terms of the mod ℓ Howe correspondence.

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Notation

Let ℓ be a prime number. For a finite abelian group A, let $A^\vee$ denote the character group $\mathrm{Hom}(A, \overline{\mathbb{Q}}_\ell^\times)$. For a finite group G and a finite-dimensional representation π and an irreducible representation ρ of G over $\overline{\mathbb{Q}}_\ell$, let $\pi[\rho]$ denote the ρ-isotypic part of π. For a trivial representation 1 of G, we often write π^G for $\pi[1]$.

Every scheme is equipped with the reduced scheme structure. For an integer $i \geq 0$, we write $\mathbb{A}^i$ and $\mathbb{P}^i$ for the i-dimensional affine space over $\overline{\mathbb{F}}_q$ and the i-dimensional projective space over $\overline{\mathbb{F}}_q$, respectively. We set $\mathbb{G}_\mathrm{m} = \mathbb{A}^1 \setminus \{0\}$. For a scheme X over a field k and a field extension l/k, let X_l denote the base change of X to l.

2 Preliminaries

2.1 Weil representation of unitary group

In this subsection, we recall several facts proved in [IT23]. Let q be a power of a prime number p. For a positive integer m prime to p, we put

$$\mu_m = \{a \in \overline{\mathbb{F}}_q \mid a^m = 1\}.$$

Let n be a positive integer. Let U_n be the unitary group over $\mathbb{F}_q$ defined by the hermitian form

$$\mathbb{F}_{q^2}^n \times \mathbb{F}_{q^2}^n \to \mathbb{F}_{q^2};\ ((x_i), (x'_i)) \mapsto \sum_{i=1}^n x_i^q x'_i.$$

We consider the Fermat hypersurface S_n defined by $\sum_{i=1}^n x_i^{q+1} = 0$ in $\mathbb{P}_{\mathbb{F}_q}^{n-1}$. Let $Y_n = \mathbb{P}_{\mathbb{F}_q}^{n-1} \setminus S_n$. Let $\mathrm{U}_n(\mathbb{F}_q)$ act on $\mathbb{P}_{\mathbb{F}_{q^2}}^{n-1}$ by left multiplication. Let $\widetilde{Y}_n$ be the affine smooth variety defined by $\sum_{i=1}^n x_i^{q+1} = 1$ in $\mathbb{A}_{\mathbb{F}_q}^n$. Similarly, $\mathrm{U}_n(\mathbb{F}_q)$ acts on $\widetilde{Y}_{n,\mathbb{F}_{q^2}}$. The morphism

$$\widetilde{Y}_n \to Y_n;\ (x_1, \ldots, x_n) \mapsto [x_1 : \cdots : x_n] \tag{2.1}$$

is a μ_{q+1}-torsor and $\mathrm{U}_n(\mathbb{F}_q)$-equivariant.

Let $\ell \neq p$ be a prime number. Let ω_{U_n} denote the Weil representation of $\mathrm{U}_n(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$ (*cf.* [Gér77, Theorem 4.9.2]).

Let $\mathcal{O}$ be the ring of integers in an algebraic extension of $\mathbb{Q}_\ell$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. We set $\mathbb{F} = \mathcal{O}/\mathfrak{m}$.

Let $\Lambda \in \{\overline{\mathbb{Q}}_\ell, \mathcal{O}, \mathbb{F}\}$. For a separated and of finite type scheme Y over $\mathbb{F}_q$ which admits a left action of a finite group G, let G act on $H_\mathrm{c}^i(Y_{\overline{\mathbb{F}}_q}, \Lambda)$ as $(g^*)^{-1}$ for $g \in G$. We put

$$V_n = H_\mathrm{c}^{n-1}(\widetilde{Y}_{n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell).$$

For $\chi \in \mathrm{Hom}(\mu_{q-1}, \Lambda^\times)$, let $\mathscr{K}_\chi$ denote the Λ-sheaf on $Y_{n,\mathbb{F}_{q^2}}$ defined by χ^{-1} and the covering (2.1). For $\chi \in \mu_{q-1}^\vee$, we have $V_n[\chi] = H_\mathrm{c}^{n-1}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)$, which is the middle degree cohomology in a Deligne–Lusztig induction by [IT23, (5.1), §6.1].

Let X_n be the affine smooth variety over $\mathbb{F}_{q^2}$ defined by

$$z^q + z = \sum_{i=1}^{n} x_i^{q+1}$$

in $\mathbb{A}^{n+1}_{\mathbb{F}_{q^2}} = \operatorname{Spec} \mathbb{F}_{q^2}[x_1, \ldots, x_n, z]$. Let $\mathrm{U}_n(\mathbb{F}_q)$ act on X_n by

$$X_n \to X_n;\ (v, z) \mapsto (gv, z) \quad \text{for } g \in \mathrm{U}_n(\mathbb{F}_q),$$

where we regard $v = (x_i)$ as a column vector. We put $\mathbb{F}_{q,\varepsilon} = \{a \in \mathbb{F}_{q^2} \mid a + \varepsilon a^q = 0\}$. We sometimes abbreviate ± 1 as $\pm$. Let $\mathbb{F}_{q,+}$ act on X_n by $z \mapsto z + a$ for $a \in \mathbb{F}_{q,+}$.

Let $\psi \in \operatorname{Hom}(\mathbb{F}_{q,\varepsilon}, \Lambda^\times) \setminus \{1\}$. Let $\mathscr{L}_\psi$ denote the Λ-sheaf associated to ψ^{-1} and $z^q + \varepsilon z = t$ on $\mathbb{A}^1_{\mathbb{F}_q} = \operatorname{Spec} \mathbb{F}_q[t]$. We consider the morphism

$$\pi \colon \mathbb{A}^n_{\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q};\ (x_i)_{1 \le i \le n} \mapsto \sum_{i=1}^{n} x_i^{q+1}.$$

Then we have an isomorphism

$$H^i_{\mathrm{c}}(X_{n,\overline{\mathbb{F}}_q}, \Lambda)[\psi] \simeq H^i_{\mathrm{c}}(\mathbb{A}^n, \pi^* \mathscr{L}_\psi) \tag{2.2}$$

for $i \geq 0$.

Proposition 2.1. *Assume that $n \geq 2$.*

(1) We have $V_n \simeq \omega_{\mathrm{U}_n}$ as representations of $\mathrm{U}_n(\mathbb{F}_q)$.

(2) For $\chi \in \mu_{q+1}^\vee$, we have

$$\dim V_n[\chi] = \begin{cases} \dfrac{q^n + (-1)^n q}{q+1} & \text{if } \chi = 1, \\ \dfrac{q^n - (-1)^n}{q+1} & \text{if } \chi \neq 1. \end{cases}$$

The $\mathrm{U}_n(\mathbb{F}_q)$-representations $\{V_n[\chi]\}_{\chi \in \mu_{q+1}^\vee}$ are irreducible and distinct. Moreover, only $V_n[1]$ is unipotent as a $\mathrm{U}_n(\mathbb{F}_q)$-representation.

Proof. We have

$$V_n \simeq \bigoplus_{\chi \in \mu_{q+1}^\vee} H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi) \simeq H^n_{\mathrm{c}}(\mathbb{A}^n, \pi^* \mathscr{L}_\psi) \tag{2.3}$$

as representations of $\mathrm{U}_n(\mathbb{F}_q)$ by [IT23, Lemma 4.3, Corollary 4.6, Lemma 4.7 (2)]. The claim (1) follows from (2.2), (2.3) and [IT23, (2.6), Theorem 2.5]. The claim (2) follows from the claim (1), (2.3) and [IT23, Lemma 4.2, Corollary 6.2]. □

2.2 General facts on étale cohomology

We recall a basic fact on cohomology of an affine smooth variety, which will be used frequently.

Lemma 2.2. *Let X be an affine smooth variety over $\overline{\mathbb{F}}_q$ of dimension d. Let $\ell \neq p$. Let F be a finite extension of $\mathbb{Q}_\ell$. Let $\mathcal{O}_F$ be the ring of integers of F. Let κ_F be the residue field of $\mathcal{O}_F$. Let $\Lambda \in \{\mathcal{O}_F, \kappa_F\}$. Suppose that $\mathscr{F}$ is a smooth Λ-sheaf on X.*

(1) Assume $\Lambda = \kappa_F$. Then we have $H^i_{\mathrm{c}}(X, \mathscr{F}) = 0$ for $i < d$.

(2) Assume $\Lambda = \mathcal{O}_F$. The middle cohomology $H^d_{\mathrm{c}}(X, \mathscr{F})$ is a finitely generated free $\mathcal{O}_F$-module.

Proof. The first assertion follows from affine vanishing and Poincaré duality. We show the second claim. We take a uniformizer ϖ of $\mathcal{O}_F$. Then, we have an exact sequence

$$H^{d-1}_{\mathrm{c}}(X, \mathscr{F}/\varpi) \to H^d_{\mathrm{c}}(X, \mathscr{F}) \xrightarrow{\varpi} H^d_{\mathrm{c}}(X, \mathscr{F}).$$

Since we have $H^{d-1}_{\mathrm{c}}(X, \mathscr{F}/\varpi) = 0$ by the first claim, the ϖ-multiplication map is injective. Since $H^d_{\mathrm{c}}(X, \mathscr{F})$ is a finitely generated $\mathcal{O}_F$-module, the second claim follows. □

We recall a well-known fact, which will be used in the proof of Proposition 4.4.

Lemma 2.3. *Let the notation be as in Lemma 2.2. Let G be a finite group. Let $X \to Y$ be a G-torsor between d-dimensional affine smooth varieties over $\overline{\mathbb{F}}_q$.*

(1) We have an isomorphism $H^d_{\mathrm{c}}(X, \kappa_F)^G \simeq H^d_{\mathrm{c}}(Y, \kappa_F)$.

(2) Assume $\ell \nmid |G|$. Then we have an isomorphism $H^i_{\mathrm{c}}(X, \kappa_F)^G \simeq H^i_{\mathrm{c}}(Y, \kappa_F)$ for any i.

Proof. As in [Ill81, Lemma 2.2], we have a spectral sequence

$$E_2^{p,q} = H^p(G, H^q_{\mathrm{c}}(X, \kappa_F)) \Longrightarrow E^{p+q} = H^{p+q}_{\mathrm{c}}(Y, \kappa_F).$$

Since we have $H^i_{\mathrm{c}}(X, \kappa_F) = 0$ for $i < d$ by Lemma 2.2 (1), we have an isomorphism $E_2^{0,d} \simeq E^d$. Hence the first claim follows.

We show the second claim. For any $\kappa_F[G]$-module M, we have $H^i(G, M) = 0$ for any $i > 0$ by $\ell \nmid |G|$. Hence the second claim follows from the above spectral sequence. □

2.3 Lusztig series and ℓ-blocks

We briefly recall several facts on Lusztig series (*cf.* [Lus77, §7]). We mainly follow [DM91, Chapter 13].

Let G be a connected reductive group over $\mathbb{F}_q$. Let $\mathrm{Irr}(G(\mathbb{F}_q))$ denote the set of irreducible characters of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$. Let G^* be a connected reductive group over $\mathbb{F}_q$ which is the dual of G in the sense of [DM91, 13.10 Definition].

We fix an isomorphism $\overline{\mathbb{F}}_q^\times \simeq (\mathbb{Q}/\mathbb{Z})_{p'}$ and an embedding $\overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}_\ell^\times$, where $(\mathbb{Q}/\mathbb{Z})_{p'}$ denotes the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of the elements of order prime to p. Let (s) be a geometric conjugacy class of a semisimple element $s \in G^*(\mathbb{F}_q)$. As in [DM91, 13.16 Definition], let $\mathcal{E}(G, (s))$ be the subset of $\mathrm{Irr}(G(\mathbb{F}_q))$ which consists of irreducible constitutes of a Deligne–Lusztig character $R_T^G(\theta)$, where (T, θ) is of the geometric conjugacy class associated to (s) in the sense of [DM91, 13.2 Definition, 13.12 Proposition]. The subset $\mathcal{E}(G, (s))$ is called a Lusztig series associated to (s). By [DM91, 13.17 Proposition], $\mathrm{Irr}(G(\mathbb{F}_q))$ is partitioned into Lusztig series. The following fact is well-known.

Lemma 2.4. *Let $f\colon G \to G'$ be a morphism between connected reductive groups over $\mathbb{F}_q$ with a central connected kernel such that the image of f contains the derived group of G'. Let G^* and G'^* be the dual groups of G and G'. Let $s \in G^*(\mathbb{F}_q)$ be the image of a semisimple element s' in $G'^*(\mathbb{F}_q)$. Then the irreducible constituents of the inflations under f of elements in $\mathcal{E}(G', (s'))$ are in $\mathcal{E}(G, (s))$.*

Proof. This follows from [DM91, 13.22 Proposition]. □

For a semisimple ℓ'-element $s \in G^*(\mathbb{F}_q)$, we define

$$\mathcal{E}_\ell(G, (s)) = \bigcup_{t \in (C_{G^*(\mathbb{F}_q)}(s))_\ell} \mathcal{E}(G, (st)).$$

It is known that this set is a union of ℓ-blocks by [BM89, 2.2 Théorème]. Any block contained in $\mathcal{E}_\ell(G, (1))$ is called a unipotent ℓ-block.

Lemma 2.5. *Let s and s' be semisimple ℓ'-elements of $G^*(\mathbb{F}_q)$. Assume that s and s' are not geometrically conjugate. Let $\rho \in \mathcal{E}(G, (s))$ and $\rho' \in \mathcal{E}(G, (s'))$. Then $\rho \notin \mathcal{E}_\ell(G, (s'))$. In particular, ρ and ρ' are in different ℓ-blocks.*

Proof. Assume $\rho \in \mathcal{E}_\ell(G, (s'))$. Then there exists an element $s'' \in C_{G^*(\mathbb{F}_q)}(s')$ of ℓ-power such that $\rho \in \mathcal{E}(G, (s's''))$. Since $\rho \in \mathcal{E}(G, (s))$, the elements s and $s's''$ are geometrically conjugate by [DM91, 13.17 Proposition]. Let ℓ^b be the order of s''. By $s'' \in C_{G^*(\mathbb{F}_q)}(s')$, the elements s^{ℓ^b} and s'^{ℓ^b} are geometrically conjugate. Then s and s' are geometrically conjugate, since s and s' are ℓ'-elements. This is a contradiction. □

For an irreducible representation π of a finite group, let $\overline{\pi}$ denote the Brauer character associated to a mod ℓ reduction of π. For any integer $m \geq 1$, let m_p be the largest power of p dividing m and $m_{p'} = m/m_p$.

Lemma 2.6. *Let ρ be an irreducible ordinary character of G^F. Assume $\rho \in \mathcal{E}_\ell(G, (s))$ for some semisimple ℓ'-element s of $G^*(\mathbb{F}_q)$. If*

$$\frac{|G(\mathbb{F}_q)|_{p'}}{|C_{G^*(\mathbb{F}_q)}(s)|_{p'}} = \rho(1),$$

then $\overline{\rho}$ is an irreducible Brauer character.

Proof. Assume that $\overline{\rho}$ is not irreducible and take an irreducible Brauer subcharacter χ of $\overline{\rho}$. Then

$$\frac{|G(\mathbb{F}_q)|_{p'}}{|C_{G^*(\mathbb{F}_q)}(s)|_{p'}} \leq \chi(1) < \rho(1)$$

by [HM01, Proposition 1]. This is a contradiction. □

We recall some results in [HM01, §6]. Take a nonisotropic vector c in the standard representation of $\mathrm{U}_n(\mathbb{F}_q)$. Let $\hat{S}$ be the subgroup of $\mathrm{U}_n(\mathbb{F}_q)$ fixing the line $\langle c \rangle$ and inducing the identity on the orthogonal complement of $\langle c \rangle$. Let S be the image of $\hat{S}$ in $\mathrm{PU}_n(\mathbb{F}_q)$. Then $\hat{S}$ and S are cyclic groups of order $q + 1$. We follow [HM01, §6] for a definition of the Weil characters of $\mathrm{SU}_n(\mathbb{F}_q)$.

Lemma 2.7. *Assume that $n \geq 3$. The Weil characters of $\mathrm{SU}_n(\mathbb{F}_q)$ consist of one unipotent character $\chi_{(n-1,1)} \in \mathcal{E}(\mathrm{SU}_n, (1))$ of degree $(q^n + (-1)^n q)/(q+1)$ and q non-unipotent characters $\chi_{s,(n-1)} \in \mathcal{E}(\mathrm{SU}_n, (s))$ of degree $(q^n - (-1)^n)/(q+1)$ for $s \in S \setminus \{1\}$, where the elements of S give different geometrically conjugacy classes. We put*

$$N = \min\left\{ \frac{q^n + (-1)^n q}{q+1}, \frac{q^n - (-1)^n}{q+1} \right\}$$

Let V be a Weil character of $\mathrm{SU}_n(\mathbb{F}_q)$. If the degree of V is N, then $\overline{V}$ is irreducible. If the degree of V is $N+1$, then $\overline{V}$ is irreducible or has two irreducible constituents, one of which is trivial. Further we have the following:

(1) Let $n \geq 4$ be even. Then $\overline{\chi}_{(n-1,1)}$ is irreducible if and only if $\ell \nmid q+1$.

(2) Let $n \geq 3$ be odd. Then $\overline{\chi}_{s,(n-1)}$ is irreducible if and only if the order of s is not a power of ℓ.

Proof. Everything is explained in [HM01, p. 755] except that the elements of S give different geometrically conjugacy classes. Assume that different elements s and s' in S are geometrically conjugate. Then their lifts $\hat{s}$ and $\hat{s}'$ in $\hat{S}$ are geometrically conjugate modulo the center. Considering the eigenvalues of $\hat{s}$ and $\hat{s}'$, we have a contradiction. □

Lemma 2.8. *Let $\chi \in \mu_{q+1}^{\vee} \setminus \{1\}$. We view χ as a character of the diagonal torus $\mathrm{U}_1(\mathbb{F}_q)^n$ of $\mathrm{U}_n(\mathbb{F}_q)$ under the first projection. Let $\hat{s} \in \hat{S}$ be the element corresponding to χ by [DM91, 13.12 Proposition]. Let $s \in S$ be the image of $\hat{s}$. Then we have $V_n[1]|_{\mathrm{SU}_n(\mathbb{F}_q)} = \chi_{(n-1,1)}$ and $V_n[\chi]|_{\mathrm{SU}_n(\mathbb{F}_q)} = \chi_{s,(n-1)}$.*

Proof. This follows from Lemma 2.4 and Lemma 2.7. □

3 Cohomology as representation of unitary group

In this section, we investigate mainly $H_{\mathrm{c}}^n(Y_{n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)$ as an $\overline{\mathbb{F}}_\ell[\mathrm{U}_n(\mathbb{F}_q)]$-module.

In this section, we often ignore Tate twists when it is not necessary to consider Frobenius action. For an $\mathcal{O}$-module M, let $M[\mathfrak{m}]$ denote the $\mathcal{O}$-submodule of M consisting of elements annihilated by any element of $\mathfrak{m}$.

Lemma 3.1. *(1) The cohomology $H^i(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O})$ is free as an $\mathcal{O}$-module for any i.*

(2) The cohomology $H^i(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O})$ is zero if $i \neq n-2$ and i is odd, and is free of rank one as an $\mathcal{O}$-module if $0 \leq i \leq 2(n-2)$ is even and $i \neq n-2$.

(3) We have an isomorphism $H^i(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \simeq H^i(S_{n,\overline{\mathbb{F}}_q}, \mathbb{F})$ for any i.

Proof. We denote by $S_{n,\mathbb{Q}}$ the Fermat variety defined by the same equation as S_n in $\mathbb{P}_{\mathbb{Q}}^{n-1}$. Let i be an integer. We have isomorphisms

$$H^i(S_{n,\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O} \simeq H^i(S_{n,\mathbb{C}}^{\mathrm{an}}, \mathcal{O}) \simeq H^i(S_{n,\mathbb{C}}, \mathcal{O}),$$

where the first isomorphism follows from that $\mathcal{O}$ is flat over $\mathbb{Z}$, and the second one follows from the comparison theorem between singular and étale cohomology. By taking an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$, we have an isomorphism $H^i(S_{n,\mathbb{C}}, \mathcal{O}) \simeq H^i(S_{n,\overline{\mathbb{Q}}_p}, \mathcal{O})$. Since $S_{n,\mathbb{Q}}$ has good reduction at p and the reduction equals S_n, we have an isomorphism

$$H^i(S_{n,\overline{\mathbb{Q}}_p}, \mathcal{O}) \simeq H^i(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O})$$

by the proper base change theorem. As a result, we have

$$H^i(S^{\mathrm{an}}_{n,\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O} \simeq H^i(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O}). \tag{3.1}$$

Hence the first claim follows, because $H^i(S^{\mathrm{an}}_{n,\mathbb{C}}, \mathbb{Z})$ is a free $\mathbb{Z}$-module by [Dim92, Proposition (B32) (ii)]. The second claim follows from (3.1) and [Dim92, Theorem (B22)]. We have a short exact sequence

$$0 \to H^i(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \to H^i(S_{n,\overline{\mathbb{F}}_q}, \mathbb{F}) \to H^{i+1}(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O})[\mathfrak{m}] \to 0.$$

Hence the third claim follows from the first one. □

In the sequel, we always assume that $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbb{Q}_\ell(\mu_{p(q+1)})$. Every homomorphism is $\mathrm{U}_n(\mathbb{F}_q)$-equivariant. Let the notation be as in §4.1. We have a long exact sequence

$$\begin{aligned} \cdots &\to H^i_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \to H^i(\mathbb{P}^{n-1}, \mathcal{O}) \to H^i(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \\ &\to H^{i+1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \to H^{i+1}(\mathbb{P}^{n-1}, \mathcal{O}) \to H^{i+1}(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \to \cdots . \end{aligned} \tag{3.2}$$

By Lemma 2.2 (1), the restriction map

$$H^i(\mathbb{P}^{n-1}, \mathcal{O}) \to H^i(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O})$$

is an isomorphism for $i < n-2$. By Lemma 3.1 (1) and Poincaré duality, we obtain an isomorphism

$$f_i \colon H^i(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \xrightarrow{\sim} H^{i+2}(\mathbb{P}^{n-1}, \mathcal{O})$$

for $n-1 \leq i \leq 2n-4$. Let $\kappa = [\mathbb{P}^{n-2}] \in H^2(\mathbb{P}^{n-1}, \mathcal{O})$ be the cycle class of the hyperplane $\mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$, and $\kappa^i \in H^{2i}(\mathbb{P}^{n-1}, \mathcal{O})$ the cup product of κ. For $1 \leq i \leq n-1$, the map $\kappa^i \colon \mathcal{O} \to H^{2i}(\mathbb{P}^{n-1}, \mathcal{O});\ 1 \mapsto \kappa^i$ is an isomorphism. Let $[S_{n,\overline{\mathbb{F}}_q}] \in H^2(\mathbb{P}^{n-1}, \mathcal{O})$ be the cycle class of $S_{n,\overline{\mathbb{F}}_q} \subset \mathbb{P}^{n-1}$. For $n-1 \leq 2i \leq 2n-4$, the composite

$$H^{2i}(\mathbb{P}^{n-1}, \mathcal{O}) \xrightarrow{\text{rest.}} H^{2i}(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \xrightarrow[\simeq]{f_{2i}} H^{2i+2}(\mathbb{P}^{n-1}, \mathcal{O})$$

equals the map induced by the cup product by $[S_{n,\overline{\mathbb{F}}_q}]$. Clearly, we have $[S_{n,\overline{\mathbb{F}}_q}] = (q+1)\kappa$ in $H^2(\mathbb{P}^{n-1}, \mathcal{O})$. We write $q+1 = \ell^a r$ with $(\ell, r) = 1$. By Lemma 3.1 (2), we have

$$H^i_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \simeq \begin{cases} 0 & \text{if } i \text{ is even,} \\ \mathcal{O}/\ell^a & \text{if } i \text{ is odd} \end{cases} \tag{3.3}$$

for $n \leq i < 2n-2$.

For a character $\chi \colon \mu_{q+1} \to \mathcal{O}^\times$, we write $\overline{\chi}$ for the composite of χ and the reduction map $\mathcal{O}^\times \to \mathbb{F}^\times$.

We have a short exact sequence

$$0 \to H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi) \otimes_{\mathcal{O}} \mathbb{F} \to H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\overline{\chi}}) \to H^n_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)[\mathfrak{m}] \to 0. \tag{3.4}$$

By (3.3), we have

$$\dim_{\mathbb{F}} H^n_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O})[\mathfrak{m}] = \begin{cases} 0 & \text{if } a = 0, \\ \frac{1-(-1)^n}{2} & \text{if } a \geq 1. \end{cases} \tag{3.5}$$

By Proposition 2.1, Lemma 2.2 (2), (3.4) with $\chi = 1$ and (3.5), we have

$$\dim_{\mathbb{F}} H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathbb{F}) = \begin{cases} \frac{q^n+(-1)^n q}{q+1} & \text{if } a = 0, \\ \frac{q^n-(-1)^n}{q+1} + \frac{1+(-1)^n}{2} & \text{if } a \geq 1. \end{cases} \tag{3.6}$$

Hence, we have

$$\dim_{\mathbb{F}} H^n_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)[\mathfrak{m}] = \frac{1+(-1)^n}{2} \quad \text{if } \chi \neq 1 \text{ and } \overline{\chi} = 1 \tag{3.7}$$

again by Proposition 2.1, Lemma 2.2 (2) and (3.4) with χ. Note that a character $\chi \neq 1$ such that $\overline{\chi} = 1$ does not exist if $a = 0$.

In the following, we investigate (3.4) when $\overline{\chi} \neq 1$. We set $Y_{n,r} = \widetilde{Y}_{n,\overline{\mathbb{F}}_q}/\mu_{\ell^a}$. We have a decomposition $\mu_{q+1} = \mu_{\ell^a} \times \mu_r$. We write as $\chi = \chi_{\ell^a}\chi_r$ with $\chi_{\ell^a} \in \mathrm{Hom}(\mu_{\ell^a}, \mathcal{O}^\times)$ and $\chi_r \in \mathrm{Hom}(\mu_r, \mathcal{O}^\times)$. We have

$$0 \to H^{n-1}_{\mathrm{c}}(Y_{n,r}, \mathscr{K}_{\chi_{\ell^a}}) \otimes_{\mathcal{O}} \mathbb{F} \to H^{n-1}_{\mathrm{c}}(Y_{n,r}, \mathbb{F}) \to H^n_{\mathrm{c}}(Y_{n,r}, \mathscr{K}_{\chi_{\ell^a}})[\mathfrak{m}] \to 0. \tag{3.8}$$

The natural morphism $Y_{n,r} \to Y_{n,\overline{\mathbb{F}}_q}$ is a μ_r-torsor. By $(\ell, r) = 1$, we have isomorphisms

$$\begin{aligned} H^i_{\mathrm{c}}(Y_{n,r}, \mathscr{K}_{\chi_{\ell^a}}) &\simeq \bigoplus_{\chi_r \in \mathrm{Hom}(\mu_r, \mathcal{O}^\times)} H^i_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\chi_{\ell^a}\chi_r}), \\ H^i_{\mathrm{c}}(Y_{n,r}, \mathbb{F}) &\simeq \bigoplus_{\overline{\chi}_r \in \mathrm{Hom}(\mu_r, \mathbb{F}^\times)} H^i_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\overline{\chi}_r}) \end{aligned} \tag{3.9}$$

for any integer i. By Proposition 2.1, Lemma 2.2 (2) and the first isomorphism in (3.9), we have

$$\mathrm{rank}_{\mathcal{O}}\, H^{n-1}_{\mathrm{c}}(Y_{n,r}, \mathscr{K}_{\chi_{\ell^a}}) = \begin{cases} \frac{q^n-(-1)^n}{q+1} r & \text{if } \chi_{\ell^a} \neq 1, \\ \frac{q^n-(-1)^n}{q+1} r + (-1)^n & \text{if } \chi_{\ell^a} = 1. \end{cases} \tag{3.10}$$

We show the following lemma by using a comparison theorem between singular and étale cohomology and applying results on weighted hypersurfaces in [Dim92].

Lemma 3.2. *The pull-back $H^i_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \to H^i_{\mathrm{c}}(Y_{n,r}, \mathcal{O})$ is an isomorphism for $n \leq i < 2n-2$.*

Proof. Let $f\colon Y_{n,r} \to Y_{n,\overline{\mathbb{F}}_q}$ be the natural finite morphism. We have the trace map $f_*\colon H^i_{\mathrm{c}}(Y_{n,r}, \mathcal{O}) \to H^i_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O})$. The pull-back map $f^*\colon H^i_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \to H^i_{\mathrm{c}}(Y_{n,r}, \mathcal{O})$ is injective by $(\ell, r) = 1$, because the composite $f_* \circ f^*$ is the r-multiplication map. Therefore it suffices to show that the map is surjective.

We regard S_n as a closed subscheme of S_{n+1} defined by $x_{n+1} = 0$. We take $\xi \in \mathbb{F}_{q^2}$ such that $\xi^{q+1} = -1$. Then $\widetilde{Y}_n$ is isomorphic to the complement $S_{n+1} \setminus S_n$ over $\mathbb{F}_{q^2}$ by

$$\widetilde{Y}_{n,\mathbb{F}_{q^2}} \xrightarrow{\sim} (S_{n+1} \setminus S_n)_{\mathbb{F}_{q^2}};\ (x_i)_{1 \leq i \leq n} \mapsto [x_1 : \cdots : x_n : \xi].$$

Let μ_{q+1} act on $S_{n+1,\mathbb{F}_{q^2}}$ by $[x_1 : \cdots : x_{n+1}] \mapsto [x_1 : \cdots : x_n : \zeta^{-1}x_{n+1}]$ for $\zeta \in \mu_{q+1}$. We have the well-defined morphism $\pi\colon S_{n+1} \to \mathbb{P}^{n-1}_{\mathbb{F}_{q^2}}$; $[x_1 : \cdots : x_{n+1}] \mapsto [x_1 : \cdots : x_n]$. We have a commutative diagram

$$\begin{array}{ccccc} \widetilde{Y}_{n,\mathbb{F}_{q^2}} & \longrightarrow & S_{n+1,\mathbb{F}_{q^2}} & \longleftarrow & S_{n,\mathbb{F}_{q^2}} \\ \downarrow & & \downarrow \pi & & \downarrow \simeq \\ Y_{n,\mathbb{F}_{q^2}} & \longrightarrow & \mathbb{P}^{n-1}_{\mathbb{F}_{q^2}} & \longleftarrow & S_{n,\mathbb{F}_{q^2}}. \end{array}$$

By considering the base change of this to $\overline{\mathbb{F}}_q$ and taking the quotients on the upper line by μ_{ℓ^a}, we obtain a commutative diagram

$$\begin{array}{ccccc} Y_{n,r} & \longrightarrow & S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a} & \longleftarrow & S_{n,\overline{\mathbb{F}}_q} \\ \downarrow f & & \downarrow & & \downarrow \simeq \\ Y_{n,\overline{\mathbb{F}}_q} & \longrightarrow & \mathbb{P}^{n-1} & \longleftarrow & S_{n,\overline{\mathbb{F}}_q}. \end{array} \tag{3.11}$$

Let $\mathbf{w} = (1, \ldots, 1, \ell^a) \in \mathbb{Z}^{n+1}_{\geq 1}$ and $\mathbb{P}(\mathbf{w})$ be the weighted projective space associated to $\mathbf{w}$ over $\overline{\mathbb{F}}_q$. Then the quotient $S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a}$ is isomorphic to the weighted hypersurface defined by

$$\sum_{i=1}^{n} X_i^{q+1} + X_{n+1}^r = 0 \tag{3.12}$$

in $\mathbb{P}(\mathbf{w}) \simeq \mathbb{P}^n/\mu_{\ell^a}$. Let $U_i \subset S_{n+1,\overline{\mathbb{F}}_q}$ be the open subscheme defined by $x_i \neq 0$ for $1 \leq i \leq n$. Then we have $S_{n+1,\overline{\mathbb{F}}_q} = \bigcup_{i=1}^n U_i$. For each $1 \leq i \leq n$, the quotient U_i/μ_{ℓ^a} is defined by

$$1 + s_1^{q+1} + \cdots + s_{i-1}^{q+1} + s_{i+1}^{q+1} + \cdots + s_n^{q+1} + t_i^r = 0$$

in $\mathbb{A}^n$. This is smooth over $\overline{\mathbb{F}}_q$ by $p \nmid q+1$ and the Jacobian criterion. Hence $S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a}$ is smooth over $\overline{\mathbb{F}}_q$.

We consider the smooth hypersurface defined by the same equation as (3.12) in the weighted projective space $\mathbb{P}(\mathbf{w})$ over $\mathbb{Q}$, which we denote by S'. Then $S'^{\mathrm{an}}_{\mathbb{C}}$ is strongly smooth as in [Dim92, Example (B31)]. Hence the integral cohomology algebra of it is torsion-free by [Dim92, Proposition (B32) (ii)]. Clearly, S' has good reduction at p, and the reduction is isomorphic to $S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a}$ over $\overline{\mathbb{F}}_q$. In the same manner as (3.1), we have an isomorphism

$$H^i(S'^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O} \simeq H^i(S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a}, \mathcal{O}).$$

Hence, $H^i(S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a}, \mathcal{O})$ is a free $\mathcal{O}$-module of rank one for any even integer $i \neq n-1$ and $H^i(S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a}, \mathcal{O}) = 0$ for any odd integer $i \neq n-1$ by [Dim92, (B33)]. By (3.11), we have a commutative diagram

$$\begin{array}{ccccccc} H^{2i}(\mathbb{P}^{n-1}, \mathcal{O}) & \longrightarrow & H^{2i}(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) & \longrightarrow & H_{\mathrm{c}}^{2i+1}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) & \longrightarrow & 0 \\ \downarrow & & \downarrow \simeq & & \downarrow & & \\ H^{2i}(S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a}, \mathcal{O}) & \longrightarrow & H^{2i}(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) & \longrightarrow & H_{\mathrm{c}}^{2i+1}(Y_{n,r}, \mathcal{O}) & \longrightarrow & 0 \end{array}$$

for $n-1 \leq 2i < 2n-3$, where the horizontal lines are exact. Hence the right vertical map is surjective. Hence the claim for any odd integer i follows.

By (3.3), it suffices to show $H^{2i}_{\mathrm{c}}(Y_{n,r},\mathcal{O})=0$ for $n\le 2i<2n-2$. We have an exact sequence

$$0\to H^{2i}_{\mathrm{c}}(Y_{n,r},\mathcal{O})\xrightarrow{g_i} H^{2i}(S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a},\mathcal{O})\to H^{2i}(S_{n,\overline{\mathbb{F}}_q},\mathcal{O})$$

for $n\le 2i<2n-2$ by Lemma 3.1 (2). If $H^{2i}_{\mathrm{c}}(Y_{n,r},\mathcal{O})\neq 0$, the cokernel of g_i is torsion, because $H^{2i}(S_{n+1,\overline{\mathbb{F}}_q}/\mu_{\ell^a},\mathcal{O})$ is a free $\mathcal{O}$-module of rank one. Since $H^{2i}(S_{n,\overline{\mathbb{F}}_q},\mathcal{O})$ is a free $\mathcal{O}$-module by Lemma 3.1 (1), we obtain $H^{2i}_{\mathrm{c}}(Y_{n,r},\mathcal{O})=0$. □

We show a fundamental proposition through the paper.

Proposition 3.3. *(1) Assume $\ell\nmid q+1$. Let $\chi\in\mathrm{Hom}(\mu_{q+1},\mathcal{O}^{\times})$. We have an isomorphism*

$$H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\chi})\otimes_{\mathcal{O}}\mathbb{F}\xrightarrow{\sim} H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\overline{\chi}})$$

as $\mathbb{F}[\mathrm{U}_n(\mathbb{F}_q)]$-modules.

(2) Assume $\ell\mid q+1$. We have a short exact sequence

$$0\to H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\chi_{\ell^a}\chi_r})\otimes_{\mathcal{O}}\mathbb{F}\to H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\overline{\chi}_r})\to H^{n}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\chi_{\ell^a}\chi_r})[\mathfrak{m}]\to 0$$

as $\mathbb{F}[\mathrm{U}_n(\mathbb{F}_q)]$-modules. Furthermore, we have

$$\dim_{\mathbb{F}} H^{n}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\chi_{\ell^a}\chi_r})[\mathfrak{m}]=\begin{cases}0 & \text{if } \chi_r\neq 1,\\ \frac{1+(-1)^n}{2} & \text{if } \chi_r=1 \text{ and } \chi_{\ell^a}\neq 1,\\ \frac{1-(-1)^n}{2} & \text{if } \chi_r=1 \text{ and } \chi_{\ell^a}=1.\end{cases}$$

Proof. By (3.3) and Lemma 3.2, we have

$$H^{i}_{\mathrm{c}}(Y_{n,r},\mathcal{O})\xleftarrow{\sim} H^{i}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathcal{O})\simeq\begin{cases}0 & \text{if } i \text{ is even},\\ \mathcal{O}/\ell^a & \text{if } i \text{ is odd}\end{cases}\tag{3.13}$$

for $n\le i<2n-2$. Hence, by (3.8) with $\chi_{\ell^a}=1$ and (3.10), we have

$$\dim_{\mathbb{F}} H^{n-1}_{\mathrm{c}}(Y_{n,r},\mathbb{F})=\begin{cases}\frac{q^n-(-1)^n}{q+1}r+(-1)^n & \text{if } a=0,\\ \frac{q^n-(-1)^n}{q+1}r+\frac{1+(-1)^n}{2} & \text{if } a\ge 1.\end{cases}$$

Hence we have

$$\dim_{\mathbb{F}} H^{n}_{\mathrm{c}}(Y_{n,r},\mathscr{K}_{\chi_{\ell^a}})[\mathfrak{m}]=\begin{cases}0 & \text{if } a=0,\\ \frac{1+(-1)^n}{2} & \text{if } a\ge 1 \text{ and } \chi_{\ell^a}\neq 1,\\ \frac{1-(-1)^n}{2} & \text{if } a\ge 1 \text{ and } \chi_{\ell^a}=1\end{cases}$$

by (3.8) and (3.10). According to (3.5) and (3.7), we have the same formula for $\dim_{\mathbb{F}} H^{n}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\chi_{\ell^a}})[\mathfrak{m}]$. Hence we have

$$H^{n}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\chi_{\ell^a}\chi_r})[\mathfrak{m}]=0\quad\text{if } \chi_r\neq 1\tag{3.14}$$

by (3.9). Hence the latter claim in the claim (2) is proved. The former one is (3.4). By (3.4) and (3.14), we have an isomorphism

$$H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\chi_{\ell^a}\chi_r})\otimes_{\mathcal{O}}\mathbb{F}\simeq H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q},\mathscr{K}_{\overline{\chi}_r})\quad\text{if } \chi_r\neq 1.\tag{3.15}$$

The claim (1) for $\chi=1$ follows from (3.4) with $\chi=1$ and (3.5), and the one for $\chi\neq 1$ follows from (3.15). □

Lemma 3.4. *Assume $\ell \mid q+1$.*

(1) Assume that n is odd. Then $H^n_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O})[\mathfrak{m}]$ is a trivial representation of $\mathrm{U}_n(\mathbb{F}_q)$.

(2) Assume that $n \geq 4$ is even and $\ell \neq 2$. Let $\chi_{\ell^a} \in \mathrm{Hom}(\mu_{q+1}, \mathcal{O}^\times)$ be a non-trivial character of ℓ-power order. Then $H^n_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\chi_{\ell^a}})[\mathfrak{m}]$ is a trivial representation of $\mathrm{U}_n(\mathbb{F}_q)$.

Proof. Assume that n is odd. Since $H^{n-1}_{\mathrm{c}}(S_{n,\overline{\mathbb{F}}_q}, \mathcal{O})$ is a trivial $\mathrm{U}_n(\mathbb{F}_q)$-representation by [HM78, Theorem 1], $H^n_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O})$ is so by (3.2). Hence, the first assertion follows.

We show the second claim. By Proposition 3.3 (2), we have isomorphisms

$$\begin{aligned} H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} &\simeq H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathbb{F}), \\ H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathcal{O}) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_\ell &\simeq H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) \end{aligned} \tag{3.16}$$

and a short exact sequence

$$0 \to H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\chi_{\ell^a}}) \otimes_{\mathcal{O}} \mathbb{F} \to H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathbb{F}) \to H^n_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\chi_{\ell^a}})[\mathfrak{m}] \to 0.$$

By these and Lemma 2.7, $H^n_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\chi_{\ell^a}})[\mathfrak{m}]$ is a trivial $\mathbb{F}[\mathrm{SU}_n(\mathbb{F}_q)]$-module. Hence, $\mathrm{U}_n(\mathbb{F}_q)$ on it factoring through det by $n \geq 4$ and [Gér77, (1), (8) in the proof of Theorem 3.3]. In the sequel, we need the assumption $\ell \neq 2$, because we apply [FS82]. Let $\chi \in \mu_r^\vee \setminus \{1\}$. Then there exists a semisimple ℓ'-element s_χ in the center of $\mathrm{U}_n(\mathbb{F}_q)$ such that the character $\chi \circ \det$ of $\mathrm{U}_n(\mathbb{F}_q)$ belongs to the ℓ-block corresponding to $s_\chi^{\mathrm{U}_n(\mathbb{F}_q)}$ in the notation in [FS82, the first paragraph of §6]. Then s_χ is non-trivial by [FS82, p. 116, Theorem (6A)] using the fact that the 1-dimensional unipotent representation of $\mathrm{U}_n(\mathbb{F}_q)$ is trivial. Recall that $H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ is a unipotent $\mathrm{U}_n(\mathbb{F}_q)$-representation by Proposition 2.1. Hence it belongs to the block corresponding to $1^{\mathrm{U}_n(\mathbb{F}_q)}$. The blocks corresponding to $s_\chi^{\mathrm{U}_n(\mathbb{F}_q)}$ and $1^{\mathrm{U}_n(\mathbb{F}_q)}$ are distinct by [FS82, Theorem (5D)]. Hence, $\overline{\chi} \circ \det$ can not appear as a quotient of $H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathbb{F})$ by (3.16). Therefore, the claim follows. □

Corollary 3.5. *We have $\overline{V_n[\chi]} = H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\overline{\chi}})$ for any $\chi \in \mu_{q+1}^\vee$ if $\ell \nmid q+1$, and*

$$\begin{aligned} \overline{V_n[\chi_{\ell^a}]} + \frac{1+(-1)^n}{2} &= \overline{V_n[1]} + \frac{1-(-1)^n}{2} = H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell) \text{ for any } \chi_{\ell^a} \in \mu_{\ell^a}^\vee \setminus \{1\}, \\ \overline{V_n[\chi_{\ell^a}\chi_r]} &= \overline{V_n[\chi_r]} = H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\overline{\chi_r}}) \quad \text{for any } \chi_{\ell^a} \in \mu_{\ell^a}^\vee \text{ and } \chi_r \in \mu_r^\vee \setminus \{1\} \end{aligned}$$

if $\ell \mid q+1$ as Brauer characters of $\mathrm{U}_n(\mathbb{F}_q)$.

Proof. The claims follow from Proposition 3.3 and Lemma 3.4. □

We deduce the following proposition by combining the above theory with Corollary 3.5.

Proposition 3.6. *We assume that $n \geq 3$.*

(1) Assume $\ell \nmid q+1$. The $\mathrm{U}_n(\mathbb{F}_q)$-representations

$$H^{n-1}_{\mathrm{c}}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_\xi) \quad \text{for } \xi \in \mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{F}}_\ell^\times)$$

are irreducible. Moreover, these are distinct.

(2) *Assume $\ell \mid q+1$. Moreover, we suppose $\ell \neq 2$ if n is even. The middle cohomology $H_c^{n-1}(Y_{n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)$ has two irreducible constitutes one of which is a trivial character. The $\mathrm{U}_n(\mathbb{F}_q)$-representations*

$$H_c^{n-1}(Y_{n,\overline{\mathbb{F}}_q}, \mathscr{K}_\xi) \quad \text{for } \xi \in \mathrm{Hom}(\mu_r, \overline{\mathbb{F}}_\ell^\times) \setminus \{1\}$$

are irreducible and distinct.

Proof. Let S be as in §2.3. Let $S' \subset S$ denote the subgroup of order r. We have

$$\{V_n[\chi_r]|_{\mathrm{SU}_n(\mathbb{F}_q)} \mid \chi_r \in \mu_r^\vee \setminus \{1\}\} = \{\chi_{s,(n-1)} \mid s \in S' \setminus \{1\}\}$$

by Lemma 2.8. Hence all the claims other than distinction follow from Proposition 2.1, Lemma 2.7 and Corollary 3.5.

It remains to show that $\overline{\chi}_{(n-1,1)}$ and $\overline{\chi}_{s,(n-1)}$ for $s \in S' \setminus \{1\}$ are all different. This follows from Lemma 2.5 and Lemma 2.7. □

4 Cohomology as representation of symplectic group

4.1 Geometric setting

Let n be a positive integer. The variety S_{2n} is isomorphic to the projective variety S'_{2n} defined by $\sum_{i=1}^n (x_i^q y_i - x_i y_i^q) = 0$ in $\mathbb{P}^{2n-1}_{\mathbb{F}_q}$. We set $Y'_{2n} = \mathbb{P}^{2n-1}_{\mathbb{F}_q} \setminus S'_{2n}$. Then we have $Y_{2n} \simeq Y'_{2n,\mathbb{F}_{q^2}}$. Let

$$J = \begin{pmatrix} \mathbf{0}_n & E_n \\ -E_n & \mathbf{0}_n \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{F}_q).$$

Let Sp_{2n} be the symplectic group over $\mathbb{F}_q$ defined by the symplectic form

$$\mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q;\ (v, v') \mapsto {}^t v J v'.$$

Let $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ act on $\mathbb{P}^{2n-1}_{\mathbb{F}_q}$ by left multiplication. This action stabilizes Y'_{2n}. Let $\widetilde{Y}'_{2n}$ be the affine smooth variety defined by $\sum_{i=1}^n (x_i^q y_i - x_i y_i^q) = 1$ in $\mathbb{A}^{2n}_{\mathbb{F}_q}$. This affine variety admits a similar action of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. Similarly to (2.1), we have the $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-equivariant μ_{q+1}-covering

$$\widetilde{Y}'_{2n} \to Y'_{2n};\ (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto [x_1 : \cdots : x_n : y_1 : \cdots : y_n]. \tag{4.1}$$

Let $\mathrm{Fr}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the geometric Frobenius automorphism defined by $x \mapsto x^{q^{-1}}$ for $x \in \overline{\mathbb{F}}_q$. For a separated and of finite type scheme Z over $\mathbb{F}_q$, let Fr_q denote the pull-back of Fr_q on $H_c^i(Z_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$.

Let X'_{2n} be the affine smooth variety defined by

$$z^q - z = \sum_{i=1}^n (x_i y_i^q - x_i^q y_i)$$

in $\mathbb{A}^{2n+1}_{\mathbb{F}_q} = \mathrm{Spec}\, \mathbb{F}_q[x_1, \ldots, x_n, y_1, \ldots, y_n, z]$. We write $v = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. Let U'_{2n} be the unitary group over $\mathbb{F}_q$ defined by the skew-hermitian form

$$\mathbb{F}_{q^2}^n \times \mathbb{F}_{q^2}^n \to \mathbb{F}_{q^2};\ (v, v') \mapsto {}^t \overline{v} J v'.$$

The group $\mathrm{U}'_{2n}(\mathbb{F}_q)$ acts on $X'_{2n,\mathbb{F}_{q^2}}$ by $(v,z)\mapsto(gv,z)$ for $g\in\mathrm{U}'_{2n}(\mathbb{F}_q)$. Let $\mathbb{F}_q$ act on X'_{2n} by $z\mapsto z+\eta$ for $\eta\in\mathbb{F}_q$.

We put

$$W_{n,\psi}=H^{2n}_{\mathrm{c}}(X'_{2n,\overline{\mathbb{F}}_q},\overline{\mathbb{Q}}_\ell(n))[\psi].$$

We identify μ_{q+1} with the center of $\mathrm{U}'_{2n}(\mathbb{F}_q)$. By [IT23, Lemma 3.4], the geometric Frobenius Fr_q stabilizes $W_{n,\psi}[\chi]$ for $\chi\in\mu_{q+1}^\vee$ such that $\chi^2=1$ and acts on it as an involution. Let $\kappa\in\{\pm\}$. For such χ, let $W_{n,\psi}[\chi]^\kappa$ denote the κ-eigenspace of Fr_q.

Let ν be the quadratic character of μ_{q+1} if $p\neq 2$.

Lemma 4.1 ([IT23, Lemma 7.1])**.** *Let $n\geq 1$. We have*

$$\dim W_{n,\psi}[1]^\kappa=\frac{(q^n+\kappa)(q^n+\kappa q)}{2(q+1)},\quad \dim W_{n,\psi}[\chi]=\frac{q^{2n}-1}{q+1}$$

for $\kappa\in\{\pm\}$ and $\chi\in\mu_{q+1}^\vee\setminus\{1\}$. Further,

$$\dim W_{n,\psi}[\nu]^\kappa=\frac{q^{2n}-1}{2(q+1)}$$

for $\kappa\in\{\pm\}$ if $p\neq 2$.

Let $\Lambda\in\{\overline{\mathbb{Q}}_\ell,\mathbb{F}\}$ and $\psi\in\mathrm{Hom}(\mathbb{F}_q,\Lambda^\times)\setminus\{1\}$. Let

$$\pi'\colon\mathbb{A}^{2n}_{\mathbb{F}_q}\to\mathbb{A}^1_{\mathbb{F}_q};\ ((x_i)_{1\leq i\leq n},(y_i)_{1\leq i\leq n})\mapsto\sum_{i=1}^n(x_iy_i^q-x_i^qy_i).$$

Then we have a natural isomorphism

$$H^{2n}_{\mathrm{c}}(X'_{2n,\overline{\mathbb{F}}_q},\Lambda)[\psi]\simeq H^{2n}_{\mathrm{c}}(\mathbb{A}^{2n},\pi'^*\mathscr{L}_\psi). \tag{4.2}$$

Let $\mathbf{0}\in\mathbb{A}^{2n}_{\mathbb{F}_q}$ be the zero section. Let $U'=\pi'^{-1}(\mathbb{G}_{\mathrm{m},\mathbb{F}_q})$, $Z'=\pi'^{-1}(0)$ and $Z'^0=Z'\setminus\{\mathbf{0}\}$. In the following, for a μ_{q+1}-representations M over Λ, let $M[1]$ denote the μ_{q+1}-fixed part of M.

Lemma 4.2. *Assume that $q+1$ is invertible in Λ and $n\geq 2$. Then we have*

$$H^{2n}_{\mathrm{c}}(Y'_{2n,\overline{\mathbb{F}}_q},\Lambda)=0,\quad H^{2n+1}_{\mathrm{c}}(U'_{\overline{\mathbb{F}}_q},\pi'^*\mathscr{L}_\psi)[1]=0.$$

Proof. The first claim follows from (3.3) using the isomorphism $Y_{2n,\overline{\mathbb{F}}_q}\simeq Y'_{2n,\overline{\mathbb{F}}_q}$ and the assumption that $q+1$ is invertible in Λ. The second claim follows from the first one and [IT23, Lemma 4.3, Remark 7.11]. □

Lemma 4.3. *Assume that $q+1$ is invertible in Λ. We have an isomorphism*

$$H^{2n}_{\mathrm{c}}(\mathbb{A}^{2n},\pi'^*\mathscr{L}_\psi)[1]\simeq H^{2n-1}_{\mathrm{c}}(Y'_{2n,\overline{\mathbb{F}}_q},\Lambda(-1))$$

as representations of $\mathrm{U}'_{2n}(\mathbb{F}_q)$ and $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Proof. Since $\pi'^*\mathscr{L}_\psi|_{Z'} = \Lambda$, we have an exact sequence

$$H^{2n}_{\mathrm{c}}(\mathbb{A}^{2n}, \pi'^*\mathscr{L}_\psi)[1] \longrightarrow H^{2n}_{\mathrm{c}}(Z'_{\overline{\mathbb{F}}_q}, \Lambda)[1] \to 0$$

by Lemma 4.2 and the assumption that $q+1$ is invertible in Λ. By $n \geq 1$ and [IT23, Lemma 4.4 (3), Remark 7.11], we have

$$H^{2n}_{\mathrm{c}}(Z'_{\overline{\mathbb{F}}_q}, \Lambda)[1] \simeq H^{2n}_{\mathrm{c}}(Z'^0_{\overline{\mathbb{F}}_q}, \Lambda)[1] = H^{2n}_{\mathrm{c}}(Z'^0_{\overline{\mathbb{F}}_q}, \Lambda).$$

We have a morphism

$$H^{2n}_{\mathrm{c}}(Z'^0_{\overline{\mathbb{F}}_q}, \Lambda) \longrightarrow H^{2n-2}(S'_{2n,\overline{\mathbb{F}}_q}, \Lambda(-1))$$

by [IT23, Lemma 4.4, Remark 7.11], whose cokernel is a sum of trivial representations. Further we have a surjective morphism

$$H^{2n-2}(S'_{2n,\overline{\mathbb{F}}_q}, \Lambda(-1)) \longrightarrow H^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \Lambda(-1))$$

by the long exact sequence for $Y'_{2n} = \mathbb{P}^{2n-1}_{\mathbb{F}_q} \setminus S'_{2n}$. Consider the composition of the above morphisms

$$H^{2n}_{\mathrm{c}}(\mathbb{A}^{2n}, \pi'^*\mathscr{L}_\psi)[1] \longrightarrow H^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \Lambda(-1)). \tag{4.3}$$

We have

$$\dim H^{2n}_{\mathrm{c}}(\mathbb{A}^{2n}, \pi'^*\mathscr{L}_\psi)[1] = \dim H^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \Lambda(-1)) = \frac{q^{2n}+q}{q+1}. \tag{4.4}$$

by [IT23, (2.6), Proposition 2.6, Lemma 4.2], Proposition 2.1 (2), (3.4) and (3.5). The $\mathrm{U}_{2n}(\mathbb{F}_q)$-representation $H^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \Lambda(-1))$ is irreducible of dimension greater than 1 by Proposition 3.6 (1) and (4.4). Hence (4.3) is surjective, since the cokernel of (4.3) is a sum of trivial representations. Therefore (4.3) is an isomorphism by (4.4). □

4.2 Invariant part

In this subsection, we study some invariant parts of $H^{2n-1}_{\mathrm{c}}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathbb{F})$.

Let U be the unipotent radical of the Borel subgroup of SL_2 consisting of upper triangular matrices. Recall that we have the isomorphisms

$$\begin{aligned} &\mathbb{A}^2_{\mathbb{F}_q}/U(\mathbb{F}_q) \xrightarrow{\sim} \mathbb{A}^2_{\mathbb{F}_q};\ (x,y) \mapsto (x^q - xy^{q-1}, y), \\ &\mathbb{A}^2_{\mathbb{F}_q}/\mathrm{SL}_2(\mathbb{F}_q) \xrightarrow{\sim} \mathbb{A}^2_{\mathbb{F}_q};\ (x,y) \mapsto \left(x^q y - xy^q, \frac{x^{q^2}y - xy^{q^2}}{x^q y - xy^q}\right) \end{aligned} \tag{4.5}$$

(*cf.* [Bon11, Exercise 2.2 (b), (e)]).

We regard a product group $\mathrm{SL}_2(\mathbb{F}_q)^n$ as a subgroup of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ by the injective homomorphism

$$\mathrm{SL}_2(\mathbb{F}_q)^n \hookrightarrow \mathrm{Sp}_{2n}(\mathbb{F}_q);\ \left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\right)_{1 \leq i \leq n} \mapsto \begin{pmatrix} \mathrm{diag}(a_1,\ldots,a_n) & \mathrm{diag}(b_1,\ldots,b_n) \\ \mathrm{diag}(c_1,\ldots,c_n) & \mathrm{diag}(d_1,\ldots,d_n) \end{pmatrix}.$$

We understand the quotients $\widetilde{Y}'_{2n}/U(\mathbb{F}_q)^n$ and $\widetilde{Y}'_{2n}/\operatorname{SL}_2(\mathbb{F}_q)^n$, respectively. By (4.5), we have the isomorphisms

$$\begin{aligned}
&\widetilde{Y}'_{2n}/U(\mathbb{F}_q)^n \xrightarrow{\sim} \left\{((s_i)_{1\le i\le n},(t_i)_{1\le i\le n}) \in \mathbb{A}^{2n}_{\mathbb{F}_{q^2}} \;\middle|\; \sum_{i=1}^{n} s_i t_i = 1\right\}; \\
&((x_i)_{1\le i\le n},(y_i)_{1\le i\le n}) \mapsto \left((x_i^q - x_i y_i^{q-1})_{1\le i\le n},(y_i)_{1\le i\le n}\right), \\
&\widetilde{Y}'_{2n}/\operatorname{SL}_2(\mathbb{F}_q)^n \xrightarrow{\sim} \left\{((s_i)_{1\le i\le n},(t_i)_{1\le i\le n}) \in \mathbb{A}^{2n}_{\mathbb{F}_{q^2}} \;\middle|\; \sum_{i=1}^{n} s_i = 1\right\} \simeq \mathbb{A}^{2n-1}_{\mathbb{F}_{q^2}}; \\
&((x_i)_{1\le i\le n},(y_i)_{1\le i\le n}) \mapsto \left((x_i^q y_i - x_i y_i^q)_{1\le i\le n}, \left(\frac{x_i^{q^2} y_i - x_i y_i^{q^2}}{x_i^q y_i - x_i y_i^q}\right)_{1\le i\le n}\right).
\end{aligned} \tag{4.6}$$

The actions of $U(\mathbb{F}_q)^n$ and $\operatorname{SL}_2(\mathbb{F}_q)^n$ on $\widetilde{Y}'_{2n}$ are not free if $n \ge 2$.

The following proposition plays a key role to show Proposition 5.8 and corresponding results in the case where $p \ne 2$.

Proposition 4.4. *(1) We have*

$$H_{\mathrm{c}}^{2n-1}(\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})^{\operatorname{SL}_2(\mathbb{F}_q)^n} = 0, \quad H_{\mathrm{c}}^{2n-1}(\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})^{U(\mathbb{F}_q)^n} \simeq \mathbb{F}.$$

(2) We have

$$H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})^{\operatorname{Sp}_{2n}(\mathbb{F}_q)} = 0, \quad H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})^{U(\mathbb{F}_q)^n} \simeq \mathbb{F}.$$

Proof. We show the claim (1) by induction on n. The action of $\operatorname{SL}_2(\mathbb{F}_q)$ on $\widetilde{Y}'_2$ is free by [Bon11, Proposition 2.1.2]. Hence, the claim for $n=1$ follows from Lemma 2.3 (1), (4.6) and $H_{\mathrm{c}}^1(\mathbb{G}_{\mathrm{m}},\mathbb{F}) \simeq \mathbb{F}$.

Assume $n \ge 2$. We consider the closed subscheme R_{2n} of $\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q}$ defined by $y_n = 0$. This is isomorphic to $\mathbb{A}^1 \times \widetilde{Y}'_{2n-2,\overline{\mathbb{F}}_q}$. Let $Q_{2n} = \widetilde{Y}'_{2n,\overline{\mathbb{F}}_q} \setminus R_{2n}$. Similarly to (4.6), the quotient $Q_{2n}/U(\mathbb{F}_q)$ is isomorphic to $\mathbb{A}^{2n-2} \times \mathbb{G}_{\mathrm{m}}$. Therefore, $H_{\mathrm{c}}^i(Q_{2n},\mathbb{F})^{U(\mathbb{F}_q)}$ is zero for $i = 2n-1, 2n$ by $n \ge 2$. Hence, we have isomorphisms

$$H_{\mathrm{c}}^{2n-1}(\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})^{U(\mathbb{F}_q)} \xrightarrow{\sim} H_{\mathrm{c}}^{2n-1}(R_{2n},\mathbb{F})^{U(\mathbb{F}_q)} \simeq H_{\mathrm{c}}^{2n-3}(\widetilde{Y}'_{2n-2,\overline{\mathbb{F}}_q},\mathbb{F}),$$

which are compatible with the actions of $\operatorname{SL}_2(\mathbb{F}_q)^{n-1}$. Hence, the claim follows from the induction hypothesis.

We show the claim (2). By applying Lemma 2.3 (1) to the μ_{q+1}-torsor (4.1), we have

$$H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q},\mathbb{F}) \simeq H_{\mathrm{c}}^{2n-1}(\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})^{\mu_{q+1}} \quad \text{for any } n \ge 1. \tag{4.7}$$

We have $H_{\mathrm{c}}^{2n-1}(\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})^{\operatorname{Sp}_{2n}(\mathbb{F}_q)} = 0$. By (4.7), we have the inclusion $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q},\mathbb{F}) \subset H_{\mathrm{c}}^{2n-1}(\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})$. Hence, we have $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})^{\operatorname{Sp}_{2n}(\mathbb{F}_q)} = 0$.

The action of μ_{q+1} on $\widetilde{Y}'_{2n}$ commutes with the one of $U(\mathbb{F}_q)^n$. By the above proof, we have an isomorphism

$$H_{\mathrm{c}}^{2n-1}(\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q},\mathbb{F})^{U(\mathbb{F}_q)^n} \simeq H_{\mathrm{c}}^1(\mathbb{G}_{\mathrm{m}},\mathbb{F})$$

as $\mathbb{F}[\mu_{q+1}]$-modules, where μ_{q+1} acts on $\mathbb{G}_{\mathrm{m}}$ by the usual multiplication. Hence μ_{q+1} acts on $H_{\mathrm{c}}^{2n-1}(\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q}, \mathbb{F})^{U(\mathbb{F}_q)^n}$ trivially. Therefore, we have

$$H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathbb{F})^{U(\mathbb{F}_q)^n} \simeq H_{\mathrm{c}}^{2n-1}(\widetilde{Y}'_{2n,\overline{\mathbb{F}}_q}, \mathbb{F})^{U(\mathbb{F}_q)^n \times \mu_{q+1}} \simeq \mathbb{F}$$

by (4.7). □

4.3 Trace computations

In this subsection, we assume $p \neq 2$. An aim in this subsection is to show Proposition 4.9, which implies that the Brauer characters associated to $W_{n,\psi}[\nu]^+$ and $W_{n,\psi}[\nu]^-$ are distinct in the case where $\ell \neq 2$ (*cf.* Proposition 5.5).

Let $\left(\frac{a}{\mathbb{F}_q}\right) = a^{\frac{q-1}{2}}$ for $a \in \mathbb{F}_q^{\times}$. For $\psi \in \mathrm{Hom}(\mathbb{F}_q, \Lambda^{\times}) \setminus \{1\}$, we consider the quadratic Gauss sum

$$G(\psi) = \sum_{x \in \mathbb{F}_q^{\times}} \left(\frac{x}{\mathbb{F}_q}\right) \psi(x) \in \Lambda.$$

As a well-known fact, we have $G(\psi)^2 = \left(\frac{-1}{\mathbb{F}_q}\right) q$. In particular, we have $G(\psi) \neq 0$.

Let X be the affine smooth surface defined by $z^q - z = xy^q - x^q y$ in $\mathbb{A}^3_{\mathbb{F}_q}$. We consider the projective smooth surface $\overline{X}$ defined by

$$Z_2^q Z_3 - Z_2 Z_3^q = Z_0 Z_1^q - Z_0^q Z_1$$

in $\mathbb{P}^3_{\mathbb{F}_q} = \mathrm{Proj}\, \mathbb{F}_q[Z_0, Z_1, Z_2, Z_3]$. We regard X as an open subscheme of $\overline{X}$ by $(x, y, z) \mapsto [x : y : z : 1]$. Let $D = \overline{X} \setminus X$. Let

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q), \tag{4.8}$$

which is of order p. Let F denote the Frobenius endomorphism of X over $\mathbb{F}_q$. Let $\eta \in \mathbb{F}_q$ and $\zeta \in \mu_{q+1}$. Let $f_{\eta,\zeta}$ denote the endomorphism $F\eta\zeta u$ of $X_{\overline{\mathbb{F}}_q}$. This endomorphism extends to the one of $\overline{X}_{\overline{\mathbb{F}}_q}$ given by

$$f_{\eta,\zeta} \colon \overline{X}_{\overline{\mathbb{F}}_q} \to \overline{X}_{\overline{\mathbb{F}}_q};\ [Z_0 : Z_1 : Z_2 : Z_3] \mapsto \left[(Z_0 + Z_1)^q : Z_1^q : \zeta(Z_2 + \eta Z_3)^q : \zeta Z_3^q\right].$$

This endomorphism $f_{\eta,\zeta}$ stabilizes $D_{\overline{\mathbb{F}}_q}$.

Lemma 4.5. *We have*

$$\mathrm{Tr}(f_{\eta,\zeta}; H^*(\overline{X}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)) = \begin{cases} q^2 + q + 1 & \text{if } \eta = 0, \\ 2q^2 + q + 1 & \text{if } \eta \neq 0 \text{ and } \nu(\zeta)\left(\frac{-\eta}{\mathbb{F}_q}\right) = 1, \\ q + 1 & \text{if } \eta \neq 0 \text{ and } \nu(\zeta)\left(\frac{-\eta}{\mathbb{F}_q}\right) = -1. \end{cases}$$

Proof. By the Grothendieck–Lefschetz trace formula, it suffices to count the number of the fixed points of $f_{\eta,\zeta}$ on $\overline{X}_{\overline{\mathbb{F}}_q}$ with multiplicity. The set of the fixed points of $f_{\eta,\zeta}$ equals the union of the two sets

$$\Sigma_1 = \{[x : y : z : 1] \in \mathbb{P}^3 \mid x^q - \zeta x = -\zeta y,\ y^q = \zeta y,\ y^{q+1} = -\eta,\ z^q - z = -\eta\},$$
$$\Sigma_2 = \{[0 : z : 1 : 0] \in \mathbb{P}^3 \mid z^q = \zeta z\} \cup \{[0 : 1 : 0 : 0]\}.$$

Assume that $\eta \neq 0$ and $\nu(\zeta)\left(\frac{-\eta}{\mathbb{F}_q}\right) = 1$. We have

$$\Sigma_1 = \{[x : y : z : 1] \in \mathbb{P}^3 \mid x^q - \zeta x = -\zeta y,\ y^2 = -\eta/\zeta,\ z^q - z = -\eta\}$$

and $|\Sigma_1| = 2q^2$. One can check that the multiplicity of $f_{\eta,\zeta}$ at any point of $\Sigma_1 \cup \Sigma_2$ equals one. Hence the claim in this case follows. Assume that $\eta \neq 0$ and $\nu(\zeta)\left(\frac{-\eta}{\mathbb{F}_q}\right) = -1$. Then we have $|\Sigma_1| = 0$. The claim is shown in the same way as above. The other case is computed similarly. □

We simply write f_ζ for $f_{0,\zeta}$. Let $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$. we simply write $\mathscr{L}_\psi^0$ for the pullback of $\mathscr{L}_\psi$ under $\mathbb{A}^2 \to \mathbb{A}^1;\ (x, y) \mapsto xy^q - x^q y$.

Corollary 4.6. *We have*

$$\frac{1}{q+1} \sum_{\zeta \in \mu_{q+1}} \nu(\zeta) \operatorname{Tr}(f_\zeta; H_c^2(\mathbb{A}^2, \mathscr{L}_\psi^0(1))) = G(\psi).$$

Proof. For any i, we have isomorphisms

$$H_c^i(\mathbb{A}^2, \mathscr{L}_\psi^0(1)) \simeq H_c^i(X_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell(1))[\psi] \xrightarrow{\sim} H_c^i(\overline{X}_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell(1))[\psi],$$

where the second isomorphism follows, since the group $\mathbb{F}_q$ acts on D trivially. By the Künneth formula and [IT23, Lemma 3.3], we have

$$\begin{aligned}\sum_{\zeta \in \mu_{q+1}} \nu(\zeta) \operatorname{Tr}(f_\zeta; H_c^2(\mathbb{A}^2, \mathscr{L}_\psi^0(1))) &= \sum_{\zeta \in \mu_{q+1}} \nu(\zeta) \operatorname{Tr}(f_\zeta; H_c^*(\mathbb{A}^2, \mathscr{L}_\psi^0(1))) \\ &= \sum_{\zeta \in \mu_{q+1}} \nu(\zeta) \operatorname{Tr}(f_\zeta; H^*(\overline{X}_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell(1))[\psi]) \\ &= \frac{1}{q} \sum_{\zeta \in \mu_{q+1}} \nu(\zeta) \sum_{\eta \in \mathbb{F}_q} \psi^{-1}(\eta) \operatorname{Tr}(f_{\eta,\zeta}; H^*(\overline{X}_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell(1))).\end{aligned}$$

The last term equals

$$\begin{aligned}&\frac{1}{q^2} \sum_{\zeta \in \mu_{\frac{q+1}{2}}} \left((2q^2 + q + 1) \sum_{\eta \in (\mathbb{F}_q^\times)^2} \psi(\eta) + (q+1) \sum_{\eta \notin (\mathbb{F}_q^\times)^2} \psi(\eta) \right) \\ &- \frac{1}{q^2} \sum_{\zeta \notin \mu_{\frac{q+1}{2}}} \left((2q^2 + q + 1) \sum_{\eta \notin (\mathbb{F}_q^\times)^2} \psi(\eta) + (q+1) \sum_{\eta \in (\mathbb{F}_q^\times)^2} \psi(\eta) \right) = (q+1) G(\psi)\end{aligned}$$

by Lemma 4.5 and $\sum_{\zeta \in \mu_{q+1}} \nu(\zeta) = 0$. □

Lemma 4.7. *We have*

$$\operatorname{Tr}(F\eta\zeta; H^*(\overline{X}_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)) = \begin{cases} (q+1)(q^2+1) & \text{if } \eta = 0, \\ q^2 + q + 1 & \text{if } \eta \neq 0. \end{cases}$$

Proof. The set of the fixed points of $F\eta\zeta$ on $\overline{X}_{\overline{\mathbb{F}}_q}$ is the union of the three finite sets

$$\begin{aligned}
\Sigma_1 &= \{[x:y:z:1] \in \mathbb{P}^3 \mid z^q - z = xy^q - x^q y = -\eta,\ x^q = \zeta x,\ y^q = \zeta y\},\\
\Sigma_2 &= \{[x:y:1:0] \in \mathbb{P}^3 \mid x^q = \zeta x,\ y^q = \zeta y\},\\
\Sigma_3 &= \{[Z_0:Z_1:0:0] \in \mathbb{P}^3 \mid [Z_0:Z_1] \in \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_q)\}.
\end{aligned}$$

We have $\Sigma_1 = \emptyset$ if $\eta \neq 0$ and $|\Sigma_1| = q^3$ if $\eta = 0$. The multiplicity of $F\eta\zeta$ at any point of $\bigcup_{i=1}^3 \Sigma_i$ equals one. Hence the claim follows. □

Corollary 4.8. *Let* $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$. *We have* $\mathrm{Tr}(F\zeta; H^2_{\mathrm{c}}(\mathbb{A}^2, \mathscr{L}^0_\psi(1))) = q$.

Proof. Similarly to the proof of Corollary 4.6, we have

$$\begin{aligned}
\mathrm{Tr}(F\zeta; H^2_{\mathrm{c}}(\mathbb{A}^2, \mathscr{L}^0_\psi(1))) &= \mathrm{Tr}(F\zeta; H^*_{\mathrm{c}}(\mathbb{A}^2, \mathscr{L}^0_\psi(1)))\\
&= \frac{1}{q^2} \sum_{\eta \in \mathbb{F}_q} \psi^{-1}(\eta)\, \mathrm{Tr}(F\eta\zeta; H^*(\overline{X}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell(1))) = q
\end{aligned}$$

by Lemma 4.7. □

Proposition 4.9. *Let* $g_0 = (u, 1, \ldots, 1) \in \mathrm{SL}_2(\mathbb{F}_q)^n \subset \mathrm{Sp}_{2n}(\mathbb{F}_q)$, *where* $u \in \mathrm{SL}_2(\mathbb{F}_q)$ *is as in* (4.8). *We have*

$$W_{n,\psi}[\nu]^+(g_0) - W_{n,\psi}[\nu]^-(g_0) = q^{n-1} G(\psi).$$

In particular, we have $W_{n,\psi}[\nu]^+(g_0) \neq W_{n,\psi}[\nu]^-(g_0)$.

Proof. Let $\kappa \in \{\pm\}$. Let ν_κ be the character of $\mu_{q+1} \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ extending ν by the condition $\nu_\kappa(\mathrm{Fr}_q) = \kappa$. For $g \in \mathrm{Sp}_{2n}(\mathbb{F}_q)$, the trace $W_{n,\psi}[\nu]^\kappa(g) = W_{n,\psi}[\nu_\kappa](g)$ equals

$$\begin{aligned}
&\frac{1}{|\mu_{q+1} \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)|} \sum_{h \in \mu_{q+1} \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)} \nu_\kappa(h)^{-1} W_{n,\psi}(hg)\\
&= \frac{1}{2(q+1)} \sum_{\zeta \in \mu_{q+1}} \nu(\zeta) \left(\mathrm{Tr}(\zeta g; H^{2n}_{\mathrm{c}}(\mathbb{A}^{2n}, \pi'^* \mathscr{L}_\psi(n))) + \kappa\, \mathrm{Tr}(F\zeta g; H^{2n}_{\mathrm{c}}(\mathbb{A}^{2n}, \pi'^* \mathscr{L}_\psi(n))) \right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&W_{n,\psi}[\nu]^+(g_0) - W_{n,\psi}[\nu]^-(g_0)\\
&= \frac{1}{q+1} \sum_{\zeta \in \mu_{q+1}} \nu(\zeta)\, \mathrm{Tr}(F\zeta g_0; H^{2n}_{\mathrm{c}}(\mathbb{A}^{2n}, \pi'^* \mathscr{L}_\psi(n)))\\
&= \frac{1}{q+1} \sum_{\zeta \in \mu_{q+1}} \nu(\zeta)\, \mathrm{Tr}(F\zeta u; H^2_{\mathrm{c}}(\mathbb{A}^2, \mathscr{L}^0_\psi(1)))\, \mathrm{Tr}(F\zeta; H^2_{\mathrm{c}}(\mathbb{A}^2, \mathscr{L}^0_\psi(1)))^{n-1}\\
&= \frac{q^{n-1}}{q+1} \sum_{\zeta \in \mu_{q+1}} \nu(\zeta)\, \mathrm{Tr}(F\zeta u; H^2_{\mathrm{c}}(\mathbb{A}^2, \mathscr{L}^0_\psi(1))) = q^{n-1} G(\psi),
\end{aligned}$$

where the second equality follows from the Künneth formula, the third one follows from Corollary 4.8 and the last one follows from Corollary 4.6. □

5 Representation of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$

5.1 Weil representation of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$

Assume that $p \neq 2$ in this subsection. Let $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$. A Weil representation of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ associated to ψ is studied in [Gér77] and [How73], which we denote by ω_ψ. This has dimension q^n, and splits to two irreducible representations $\omega_{\psi,+}$ and $\omega_{\psi,-}$, which are of dimensions $(q^n+1)/2$ and $(q^n-1)/2$, respectively by [Gér77, Corollary 4.4 (a)]. For $\psi \in \mathbb{F}_q^\vee$ and $a \in \mathbb{F}_q$, let ψ_a denote the character of $\mathbb{F}_q$ defined by $x \mapsto \psi(ax)$ for $x \in \mathbb{F}_q$. For $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$, it is known that $\omega_{\psi,\kappa} \simeq \omega_{\psi_a,\kappa}$ if and only if $a \in (\mathbb{F}_q^\times)^2$ by [Shi80, Corollary 2.12].

For an element $s \in \mathrm{SO}_{2n+1}(\mathbb{F}_q)$, let $\mathrm{Spec}(s)$ denote the set of the eigenvalues of s as an element of $\mathrm{GL}_{2n+1}(\mathbb{F}_q)$. For $\kappa \in \{\pm\}$, let $s_\kappa \in \mathrm{SO}_{2n+1}(\mathbb{F}_q)$ be a semisimple element such that $\mathrm{Spec}(s_\kappa) = \{1, -1, \ldots, -1\}$ and $C_{\mathrm{SO}_{2n+1}(\mathbb{F}_q)}(s_\kappa) = \mathrm{O}_{2n}^\kappa(\mathbb{F}_q)$.

Lemma 5.1. *We have $\omega_{\psi,\kappa} \in \mathcal{E}(\mathrm{Sp}_{2n}, (s_\kappa))$ for $\kappa \in \{\pm\}$.*

Proof. We know that there are two irreducible representations in $\mathcal{E}(\mathrm{Sp}_{2n}, (s_\kappa))$ of degree $(q^n+\kappa)/2$ by [DM91, 13.23 Theorem, 13.24 Remark]. Hence the claim follows from [LOST10, Lemma 4.9]. □

By Lemma 2.6, these $\omega_{\psi,\kappa}$ remain irreducible after mod ℓ reduction. These mod ℓ irreducible modules of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ are called *Weil modules* in [GMST02, §5]. We will use this terminology later. There are just two Weil modules for each dimension (*cf.* [GMST02, p. 305]).

Lemma 5.2. *Assume that $p \neq 2$. We have $\omega_{\psi,\kappa} \notin \mathcal{E}_\ell(\mathrm{Sp}_{2n}, (1))$ for any $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$ and $\kappa \in \{\pm\}$.*

Proof. This follows from Lemma 2.5, Lemma 5.1 and $\ell \neq 2$. □

Remark 5.3. *If $n = 2$, $p \neq 2$ and $\ell = 2$, the representation $\omega_{\psi,\kappa}$ belongs to the principal block by [Whi90, p. 710].*

5.2 Frobenius action

In the sequel, every cohomology is an $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation, and every homomorphism is $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-equivariant. Let $\Lambda \in \{\overline{\mathbb{Q}}_\ell, \mathcal{O}, \overline{\mathbb{F}}_\ell\}$. Let $\psi \in \mathrm{Hom}(\mathbb{F}_q, \Lambda^\times) \setminus \{1\}$ and $\chi \in \mathrm{Hom}(\mu_{q+1}, \Lambda^\times)$ such that $\chi^2 = 1$. We set

$$H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)^\kappa = \begin{cases} H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \Lambda)^{\mathrm{Fr}_q = \kappa q^{n-1}} & \text{if } \chi = 1, \\ H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\nu)^{\mathrm{Fr}_q = -\kappa q^n G(\psi)^{-1}} & \text{if } \chi = \nu \end{cases}$$

for $\kappa \in \{\pm\}$.

Lemma 5.4. *(1) We have a decomposition*

$$H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi) \simeq H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)^+ \oplus H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)^-.$$

(2) If $\Lambda = \mathcal{O}$, we have isomorphisms

$$\begin{aligned} H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)^\kappa \otimes_{\mathcal{O}} \overline{\mathbb{F}}_\ell &\simeq H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\overline{\chi}})^\kappa, \\ H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\chi)^\kappa \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_\ell &\simeq W_{n,\psi}[\chi]^\kappa \end{aligned} \tag{5.1}$$

for $\kappa \in \{\pm\}$.

Proof. The claim (1) for $\Lambda = \overline{\mathbb{Q}}_\ell$ and the second isomorphism in (5.1) follow from Lemma 4.3 and [IT23, Lemma 3.4, Lemma 4.3, Corollary 4.6]. Then the claim (1) for $\Lambda = \mathcal{O}$ follows from Lemma 2.2 (2). Further, the claim (1) for $\Lambda = \overline{\mathbb{F}}_\ell$ and the first isomorphism in (5.1) follow from Proposition 3.3. □

5.3 Non-unipotent representation

In the following we assume that $\ell \neq 2$. For $\chi \in \mu_{q+1}^\vee$, let $s_\chi \in \mathrm{SO}_{2n+1}(\mathbb{F}_q)$ be a semisimple element corresponding to χ such that $\mathrm{Spec}(s_\chi) = \{1, \dots, 1, \zeta_\chi, \zeta_\chi^{-1}\}$ for $\zeta_\chi \in \mu_{q+1}$ and

$$C_{\mathrm{SO}_{2n+1}(\mathbb{F}_q)}(s_\chi) = \begin{cases} \mathrm{SO}_{2n-1}(\mathbb{F}_q) \times \mathrm{U}_1(\mathbb{F}_q) & \text{if } \chi^2 \neq 1, \\ \mathrm{SO}_{2n-1}(\mathbb{F}_q) \times \mathrm{O}_2^-(\mathbb{F}_q) & \text{if } p \neq 2 \text{ and } \chi = \nu. \end{cases} \tag{5.2}$$

We have

$$W_{n,\psi}[\chi] \in \mathcal{E}(\mathrm{Sp}_{2n}, (s_\chi)) \quad \text{if } \chi^2 \neq 1, \quad W_{n,\psi}[\chi]^\kappa \in \mathcal{E}(\mathrm{Sp}_{2n}, (s_\chi)) \quad \text{if } \chi^2 = 1 \tag{5.3}$$

by [IT23, Proposition 7.12]. We write as $q + 1 = \ell^a r$ with $(\ell, r) = 1$.

Proposition 5.5. *Let $\chi \in \mu_{q+1}^\vee$. We write as $\chi = \chi_{\ell^a}\chi_r$ as before.*

(1) If $\chi_r^2 \neq 1$, the Brauer character $\overline{W_{n,\psi}[\chi]}$ is irreducible.

(2) Assume $p \neq 2$. For $\kappa \in \{\pm\}$, the Brauer character $\overline{W_{n,\psi}[\nu]^\kappa}$ is irreducible.

Proof. To show the claim (1), we may assume that $\chi_{\ell^a} = 1$ by Corollary 3.5. Then the claims follow from Lemma 2.6 using Lemma 4.1, (5.2) and (5.3). □

Lemma 5.6. *The Brauer characters $\overline{W_{n,\psi}[\nu]^+}$ and $\overline{W_{n,\psi}[\nu]^-}$ are different.*

Proof. It suffices to show that the characters $W_{n,\psi}[\nu]^+$ and $W_{n,\psi}[\nu]^-$ are distinct restricted to the subset consisting of ℓ-regular elements of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. The element g_0 in Proposition 4.9 is of order p and ℓ-regular by $(p, \ell) = 1$. Hence the claim follows from Proposition 4.9. □

5.4 Unipotent representation

Lemma 5.7. *Assume that $n \geq 2$. Then the $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^-$ is irreducible.*

Proof. The $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation $W_n[1]^-$ is irreducible modulo ℓ by Lemma 4.1 and [GT04, (6), Corollary 7.5] if $p = 2$ and [GMST02, Corollary 7.4] if $p \neq 2$. Hence the claim follows from (5.1). □

Assume that $\ell \mid q+1$ and $n \geq 2$. By Proposition 3.3 (2) and Lemma 3.4 (2), we have a short exact sequence

$$0 \to H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_{\chi_{\ell^a}}) \otimes_{\mathcal{O}} \overline{\mathbb{F}}_\ell \to H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell) \xrightarrow{\delta} \mathbf{1} \to 0 \tag{5.4}$$

for any non-trivial character χ_{ℓ^a}. By Lemma 5.7, the restriction of δ to $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^-$ is a zero map. We denote by δ^+ the restriction of δ to $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+$. Then we have a short exact sequence

$$0 \to \ker \delta^+ \to H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+ \to \mathbf{1} \to 0. \tag{5.5}$$

Proposition 5.8. *Assume that $p = 2$ and $n \geq 2$. If $\ell \nmid q+1$, the $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+$ is irreducible.*

Assume $\ell \mid q+1$. Then, the $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation $\ker \delta^+$ is irreducible. The representation $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+$ is indecomposable of length two with irreducible constitutes $\mathbf{1}$ and $\ker \delta^+$.

Proof. We have

$$W_{n,\psi}[1]^- = \alpha_n, \quad W_{n,\psi}[1]^+ = \beta_n$$

in the notation of [GT04, Definition (6)] by Lemma 4.1, [GT04, (4)] and [TZ97, Lemma 4.1].

Assume $\ell \nmid q+1$. By Proposition 3.3 (1), we have

$$H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathcal{O}) \otimes_{\mathcal{O}} \overline{\mathbb{F}}_\ell \simeq H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell).$$

The representation $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+$ is irreducible by (5.1) and [GT04, Corollary 7.5 (i)].

Assume $\ell \mid q+1$. By (5.1) and [GT04, Corollary 7.5 (i)], $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+$ has two irreducible constitutes. Hence, $\ker \delta^+$ is irreducible by (5.5).

We show that $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+$ is indecomposable. Assume that it is not so. Then it is completely reducible, and is isomorphic to a direct sum of $\ker \delta^+$ and $\mathbf{1}$ by the Jordan–Hölder theorem. This is contrary to Proposition 4.4 (2). Hence, we obtain the claim. □

Proposition 5.9. *Assume that $n \geq 2$, $p \neq 2$ and $\ell \mid q+1$. The $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation $\ker \delta^+$ is irreducible. The $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+$ is indecomposable of length two with irreducible constitutes $\mathbf{1}$ and $\ker \delta^+$.*

Proof. Let U be as in §4.2. Since $U(\mathbb{F}_q)^n$ is a p-group and $\ell \neq p$, any $U(\mathbb{F}_q)^n$-representation over $\overline{\mathbb{F}}_\ell$ is semisimple. By Proposition 4.4 (2) and (5.5), we have

$$\dim(H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+)^{U(\mathbb{F}_q)^n} = 1. \tag{5.6}$$

We set $m = (q^n - q)(q^n - 1)/(2(q+1))$. We assume that $\ker \delta^+$ has more than one irreducible constitutes. By the assumption $p \neq 2$, $\ell \neq 2$ and $\ell \mid q+1$, we have $q > 3$. We have $\dim \ker \delta^+ = m + q^n - 1 < 2m$. Hence we can take an irreducible constitute of $\ker \delta^+$ whose dimension is less than m, for which we write β. By (5.5) and (5.6), we have $\dim \beta^{U(\mathbb{F}_q)^n} = 0$. Hence, β is not a trivial representation of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. By [GMST02, Theorem 2.1], β must be a Weil module. The $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)^+$ is unipotent by (4.2), Lemma 4.3 and [IT23, Corollary 7.13]. Hence $\ker \delta^+$ belongs to a unipotent block. Since β and $\ker \delta^+$ belong to the same block, this is contrary to Lemma 5.2. Hence, we obtain the first claim.

By (5.5) and the first claim, $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+$ has length two. The sequence (5.5) is non-split by Proposition 4.4 (2). Hence the second claim follows. □

Lemma 5.10. *Let $\psi \in \mathrm{Hom}(\mathbb{F}_{q,+}, \mathbb{F}^\times) \setminus \{1\}$. The canonical map*

$$H_{\mathrm{c}}^n(\mathbb{A}^n, \pi^*\mathscr{L}_\psi) \to H^n(\mathbb{A}^n, \pi^*\mathscr{L}_\psi)$$

is an isomorphism.

Proof. Let C be the affine curve over $\mathbb{F}_q$ defined by $z^q + z = t^{q+1}$ in $\mathbb{A}^2_{\mathbb{F}_q}$. We have $H_{\mathrm{c}}^1(C_{\overline{\mathbb{F}}_q}, \mathbb{F})[\psi] \simeq H_{\mathrm{c}}^1(\mathbb{A}^1, \mathscr{L}_\psi)$. Hence by the Künneth formula, we have

$$H_{\mathrm{c}}^n(\mathbb{A}^n, \pi^*\mathscr{L}_\psi) \simeq (H_{\mathrm{c}}^1(C_{\overline{\mathbb{F}}_q}, \mathbb{F})[\psi])^{\otimes n}, \quad H^n(\mathbb{A}^n, \pi^*\mathscr{L}_\psi) \simeq (H^1(C_{\overline{\mathbb{F}}_q}, \mathbb{F})[\psi])^{\otimes n}.$$

Hence it suffices to show that the canonical map $H_{\mathrm{c}}^1(C_{\overline{\mathbb{F}}_q}, \mathbb{F}) \to H^1(C_{\overline{\mathbb{F}}_q}, \mathbb{F})$ is an isomorphism. The curve C has the smooth compactification $\overline{C}$ defined by $X^qY + XY^q = Z^{q+1}$ in $\mathbb{P}^2_{\mathbb{F}_q}$. The complement $\overline{C} \setminus C$ consists of an $\mathbb{F}_q$-valued point. Hence, the claim follows. □

Lemma 5.11. *Let $\psi \in \mathrm{Hom}(\mathbb{F}_q, \mathbb{F}^\times) \setminus \{1\}$. Then $H_{\mathrm{c}}^{2n}(\mathbb{A}^{2n}, \pi'^*\mathscr{L}_\psi)[1]$ is a self-dual representation of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$.*

Proof. By Poincaré duality, we have an isomorphism

$$H^{2n}(\mathbb{A}^{2n}, \pi'^*\mathscr{L}_\psi) \simeq H_{\mathrm{c}}^{2n}(\mathbb{A}^{2n}, \pi'^*\mathscr{L}_{\psi^{-1}})^\vee.$$

Hence the claim follows from (2.2), (4.2), Lemma 5.10 and [IT23, Remark 3.2]. □

Proposition 5.12. *Assume that $n \geq 2$, $p \neq 2$ and $\ell \nmid q+1$.*

(1) The $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^+$ is irreducible.

(2) For each $\kappa \in \{\pm\}$, the $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^\kappa$ is self-dual.

Proof. For $\kappa \in \{\pm\}$, we simply write W^κ for $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_\ell)^\kappa$.

We show the first claim. Assume $(n, q) = (2, 3)$. We have $|\mathrm{Sp}_4(\mathbb{F}_3)| = 2^7 \cdot 3^4 \cdot 5$ and $\dim W^+ = 15$ by Lemma 4.1. By the assumption, we have $\ell \neq 2, 3$. Hence, the claim in this case follows from the Brauer–Nesbitt theorem.

Assume $(n, q) \neq (2, 3)$. Let m be as in the proof of Proposition 5.9. Assume that W^+ is not irreducible. By $(n, q) \neq (2, 3)$, we can take an irreducible component β of W^+ whose dimension is less than m. Then β is a trivial module or a Weil module by [GMST02, Theorem 2.1]. We know that β is a trivial module by Lemma 5.2. By the last isomorphism in Proposition 4.4 (2), W^+ has at most one trivial module as irreducible constitutes. Hence, W^+ must have length two by a similar argument as above. By $(W^+)^{\mathrm{Sp}_{2n}(\mathbb{F}_q)} = 0$ as in Proposition 4.4 (2), we have a non-split surjective homomorphism $W^+ \twoheadrightarrow \mathbf{1}$. We set $W = W^+ \oplus W^-$. Since W is self-dual by Lemma 4.3 and Lemma 5.11, we have an injective homomorphism $\mathbf{1} \hookrightarrow W$. By taking the $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-fixed part of this, we have $W^{\mathrm{Sp}_{2n}(\mathbb{F}_q)} \neq 0$, which is contrary to Proposition 4.4 (2). Hence W^+ is irreducible.

We show the second claim. For each $\kappa \in \{\pm\}$, the $\mathrm{Sp}_{2n}(\mathbb{F}_q)$-representation W^κ is irreducible by Lemma 5.7 and the first claim. Since the dimensions of W^+ and W^- are different, we obtain the claim by the self-duality of W. □

6 Mod ℓ Howe correspondence

We formulate a mod ℓ Howe correspondence for $(\mathrm{Sp}_{2n}, \mathrm{O}_2^-)$ using mod ℓ cohomology of Y'_{2n}, and show that it is compatible with the ordinary Howe correspondence.

6.1 Representation of $\mathrm{O}_2^-(\mathbb{F}_q)$

Let $W = \mathbb{F}_{q^2}$. We consider the quadratic form $Q\colon W \to \mathbb{F}_q;\ x \mapsto x^{q+1}$. Recall that O_2^- is the orthogonal group over $\mathbb{F}_q$ defined by Q. Clearly, we have $Q(\zeta x) = Q(x)$ for any $x \in W$ and $\zeta \in \mu_{q+1}$. Hence, we have a natural inclusion $\mu_{q+1} \hookrightarrow \mathrm{O}_2^-(\mathbb{F}_q)$. We regard $F_W\colon W \to W;\ x \mapsto x^q$ as an element of $\mathrm{O}_2^-(\mathbb{F}_q)$. We can easily check that $\mu_{q+1} \cap \langle F_W \rangle = \{1\}$. This group $\mathrm{O}_2^-(\mathbb{F}_q)$ is isomorphic to the dihedral group of order $2(q+1)$ by [KL90, Proposition 2.9.1]. We fix the isomorphism

$$\mu_{q+1} \rtimes (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \mathrm{O}_2^-(\mathbb{F}_q);\ (\zeta, i) \mapsto \zeta F_W^i.$$

For a pair $(\xi, \kappa) \in \mathrm{Hom}(\mu_{q+1}, \mu_2(\overline{\mathbb{F}}_\ell)) \times \{\pm\}$ such that $\chi_0^2 = 1$, the map

$$(\xi, \kappa)\colon\ \mathrm{O}_2^-(\mathbb{F}_q) \to \mu_2(\overline{\mathbb{F}}_\ell);\ (x, k) \mapsto \kappa^k \chi_0(x)$$

for $x \in \mu_{q+1}$ and $k \in \mathbb{Z}/2\mathbb{Z}$ is a character. For a character $\xi \in \mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{F}}_\ell^\times)$ such that $\xi^2 \neq 1$, the two-dimensional representation $\sigma_\xi = \mathrm{Ind}_{\mu_{q+1}}^{\mathrm{O}_2^-(\mathbb{F}_q)} \xi$ is irreducible. Note that $\sigma_\xi \simeq \sigma_{\xi^{-1}}$ as $\mathrm{O}_2^-(\mathbb{F}_q)$-representations. Any irreducible representation of $\mathrm{O}_2^-(\mathbb{F}_q)$ is isomorphic to the one of these representations.

6.2 Formulation

Let $\mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(\mathrm{O}_2^-(\mathbb{F}_q))$ be the set of irreducible representations of $\mathrm{O}_2^-(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_\ell$. Let $1 \in \mathbb{Z}/2\mathbb{Z}$ act on $\mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{F}}_\ell^\times)$ by $\xi \mapsto \xi^{-1}$. Then $\mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(\mathrm{O}_2^-(\mathbb{F}_q))$ is parametrized by

$$\{\xi \in \mathrm{Hom}(\mu_{q+1}, \overline{\mathbb{F}}_\ell^\times) \mid \xi^2 \neq 1\}/(\mathbb{Z}/2\mathbb{Z}) \cup \{(\xi, \kappa) \mid \xi \in \mathrm{Hom}(\mu_{q+1}, \mu_2(\overline{\mathbb{F}}_\ell)), \kappa \in \{\pm\}\}$$

as in §6.1.

Assume $n \geq 2$. We define a mod ℓ Howe correspondence

$$\Theta_\ell\colon\ \mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(\mathrm{O}_2^-(\mathbb{F}_q)) \to \{\text{the representations of } \mathrm{Sp}_{2n}(\mathbb{F}_q) \text{ over } \overline{\mathbb{F}}_\ell\}$$

by

$$[\xi] \mapsto H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\xi), \quad (\xi, \kappa) \mapsto H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\xi)^\kappa.$$

Theorem 6.1. *Let τ be an irreducible representation of $\mathrm{O}_2^-(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_\ell$. Then $\Theta_\ell(\tau)$ is irreducible except the case where $\ell \mid q+1$ and τ corresponds to $(1, +)$, in which case $\Theta_\ell(\tau)$ is a non-trivial extension of the trivial representation by an irreducible representation. Furthermore, if $\tau, \tau' \in \mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(\mathrm{O}_2^-(\mathbb{F}_q))$ are different, $\Theta_\ell(\tau)$ and $\Theta_\ell(\tau')$ have no irreducible constituent in common.*

Proof. The first claim follows from Proposition 5.5, Lemma 5.7, Proposition 5.8, Proposition 5.9 and Proposition 5.12.

By Lemma 2.5 and (5.3) for $\chi \in \mu_r^\vee$, the representations $H_{\mathrm{c}}^{2n-1}(Y'_{2n,\overline{\mathbb{F}}_q}, \mathscr{K}_\xi)$ of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ for $\xi \in \mathrm{Hom}(\mu_r, \overline{\mathbb{F}}_\ell^\times)$ have no irreducible constituent in common. Therefore the second claim follows from Lemma 4.1, (5.1) and Lemma 5.6. □

We extend Θ_ℓ to the set of finite-dimensional semisimple representations of $\mathrm{O}_2^-(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_\ell$ by additivity. Let Θ be the Howe correspondence for $\mathrm{Sp}_{2n}(\mathbb{F}_q) \times \mathrm{O}_2^-(\mathbb{F}_q)$ (*cf.* [IT23, §7.2]).

Proposition 6.2. *Let π be an irreducible representation of $\mathrm{O}_2^-(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$. We have an injection*

$$\overline{\Theta(\pi)}^{\mathrm{ss}} \hookrightarrow \Theta_\ell(\overline{\pi}^{\mathrm{ss}}),$$

where $\overline{(-)}^{\mathrm{ss}}$ denotes the semi-simplification of a mod ℓ reduction. The injection is an isomorphism except the cases where π corresponds to $\chi = \chi_r \chi_{\ell^a}$ and we have $\chi_r = 1$, $\chi_{\ell^a} \neq 1$.

Proof. This follows from (5.1). □

Remark 6.3. *A mod ℓ Howe correspondence is studied in [Aub94] in a different way and in a general setting under $p \neq 2$ up to semi-simplifications.*

References

[Aub94] A.-M. Aubert, Séries de Harish-Chandra de modules et correspondance de Howe modulaire, J. Algebra 165 (1994), no. 3, 576–601.

[BM89] M. Broué and J. Michel, Blocs et séries de Lusztig dans un groupe réductif fini, J. Reine Angew. Math. 395 (1989), 56–67.

[Bon11] C. Bonnafé, Representations of $\mathrm{SL}_2(\mathbb{F}_q)$, vol. 13 of Algebra and Applications, Springer-Verlag London, Ltd., London, 2011.

[Dim92] A. Dimca, Singularities and topology of hypersurfaces, Universitext, Springer-Verlag, New York, 1992.

[DM91] F. Digne and J. Michel, Representations of finite groups of Lie type, vol. 21 of London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 1991.

[FS82] P. Fong and B. Srinivasan, The blocks of finite general linear and unitary groups, Invent. Math. 69 (1982), no. 1, 109–153.

[Gér77] P. Gérardin, Weil representations associated to finite fields, J. Algebra 46 (1977), no. 1, 54–101.

[GMST02] R. M. Guralnick, K. Magaard, J. Saxl and P. H. Tiep, Cross characteristic representations of symplectic and unitary groups, J. Algebra 257 (2002), no. 2, 291–347.

[GT04] R. M. Guralnick and P. H. Tiep, Cross characteristic representations of even characteristic symplectic groups, Trans. Amer. Math. Soc. 356 (2004), no. 12, 4969–5023.

[HM78] R. Hotta and K. Matsui, On a lemma of Tate-Thompson, Hiroshima Math. J. 8 (1978), no. 2, 255–268.

[HM01] G. Hiss and G. Malle, Low-dimensional representations of special unitary groups, J. Algebra 236 (2001), no. 2, 745–767.

[How73] R. E. Howe, On the character of Weil's representation, Trans. Amer. Math. Soc. 177 (1973), 287–298.

[Ill81] L. Illusie, Théorie de Brauer et caractéristique d'Euler-Poincaré (d'après P. Deligne), in The Euler-Poincaré characteristic (French), vol. 82 of Astérisque, pp. 161–172, Soc. Math. France, Paris, 1981.

[IT19] N. Imai and T. Tsushima, Shintani lifts for Weil representations of unitary groups over finite fields, 2019, arXiv:1906.03615, to appear in Math. Res. Lett.

[IT23] N. Imai and T. Tsushima, Geometric construction of Heisenberg-Weil representations for finite unitary groups and Howe correspondences, Eur. J. Math. 9 (2023), no. 2, Paper No. 31, 34.

[KL90] P. Kleidman and M. Liebeck, The subgroup structure of the finite classical groups, vol. 129 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1990.

[LOST10] M. W. Liebeck, E. A. O'Brien, A. Shalev and P. H. Tiep, The Ore conjecture, J. Eur. Math. Soc. (JEMS) 12 (2010), no. 4, 939–1008.

[Lus77] G. Lusztig, Irreducible representations of finite classical groups, Invent. Math. 43 (1977), no. 2, 125–175.

[Sai72] M. Saito, Représentations unitaires des groupes symplectiques, J. Math. Soc. Japan 24 (1972), 232–251.

[Shi80] K. Shinoda, The characters of Weil representations associated to finite fields, J. Algebra 66 (1980), no. 1, 251–280.

[TZ97] P. H. Tiep and A. E. Zalesskii, Some characterizations of the Weil representations of the symplectic and unitary groups, J. Algebra 192 (1997), no. 1, 130–165.

[Wei64] A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math. 111 (1964), 143–211.

[Whi90] D. L. White, Decomposition numbers of $\mathrm{Sp}(4, q)$ for primes dividing $q \pm 1$, J. Algebra 132 (1990), no. 2, 488–500.

Naoki Imai
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
naoki@ms.u-tokyo.ac.jp

Takahiro Tsushima
Department of Mathematics and Informatics, Faculty of Science, Chiba University 1-33 Yayoi-cho, Inage, Chiba, 263-8522, Japan
tsushima@math.s.chiba-u.ac.jp