Langlands parameters for reductive groups over finite fields

Naoki Imai and David A. Vogan, Jr.

Abstract

We define Langlands parameters for connected reductive groups over finite fields and formulate the Langlands correspondence for finite fields using these parameters.

1 Introduction

The goal of this paper is try to formulate a parametrization of representations of a finite Chevalley group $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}}_\ell$ in terms of the Langlands dual group over $\overline{\mathbb{Q}}_\ell$ of G. One motivation is to relate this parametrization for $G(\mathbb{F}_q)$ to the (still conjectural!) Langlands parametrization of irreducible representations of groups over local fields. The reason for using the dual group over $\overline{\mathbb{Q}}_\ell$ is that Langlands' philosophy suggests that representations of a reductive group G on $k = \overline{k}$ vector spaces ought to be related to (maps of a Weil group to) a k-dual group. Such a parametrization was given for representations of $GL_n(\mathbb{F}_q)$ by Ian Macdonald in [Mac80].

For more general Chevalley groups, parametrizations involving the dual group over $\overline{\mathbb{F}}_q$ may be found in [DL76], [Lus84a], and [Lus88]. A formulation using the complex dual group is stated in [Lus84b], but even this formulation is not so well connected to Langlands parameters. The reason is that the Deligne–Lusztig and Lusztig formulations are stated in terms of a single *element* of the dual group. In the most fundamental and simplest example where $G(\mathbb{F}_q) = \mathbb{F}_q^\times$, a representation is simply a character of the multiplicative group $\mathbb{F}_q^\times$. The dual group over $\overline{\mathbb{F}}_q$ is $\overline{\mathbb{F}}_q^\times$. The special dual group elements that Deligne and Lusztig consider in [DL76] and [Lus88] are elements of order $q-1$; that is, elements of $\mathbb{F}_q^\times$. They are therefore seeking to parametrize *characters* of $\mathbb{F}_q^\times$ by *elements* of $\mathbb{F}_q^\times$. Because the multiplicative group of a finite field is cyclic, such a parametrization is possible, but it is never natural. The choices required appear in [DL76, (5.0.1)–(5.0.2)]: isomorphisms

$$\begin{aligned} \overline{\mathbb{F}}_q^\times &\xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_{p'} \\ \text{roots of 1, order prime to } p \text{ in } \overline{\mathbb{Q}}_\ell^\times &\xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_{p'}. \end{aligned}$$

The first of these choices appears also in [Lus84a, (8.4.3)]. On the right in both isomorphisms is the additive group of elements of order prime to p in $\mathbb{Q}/\mathbb{Z}$. The isomorphisms exist essentially because all cyclic groups of the same order are isomorphic; but they cannot be chosen naturally. The field $\overline{\mathbb{Q}}_\ell$ appears in the second because the methods of étale cohomology employed by Deligne and Lusztig produce representations not on complex vector spaces but rather on vector spaces over $\overline{\mathbb{Q}}_\ell$.

In this paper, we define the Weil–Deligne group for a finite field, and use these to formulate the Langlands correspondence for the finite field. Each fiber of this correspondence should be parametrized by irreducible representations of a finite group attached to a Langlands parameter, just as for other Langlands correspondences. In a subsequent paper by the first author, we construct the Langlands correspondence for finite fields under the good prime assumption, and discuss a conjectural relation between the Langlands correspondence for finite fields and the categorical local Langlands correspondence.

Acknowledgements

The origin of the ideas in this paper was presented by the second author in his talk at the workshop "New Developments in Representation Theory" at IMS, Singapore in March 2016, which the first author attended. We thank the organizers of the workshop for providing that opportunity.

2 Langlands dual group

The point of root data is that they provide a combinatorial way to specify a reductive group. Here is a statement.

Theorem 2.1. *Let k be a field with separable closure k^{sep}, and let G and be a connected reductive algebraic group defined over k. Fix a Borel pair (T, B) (not necessarily defined over k). Write $\mathcal{BR}$ for the corresponding based root datum ([ABV92, Definition 2.10]).*

(1) There is a natural action of the Galois group $\Gamma = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ on the based root datum. This action depends only on the inner class of the k-rational form G. The action factors through the Galois group of a finite Galois extension E/k.

(2) Suppose $T' \subset B' \subset G'$ is a Borel pair in another connected reductive algebraic group over k^{sep}, with based root datum $\mathcal{BR}'$, and that

$$\Xi\colon \mathcal{BR} \to \mathcal{BR}'$$

is an isomorphism of based root data, then Ξ is induced by an isomorphism of algebraic groups over k^{sep}

$$\xi\colon (T \subset B \subset G) \to (T' \subset B' \subset G')$$

The isomorphism ξ is uniquely determined up to pre-composition with $\mathrm{Ad}(t)$ (for some $t \in T$), or post-composition with $\mathrm{Ad}(t')$ (for some $t' \in T'$).

(3) Suppose that we have pinnings $\mathcal{P}$ for $T \subset B \subset G$ and $\mathcal{P}'$ for $T' \subset B' \subset G'$. Then Ξ is induced by a unique *isomorphism of algebraic groups*

$$\xi_{\mathcal{P},\mathcal{P}'}\colon (G, \mathcal{P}) \to (G', \mathcal{P}').$$

(4) *Suppose $\mathcal{BR}''$ is any based root datum. Then there is a Borel pair $T'' \subset B''$, with a pinning $\mathcal{P}''$, in a reductive algebraic group G'' over k^{sep} with the property that the corresponding based root datum is $\mathcal{BR}''$. Because of (3), the pair $(G'', \mathcal{P}'')$ is unique up to a unique isomorphism.*

(5) *In the setting of (4), suppose in addition that $\mathcal{BR}''$ is endowed with an action of $\Gamma = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ that factors through the Galois group of a finite Galois extension E/k. Then there is a unique definition of G'' over k with the following properties:*

   (a) *The torus T'' is defined over k, and the corresponding action of Γ on $X^*(T'')$ is the given one on $\mathcal{BR}''$.*

   (b) *Each map $\phi_{\alpha''}\colon \mathrm{SL}_2 \to G''$ in the pinning $\mathcal{P}''$ is defined over E (using the standard definition of SL_2 over E).*

   As a consequence of these properties, B'' is also defined over k, so that G'' is quasisplit.

Corollary 2.2. *Suppose $(G, \mathcal{P})$ is a connected reductive algebraic group with a pinning over a separably closed field k^{sep}, and $\mathcal{BR}$ is the corresponding based root datum. Write $\mathrm{Aut}(G, \mathcal{P})$ for the group of algebraic automorphisms of G preserving the pinning (in the weak sense of permuting the collection of maps from SL_2 to G). Then there is a natural isomorphism*

$$\mathrm{Aut}(G, \mathcal{P}) \simeq \mathrm{Aut}(\mathcal{BR}):$$

that is, every automorphism of the based root datum of G arises from a unique algebraic group automorphism preserving the pinning.

Every algebraic automorphism of G differs by an inner automorphism from one preserving $\mathcal{P}$; and the only inner automorphism preserving $\mathcal{P}$ is the identity. Consequently there is a semidirect product decomposition

$$\mathrm{Aut}(G) \simeq \mathrm{Int}(G) \rtimes \mathrm{Aut}(G, \mathcal{P}).$$

Definition 2.3. *Suppose G is a reductive algebraic group defined over the field k, and that $T \subset B \subset G$ is a Borel pair in $G_{k^{\mathrm{sep}}}$; we do not require that T or B be defined over k. Let $\Gamma = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ act on the based root datum $\mathcal{BR}$ as in Theorem 2.1(1). Suppose $K = K^{\mathrm{sep}}$ is another field, assumed to be separably closed. A* dual group to G over K *is a pinned reductive algebraic group $(G^\vee, \mathcal{P}^\vee)$ with based root datum equal to the dual based root datum*

$$\mathcal{BR}^\vee = (X_*, \Pi^\vee, X^*, \Pi).$$

(According to Theorem 2.1(4), the pinned group $(G^\vee, \mathcal{P}^\vee)$ is unique up to a unique isomorphism.)

We let Γ act on the dual based root datum $\mathcal{BR}^\vee$ by the inverse transpose of its action on $\mathcal{BR}$. Because of the uniqueness of $(G^\vee, \mathcal{P}^\vee)$, Γ acts on $(G^\vee, \mathcal{P}^\vee)$ by transport of structure. (This is Corollary 2.2.) The L-group of G over K *is the semidirect product*

$$^LG = \mathrm{Gal}(k^{\mathrm{sep}}/k) \ltimes G^\vee(K).$$

This action is by algebraic automorphisms, so ${}^LG$ is a pro-algebraic group over K. It is the inverse limit of the algebraic groups

$$\mathrm{Gal}(E/k) \ltimes G^\vee(K)$$

with E a finite Galois extension of k. The L-group of G depends only on the inner class of the k-rational form G.

The general shape of a "Langlands conjecture" for group representations is

$$\text{irreducible representations of } G(k) \text{ on } K\text{-vector spaces (up to equivalence) should fall into disjoint } \textit{packets } \Pi_\phi \text{ indexed by Langlands parameters } \phi \text{ (up to conjugation by } G^\vee(K)). \tag{2.1}$$

In this conjecture, a *Langlands parameter* is a group homomorphism

$$\phi \colon W_k \to {}^LG$$

subject to requirements including

(1) $\mathrm{im}(\phi)$ is semisimple;

(2) ϕ is compatible with the natural projections to $\mathrm{Gal}(k^{\mathrm{sep}}/k)$.

In this definition of Langlands parameter, the group W_k is a *Weil group* for the field k. (Weil groups were defined for local and global fields in [Wei51]. We have not tried to determine whether Weil's motivation in that paper, a good formulation of class field theory, can be made to suggest anything about the case of finite fields that we are now interested in.) A Weil group is required to be equipped with a natural homomorphism

$$\pi_k \colon W_k \to \mathrm{Gal}(k^{\mathrm{sep}}/k)$$

(so that condition (2) in the definition of Langlands parameter makes sense).

Recall that for a local field E, a Weil group W_E is a modified Galois group. In particular, there is always a homomorphism

$$\pi_E \colon W_E \to \mathrm{Gal}(E^{\mathrm{sep}}/E),$$

with dense image, whose kernel is an abelian subgroup of W_E.

3 Weil groups of finite fields

Suppose $\mathbb{F}_q$ is a finite field, and that $\overline{\mathbb{F}}_q$ is an algebraic closure. We know that

$$\Gamma = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \varprojlim_m \mathbb{Z}/m\mathbb{Z};$$

the generator of this group is the *arithmetic Frobenius*

$$\sigma_q \colon \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q, \qquad \sigma_q(x) = x^q.$$

For an algebraic variety X over $\mathbb{F}_q$, let $F \colon X \to X$ be the q-th power geometric Frobenius morphism. We will try to write σ_q for anything related to a Galois *group* action (so that σ_q is invertible, and typically only $\mathbb{F}_q$-linear) and F for anything related to an $\overline{\mathbb{F}}_q$-morphism (so often *not* invertible).

The following definition is motivated by [Mac80, §3].

Definition 3.1. *We put*

$$I_k = \varprojlim_m \mathbb{F}_{q^m}^\times$$

where the limit is taken over the norm maps

$$N_{md,m} \colon \mathbb{F}_{q^{md}}^\times \to \mathbb{F}_{q^m}^\times.$$

We define the Weil group of k *by*

$$W_k = I_k \rtimes \langle \sigma_q \rangle$$

where the conjugation by σ_q acts on I_k as q-th power.

Remark 3.2. *There is a natural system $\{\mathbb{F}_{q^m}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\}_{m\geq 0}$ of finite quotients of W_k. These look similar to $W_\mathbb{R} = \mathbb{C}^\times \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.*

Definition 3.3. *Suppose $k = \mathbb{F}_q$ is a finite field. Suppose G is a reductive algebraic group over $\mathbb{F}_q$, K is an algebraically closed field, and ${}^LG$ is the L-group of G over K (Definition 2.3). A* Weil L-parameter *is a group homomorphism*

$$\phi \colon W_k \to {}^LG$$

such that

1. *the map ϕ is compatible with projections to $\mathrm{Gal}(\overline{k}/k)$;*
2. $\mathrm{im}(\phi)$ *is semisimple; and*
3. *$\phi|_{I_k}$ factors to some finite quotient $\mathbb{F}_{q^m}^\times$ of I_k.*

For a Weil L-parameter ϕ, write ϕ_0 for $\phi|_{I_k}$. The (pointwise) stabilizer *of ϕ_0 is*

$$Z_{G^\vee}(\phi_0) = \{y \in G^\vee \mid \mathrm{Ad}(y)(\phi(w)) = \phi(w) \quad (w \in I_k)\};$$

this is a (possibly disconnected) equal rank reductive subgroup of $G^\vee$. The Dynkin diagram of $(G^\vee)^\phi$ is obtained from the extended Dynkin diagram of $G^\vee$ by deleting a nonempty set of vertices in each simple factor.

We say that two Weil L-parameters ϕ and ϕ' are equivalent if the following condition is satisfied: there is $g \in G^\vee$ such that $\mathrm{Ad}(g)(\phi_0) = \phi'_0$, and the images of $\mathrm{Ad}(g)(\phi(\sigma_q))$ and $\phi'(\sigma_q)$ in ${}^LG/Z_{G^\vee}(\phi'_0)$ are same.

We write

$$\Phi_{\mathbb{F}_q}(G)$$

for the set of equivalence classes of Weil L-parameters.

Definition 3.4. *We say that*

$$\phi_0 \colon I_k \to {}^LG$$

is an inertial L-parameter *if it is the restriction of a Weil L-parameter to I_k. We say that two inertial L-parameters ϕ_0 and ϕ'_0 are* equivalent *if they are conjugate by some $g \in G^\vee$.*

Proposition 3.5. *A torus T defined over $\mathbb{F}_q$ is the same thing as a lattice automorphism*

$$\sigma_q^*\colon X^*(T) \to X^*(T)$$

of finite order; or equivalently the inverse transpose automorphism

$$(\sigma_q)_*\colon X_*(T) \to X_*(T), \qquad (\sigma_q)_* = {}^t(\sigma_q^*)^{-1}.$$

The algebraic functions on T are $\overline{k}$-linear combinations of the rational characters in $X^*(T)$. The geometric Frobenius morphism (of algebraic groups defined over $\mathbb{F}_q$) $F\colon T \to T$ sends the character λ to $q(\sigma_q^*)^{-1}(\lambda)$.

We need also

Proposition 3.6. *Suppose T is a torus over $\mathbb{F}_q$ with geometric Frobenius map $F\colon T \to T$. Then*

(1) $T(\overline{\mathbb{F}}_q) = \bigcup_{m\geq 1} \mathbb{F}_{q^m}^\times \otimes_{\mathbb{Z}} X_*(T)$;

(2) $T(\overline{\mathbb{F}}_q) = \bigcup_{m\geq 1} T(\mathbb{F}_{q^m}) = \bigcup_{m\geq 1} T^{F^m}$.

(3) The norm map

$$\begin{aligned} N_{md,m}&\colon T(\mathbb{F}_{q^{md}}) \to T(\mathbb{F}_{q^m}),\\ &N_{md,m}(t) = t \cdot F^m(t) \cdot F^{2m}(t) \cdots F^{(d-1)m}(t) \end{aligned}$$

is surjective, with kernel equal to $(F^m - 1)T(\mathbb{F}_{q^{md}})$.

Proof. The first assertion is trivial. The second says that every point over the algebraic closure is defined over some finite extension. (The only issue requiring care is that the subgroups defined in the first assertion are *not* the points over $\mathbb{F}_{q^m}$ unless the torus is split.) The third assertion is a consequence of Proposition 3.2.2 of [Car85]. $\square$

Corollary 3.7. *Suppose T is a torus over $\mathbb{F}_q$ with geometric Frobenius map $F\colon T \to T$, and suppose K is an algebraically closed field. Write*

$$\widehat{T(\mathbb{F}_{q^m})} = \mathrm{Hom}(T(\mathbb{F}_{q^m}), K^\times)$$

for the group of one-dimensional characters over K. Write F also for the automorphism of $\widehat{T(\mathbb{F}_{q^m})}$ induced by the geometric Frobenius. Then

$$\begin{aligned} \widehat{T(\mathbb{F}_q)} &\simeq \text{characters of } T(\mathbb{F}_{q^m}) \text{ factoring through } N_{m,1}\\ &\simeq \widehat{T(\mathbb{F}_{q^m})}^F. \end{aligned}$$

Corollary 3.7 will be the key to formulating a Langlands parametrization of torus characters.

Proposition 3.8. *Suppose T is a torus over $\mathbb{F}_q$, K is an algebraically closed field, and ${}^LT$ is the L-group of T over K. Then there is a natural bijection*

$$\Phi_{\mathbb{F}_q}(T) \longleftrightarrow \widehat{T(\mathbb{F}_q)}, \qquad \phi \longleftrightarrow \xi(\phi)$$

between the equivalence classes of Weil L-parameters of T and characters of $T(\mathbb{F}_q)$.

Here both the characters and the L-groups are defined using the algebraically closed field K. We do *not* make any assumption on the characteristic of K: both sides of the bijection may be smaller if the characteristic of the representing field K divides the order of $T(\mathbb{F}_q)$.

Proof. Corollary 3.7 provides natural inclusions

$$\widehat{T(\mathbb{F}_{q^m})} \subset \widehat{T(\mathbb{F}_{q^{dm}})}$$

defined by norm maps; so we can consider the increasing "union"

$$\widehat{T(\mathbb{F}_{q^\infty})} = \varinjlim_m \widehat{T(\mathbb{F}_{q^m})}.$$

Here "increasing" for positive integers is defined by divisibility.

We use the geometric Frobenius morphism F of T described in Proposition 3.6. Fix a positive integer m_0 so that F^{m_0} is equal to multiplication by q^{m_0} (on the lattice $X^*(T)$, for example). (We can take for m_0 the order of the arithmetic Frobenius automorphism σ_q^* of Proposition 3.5.) According to Proposition 3.6, for any m divisible by m_0,

$$T(\mathbb{F}_{q^m}) = \mathrm{Hom}_{\mathbb{Z}}(X^*(T), \mathbb{F}_{q^m}^\times),$$

so

$$\begin{aligned} \widehat{T(\mathbb{F}_{q^m})} &= \mathrm{Hom}(T(\mathbb{F}_{q^m}), K^\times) \\ &= \mathrm{Hom}(\mathrm{Hom}_{\mathbb{Z}}(X^*(T), \mathbb{F}_{q^m}^\times), K^\times) \\ &= \mathrm{Hom}(\mathbb{F}_{q^m}^\times, X^*(T) \otimes_{\mathbb{Z}} K^\times) \\ &= \mathrm{Hom}(\mathbb{F}_{q^m}^\times, T^\vee(K)) \end{aligned} \qquad (m \text{ divisible by } m_0). \tag{3.1}$$

By (3.1), we conclude

$$\widehat{T(\mathbb{F}_{q^\infty})} = \mathrm{Hom}(I_k, T^\vee(K)),$$

the continuous homomorphisms that factor to some quotient $\mathbb{F}_{q^m}$ of I_k. The arithmetic Frobenius f acts on these homomorphisms by acting on the domain by σ_q^{-1} and on the range by σ_q^*. (To see that this is the correct action, one can follow the Galois action of the arithmetic Frobenius σ_q through the isomorphisms of (3.1)). The fixed points of σ_q^m under this action are the characters of $T(\mathbb{F}_{q^m})$:

$$\widehat{T(\mathbb{F}_{q^m})} = \mathrm{Hom}(I_k, T^\vee(K))^{\sigma_q^m} \qquad (m \geq 1). \tag{3.2}$$

Here there is no divisibility requirement on m. We considered first very divisible m to get a simple computation of characters; but Corollary 3.7 then gives a result for *all* $m \geq 1$.)

On the other hand, the right side of (3.2) is exactly

$$\mathrm{Hom}(I_k, T^\vee(K))^{\sigma_q^m} = \Phi_{\mathbb{F}_{q^m}}(T) \qquad (m \geq 1).$$

□

We are more or less ready to state a Langlands classification for finite groups of Lie type based on [DL76]. In order to make it more explicit, we need one more definition.

Definition 3.9. *In the setting of Definition 3.3, a* rigid Weil L-parameter *is a pair* $(\phi, T^\vee)$ *such that*

(1) $\phi\colon W_k \to {}^LG$ *is a Weil L-parameter;*

(2) $T^\vee$ *is a maximal torus in* $G^\vee$ *such that* $\phi(I_k) \subset T^\vee$ *and* $\operatorname{im}\phi \subset N_{^LG}(T^\vee)$.

We say that two rigid Weil L-parameters $(\phi, T^\vee)$ *and* $(\phi', T'^\vee)$ *are* equivalent *if the following condition is satisfied: there is* $g \in G^\vee$ *such that* $\operatorname{Ad}(g)(\phi_0) = \phi'_0$, $\operatorname{Ad}(g)(T^\vee) = T'^\vee$*, and the images of* $\operatorname{Ad}(g)(\phi(\sigma_q))$ *and* $\phi'(\sigma_q)$ *in* ${}^LG/T'^\vee$ *are same.*

Proposition 3.10. *Suppose we are in the setting of Definitions 3.3 and 3.9.*

(1) Any Weil L-parameter ϕ *is equivalent to the first term of a rigid Weil L-parameter* $(\phi', T^\vee)$.

(2) Assume that two rigid Weil L-parameters $(\phi, T^\vee)$ *and* $(\phi', T^\vee)$ *satisfy* $\phi|_{I_k} = \phi'|_{I_k}$*. Then there is* $g \in N_{Z_{G^\vee}(\phi_0)}(T^\vee)$ *such that* $\phi'(\sigma_q) = \phi(\sigma_q)g$*. They are equivalent if and only if* $g \in N_{Z_{G^\vee}(\phi_0)_0}(T^\vee)$.

Proof. For the first, the subgroup $\phi(I_k) \subset G^\vee$ is cyclic and semisimple, and therefore contained in a maximal torus $T^\vee$ of $G^\vee$; then automatically $T^\vee \subset Z_{G^\vee}(\phi_0)$. Then

$$\operatorname{Ad}(\phi(\sigma_q)^{-1})(T^\vee) = T_0^\vee$$

is another torus of $Z_{G^\vee}(\phi_0)$; so there is an element $g_0 \in Z_{G^\vee}(\phi_0)_0$ so that

$$\operatorname{Ad}(g_0)(T^\vee) = T_0^\vee.$$

If we define $g = \phi(\sigma_q)g_0$, then the conclusion is that

$$\operatorname{Ad}(g)(T^\vee) = T^\vee,$$

so that $g \in N_{^LG}(T^\vee)$. Because $g_0 \in G^\vee$, g has the same image $\sigma_q \in \operatorname{Gal}(\overline{k}/k)$ as $\phi(\sigma_q)$. Similarly, because g_0 centralizes the image of ϕ_0, we have

$$\operatorname{Ad}(g)(\phi(w)) = \operatorname{Ad}(\phi(\sigma_q))(\phi(w)) = \phi(w)^q$$

for $w \in I_k$. We define ϕ' by $\phi'|_{I_k} = \phi|_{I_k}$ and $\phi'(\sigma_q) = g$. Then $(\phi', T^\vee)$ is a rigid Weil L-parameter, proving (1).

The second assertion follows easily from the definition. □

Because it is so central to this paper, we essentially repeat the proof of the proposition by explaining how to list all rigid Langlands parameters. Write

$$T_0^\vee \subset B_0^\vee$$

for the Borel pair specified by the pinning in the definition of $G^\vee$ (Definition 2.3). Since equivalence of rigid parameters is conjugation by $G^\vee$, every rigid parameter has a representative with torus part $T_0^\vee$; so we seek to enumerate these. Now

$$\begin{aligned} N_{{}^LG}(T_0^\vee) &= \mathrm{Gal}(\overline{k}/k) \ltimes N_{{}^\vee G}(T_0^\vee), \\ N_{{}^LG}(T_0^\vee)/T_0^\vee &= \mathrm{Gal}(\overline{k}/k) \ltimes W(G^\vee, T_0^\vee). \end{aligned} \tag{3.3}$$

We write

$$W = W(G^\vee, T_0^\vee).$$

It is now clear that equivalence classes of rigid Weil L-parameters are exactly the same thing as W-orbits of pairs

$$(\phi_0, x_0), \qquad \phi_0 \colon I_k \to T_0^\vee, \quad x_0 \in \sigma_q \cdot W \subset \mathrm{Gal}(\overline{k}/k) \ltimes W$$

subject to the requirement that

$$\begin{aligned} \mathrm{Ad}(x)(\phi_0(w)) &= \phi_0(w^q), \\ \mathrm{Ad}(x^m)(\phi_0(w)) &= \phi_0(w^{q^m}). \end{aligned} \tag{3.4}$$

If m_0 is a positive integer divisible by the order of every element of $\sigma_q W$, then (3.4) implies that the image of ϕ_0 consists of elements of order dividing $q^{m_0} - 1$. This means in particular that ϕ_0 must factor to $\mathbb{F}_{q^{m_0}}^\times$. If we choose a multiplicative generator η of this group, then ϕ_0 is determined by the single element $y_0 = \phi_0(\eta) \in T^\vee$, which is required only to satisfy

$$x_0 y_0 x_0^{-1} = y_0^q. \tag{3.5}$$

That is, *for each element $x_0 \in \sigma_q \cdot W$, the set of rigid Langlands parameters* (ϕ_0, x_0) *may be identified with the finite subgroup of elements $y_0 \in T^\vee$ (necessarily of order dividing $q^m - 1$) satisfying* (3.5).

Therefore the equivalence classes of rigid Weil L-parameters are partitioned by W-conjugacy classes in the coset $\sigma_q \cdot W$; and if a conjugacy class has representative x_0, then the corresponding set of parameters may be labelled (not canonically) by the finite group defined by (3.5).

With these explicit descriptions of L-parameters in hand, we can relate them to our finite Chevalley group $G(k)$.

Proposition 3.11. *Suppose G is a connected reductive algebraic group defined over the finite field $k = \mathbb{F}_q$, $B_0 \subset G$ is a Borel subgroup defined over k, and $T_0 \subset B_0$ is a maximal torus defined over k. Let* $\mathrm{Gal}(\overline{k}/k)$ *act on* $W(G, T_0)$ *as in Theorem 2.1, and form the semidirect product*

$$\mathrm{Gal}(\overline{k}/k) \ltimes W(G, T_0).$$

Fix also a second algebraically closed field K, over which we define L-groups and represent k-groups.

(1) The semidirect product above is naturally isomorphic to (3.3).

(2) *The $G(k)$-conjugacy classes of maximal tori in G defined over k are naturally indexed (by the Frobenius action) by W-conjugacy classes in $\sigma_q \cdot W$. This bijection is given by sending $\mathrm{Ad}(g)T_0$ to the image of $g^{-1}F(g) \in N_G(T_0)$ in $W \cong \sigma_q \cdot W$.*

(3) *If $x_0 \in \sigma_q \cdot W$ and T_{x_0} is a corresponding maximal torus defined over k, then there is a natural isomorphism*

$$N_G(T_{x_0})(k)/T_{x_0}(k) \simeq W(G^\vee, T^\vee)^{x_0}.$$

(4) *In the setting of (3), there is a natural bijection*

$$\begin{aligned}&\widehat{T_{x_0}(k)}/(N_G(T_{x_0})(k)/T_{x_0}(k))\\ &\quad\simeq \{\textit{rigid Weil L-parameters } (\phi, \widehat{T}) \textit{ such that } \phi(\sigma_q) \textit{ is a lift of } x_0\}/\sim.\end{aligned}$$

(5) *In the bijection of (4), suppose two characters $\theta_1 \in \widehat{T_{x_1}(k)}$ and $\theta_2 \in \widehat{T_{x_2}(k)}$ correspond to the rigid parameters (ϕ_i, x_i). Then the pair are geometrically conjugate ([DL76, Definition 5.5]) if and only if the inertial L-parameters $\phi_1|_{I_k}$ and $\phi_2|_{I_k}$ are equivalent.*

Proof. The claim (1) follows from the construction of the L-group in Definition 2.3. The claim (2) is [DL76, Corollary 1.14]. As for (3), we have

$$N_G(T_{x_0})(k)/T_{x_0}(k) \simeq W(G, T_{x_0})^F \simeq W(G, T_0)^{x_0} \simeq W(G^\vee, T^\vee)^{x_0}$$

by [DM20, Proposition 4.4.1]. The claim (4) is Proposition 3.8. For the claim (5), suppose $m \geq 1$; consider the (surjective) norm homomorphisms

$$N \colon T_{x_i}(\mathbb{F}_{q^m}) \to T_{x_i}(\mathbb{F}_q)$$

of Proposition 3.6. In the bijections of (3), the pairs $(T_i(\mathbb{F}_{q^m}), \theta_i \circ N)$ clearly correspond to the rigid Weil L-parameters $(\phi_i|_{W_{\mathbb{F}_{q^m}}}, \widehat{T})$. We choose m so that $\mathrm{Ad}(x_i^m)$ is trivial on $T^\vee$; so equivalence of the rigid Weil L-parameters is the same as equivalence of $\phi_1|_{I_k}$ and $\phi_2|_{I_k}$ by (4). □

The proposition says that equivalence classes of rigid Weil L-parameters are in one-to-one correspondence with $G(k)$-conjugacy classes of pairs (T, θ), with T a maximal torus in G defined over k, and $\theta \in \widehat{T(k)}$. A version of this is in [DL76, (5.21.5)].

The main results of [DL76] concern the case

$$K = \overline{\mathbb{Q}}_\ell,$$

with ℓ any prime not equal to p. In that setting, Deligne and Lusztig define a virtual K-representation $R_{T_{x_0}}(\theta)$ of $G(k)$ for every (T_{x_0}, θ) as in the proposition.

Here is a way to write the Deligne–Lusztig results as a Langlands classification for finite groups of Lie type.

Theorem 3.12 (Deligne–Lusztig [DL76]). *Suppose G is a connected reductive algebraic group defined over the finite field $k = \mathbb{F}_q$. Consider*

$$K = \overline{\mathbb{Q}}_\ell,$$

with ℓ any prime not equal to p. Suppose $(\phi, T^\vee)$ is a rigid Weil L-parameter (Definition 3.9). Let (T, θ) be a corresponding pair consisting of a maximal torus defined over k and a $K^\times$-valued character of $T(k)$ (Proposition 3.11(3)). Define

$$R_T(\theta) = \textit{virtual } K\textit{-representation of } G(k)$$

as in [DL76].

(1) The virtual representations $R_{T_1}(\theta_1)$ and $R_{T_2}(\theta_2)$ have irreducible summands in common only if (T_1, θ_1) and (T_2, θ_2) are geometrically conjugate; that is, only if the corresponding rigid Weil L-parameters have equivalent underlying inertial L-parameters.

(2) Every irreducible $G(k)$ representation over K appears as an irreducible summand of some $R_T(\theta)$.

Definition 3.13. *In the setting of Theorem 5.9 (so that $K = \overline{\mathbb{Q}}_\ell$) write* $\Pi(G(k))$ *for the set of irreducible K-representations of $G(k)$.*

Suppose ϕ_0 is an inertial L-parameter. The L-packet of ϕ_0 *is*

$$\Pi_{\phi_0}(G(k)) = \{\pi \in \Pi(G(k)) \mid \pi \textit{ appears in } R_T(\theta)\}$$

for some character θ of some rational torus corresponding to a rigid Weil L-parameter $(\phi, T^\vee)$ such that $\phi|_{I_k}$ is equivalent to ϕ_0.

According to Theorem 3.12, the L-packets partition $\Pi(G(k))$.

What we want next is a more explicit description of the packets Π_{ϕ_0}. To begin, we ask how large these packets are.

Theorem 3.14 (Deligne–Lusztig [DL76, Theorem 6.8]). *In the setting of Theorem 3.12, write the decomposition of the virtual representation $R_T(\theta)$ into irreducible representations as*

$$R_T(\theta) = \sum_{\pi \in \Pi_{\phi_0}(G(k))} m(\pi)\pi,$$

with each multiplicity $m(\pi)$ an integer. Then

$$\sum_{\pi \in \Pi_{\phi_0}(G(k))} m(\pi)^2 = |(W(G,T)^F)^\theta| = |W((G^\vee)^{\phi_0}, T^\vee)^x|.$$

Here the two Weyl groups appearing are identified by Proposition 3.11(3).

Corollary 3.15. *In the setting of Theorem 3.12, suppose that the stabilizer $(G^\vee)^{\phi_0}$ of the inertial L-parameter ϕ_0 is a maximal torus in $G^\vee$. Then $\Pi_{\phi_0}(G(k))$ is a single irreducible representation, namely $\pm R_T(\theta)$.*

Proof. The hypothesis is equivalent to the triviality of $W((G^\vee)^{\phi_0}, T^\vee)$. □

Theorem 3.14 and Corollary 3.15 suggest that the size of the L-packet $\Pi_{\phi_0}(G(k))$ is controlled by the failure of the (possibly disconnected) reductive group $(G^\vee)^{\phi_0}$ to be a torus. The first step is to enlarge the Weil group to the Weil–Deligne group, which we introduce in the next section.

4 Weil–Deligne groups and a Langlands correspondence for finite fields

We want to refine the partition of the representations $\Pi(G(k))$ in (2.1). We will turn next to a rough outline of the idea introduced by Deligne for doing that.

The general shape of a Deligne's modified version of "Langlands conjecture" for group representations is

> irreducible representations of $G(k)$ on K-vector spaces (up to equivalence) should fall into disjoint *packets* Π_φ indexed by L-parameters φ of Weil–Deligne type (up to conjugation by $G^\vee(K)$).

The conjecture includes an idea about the structure of an L-packet:

> representations in a packet Π_φ should be indexed approximately by some irreducible $G^\vee(K)$-equivariant local systems of K-vector spaces on $G^\vee(K) \cdot \varphi$; that is, by irreducible K-representations of the group of connected components $G^\vee(K)^\varphi / G^\vee(K)^\varphi_0$.

An even more optimistic version (proven for real groups in [ABV92]) is

> the category of $G(k)$ representations built from a packet Π_φ should be in duality with the category of $G^\vee(K)$-equivariant perverse sheaves on $G^\vee(K) \cdot \varphi$.

In this conjecture, an *L-parameter of Weil–Deligne type* is an algebraic group homomorphism

$$\varphi\colon WD_k \to {}^L G$$

subject to requirements including

(1) $\mathrm{im}(\varphi|_{W_k})$ is semisimple;

(2) φ is compatible with the natural projections to $\mathrm{Gal}(k^{\mathrm{sep}}/k)$.

In this definition of L-parameter of Weil–Deligne type, the group WD_k is a *Weil–Deligne group* for the field k. We will not state general requirements for a Weil–Deligne group, because we do not know how to formulate them in a way consistent with our fond hope: that there is a definition of something like an archimedean Weil–Deligne "group" (or at least of its representations) that incorporates the modified notion of "Langlands parameter" introduced in [ABV92]. The same fond hope asks also that this archimedean definition be made consistent with Deligne's non-archimedean definition in [Del73, 8.3.6].

Here at any rate is a definition of a Weil–Deligne group for a finite field. Each Weil L-parameter ϕ will have several extensions to an L-parameter φ; we will try to arrange a corresponding partition of each L-packet Π_ϕ into several smaller packets.

Definition 4.1. *The* Weil–Deligne group of the finite field k *is the locally pro-algebraic group scheme*

$$WD_k = \mathbb{G}_{\mathrm{a}} \rtimes W_k$$

over $\mathbb{Z}[1/p]$, *where* $(\sigma_q^n, w) \in W_k$ *acts on* $\mathbb{G}_{\mathrm{a}}$ *by the multiplication by* q^n.

Assume that the characteristic of K *is not* p. *An* L-parameter of Weil–Deligne type *is a locally pro-algebraic group homomorphism*

$$\varphi\colon WD_k(K) \to {}^L G$$

satisfying the following conditions:

(1) *The map* φ *is compatible with projections to* $\mathrm{Gal}(\overline{k}/k)$.

(2) $\varphi|_{I_k}$ *factors to some finite quotient* $\mathbb{F}_{q^m}^{\times}$ *of* I_k.

We say that the L-parameter φ *is* special *if* $\varphi|_{\mathbb{G}_{\mathrm{a}}(K)}(1)$ *is a special unipotent element of* $(G^{\vee})^{\varphi(I_k)}$. *We say that* φ *is Frobenius semisimple if* $\varphi(\sigma_q)$ *is semisimple in* ${}^L G$.

Lemma 4.2. *Let* $g = su \in G^{\vee}$ *be the Jordan decomposition. Then we have*

$$\pi_0\left(Z_{Z_{G^{\vee}}(s)^{\circ}}(u)/Z\left(Z_{G^{\vee}}(s)^{\circ}\right)\right) \cong \pi_0\left(Z_{Z_{G^{\vee}}(s)^{\circ}}(u)/Z(G^{\vee})\right).$$

Proof. We take a Borel pair $T \subset B$ in $G_{k^{\mathrm{sep}}}$. We may assume that $s \in T^{\vee}$. Let Δ be the set of simple root of $G_{k^{\mathrm{sep}}}$ with respect to $T \subset B$. Let $I \subset \Delta$ be the subset consisting of elements which are the simple coroots of $Z_{G^{\vee}}(s)^{\circ}$. Since $\mathbb{Z}\Delta/\mathbb{Z}I$ has no torsion, the natural map

$$(X^*(T)/\mathbb{Z}I)_{\mathrm{tor}} \to (X^*(T)/\mathbb{Z}\Delta)_{\mathrm{tor}}$$

is injective. Then $\pi_0(Z(G^{\vee})) \to \pi_0(Z(Z_{G^{\vee}}(s)^{\circ}))$ is surjective, since it is identified with

$$(X^*(T)/\mathbb{Z}\Delta)_{\mathrm{tor}}^{\vee} \to (X^*(T)/\mathbb{Z}I)_{\mathrm{tor}}^{\vee}.$$

Hence, we have

$$\begin{aligned}
\pi_0(Z_{Z_{G^{\vee}}(s)^{\circ}}(u)/Z(Z_{G^{\vee}}(s)^{\circ})) &\cong \pi_0(Z_{Z_{G^{\vee}}(s)^{\circ}}(u))/\pi_0(Z(Z_{G^{\vee}}(s)^{\circ})) \\
&\cong \pi_0(Z_{Z_{G^{\vee}}(s)^{\circ}}(u))/\pi_0(Z(G^{\vee})) \\
&\cong \pi_0(Z_{Z_{G^{\vee}}(s)^{\circ}}(u)/Z(G^{\vee})),
\end{aligned}$$

where we use the surjectivity of $\pi_0(Z(G^{\vee})) \to \pi_0(Z(Z_{G^{\vee}}(s)^{\circ}))$ at the second equality. □

We put

$$A_{Z_{G^{\vee}}(\varphi(I_k))^{\circ}}(\varphi(\mathbb{G}_{\mathrm{a}})) = \pi_0(Z_{Z_{G^{\vee}}(\varphi(I_k))^{\circ}}(\varphi(\mathbb{G}_{\mathrm{a}}))/Z(Z_{G^{\vee}}(\varphi(I_k))^{\circ})).$$

We define *Lusztig's canonical quotient* $\overline{A}_{Z_{G^{\vee}}(\varphi(I_k))^{\circ}}(\varphi(\mathbb{G}_{\mathrm{a}}))$ *of* $A_{Z_{G^{\vee}}(\varphi(I_k))^{\circ}}(\varphi(\mathbb{G}_{\mathrm{a}}))$ as in [Lus84a, 13.1] using the isomorphism given by Lemma 4.2. We put $\varphi_0 = \varphi|_{\mathbb{G}_{\mathrm{a}} \times I_k}$ and

$$A(\varphi_0) = \pi_0\left(Z_{G^{\vee}}(\varphi_0)/Z(G^{\vee})\right).$$

Further we put

$$\overline{A}(\varphi_0) = A(\varphi_0)/\operatorname{Ker}(A_{Z_{G^\vee}(\varphi(I_k))^\circ}(\varphi(\mathbb{G}_\mathrm{a})) \to \overline{A}_{Z_{G^\vee}(\varphi(I_k))^\circ}(\varphi(\mathbb{G}_\mathrm{a}))).$$

To obtain an enlargement of $\overline{A}(\varphi_0)$, we put

$$\widetilde{Z}(\varphi_0) = \{(g, \sigma_q^m) \in {}^L G \mid \operatorname{Ad}((g, \sigma_q^m))(\varphi_0(x)) = \varphi_0(\operatorname{Ad}(\sigma_q^m)(x)) \text{ for all } x \in \mathbb{G}_\mathrm{a} \times I_k\}$$

and

$$\widetilde{A}(\varphi_0) = \widetilde{Z}(\varphi_0)/\operatorname{Ker}(Z_{G^\vee}(\varphi_0) \to \overline{A}(\varphi_0)).$$

We have $\varphi(\sigma_q) \in \widetilde{Z}(\varphi_0)$. Let $\overline{\varphi(\sigma_q)}$ be the image of $\varphi(\sigma_q)$ under the natural projection

$$\widetilde{Z}(\varphi_0) \to \widetilde{A}(\varphi_0).$$

We say that two L-parameters φ and φ' of Weil–Deligne type are equivalent if the following condition is satisfied: there is $g \in G^\vee$ such that $\operatorname{Ad}(g)(\varphi_0) = \varphi'_0$ and $\overline{\varphi(\sigma_q)}$ corresponds to $\overline{\varphi'(\sigma_q)}$ under the bijection

$$\widetilde{A}(\varphi_0) \cong \widetilde{A}(\varphi'_0)$$

induced by $\operatorname{Ad}(g)$, where $\varphi_0 = \varphi|_{\mathbb{G}_\mathrm{a} \times I_k}$ and $\varphi'_0 = \varphi'|_{\mathbb{G}_\mathrm{a} \times I_k}$. Let $\Phi_K(G)$ be the equivalence classes of Frobenius semisimple L-parameters over K of G. We write $\Phi_K(G)_\mathrm{sp} \subset \Phi_K(G)$ for the equivalence classes of special ones.

We put

$$A_\varphi = Z_{\overline{A}(\varphi_0)}(\overline{\varphi(\sigma_q)}).$$

The following is a formulation of the Langlands correspondence for finite fields.

Conjecture 4.3. *There is a natural map*

$$\mathcal{L}_G \colon \operatorname{Irr}_{\overline{\mathbb{Q}}_\ell}(G(k)) \to \Phi_{\overline{\mathbb{Q}}_\ell}(G)_\mathrm{sp}$$

such that, for $\varphi \in \Phi_{\overline{\mathbb{Q}}_\ell}(G)_\mathrm{sp}$, we have a bijection between $\mathcal{L}_G^{-1}(\varphi)$ and $\operatorname{Irr}_{\overline{\mathbb{Q}}_\ell}(A_\varphi)$.

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Naoki Imai
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
naoki@ms.u-tokyo.ac.jp

David A. Vogan, Jr.
2-355, Department of Mathematics MIT, Cambridge, MA 02139, USA
dav@math.mit.edu