THE JACOBSON–MOROZOV MORPHISM FOR LANGLANDS PARAMETERS IN THE RELATIVE SETTING

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ABSTRACT. We construct a moduli space LP_G of SL_2-parameters over $\mathbb{Q}$, and show that it has good geometric properties (e.g. explicitly parametrized geometric connected components and smoothness). We construct a *Jacobson–Morozov morphism* $\mathsf{JM}\colon \mathsf{LP}_G \to \mathsf{WDP}_G$ (where WDP_G is the moduli space of Weil–Deligne parameters considered by several other authors). We show that JM is an isomorphism over a dense open of WDP_G, that it induces an isomorphism between the discrete loci $\mathsf{LP}_G^{\mathrm{disc}} \to \mathsf{WDP}_G^{\mathrm{disc}}$, and that for any $\mathbb{Q}$-algebra A it induces a bijection between Frobenius semi-simple equivalence classes in $\mathsf{LP}_G(A)$ and Frobenius semi-simple equivalence classes in $\mathsf{WDP}_G(A)$ with constant (up to conjugacy) monodromy operator.

CONTENTS

1. INTRODUCTION

Motivation. A problem of fundamental importance in the study of harmonic analysis is the classification of irreducible complex admissible representations of $G(F)$ where F is a non-archimedean local field, and G is a reductive group over F. The local Langlands correspondence, a guiding principle for many areas of number theory in the last 40 years, posits a parameterization of such admissible representations in terms of equivalence classes of parameters related to the Galois theory of F. These parameters come in several forms. Chief amongst these are the *complex L-parameters* which are homomorphisms $\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(\mathbb{C})$ satisfying certain properties (cf. [SZ18, §3]), and *complex Weil–Deligne parameters* which are pairs (φ, N) where $\varphi\colon W_F \to {}^L G(\mathbb{C})$ is a homomorphism and N is a nilpotent element of the Lie algebra of $\widehat{G}(\mathbb{C})$, satisfying certain properties (cf. [GR10, §2.1]). The notion of equivalence in both cases is that of $\widehat{G}(\mathbb{C})$-conjugacy.

The classical theorem of Jacobson–Morozov (cf. [Jac79, §III.11, Theorem 17]) asserts that the *Jacobson–Morozov map* $\theta \mapsto d\theta\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right)$ gives a surjection

$$\mathsf{JM}\colon \left\{\begin{matrix}\text{Algebraic homomorphisms} \\ \theta\colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})\end{matrix}\right\} \to \left\{\begin{matrix}\text{Nilpotent elements} \\ N \in \mathrm{Lie}(\widehat{G}(\mathbb{C}))\end{matrix}\right\},$$

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which becomes a bijection on the level of $\widehat{G}(\mathbb{C})$-quotients. One may extend this to a *Jacobson–Morozov map*

$$\mathsf{JM}\colon \left\{ \begin{array}{c} \text{Complex } L\text{-parameters} \\ \psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(\mathbb{C}) \end{array} \right\} \to \left\{ \begin{array}{c} \text{Complex Weil–Deligne parameters} \\ (\varphi, N) \end{array} \right\}.$$

This map is not a bijection, even up to equivalence and, in fact, is not even surjective (see Example 3.5). That said, the Jacobson–Morozov map *does* give a bijection between equivalence classes of Frobenius semi-simple parameters (see [GR10, Proposition 2.2] or [Ima20, Proposition 1.13]), those which feature most prominently in the local Langlands correspondence. Therefore, in practice the Jacobson–Morozov map allows one to pass fairly freely between these two notions of parameter and to treat them as essentially equivalent. This is useful as each of these perspectives has its own advantages (e.g. as illustrated quite well in [GR10]).

The goal of this article is to put the above results on a *moduli-theoretic footing*. Namely we define and study a moduli space of L-parameters, and construct a *Jacobson–Morozov morphism*

$$\mathsf{JM}\colon \mathsf{LP}_G \to \mathsf{WDP}_G$$

between the moduli space of L-parameters and the moduli space of Weil–Deligne parameters. We then show that there is a natural stratification of the moduli space of Weil–Deligne parameters with the property that over each stratum the Jacobson–Morozov morphism takes a particularly simple form. Using this, we show that the Jacobson–Morozov morphism satisfies some birational-like properties, is an isomorphism over the discrete locus, and that a version of the above bijection between equivalence classes of complex Frobenius semi-simple parameters has an analogue over an arbitrary $\mathbb{Q}$-algebra.[1]

Statement of main results. Let F be a non-archimedean local field and G a reductive group over F. In §6.1 we define the *moduli space of L-parameters* for G which we denote LP_G.

Proposition 1 (see Corollary 6.8)**.** *The moduli space LP_G is smooth over $\mathbb{Q}$ and has explicitly parameterized affine connected components.*

On the other hand, let WDP_G denote the moduli space of Weil–Deligne parameters (e.g. as in [Zhu20, §3.1]). In §6.3 we define the *Jacobson–Morozov* morphism

$$\mathsf{JM}\colon \mathsf{LP}_G \to \mathsf{WDP}_G.$$

Our major result may then be stated as follows.

Theorem 1 (see Theorem 7.9 and Theorem 7.13)**.** *The Jacobson–Morozov morphism is weakly birational and induces an isomorphism $\mathsf{LP}_G^{\mathrm{disc}} \xrightarrow{\sim} \mathsf{WDP}_G^{\mathrm{disc}}$ over the discrete loci.*

Here we say a morphism of schemes $f\colon Y \to X$ is *weakly birational* if there exists a dense open subset $U \subseteq X$ such that $f\colon f^{-1}(U) \to U$ is an isomorphism. A weakly birational map f is birational if and only if f induces a bijection at the level of irreducible components. Also, the discrete loci inside of LP_G and WDP_G are defined, at least when G is semi-simple, as the locus of points where the centralizer of the universal parameter is quasi-finite over the base (see Definition 7.3 and Definition 7.11 for general definitions).

To prove Theorem 1 we stratify WDP_G by its nilpotent orbits. Denote by $\widehat{\mathcal{N}}$ the nilpotent variety for $\widehat{G}$ and form the stratification $\widehat{\mathcal{N}}^{\sqcup} := \bigsqcup_N \mathcal{O}_N$ by its $\widehat{G}$-orbits which we treat as a disconnected scheme over $\mathbb{Q}$. We then obtain a stratification $\mathsf{WDP}_G^{\sqcup}$ by pulling back $\widehat{\mathcal{N}}^{\sqcup}$ along the natural map $\mathsf{WDP}_G \to \widehat{\mathcal{N}}$. We give an explicit description of the structure of this variety.

Proposition 2 (see Corollary 5.17)**.** *The moduli space $\mathsf{WDP}_G^{\sqcup}$ is smooth over $\mathbb{Q}$ and has explicitly parameterized connected components.*

[1] The reason we do not restrict our attention to semi-simple parameters is that they do not form a representable presheaf. Thus, to do geometry we are required to work with arbitrary parameters.

The Jacobson–Morozov morphism factorizes through $\mathsf{WDP}_G^{\sqcup}$ and interacts well with the explicit decompositions indicated in Proposition 1 and Proposition 2. Utilizing this we show the following, which implies the weakly birational portion of Theorem 1.

Proposition 3 (see Theorem 7.9)**.** *The morphism* $\mathsf{JM}\colon \mathsf{LP}_G \to \mathsf{WDP}_G^{\sqcup}$ *is birational.*

A key component of our proof of Proposition 3 is a relative version of the bijection between equivalence classes of complex Frobenius semi-simple parameters. Here, Frobenius semi-simplicity is somewhat delicate and defined in Definition 5.10 and Definition 6.11.

Theorem 2 (see Theorem 6.16)**.** *For any* $\mathbb{Q}$*-algebra* A *the map*

$$\mathsf{JM}\colon \mathsf{LP}_G(A)/\widehat{G}(A) \to \mathsf{WDP}_G^{\sqcup}(A)/\widehat{G}(A)$$

is a bijection on Frobenius semi-simple elements.

We finally mention that another important ingredient in our proof of Proposition 3 is a result which may be interpreted as a stronger version of the isomorphy of the Jacobson–Morozov morphism over the discrete loci, as stated in Theorem 1. Namely, in Proposition 7.8 we show that the Jacobson–Morozov morphism is an isomorphism over the locus of points of WDP_G whose centralizer has reductive identity component. The relationship to birationality comes from Proposition 7.7 which shows that the locus of such points is dense in WDP_G and thus, a fortiori, dense in $\mathsf{WDP}_G^{\sqcup}$ (the same holds true for LP_G).

As the moduli space of Weil–Deligne parameters has featured quite prominently in recent developments in the Langlands program and adjacent fields (e.g. see [BG19], [DHKM20], [Zhu20] and [FS21]) we feel that these results will be valuable in the study of the fine structure of the space WDP_G. In particular, one may in theory reduce many questions involving 'generic' geometric structure of WDP_G to the study of LP_G. More specifically, we have stratified the geometric space WDP_G into pieces such that each stratum is smooth and (essentially) like a homogenous space for a group, and thus simple geometrically (cf. Theorem 5.16). Moreover, each of these strata is birational to similarly defined strata in the representation-theoretically simpler space LP_G. In fact, such ideas have already implicitly appeared in several important geometric results concerning WDP_G (e.g. see [BG19, §2.3]).

In addition to its potential uses to study the geometry of WDP_G, we believe that these moduli-theoretic results are clarifying in several other ways. Namely, the weak birationality of the Jacobson–Morozov morphism helps qualify in the classical setting that almost every complex Weil–Deligne parameter is in the image of the Jacobson–Morozov map. Moreover, the isomorphy over the discrete locus may also be used to deduce results of interest even in this classical case (e.g. see Proposition 3.18). Finally, we feel that our explicit description of the moduli space of L-parameters (e.g. its set of connected components) helps explain some phenomena differentiating LP_G from WDP_G as previously observed by others (c.f. the introduction to [DHKM20]).

Future directions. While our results are written over $\mathbb{Q}$, it is clear that they extend over $\mathbb{Z}[\frac{1}{N}]$ for sufficiently divisible N. Evidently one cannot hope to extend our results over all of $\mathbb{Z}[\frac{1}{p}]$ as currently written. But, as in op. cit. (and [Hel20]), the correct analogue of WDP_G over $\mathbb{Z}[\frac{1}{p}]$ does not directly involve Weil–Deligne paramters but, instead, involves a scheme of 1-cocycles for the discretization W_F^0 of the tame inertia group. One may then ask whether there is an analogous description of LP_G which allows our results to work over $\mathbb{Z}[\frac{1}{p}]$.

Also, as the morphism $\mathsf{JM}\colon \mathsf{LP}_G \to \mathsf{WDP}_G$ is weakly birational there exists a dense open subset U of WDP_G such that $\mathsf{JM}\colon \mathsf{JM}^{-1}(U) \to U$ is an isomorphism. In Proposition 3.15 below, we essentially show that the analytication $\mathsf{JM}^{-1}(U)^{\mathrm{an}}_{\mathbb{C}}$ contains all (essentially) tempered L-parameters. From a geometric perspective (e.g. from the perspective of [FS21]) it is more natural to consider ℓ-adic L-parameters instead of complex ones. One is then naturally led to the ask whether $\mathsf{JM}^{-1}(U)^{\mathrm{an}}_{\mathbb{Q}_\ell}$ contains the analogue of (essentially) tempered representations, which are the (essentially) ν-tempered representations of Dat (see [Dat05]).

Notation and conventions.

- F is a non-archimedean local field with residue field of characteristic p and size q,
- W_F is the Weil group of F,
- for a Galois extension of fields k'/k, we write the Galois group as $\Gamma_{k'/k}$ and we write Γ_k for the absolute Galois group of k,
- for a ring R we shall denote by $\mathbf{Alg}_R$ the category of R-algebras,
- we shall frequently abuse terminology and call a covariant functor $\mathbf{Alg}_R \to \mathcal{C}$ a $\mathcal{C}$-valued presheaf,
- a reductive group S-scheme H will always have connected fibers,
- for a set X we shall denote by $\underline{X}$ the associated constant scheme over $\mathbb{Q}$.

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2. Some group theoretic preliminaries

In this section we establish some notation, definitions, and basic well-known results that we shall often use without comment in the sequel. We encourage the reader to skip this section on first reading, referring back only when necessary.

2.1. **The nilpotent variety, unipotent variety, and exponential map.** Let us fix k to be a field of characteristic 0 and H to be a reductive group over k. We denote by $\mathfrak{h}$ the Lie algebra of H thought of both as a vector k-space and as a k-scheme.

Let A be a k-algebra and x an element of $\mathfrak{h}_A$. Recall then that as in [DG70, II, §6, №3] one may associate an element $\exp(Tx)$ in $H(A[\![T]\!])$ to x. We then say that x is *nilpotent* if it satisfies any of the following equivalent conditions.

Proposition 2.1. *The following are equivalent:*

(1) for all finite-dimensional representations $\rho\colon H \to \mathrm{GL}(V)$ the endomorphism $d\rho(x)$ of V_A is nilpotent,
(2) there exists a faithful finite-dimensional representation $\rho\colon H \to \mathrm{GL}(V)$ such that the endomorphism $d\rho(x)$ of V_A is nilpotent,
(3) $\exp(Tx)$ belongs to $H(A[T])$,
(4) there exists a morphism of group A-schemes $\alpha\colon \mathbb{G}_{a,A} \to H_A$ such that $x = d\alpha(1)$,

if A is in addition reduced, then (1)-(4) are equivalent to

(5) x belongs to $\mathfrak{h}_A^{\mathrm{der}}$ and $\mathrm{ad}(x)$ is a nilpotent transformation of $\mathfrak{h}_A^{\mathrm{der}}$.

Proof. The equivalence of (1)-(4) is given by [DG70, II, §6, №3, Corollaire 3.5]. To see the equivalence of (1) and (5), in the case when A is reduced, we may assume that A is a field. Let $\sigma\colon H/Z(H^{\mathrm{der}}) \to \mathrm{GL}(W)$ be the faithful representation given by taking a direct sum of $\mathrm{Ad}\colon H \to \mathrm{GL}(\mathfrak{h}^{\mathrm{der}})$ and the composition of $H \to H^{\mathrm{ab}}$ with a faithful representation of H^{ab}. It is clear that applying (1) to σ shows that (5) holds. Conversely, suppose that (5) holds, so then $d\sigma(x)$ is nilpotent. Let ρ be as in (1). We may assume that ρ is irreducible. We put $n = |Z(H^{\mathrm{der}})|$. Then $\rho^{\otimes n}\colon H \to \mathrm{GL}(V^{\otimes n})$ factors through $H/Z(H^{\mathrm{der}})$. Hence by [Del82, Proposition 3.1 (a)] $d\rho^{\otimes n}(x)$ is nilpotent. This implies that $d\rho(x)$ is nilpotent. □

Let us consider the symmetric algebra on $\mathfrak{h}^*$ (resp. the graded ideal of positive degree tensors)

$$S(\mathfrak{h}^*) = \bigoplus_{d\geqslant 0} S^d(\mathfrak{h}^*) = \mathrm{Hom}(\mathfrak{h}, \mathbb{A}^1_k), \qquad \left(\text{resp. } S^+(\mathfrak{h}^*) := \bigoplus_{d>0} S^d(\mathfrak{h}^*)\right).$$

Let $S(\mathfrak{h}^*)^H$ be the k-subalgebra of $S(\mathfrak{h}^*)$ which is invariant for the adjoint action of H on $\mathfrak{h}$ (in the sense of [MFK94, Definition 0.5 i)]). Let us then consider the radical ideal

$$S^+(\mathfrak{h}^*)^H := S^+(\mathfrak{h}^*) \cap S(\mathfrak{h}^*)^H.$$

The *nilpotent variety* of H is the closed subshceme of $\mathfrak{h}$ given by $\mathcal{N} := V\left(S^+(\mathfrak{h}^*)^H\right)$ (or $\mathcal{N}_H$ when we want to emphasize H). This is not a misnomer as for any extension k' of k we have

$$\mathcal{N}(k') = \{x \in \mathfrak{h}_{k'} : x \text{ is nilpotent}\}$$

(cf. [Jan04, §6.1, Lemma]). In particular, $\mathcal{N}$ is the unique reduced subscheme of $\mathfrak{h}$ whose $\overline{k}$-points consist of the nilpotent elements of $\mathfrak{h}_{\overline{k}}$.

The nilpotent variety $\mathcal{N}$ is an integral (cf. [Jan04, §6.2, Lemma]) finite type affine k-scheme of dimension $\dim(H) - r$ where r is the geometric rank of H (see [Jan04, §6.4]). In fact, as k is of characteristic 0, it is normal by the results of [Kos63]. Observe that the nilpotent variety is stable under the adjoint action of H. Also observe that if $f\colon H \to H'$ is a morphism of reductive groups over k it induces a morphism $df\colon \mathcal{N}_H \to \mathcal{N}_{H'}$ and satisfies $df(\mathrm{Ad}(h)(x)) = \mathrm{Ad}(f(h))(df(x))$.

Example 2.2. Let $\mathrm{Mat}_{n,k}$ be the scheme of n-by-n matrices over k, and let $I \subseteq \mathcal{O}(\mathrm{Mat}_{n,k})$ be generated by those polynomials corresponding to $(a_{ij})^n = 0$. Then, $\mathcal{N}_{\mathrm{GL}_{n,k}} = V(\sqrt{I})$.

From this example, and the functoriality of the nilpotent variety, it's easy to see that if A is a k-algebra, then one has the containment

$$\mathcal{N}(A) \subseteq \{x \in \mathfrak{h}_A : x \text{ is nilpotent}\},$$

which is an equality if A is reduced, but can differ otherwise. That said, from this containment we see that for any element x of $\mathcal{N}(A)$ we may define an element $\exp(x)$ of $H(A)$ as in [DG70, II, §6, №3, 3.7]. As this association is functorial we obtain an H-equivariant morphism of schemes $\mathcal{N} \to H$ called the *exponential morphism* and denoted by exp (or $\exp_H$ when we want to emphasize H) which is functorial in H. We would now like to describe the image of exp.

To this end, note that there exists a unique reduced closed subscheme $\mathcal{U}$ (or $\mathcal{U}_H$ when we want to emphasize H) of H such that

$$\mathcal{U}(k') = \left\{h \in H(k') : h \text{ is unipotent}\right\},$$

for all extensions k' of k (see [Spr69, Proposition 1.1]). We call $\mathcal{U}$ the *unipotent variety* associated to H. It is an integral finite type affine k-scheme of dimension $\dim(H) - r$ which is stable under the conjugation action of H (see loc. cit.). Moreover, as k is of characteristic 0, it is normal (see [Spr69, Proposition 1.3]). We observe that $\mathcal{U}$ is stable under the conjugation action of H.

Observe that exp factorizes through $\mathcal{U}$, as both are reduced, and so this may be checked on the level of $\overline{k}$-points. We have the following ombnibus result concerning the exponential morphism.

Proposition 2.3. *Let H be a reductive group over a characteristic 0 field k. Then,*

(1) the exponential map $\exp\colon \mathcal{N}_H \to \mathcal{U}_H$ is an H-equivariant isomorphism,
(2) for any k-algebra A and any x in $\mathcal{N}_H(A)$, $\mathrm{Ad}(\exp(x))$ is equal to $\sum_{i=0}^{\infty} \frac{1}{i!} \mathrm{ad}(x)^i$,
(3) for any k-algebra A and any nilpotent Lie subalgebra $\mathfrak{n}$ of $\mathfrak{h}_A$ contained in $\mathcal{N}(A)$ the subset $\exp(\mathfrak{n}) \subseteq H(A)$ is a subgroup. If the functor $\mathfrak{n} \mapsto \mathfrak{n} \otimes_A B$ is representable by a closed subgroup scheme of $\mathcal{N}_A$ then $\exp(\mathfrak{n})$ is actually a closed subgroup scheme of H_A such that $\exp(\mathfrak{n})_x$ is unipotent for all x in $\mathrm{Spec}(A)$.

Proof. For (1), as $\mathcal{N}_H$ and $\mathcal{U}_H$ are connected and normal, and exp may be checked to be a bijection on $\overline{k}$-points, this follows from Zariski's main theorem as k is of characteristic 0. Claim (2) follows by the functoriality of the exponential map (cf. [DG70, II, §6, №3, 3.7]). Finally, (3) may be deduced by the Campbell–Hausdorff series (see [Bou72, II, §6, №4, Théorème 2]). □

2.2. **The L-group and C-group.** Fix F to be a non-archimedean local field, and let G be a reductive group over F. In this subsection we define the C-group of G, which is a modification of the L-group of G that is better suited to the theory of parameters over a general $\mathbb{Q}$-algebra.

To begin, let $\Psi(G)$ denote the canonical based root datum of $G_{\overline{F}}$ (see [Kot84, §1.1] and [Mil17, §21.42]) which comes equipped with an action of Γ_F. We fix once and for all a *Langlands dual group of* G by which we mean a pinned reductive group $(\widehat{G}, \widehat{B}, \widehat{T}, \{x_\alpha\})$ over $\mathbb{Q}$ (see [Mil17, §23.d]) together with an isomorphism between $\Psi(\widehat{G}, \widehat{B}, \widehat{T})$ and $\Psi(G)^\vee$. We denote by $\widehat{\mathfrak{g}}$ the Lie algebra of $\widehat{G}$, and by $\widehat{\mathcal{N}}$ the nilpotent variety of $\widehat{G}$.

Next, let $\mathcal{W}_F$ denote the *Weil group scheme* over $\mathbb{Q}$ associated to F as in [Tat79, (4.1)]. For a $\mathbb{Q}$-algebra A one may identify $\mathcal{W}_F(A)$ with the set of continuous maps $f\colon \pi_0(\mathrm{Spec}(A)) \to W_F$ where here $\pi_0(\mathrm{Spec}(A))$ is thought of as a profinite space (cf. [Sta21, Tag 0906]) and W_F is given its usual topology. In particular, $\mathcal{W}_F(A) = \underline{W_F}(A)$ when $\pi_0(\mathrm{Spec}(A))$ is discrete (e.g. if A is connected or Noetherian), but can differ otherwise. For w in W_F we shall occasionally abuse notation and use w to also denote its image in $\mathcal{W}_F(A)$.

Note that if $d\colon W_F \to \mathbb{Z}$ is the degree map sending a lift of arithmetic Frobenius to -1, then there is a morphism of $\mathbb{Q}$-group schemes $d\colon \mathcal{W}_F \to \underline{\mathbb{Z}}$ which takes a map f to $d \circ f$. Observe that $\underline{\mathbb{Z}}$ admits an embedding of group $\mathbb{Q}$-schemes into $\mathbb{G}_{m,\mathbb{Q}}$ corresponding to $1 \mapsto q^{-1}$ and we denote the composition of d with this map by $\|\cdot\|\colon \mathcal{W}_F \to \mathbb{G}_{m,A}$. We define $\mathcal{I}_F = \ker(\|\cdot\|)$, which is an *affine scheme* equal to $\varprojlim \underline{I_F/I_K}$ as K travels over all finite extensions of F. Note that if A is a $\mathbb{Q}$-algebra and X an A-scheme locally of finite presentation then any morphism of A-schemes $\mathcal{I}_{F,A} \to X$ must factorize through $\underline{I_F/I_K}$ for some K (cf. [Sta21, Tag 01ZC]).

Remark 2.4. One reason to prefer $\mathcal{W}_F$ over the constant group scheme $\underline{W_F}$ is that the topological group $\pi_0(\mathcal{W}_F)$ is equal to W_F with its usual topology, and similarly for $\mathcal{I}_F$.

Returning to G, note that the action of Γ_F on $\Psi(G)$ gives rise to an action of Γ_F on $(\widehat{G}, \widehat{B}, \widehat{T}, \{x_\alpha\})$ and, in particular, on $\widehat{G}$ as a group $\mathbb{Q}$-scheme. We define a finite Galois extension F^* of F characterized by the equality $\Gamma_{F^*} = \ker(W_F \to \mathrm{Aut}(\widehat{G}))$. Equivalently, F^* is the minimal field splitting G^*, the quasi-split inner form of G. We write Γ_* for $\Gamma_{F^*/F}$. As $\underline{\Gamma_*}$ acts on $\widehat{G}$ and $\mathcal{W}_F$ admits $\underline{\Gamma_*}$ as a quotient, we obtain an action of $\mathcal{W}_F$ on $\widehat{G}$. Define the *L-group scheme* of G to be the group $\mathbb{Q}$-scheme ${}^LG = \widehat{G} \rtimes \mathcal{W}_F$. Observe that there is a natural inclusion $\widehat{G} \hookrightarrow {}^LG$ which identifies $\widehat{G}$ as a normal subgroup scheme of ${}^LG$. In particular, there is a natural conjugation action of ${}^LG$ on $\widehat{G}$, which in turn induces an adjoint action of ${}^LG$ on $\widehat{\mathfrak{g}}$.

As the action of $\mathcal{W}_F$ on $\widehat{G}$ factorizes through a finite quotient, we see by Lemma 2.5 below that the group presheaf associating a $\mathbb{Q}$-algebra A to $Z_0(\widehat{G})(A) := Z(\widehat{G})(A)^{\mathcal{W}_F(A)}$ is representable.

Lemma 2.5. *Let A be a $\mathbb{Q}$-algebra, H a reductive group over A, and Σ a finite group acting on H by group A-scheme automorphisms. Then, the group functor*

$$H^\Sigma\colon \mathbf{Alg}_A \to \mathbf{Grp}, \quad B \mapsto H(B)^\Sigma$$

is represented by a subgroup scheme of H smooth over A, with H° reductive over A, and such that for all A-algebras B one has the equality $\mathrm{Lie}(H^\Sigma)(B) = \mathrm{Lie}(H)(B)^\Sigma$.

Proof. Write $H = \mathrm{Spec}(R)$, then one easily verifies that $\mathrm{Spec}(R_\Sigma)$, where R_Σ is the ring of coinvariants, represents H^Σ. As A is a $\mathbb{Q}$-algebra, it is evident that R_Σ is a direct summand of R and thus H^Σ is flat over A, and thus smooth. By [SGA3-1, Exposé VIB, Corollaire 4.4] we know that H° is representable and smooth over A, and it is then reductive by [PY02, Theorem 2.1]). The claim about Lie algebras is clear as the functor of Σ-invariants preserves kernels. □

Let X^* denote the cocharacter component of $\Psi(G)$ and R^+ the positive root component, and define δ to be the element of X^* given by the sum over the elements of R^+. By our identification between $\Psi(\widehat{G}, \widehat{B}, \widehat{T})$ and $\Psi(G)^\vee$ we see that δ corresponds to an element of $X_*(\widehat{T})$ which we also denote by δ. Let us set $z_G := \delta(-1) \in \widehat{T}(\mathbb{Q})[2]$. By the proof of [BG14, Proposition 5.39],

z_G lies in $Z_0(\widehat{G})(\mathbb{Q})$. Thus, the action of $\mathcal{W}_F$ on $\widehat{G} \times \mathbb{G}_{m,\mathbb{Q}}$ (with trivial action on the second component) fixes the pair $(z_G, -1)$. Therefore, $\mathcal{W}_F$ acts on $\check{G} := (G \times \mathbb{G}_{m,\mathbb{Q}})/\langle (z_G, -1)\rangle$. We then define the *C-group scheme* of G to be ${}^C G = \check{G} \rtimes \mathcal{W}_F$. Note that by [BG14, Proposition 5.39] there exists a central extension $\widetilde{G}$ of G such that ${}^C G$ is naturally isomorphic to ${}^L\widetilde{G}$.

The group $\widehat{G}$ admits a natural embedding into $\check{G}$, with normal image, via the first factor, and therefore we obtain a conjugation action of ${}^C G$ on $\widehat{G}$, and thus an adjoint action of ${}^C G$ on $\widehat{\mathfrak{g}}$. Also, the morphism

$$(\widehat{G} \times \mathbb{G}_{m,\mathbb{Q}}) \rtimes \mathcal{W}_F \to \mathbb{G}_{m,\mathbb{Q}} \times \mathcal{W}_F, \qquad (g, z, w) \mapsto (z^2, w)$$

annihilates $\langle (z_G, -1)\rangle$, and thus induces a morphism

$$p_C = (p_{\mathbb{G}_m}, p_{\mathcal{W}_F}) \colon {}^C G \to \mathbb{G}_{m,\mathbb{Q}} \times \mathcal{W}_F.$$

Finally, we observe that if k is an extension of $\mathbb{Q}$, and c is any element of k such that $c^2 = q$, then there is a morphism $i_c \colon {}^L G_k \to {}^C G_k$ obtained as the composition

$${}^L G_k \xrightarrow{(g,w)\mapsto(g,c^{-d(w)},w)} (\widehat{G}_k \times \mathbb{G}_{m,k}) \rtimes \mathcal{W}_{F,k} \to {}^C G_k.$$

2.3. **Scheme of homomorphisms and cross-section homomorphisms.** We establish here some terminology and basic results concering the scheme of homomorphisms as well as the scheme of cross-section homomorphisms (in the sense of [DHKM20, Appendix A]). Throughout the following we fix k to be field of characteristic 0.

Scheme of homomorphisms. Let H and H' be reductive groups over k with Lie algebras $\mathfrak{h}$ and $\mathfrak{h}'$. For a k-algebra A denote by $\mathrm{Hom}(H_A, H'_A)$ the set of group A-scheme morphisms $H_A \to H'_A$. Consider the following functor

$$\underline{\mathrm{Hom}}(H, H') \colon \mathbf{Alg}_k \to \mathbf{Set}, \qquad A \mapsto \mathrm{Hom}(H_A, H'_A),$$

and define the functor $\underline{\mathrm{Hom}}(\mathfrak{h}, \mathfrak{h}')$ similarly, both of which carry a natural H'-conjugation action.

Proposition 2.6. *The following statements hold true.*

(1) The functor $\underline{\mathrm{Hom}}(H, H')$ is representable by a smooth k-scheme for which the action map

$$\mu \colon H' \times \underline{\mathrm{Hom}}(H, H') \to \underline{\mathrm{Hom}}(H, H')$$

is smooth,

(2) if H is semi-simple then $\underline{\mathrm{Hom}}(H, H')$ is affine, and if H furthemore simply connected then the map

$$\underline{\mathrm{Hom}}(H, H') \to \underline{\mathrm{Hom}}(\mathfrak{h}, \mathfrak{h}'), \qquad f \mapsto df,$$

is an H'-equivariant isomorphism,

(3) for any k-algebra A the natural map

$$\mathrm{Hom}(H_A, H'_A) \to \mathrm{Hom}(H(A), H'(A))$$

is injective.

Proof. Statements (1) and (2) follow from [SGA3-3, Exp. XXIV, Proposition 7.3.1] and [Bri21, Theorem 2] respectively. Statement (3) follows from Proposition 2.7 below as H and H' are integral and unirational (see [Mil17, Summary 1.36, Theorem 3.23, and Theorem 17.93]). □

Proposition 2.7. *Suppose that X and Y are finite type integral k-schemes with X unirational. Then for any k-algebra A, the natural map*

$$\mathrm{Hom}(X_A, Y_A) \to \mathrm{Hom}(X(A), Y(A))$$

is injective.

Proof. One quickly reduces to the case when $X = D(w) \subseteq \mathbb{A}^n_k$ for w in $k[x_1, \dots, x_n]$, $Y = \mathbb{A}^1_k$, f lies in $A[x_1, \dots, x_n]$ and g is the zero map. As $X(F) \to X(A)$ is injective, we will be done if we can show that f does not vanish on $D(w)(k)$. If $\{a_i\}_{i \in I}$ is a basis of A as a k-vector space then we may write $f = \sum_{i \in I} a_i f_i$ where $f_i \in k[x_1, \dots, x_n]$. As f is non-zero there exists some i such that f_i is non-zero. As $D(w)(k)$ is Zariski dense in $\mathbb{A}^n_k$ as k is infinite, there then exists some x in $D(w)(k)$ such that $f_i(x) \neq 0$. Then, by setup, $f(x) \neq 0$. □

In the future, we call a homomorphism of groups $H(A) \to H'(A)$ *algebraic* if it is the map on A-points of a morphism (necessarily unique) of group A-schemes $H_A \to H'_A$.

Schemes of cross-section homomorphisms. Fix an abstract group Σ and a reductive group H over k. We then consider the presheaf

$$\underline{\mathrm{Hom}}(\Sigma, H) \colon \mathbf{Alg}_k \to \mathbf{Set}, \qquad \mathrm{Hom}(\Sigma, H(A)) = \mathrm{Hom}(\underline{\Sigma}_A, H_A).$$

This presheaf clearly carries an H-conjugation action. If, in addition, Σ acts on H by group k-scheme morphisms then for a k-algebra A we say a homomorphism $f \colon \underline{\Sigma}_A \to H_A \rtimes \underline{\Sigma}_A$ is a *cross-section homomorphism* over A if $p_2(f(\sigma)) = \sigma$ for all σ, where $p_2 \colon H_A \rtimes \underline{\Sigma}_A \to \underline{\Sigma}_A$ is the scheme-theoretic projection. We denote by $\underline{Z}^1(\Sigma, H)(A)$ the set of cross-section homomorphisms over A which is clearly a presheaf on k-algebras which carries an H-conjugation action[2].

Proposition 2.8 ([DHKM20, Lemma A.1 and Corollary A.2])**.** *Suppose that Σ is finite. Then, $\underline{\mathrm{Hom}}(\Sigma, H)$ (resp. $\underline{Z}^1(\Sigma, H)$) is represented by a finite type smooth affine k-scheme. Moreover, for all k-algebras A, and all f in $\underline{\mathrm{Hom}}(\Sigma, H)(A)$ (resp. $\underline{Z}^1(\Sigma, H)(A)$) the orbit map*

$$\mu_f \colon H_A \to \underline{\mathrm{Hom}}(\Sigma, H)_A, \qquad \left(\text{resp. } \mu_f \colon H_A \to \underline{Z}^1(\Sigma, H)_A \right)$$

is smooth.

2.4. **Transporter and centralizer schemes.** Let R be a ring, H a group-valued functor on $\mathbf{Alg}_R$, and X a set-valued functor on $\mathbf{Alg}_R$. Then, for an R-algebra S and two elements α and β of $X(S)$ we define the *transporter set* to be

$$\mathrm{Transp}_H(\alpha, \beta) := \{h \in H(S) : h \cdot \alpha = \beta\}.$$

We then define the *transporter presheaf* to be the presheaf

$$\underline{\mathrm{Transp}}_H(\alpha, \beta) \colon \mathbf{Alg}_S \to \mathbf{Set}, \qquad T \mapsto \mathrm{Transp}_H(\alpha_T, \beta_T).$$

We abbreviate $\underline{\mathrm{Transp}}_H(\beta, \beta)$ to $Z_H(\beta)$ and call it the *centralizer presheaf*, which is clearly a group presheaf. We then have the following obvious proposition.

Proposition 2.9. *Suppose that H is a group R-scheme and that X is a separated R-scheme of finite presentation. Then, for any R-algebra S and any elements α and β of $X(S)$, the presheaves $\underline{\mathrm{Transp}}_H(\alpha, \beta)$ and $Z_H(\beta)$ are representable by closed finitely presented subschemes of H_S. Moreover, for any S-algebra T one has the natural equalities*

$$\underline{\mathrm{Transp}}_H(\alpha, \beta)_T = \underline{\mathrm{Transp}}_H(\alpha_T, \beta_T), \qquad Z_H(\beta)_T = Z_H(\beta_T).$$

3. The classical setting

In this section we recall the Jacobson–Morozov theorem and the Jacobson–Morozov theorem for parameters in their classical settings. This will not only serve to emphasize the results we wish to geometrize, but will play an important role in the proof of these more general results.

[2]The notation $\underline{Z}^1(\Sigma, H)$ is used as this object is equal to the scheme of 1-cocycles in [DHKM20, Appendix A].

3.1. **The Jacobson–Morozov theorem.** Let k be a field of characteristic 0 and H an algebraic group over k such that H° is reductive. It will be useful to explicitly name the matrices

$$e_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which form a k-basis of the Lie algebra $\mathfrak{sl}_{2,k}$. We then have the Jacobson–Morozov Theorem as follows.

Theorem 3.1 (cf.[Bou75, VIII, §11, №2, Proposition 2 and Corollaire])**.** *The map*

$$\mathsf{JM}\colon \operatorname{Hom}(\mathrm{SL}_{2,k}, H) \to \mathcal{N}(k), \qquad \theta \mapsto d\theta(e_0)$$

is an $H(k)$-equivariant surjection, and induces a bijection

$$\operatorname{Hom}(\mathrm{SL}_{2,k}, H)/H(k) \to \mathcal{N}(k)/H(k).$$

Let us call a triple (e, h, f) of elements an $\mathfrak{sl}_2$-*triple* in $\mathfrak{h}$ if the following equalities hold

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Let us denote by $\mathcal{T}(k)$ (or $\mathcal{T}_H(k)$ when we want to emphasize H), the set of $\mathfrak{sl}_2$-triples in $\mathfrak{h}$. The natural adjoint action of $H(k)$ on $\mathfrak{h}$ induces an action of $H(k)$ on $\mathcal{T}(k)$.

Theorem 3.2. *The following diagram is commutative and each arrow is a bijection*

$$\begin{array}{ccc} \operatorname{Hom}(\mathrm{SL}_{2,k}, H)/H(k) & \xrightarrow{\theta \longmapsto d\theta} & \operatorname{Hom}(\mathfrak{sl}_{2,k}, \mathfrak{h})/H(k) \\ \Big\downarrow{\scriptstyle \mathsf{JM}} & & \Big\downarrow{\scriptstyle \nu \mapsto (\nu(e_0), \nu(h_0), \nu(f_0))} \\ \mathcal{N}(k)/H(k) & \xleftarrow{e \longleftarrow\!\shortmid (e,h,f)} & \mathcal{T}(k)/H(k). \end{array}$$

We end this subsection by explaining the relationship between the centralizers of θ and $N = \mathsf{JM}(\theta)$. Namely, let us set

$$\mathfrak{u}^N = \operatorname{im}(\operatorname{ad}(N)) \cap \ker(\operatorname{ad}(N)), \qquad U^N = \exp(\mathfrak{u}^N).$$

Then, we have the following Levi decomposition statement.

Proposition 3.3. *The equality $Z_H(N) = U^N \rtimes Z_H(\theta)$ holds. Further we have*

$$\operatorname{Lie}(Z_H(\theta)) = \operatorname{Lie}(Z_H(N))_0, \qquad \operatorname{Lie}(U^N) = \bigoplus_{i>0} \operatorname{Lie}(Z_H(N))_i,$$

where for an integer i we set

$$\operatorname{Lie}(Z_H(N))_i = \{x \in \operatorname{Lie} Z_H(N) : \operatorname{Ad}\left(\theta\left(\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)\right)\right) x = z^i x\}.$$

Proof. The first claim is proved in the same way as [BV85, Proposition 2.4]. The second follows from [Elk72, Lemma 5.1] by taking the derived group of H°. □

3.2. **The Jacobson–Morozov theorem for parameters.** We now recall the analogue of the Jacobson–Morozov theorem for parameters. We use the notation from §2.2.

Definition 3.4. Topologize ${}^L G(\mathbb{C})$ by giving $\widehat{G}(\mathbb{C})$ the classical topology.

(1) A *(complex) Weil–Deligne parameter* for G is a pair (φ, N) where
- $\varphi\colon W_F \to {}^L G(\mathbb{C})$ is a continuous cross-section homomorphism,
- $N \in \widehat{\mathcal{N}}(\mathbb{C})$ is such that $\operatorname{Ad}(\varphi(w))(N) = \|w\| N$ for all $w \in W_F$.

(2) A *(complex) L-parameter* for G is a map

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(\mathbb{C}),$$

such that
- $\psi|_{W_F}\colon W_F \to {}^L G(\mathbb{C})$ is a continuous cross-section homomorphism,
- $\psi|_{\mathrm{SL}_2(\mathbb{C})}\colon \mathrm{SL}_2(\mathbb{C}) \to {}^L G(\mathbb{C})$ takes values in $\widehat{G}(\mathbb{C})$ and is algebraic.

For $\tau \in \{L, \mathrm{WD}\}$ let us denote by $\Phi_G^{\tau,\square}$ the set of complex τ-parameters for G. Recall that a Weil–Deligne parameter (φ, N) (resp. an L-parameter ψ) is called *Frobenius semi-simple* if for one (equiv. for any) lift w_0 of arithmetic Frobenius the element $\varphi(w_0)$ (resp. $\psi(w_0)$) is semi-simple (in the sense of [Bor79, §8.2]). We denote by $\Phi_G^{\tau,\mathrm{ss},\square}$ the subset of Frobenius semi-simple τ-parameters. For each τ there is a natural action of $\widehat{G}(\mathbb{C})$ on $\Phi_G^{\tau,\square}$ which stabilizes the subset $\Phi_G^{\tau,\mathrm{ss},\square}$. We then define $\Phi_G^{\tau} := \Phi_G^{\tau,\square}/\widehat{G}(\mathbb{C})$ and $\Phi_G^{\tau,\mathrm{ss}} := \Phi_G^{\tau,\mathrm{ss},\square}/\widehat{G}(\mathbb{C})$. For an element ψ of $\Phi_G^{L,\square}$ we denote by θ (or θ_ψ when we want to emphasize ψ) the morphism $\psi|_{\mathrm{SL}_2(\mathbb{C})}\colon \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$.

To upgrade Theorem 3.1 to the parameter setting, we need to associate a Weil–Deligne parameter to any L-parameter. To this end, let us define a morphism of groups

$$i = (i_1, i_2)\colon W_F \to W_F \times \mathrm{SL}_2(\mathbb{C}), \qquad w \mapsto \left(w, \begin{pmatrix} \|w\|^{\frac{1}{2}} & 0 \\ 0 & \|w\|^{-\frac{1}{2}} \end{pmatrix}\right).$$

We then define the *Jacobson–Morozov map* to be the $\widehat{G}(\mathbb{C})$-equivariant map

$$\mathsf{JM}\colon \Phi_G^{L,\square} \to \Phi_G^{\mathrm{WD},\square}, \qquad \psi \mapsto (\psi \circ i, d\theta(e_0)).$$

It is easy to check that $\mathsf{JM}^{-1}(\Phi_G^{\mathrm{WD},\mathrm{ss},\square})$ is precisely $\Phi_G^{L,\mathrm{ss},\square}$. As the Jacobson–Morozov map is $\widehat{G}(\mathbb{C})$-equivariant it induces maps $\Phi_G^{L} \to \Phi_G^{\mathrm{WD}}$ and $\Phi_G^{L,\mathrm{ss}} \to \Phi_G^{\mathrm{WD},\mathrm{ss}}$.

The Jacobson–Morozov map is not a bijection as the following example illustrates.

Example 3.5. Set $G = \mathrm{GL}_4$ and as G is split we may replace ${}^L G(\mathbb{C})$ with $\widehat{G}(\mathbb{C}) = \mathrm{GL}_4(\mathbb{C})$. Consider the Weil–Deligne parameter (φ, N) given as follows

$$\varphi\colon w \mapsto \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{d(w)}, \qquad N = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose that $(\varphi, N) = \mathsf{JM}(\psi)$. Then, ψ is of the form $\rho \boxtimes \mathrm{Std}$, where Std is the standard representation of $\mathrm{SL}_2(\mathbb{C})$. Indeed, from the Jacobson–Morozov theorem one sees that as an $\mathrm{SL}_2(\mathbb{C})$ representation $\mathbb{C}^4$ is isomorphic to Std^2. One may then check that the morphism

$$\mathrm{Hom}_{\mathrm{SL}_2(\mathbb{C})}(\mathrm{Std}, \mathbb{C}^4) \boxtimes \mathrm{Std} \to \mathbb{C}^4$$

is an isomorphism of $W_F \times \mathrm{SL}_2(\mathbb{C})$-representations. That said, note that the twist of ρ by the unramified character $w \mapsto \|w\|^{-1/2}$ must be isomorphic to the representation on $\mathrm{Ker} N$ induced by φ. In particular ρ is semi-simple. Hence the Weil–Deligne parameter attached to ψ must be Frobenius semi-simple, but the original (φ, N) is not Frobenius semi-simple.

However, we have the following Jacobson–Morozov theorem for parameters.

Theorem 3.6 (see [GR10, Proposition 2.2] or [Ima20, Proposition 1.13])**.** *The Jacobson–Morzov map* $\mathsf{JM}\colon \Phi_G^{L,\mathrm{ss},\square} \to \Phi_G^{\mathrm{WD},\mathrm{ss},\square}$ *is a surjection and induces a bijection* $\Phi_G^{L,\mathrm{ss}} \to \Phi_G^{\mathrm{WD},\mathrm{ss}}$.

3.3. **Bijection over reductive centralizer locus and applications.** The Jacobson–Morozov theorem for parameters is stated at the level of $\widehat{G}(\mathbb{C})$-orbits. While this is a non-issue for now, when we attempt to geometrize this result it becomes more problematic due to the subtle nature of quotients in algebraic geometry. So, we wish to upgrade the Jacobson–Morozov theorem for parameters to a bijectivity statement before quotienting by $\widehat{G}(\mathbb{C})$.

To begin, we give an analogue of Proposition 3.3 for parameters. To state it, let (φ, N) be an element of $\Phi_G^{\mathrm{WD},\square}$ and set $U^N(\varphi) := U^N(\mathbb{C}) \cap Z_{\widehat{G}(\mathbb{C})}(\varphi)$.

Proposition 3.7. *Let* ψ *be an element of* $\Phi_G^{L,\square}$ *and set* $(\varphi, N) = \mathsf{JM}(\psi)$*. Then, the equality* $Z_{\widehat{G}(\mathbb{C})}(\varphi, N) = U^N(\varphi) \rtimes Z_{\widehat{G}(\mathbb{C})}(\psi)$ *holds.*

Proof. Given Proposition 3.3 it suffices to show that if ua belongs to $Z_{\widehat{G}(\mathbb{C})}(\varphi, N)$, where u is in $U^N(\mathbb{C})$ and a is in $Z_{\widehat{G}(\mathbb{C})}(\theta)$, then in fact u belongs to $U^N(\varphi)$ and a belongs to $Z_{\widehat{G}(\mathbb{C})}(\psi)$. To prove this, we note that conjugation by an element in the image of φ stabilizes both $U^N(\mathbb{C})$ and $Z_{\widehat{G}(\mathbb{C})}(\theta)$. Indeed, since $\mathrm{Ad}(\varphi(w))(N) = \|w\|N$, we have that conjugation by $\varphi(w)$ stabilizes $Z_{\widehat{G}(\mathbb{C})}(N)$ and hence its unipotent radical U^N. On the other hand, as $\varphi(w)$ equals $\psi(w,1)\theta(i_2(w))$, and $\psi(w,1)$ commutes with θ, one may easily check the claim that $\varphi(w)$ normalizes $Z_{\widehat{G}(\mathbb{C})}(\theta)$. Now for each $w \in W_F$, ua equals $\mathrm{Int}(\varphi(w))(u)\,\mathrm{Int}(\varphi(w))(a)$. Therefore, $\mathrm{Int}(\varphi(w))(a)a^{-1}$ equals $\mathrm{Int}(\varphi(w))(u)^{-1}u$. By what we have proven, the former is an element of $Z_{\widehat{G}(\mathbb{C})}(\theta)$ and the latter is an element of $U^N(\mathbb{C})$. Since $U^N(\mathbb{C})$ and $Z_{\widehat{G}(\mathbb{C})}(\theta)$ have trivial intersection, we have that both sides are trivial and so a and u commute with $\varphi(w)$ as desired. □

We may use this decomposition to exhibit an example of a semi-simple L-parameter ψ whose associated Weil–Deligne parameter has strictly larger centralizer.

Example 3.8. Let $G = \mathrm{GL}_3$ and consider the element ψ in $\Phi_G^{L,\mathrm{ss},\square}$ given by the following

$$\psi\left(w, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \|w\|^{-\frac{1}{2}} & 0 & 0 \\ 0 & \|w\|^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, w\right).$$

and set $(\varphi, N) = \mathsf{JM}(\psi)$. In this case, we have

$$\mathfrak{u}^N = \left\{\begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & * & 0 \end{pmatrix}\right\}.$$

Hence

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Z_{\widehat{G}(\mathbb{C})}(\varphi, N) \cap U^N(\mathbb{C}),$$

but it does not belong to $Z_{\widehat{G}(\mathbb{C})}(\psi)$ by Proposition 3.7.

Remark 3.9. We remark that although $Z_{\widehat{G}(\mathbb{C})}(\psi)$ need not equal $Z_{\widehat{G}(\mathbb{C})}(\mathsf{JM}(\psi))$, these groups are the same for the purposes of parametrizing L-packets as in [Kal16] as they have the same component groups by Proposition 3.7. More generally, one can consider the group $S_\psi^\natural$ (resp. $S_{\mathsf{JM}(\psi)}^\natural$) that is related to [Kal16, Conjecture F] and is defined by

$$Z_{\widehat{G}(\mathbb{C})}(\psi)/[Z_{\widehat{G}(\mathbb{C})}(\psi) \cap \widehat{G}(\mathbb{C})^{\mathrm{der}}]^\circ, \qquad \left(\text{resp. } Z_{\widehat{G}(\mathbb{C})}(\mathsf{JM}(\psi))/[Z_{\widehat{G}(\mathbb{C})}(\mathsf{JM}(\psi)) \cap \widehat{G}(\mathbb{C})^{\mathrm{der}}]^\circ\right).$$

These groups are equal by Proposition 3.7 as $U^N(\varphi)$ is contained in $[Z_{\widehat{G}(\mathbb{C})}(\mathsf{JM}(\psi)) \cap \widehat{G}(\mathbb{C})^{\mathrm{der}}]^\circ$.

This decomposition also allows us to give an algebraic condition for when a Weil–Deligne parameter is the image under the Jacobson–Morozov map of a semi-simple L-parameter with the same centralizer. In the rest of this section, we use Proposition 5.11, but the proof of the proposition does not depend on the rest of this section.

Proposition 3.10. *The group $Z_{\widehat{G}(\mathbb{C})}(\varphi, N)^\circ$ is reductive if and only if $(\varphi, N) = \mathsf{JM}(\psi)$ for a Frobenius semi-simple Weil–Deligne parameter ψ such that $Z_{\widehat{G}(\mathbb{C})}(\psi) = Z_{\widehat{G}(\mathbb{C})}(\varphi, N)$.*

Proof. Suppose first that $Z_{\widehat{G}(\mathbb{C})}(\varphi, N)^\circ$ is reductive. We shall show in Proposition 5.11 that this implies that (φ, N) is Frobenius semi-simple. Let ψ be any element of $\Phi_G^{L,\mathrm{ss},\square}$ such that $\mathsf{JM}(\psi) = (\varphi, N)$. By Proposition 3.7 the reductivity of $Z_{\widehat{G}(\mathbb{C})}(\varphi, N)^\circ$ implies that $U^N(\varphi)$ is trivial, and

thus $Z_{\widehat{G}(\mathbb{C})}(\psi) = Z_{\widehat{G}(\mathbb{C})}(\varphi, N)$ as desired. Conversely, if $(\varphi, N) = \mathsf{JM}(\psi)$ for an element of $\Phi_G^{L,\mathrm{ss},\square}$ and $Z_{\widehat{G}(\mathbb{C})}(\psi) = Z_{\widehat{G}(\mathbb{C})}(\varphi, N)$, then $Z_{\widehat{G}(\mathbb{C})}(\varphi, N)^\circ$ is reductive by [SZ18, Proposition 3.2] □

Let $\Phi_G^{\mathrm{WD},\mathrm{rc},\square}$ consist of those (φ, N) with $Z_{\widehat{G}(\mathbb{C})}(\varphi, N)^\circ$ reductive. We call this the *reductive centralizer locus* of $\Phi_G^{\mathrm{WD},\square}$.

Corollary 3.11. *The map* $\mathsf{JM}\colon \mathsf{JM}^{-1}\left(\Phi_G^{\mathrm{WD},\mathrm{rc},\square}\right) \to \Phi_G^{\mathrm{WD},\mathrm{rc},\square}$ *is a* $\widehat{G}(\mathbb{C})$*-equivariant bijection.*

Proof. This follows from Theorem 3.6, Proposition 3.10 and that ψ is Frobenius semi-simple if and only if $\mathsf{JM}(\psi)$ is for $\psi \in \Phi_G^{L,\square}$. □

3.4. **Essentially tempered parameters.** To make Corollary 3.11 useful, we now show that $\mathsf{JM}^{-1}(\Phi_G^{\mathrm{WD},\mathrm{rc},\square})$ contains a large class of important L-parameters. To this end, let us call an element ψ of $\Phi_G^{L,\square}$ *essentially tempered* if the projection of $\psi(W_F)$ to $\widehat{G}(\mathbb{C})/Z_0(\widehat{G})(\mathbb{C})$ is relatively compact. Let $\Phi_G^{L,\mathrm{est},\square}$ be the set consisting of essentially tempered L-parameters. We will soon show that every essentially tempered L-parameter maps into the reductive centralizer locus, but first we must establish some results concerning Frobenius semi-simple parameters.

Proposition 3.12. *Any element* ψ *of* $\Phi_G^{L,\mathrm{est},\square}$ *is Frobenius semi-simple.*

Proof. The map ψ' obtained by composing $\psi|_{W_{F^*}}$ with the projection to $\widehat{G}(\mathbb{C})/Z_0(\widehat{G})(\mathbb{C})$ is a homomorphism. By Lemma 3.13 below it suffices to show that if w_0 is an arithmetic Frobenius lift and m is divisible by $[F^* : F]$, then $\psi'(w_0^m)$ is semi-simple. But, by essentially temperedness we know that the image of $\psi'(w_0^m)$ in $\widehat{G}(\mathbb{C})/Z_0(\widehat{G})(\mathbb{C})$ is contained in a maximal compact subgroup K of $\widehat{G}(\mathbb{C})/Z_0(\widehat{G})(\mathbb{C})$. Up to conjugation, we may then assume that $K = H(\mathbb{R})$ for H a compact form of $\widehat{G}(\mathbb{C})/Z_0(\widehat{G})(\mathbb{C})$ (see [Con14, Theorem D.2.8 and Proposition D.3.2]). But, as $H(\mathbb{R})$ consists only of semi-simple elements, the claim follows. □

Lemma 3.13. *Let* (s, w) *be an element of* ${}^L G(\mathbb{C})$ *and write* $(s, w)^m = (s_m, w^m)$*. Then,* (s, w) *is Frobenius semi-simple if and only if* s_m *is semi-simple for some non-zero integer* m *divisible by* $[F^* : F]$*.*

Proof. Fix any representation $r\colon {}^L G \to \mathrm{GL}_n$. As $r((s, w)^k) = r(s, w)^k$ we see that (s, w) is semi-simple if and only if $(s, w)^k$ is for some $k > 0$. But, if m is divisible by $[F^* : F]$ then as $r((s, w)^{mk}) = r(s_m^k, 1)$ for some $k > 0$, the conclusion follows. □

The following shows that the naming of essentially tempered L-parameters is reasonable.

Proposition 3.14. *For* $\psi \in \Phi_G^{L,\square}$*, the following conditions are equivalent:*

(1) $\psi \in \Phi_G^{L,\mathrm{est},\square}$*,*

(2) there is a continuous character $\chi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to Z_0(\widehat{G})(\mathbb{C})$ *such that the projection of* $(\chi\psi)(W_F)$ *to* $\widehat{G}(\mathbb{C})$ *is relatively compact.*

Proof. It is clear that (2) implies (1). We show that (1) implies (2). Fix a Frobenius lift $w_0 \in W_F$. Set $H = Z_{\widehat{G}(\mathbb{C})}(\psi)$, which has reductive identity component by Proposition 3.12 and [SZ18, Proposition 3.2]. Let $\widehat{\psi}$ be the $\widehat{G}$-component of ψ. Taking a positive integer m to be divisible by $|\operatorname{Aut}(\psi(I_F))|$ and $[F^* : F]$ we see that $\widehat{\psi}(w_0^m) \in H$, and thus in fact $\widehat{\psi}(w_0^m) \in Z(H)$. By replacing m by a power, we may assume that $\widehat{\psi}(w_0^m) \in Z(H)^\circ$. Since $\psi \in \Phi_G^{L,\mathrm{est},\square}$, there is a compact subgroup $C \subseteq Z(H)^\circ$ such that $\widehat{\psi}(w_0^m) \in C \cdot (Z(H)^\circ \cap Z(\widehat{G})(\mathbb{C}))$. We write $\widehat{\psi}(w_0^m) = cz$ for $c \in C$ and $z \in Z(H)^\circ \cap Z(\widehat{G})(\mathbb{C})$. Since elements of $Z(H)^\circ \cap Z(\widehat{G})(\mathbb{C})$ commute with $\psi(W_F)$, we have $Z(H)^\circ \cap Z(\widehat{G})(\mathbb{C}) = Z(H)^\circ \cap Z_0(\widehat{G})(\mathbb{C})$. Replacing m again, we may assume that $z \in (Z(H)^\circ \cap Z_0(\widehat{G})^\circ(\mathbb{C}))$. We take $z_0 \in (Z(H)^\circ \cap Z_0(\widehat{G})^\circ(\mathbb{C}))$ such that $z_0^m = z$. Further we define χ as the unramified character sending w_0 to z_0^{-1}. Then the image of $(\chi\psi)(W_F)$ in $\widehat{G}(\mathbb{C})$ is contained in the image of $\bigcup_{i=0}^{m-1} \psi(I_F)(\chi\psi)(w_0^i)C$ in $\widehat{G}(\mathbb{C})$, which is compact. □

We now relate $\Phi_G^{L,\mathrm{est},\square}$ to the reductive centralizer locus of $\Phi_G^{\mathrm{WD},\square}$.

Proposition 3.15. *The containment* $\Phi_G^{L,\mathrm{est},\square} \subseteq \mathsf{JM}^{-1}(\Phi_G^{\mathrm{WD},\mathrm{rc},\square})$ *holds.*

Proof. Let ψ be an element of $\Phi_G^{L,\mathrm{est},\square}$ and set $(\varphi, N) = \mathsf{JM}(\psi)$. Then ψ is Frobenius semi-simple by Proposition 3.12. We claim that $Z_{\widehat{G}(\mathbb{C})}(\psi) = Z_{\widehat{G}(\mathbb{C})}(\varphi, N)$, from where we will be done by Proposition 3.10. By Proposition 3.7, it suffices to show that $U^N(\varphi)$ is trivial. We assume that $U^N(\varphi)$ is non-trivial and take a non-trivial weight vector v of $\mathrm{Lie}(U^N(\varphi))$ with respect to the adjoint action of $\theta|_{T_2}$, where T_2 is the standard maximal torus of $\mathrm{SL}_{2,\mathbb{C}}$. We put $u = \exp(v)$. For each $w \in W_F$ we have that $\varphi(w) = \psi(w,1)\theta(i_2(w))$. Since $\varphi(w)$ commutes with u, we see that $\mathrm{Int}(\psi(w,1)^{-1})(u)$ is equal to $\mathrm{Int}(\theta(i_2(w)))(u)$, and therefore

$$\mathrm{Ad}(\psi(w,1)^{-1})(v) = \mathrm{Ad}(\theta(i_2(w)))(v).$$

But, observe that if w_0 is a lift of arithmetic Frobenius in W_F then $i_2(w_0^{2n}) = \left(\begin{smallmatrix} q^n & 0 \\ 0 & q^{-n} \end{smallmatrix}\right)$. By Proposition 3.3, we deduce that $\mathrm{Ad}(\theta(i_2(w_0^{2n})))(v) = q^{jn}v$ for some $j \geqslant 1$. Letting n tend towards infinity, and using the fact that u is non-trivial, we deduce that the adjoint orbit of W_F on v is non-compact, which is a contradiction. □

We now state a corollary to Proposition 3.15. Before doing so, we recall an even smaller subset of $\Phi_G^{L,\mathrm{est},\square}$ that will feature prominently below. Namely, recall that (φ, N) in $\Phi_G^{\mathrm{WD},\square}$ (resp. ψ in $\Phi_G^{L,\square}$) is called *discrete* if the quotient

$$Z_{\widehat{G}(\mathbb{C})}(\varphi, N)/Z_0(\widehat{G})(\mathbb{C}) \qquad \left(\text{resp. } Z_{\widehat{G}(\mathbb{C})}(\psi)/Z_0(\widehat{G})(\mathbb{C})\right)$$

is finite. Denote by $\Phi_G^{\mathrm{WD},\mathrm{disc},\square}$ (resp. $\Phi_G^{L,\mathrm{disc},\square}$) the set of discrete parameters and $\Phi_G^{\mathrm{WD},\mathrm{disc}}$ (resp. $\Phi_G^{L,\mathrm{disc}}$) its $\widehat{G}(\mathbb{C})$-quotient. Note that $\Phi_G^{L,\mathrm{disc},\square}$ is contained in $\Phi_G^{L,\mathrm{est},\square}$ (cf. [GR10, Lemma 3.1] and [SZ18, Lemma 5.2]), and thus ψ is discrete if and only if $\mathsf{JM}(\psi)$ discrete as they have the same centralizers by Proposition 3.15 and its proof.

Corollary 3.16. *The map*

$$\mathsf{JM}\colon \Phi_G^{L,\mathrm{est},\square} \to \Phi_G^{\mathrm{WD},\square}, \qquad \left(\textit{resp. } \mathsf{JM}\colon \Phi_G^{L,\mathrm{disc},\square} \to \Phi_G^{\mathrm{WD},\mathrm{disc},\square}\right)$$

is a $\widehat{G}(\mathbb{C})$-equivariant injection (resp. bijection).

Note that implicit in the above is the following result of independent interest.

Proposition 3.17. *Any element of* $\Phi_G^{\mathrm{WD},\mathrm{disc},\square}$ *(resp.* $\Phi_G^{L,\mathrm{disc},\square}$*) is Frobenius semi-simple.*

Proof. The first claim is a special case of Proposition 5.11. The second claim follows from $\Phi_G^{L,\mathrm{disc},\square} \subseteq \Phi_G^{L,\mathrm{est},\square}$ and Proposition 3.12. □

We end this subsection by showing that one may apply Corollary 3.16 to show that the assocation of $\psi \circ i$ to ψ is injective when restricted to the set of discrete L-parameters. This result plays an important technical role in [BMY20].

Proposition 3.18. *The maps*

$$\Phi_G^{\mathrm{WD},\mathrm{disc}} \xrightarrow{(\varphi,N)\mapsto\varphi} \mathrm{Hom}(W_F, {}^LG(\mathbb{C}))/\widehat{G}(\mathbb{C}), \qquad \Phi_G^{L,\mathrm{disc}} \xrightarrow{\psi\mapsto\psi\circ i} \mathrm{Hom}(W_F, {}^LG(\mathbb{C}))/\widehat{G}(\mathbb{C})$$

are injective.

Proof. By Corollary 3.16 it suffices to show that the former map is injective. Fix λ in the set $\mathrm{Hom}(W_F, {}^LG(\mathbb{C}))$. By Proposition 3.17 it then suffices to show that (if non-empty) the set

$$P(G,\lambda) := \left\{(\varphi, N) \in \Phi_G^{\mathrm{WD},\mathrm{ss},\square} : \varphi = \lambda\right\}$$

intersects at most one $\widehat{G}(\mathbb{C})$-orbit of discrete parameters. As in [Vog93, §4], set $\widehat{G}(\mathbb{C})^\lambda$ to be $Z_{\widehat{G}(\mathbb{C})}(\lambda)$, and

$$\widehat{\mathfrak{g}}_q^{\lambda(I_F)} := \left\{ x \in \widehat{\mathfrak{g}}_{\mathbb{C}} : \begin{array}{ll} (1) & \mathrm{Ad}(\lambda(w))(x) = x \text{ for all } w \in I_F \\ (2) & \mathrm{Ad}(w_0)(x) = qx \end{array} \right\}$$

where w_0 is any lift of arithmetic Frobenius. Both $P(G,\lambda)$ and $\widehat{\mathfrak{g}}_q^{\lambda(I_F)}$ carry an action of $\widehat{G}(\mathbb{C})^\lambda$, and [Vog93, Proposition 4.5] establishes a $\widehat{G}(\mathbb{C})^\lambda$-equivariant bijection $P(G,\lambda) \to \widehat{\mathfrak{g}}_q^{\lambda(I_F)}$, and that the latter space has only finitely many orbits. Therefore, $P(G,\lambda)$ carries the structure of a vector space on which $\widehat{G}(\mathbb{C})^\lambda$ acts algebraically and with only finitely many orbits.

Suppose then that (λ, N) is a discrete element of $P(G,\lambda)$ and let $\mathcal{O} \subseteq P(G,\lambda)$ denote its $\widehat{G}(\mathbb{C})^\lambda$-orbit. Now, $\mathcal{O}$ is a locally closed subscheme of $P(G,\lambda)$ (see [Mil17, Proposition 1.65 (2)]) of dimension $\dim(\widehat{G}(\mathbb{C})^\lambda) - \dim(H)$ where H is the isotropy subgroup of (λ, N) in $\widehat{G}(\mathbb{C})^\lambda$ ([Mil17, Proposition 5.23 and Proposition 7.12]). But, note that $H = Z_{\widehat{G}(\mathbb{C})}(\lambda, N)$ and so contains $Z_0(\widehat{G})(\mathbb{C})$ as a finite index subgroup. We deduce that $\dim(\mathcal{O})$ is equal to $\dim(\widehat{G}(\mathbb{C})^\lambda) - \dim(Z_0(\widehat{G})(\mathbb{C}))$. But, as $\widehat{G}(\mathbb{C})^\lambda$ acts through $\widehat{G}(\mathbb{C})^\lambda / Z_0(\widehat{G})(\mathbb{C})$, and has finitely many (locally closed) orbits, we see that $\dim P(G,\lambda)$ is at most $\dim(\widehat{G}(\mathbb{C})^\lambda) - \dim(Z_0(\widehat{G})(\mathbb{C}))$. Thus, we deduce that $\dim(\mathcal{O}) = \dim(P(G,\lambda))$. As $\mathcal{O}$ is locally closed in $P(G,\lambda)$ we deduce that $\mathcal{O}$ is open. As $P(G,\lambda)$ is a vector space it is irreducible, so open orbits are unique, and the conclusion follows. □

4. The geometric and relative Jacobson–Morozov theorems

Before we can geometrize the Jacobson–Morozov theorem for parameters, we now first geometrize the Jacobson–Morozov theorem. After doing so, we derive a version of the Jacobson–Morozov on the level of A-points. We fix for the remainder of this section a field k of characteristic 0 and H a reductive group over k.

Remark 4.1. In this section we often assume that H is split. This will be sufficient for us as $\widehat{G}$ is a split group. That said, most of these statements admit obvious generalizations to arbitrary reductive H, with similar proofs. The exception is Theorem 4.15, but we suspect that the statement is still true and that one can employ a similar strategy to prove it.

4.1. **The orbit separation space.** Pivotal to our formulation of a geometric version of the Jacobson–Morozov theorem is a certain construction which, in a precise sense, replaces a variety with group action with the disjoint union of its orbits. Throughout this subsection we fix a reduced quasi-projective scheme X over k equipped with an action of H. We also assume that the map

$$X(k)/H(k) \to X(\overline{k})/H(\overline{k})$$

is surjective (although one may deal with the general case by Galois descent). Whenever we speak of the class of x in $X(\overline{k})/H(\overline{k})$ we assume without loss of generality that x is in $X(k)$.

For each element x of $X(k)$ let us denote by $\mathcal{O}_x$ the *orbit scheme* given as the fppf sheafification of the presheaf

$$\mathbf{Alg}_k \to \mathbf{Set}, \qquad A \mapsto \{g \cdot x : g \in H(A)\} \subseteq X(A).$$

Since X is itself an fppf sheaf, we see that $\mathcal{O}_x$ is an H-stable subsheaf of X.

Proposition 4.2. *The orbit scheme is representable by a reduced locally closed subscheme of X smooth over k. Moreover, the orbit map $\mu_x \colon H \to \mathcal{O}_x$ is smooth and surjective and identifies $\mathcal{O}_x$ as the fppf sheaf quotient $H/Z_H(x)$.*

Proof. Clearly the orbit map identifies $\mathcal{O}_x$ as the fppf sheaf quotient $H/Z_H(x)$. In [Mil17, Proposition 1.65] it is shown that $\mu_x(H)$ is a locally closed subset of H, which one may endow

with the reduced scheme structure. In [Mil17, Proposition 7.17] it is shown that $\mu_x(H)$ represents $\mathcal{O}_x$. The smoothness of the orbit map is then confirmed by [Mil17, Proposition 7.15], and the smoothness of $\mathcal{O}_x$ over k is handled by [Mil17, Corollary 5.26]. □

It will be useful to have a more explicit description of the A-points of $\mathcal{O}_x$ for a k-algebra A.

Proposition 4.3. *For any k-algebra A, there are identifications*

$$\mathcal{O}_x(A) = \{\mathbf{x} \in X(A) : x \text{ and } \mathbf{x} \text{ lie in the same } H(A)\text{-orbit étale locally on } A\},$$

and

$$\mathcal{O}_x(A)/H(A) = \ker\left(H^1_{\text{ét}}(\mathrm{Spec}(A), Z_H(x)) \to H^1_{\text{ét}}(\mathrm{Spec}(A), H)\right).$$

Proof. The first claim follows from the fact that the orbit map $\mu_x\colon H \to \mathcal{O}_x$ is a smooth surjection and [EGA4-4, Corollaire 17.16.3.(ii)]. The second claim follows by combining [Gir71, Chaptire III, Corollaire 3.2.3] with the fact that as H_A and $Z_H(x)_A$ are smooth over A, their étale cohomology functorially agrees with their fppf cohomology (cf. [Gro68, Théorème 11.7]). □

When A is a reduced k algebra, one may give a simpler description. Say an element $\mathbf{x}$ of $X(A)$ is *everywhere geometrically conjugate (egc)* to x if for all geometric points $\mathrm{Spec}(k') \to \mathrm{Spec}(A)$ one has that x and $\mathbf{x}$ have images in $X(k')$ belonging to the same $H(k')$-orbit.

Proposition 4.4. *For a reduced k-algebra A there is a functorial identification*

$$\mathcal{O}_x(A) = \{\mathbf{x} \in X(A) : \mathbf{x} \text{ is egc to } x\}.$$

Proof. Evidently any element of $\mathcal{O}_x(A)$ is egc to x. If $\mathbf{x}$ is egc to x then the morphism $\mathbf{x}\colon \mathrm{Spec}(A) \to X$ has the property that $\mathbf{x}(|\mathrm{Spec}(A)|) \subseteq |\mathcal{O}_x|$. As $\mathrm{Spec}(A)$ is reduced this implies that $\mathbf{x}$ factorizes through $\mathcal{O}_x$ as desired. □

We then assemble the spaces $\mathcal{O}_x$ into one as follows.

Definition 4.5. We define the *orbit separation* of X, denoted by $X^{\sqcup}$, to be the space

$$X^{\sqcup} := \bigsqcup_{x \in X(\overline{k})/H(\overline{k})} \mathcal{O}_x.$$

We have a tautological map $X^{\sqcup} \to X$, and we have the following omnibus result concerning its properties in the case when $X(\overline{k})/H(\overline{k})$ is finite, which is the case of most interest to us. Below, and in the sequel, we call a morphism of schemes $f\colon Y \to X$ *weakly birational* if there exists a dense open subset U of X such that $f^{-1}(U) \to U$ is an isomorphism.

Proposition 4.6. *Suppose that $X(\overline{k})/H(\overline{k})$ is finite. Then, the map $X^{\sqcup} \to X$ is a weakly birational surjective monomorphism and it is an isomorphism if and only if the action map $\mu\colon H \times X \to X$ is smooth.*

As the last condition is equivalent to the claim that $\mathcal{O}_x$ is open for each x in $X(\overline{k})$ (cf. [Bri21, Lemma 3.5] and [Sta21, Tag 05VJ]), this is a special case of Lemma 4.7 below.

Lemma 4.7. *Let $f\colon Y \to X$ be a morphism of reduced schemes finite type over k. Suppose that $Y_{\overline{k}}$ admits a scheme-theoretic decomposition $\bigsqcup_i Y_i$ such that $f|_{Y_i}$ is a locally closed immersion, and $f(Y_i(\overline{k})) \cap f(Y_j(\overline{k}))$ is empty for $i \neq j$. Then,*

(1) f is a monomorphism,
(2) f is weakly birational if and only if $f(Y(\overline{k}))$ is dense in X,
(3) f is an isomorphism if and only if $f(Y(\overline{k})) = X(\overline{k})$ and each Y_i is open in $X_{\overline{k}}$.

Proof. As all of these claims may be checked over $\overline{k}$ we may assume without loss of generality that k is algebraically closed. The final claim is clear, thus we focus on the first two claims. For the first claim, as each $f|_{Y_i}$ is a monomorphism, it suffices to show that $f(Y_i)$ and $f(Y_j)$

are disjoint for $i \neq j$. But, as $f(Y_i) \cap f(Y_j)$ is locally closed, if non-empty it would contain a k-point which is a contradiction.

To see the second claim, it suffices to show the if direction. For each irreducible component Z of X note that $\{Y_i \cap Z\}$ is a finite set of locally closed subsets with dense union. This implies that there exists some i_0 such that $Y_{i_0} \cap Z$ is open. Let C be the union of irreducible components of X which intersect Z at a proper non-empty subset. Set $U_Z := (Y_{i_0} \cap Z) - C$. Then, it is clear that if U is the union of the U_Z, then U is a dense open subset of X and as X is reduced that $f\colon f^{-1}(U) \to U$ is an isomorphism. □

Finally, observe that the orbit separation space is a functorial construction. Namely, if Y is another quasi-projective scheme over k equipped with an action of H with the same properties, then for any H-equivariant morphism $X \to Y$, the composition $X^{\sqcup} \to X \to Y$ factorizes uniquely through $Y^{\sqcup} \to Y$.

4.2. **The geometric Jacobson–Morozov theorem.** We now move to the geometrization of the Jacobson–Morozov theorem. Let us now assume that H is split. To begin, observe that one has a *Jacobson–Morozov morphism*

$$\mathsf{JM}\colon \underline{\mathrm{Hom}}(\mathrm{SL}_{2,k}, H) \to \mathcal{N}, \qquad \theta \mapsto d\theta(e_0).$$

We would like to apply the orbit separation construction from the last subsection to this map, but before we do so, we should first observe that the actions of H on $\underline{\mathrm{Hom}}(\mathrm{SL}_{2,k}, H)$ and $\mathcal{N}$ satisfy the properties used in the last section.

Proposition 4.8. *The maps*

$$\mathcal{N}(k)/H(k) \to \mathcal{N}(\overline{k})/H(\overline{k}), \qquad \mathrm{Hom}(\mathrm{SL}_{2,k}, H)/H(k) \to \mathrm{Hom}(\mathrm{SL}_{2,\overline{k}}, H_{\overline{k}})/H(\overline{k})$$

are surjections.

Proof. By Theorem 3.1 it suffices to show the first map is a surjection. Let N be an element of $\mathcal{N}(\overline{k})$. Bala–Carter theory (see [Jan04, §4]) says that there exists a Levi subgroup $\overline{L}$ of $H_{\overline{k}}$ and a parabolic subgroup $\overline{P}$ of $\overline{L}$ such that N is conjugate to an element contained in the unique open orbit of $\overline{P}$ acting on $\mathrm{Lie}(R_u(\overline{P}))$. Now, as H is split, we may assume up to conjugacy, that $\overline{L} = L_{\overline{k}}$ for a Levi subgroup L of H (see [Sol20]). As L is also split we may also assume, up to conjugacy, that $\overline{P} = P_{\overline{k}}$ for a parabolic subgroup P of L. As the unique open orbit of P acting on $\mathrm{Lie}(R_u(P))$ has a k-point, being a Zariski open of a vector k-space, we are done. □

Remark 4.9. The morphism $\mathcal{N}(k)/H(k) \to \mathcal{N}(\overline{k})/H(\overline{k})$ is rarely injective. As a concrete example, if $H = \mathrm{SL}_{2,\mathbb{Q}}$ then $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right)$ are $H(\overline{\mathbb{Q}})$-conjugate, but not $H(\mathbb{Q})$-conjugate.

Before we show that our two spaces with H-action have finitely many $H(\overline{k})$-orbits, we observe the following.

Proposition 4.10. *The morphism* $\underline{\mathrm{Hom}}(\mathrm{SL}_{2,k}, H)^{\sqcup} \to \underline{\mathrm{Hom}}(\mathrm{SL}_{2,k}, H)$ *is an isomorphism.*

Proof. It suffices to assume that k is algebraically closed. Then, by Proposition 2.6 the orbits of k-points of $\underline{\mathrm{Hom}}(\mathrm{SL}_{2,k}, H)$ are open. But, by Proposition 4.6 we deduce that the morphism under consideration is a monomorphism which is locally on the source an open embedding, so itself an open embedding. As the image contains every k-point it is an isomorphism. □

Proposition 4.11. *The sets* $\mathrm{Hom}(\mathrm{SL}_{2,\overline{k}}, H_{\overline{k}})/H(\overline{k})$ *and* $\mathcal{N}(\overline{k})/H(\overline{k})$ *are finite.*

Proof. By Theorem 3.1 these two sets are in bijection, so it suffices to prove the finiteness of either. The finiteness of the latter set is a classical result (e.g. see [Jan04, §2.8, Theorem 1]). Alternatively, one may prove the finiteness of the former set by observing that by Proposition 4.10 the sets $\mathrm{Hom}(\mathrm{SL}_{2,\overline{k}}, H_{\overline{k}})/H(\overline{k})$ and $\pi_0(\underline{\mathrm{Hom}}(\mathrm{SL}_{2,\overline{k}}, H_{\overline{k}}))$ are equipotent. But, by Proposition 2.6 the scheme $\underline{\mathrm{Hom}}(\mathrm{SL}_{2,\overline{k}}, H_{\overline{k}})$ is finite type over k and thus $\pi_0(\underline{\mathrm{Hom}}(\mathrm{SL}_{2,\overline{k}}, H_{\overline{k}}))$ is finite. □

By the functoriality of the orbit separation construction the Jacobson–Morozov morphism factors uniquely through $\mathcal{N}^{\sqcup}$ and we also denote the resulting map $\underline{\mathrm{Hom}}(\mathrm{SL}_{2,k}, H) \to \mathcal{N}^{\sqcup}$ by JM. But, unlike $\underline{\mathrm{Hom}}(\mathrm{SL}_{2,k}, H)$, the orbit separation space $\mathcal{N}^{\sqcup}$ is essentially never equal to $\mathcal{N}$.

Proposition 4.12. *The morphism $\mathcal{N}^{\sqcup} \to \mathcal{N}$ is an isomorphism if and only if H is abelian.*

Proof. If H is abelian then $\mathcal{N}$ is a single point. If $\mathcal{N}^{\sqcup} \to \mathcal{N}$ is an isomorphism then by Proposition 4.6 the orbit of 0 is open, but as it is also closed and $\mathcal{N}$ is connected we deduce that it is equal to $\mathcal{N}$. As $\dim(\mathcal{N})$ is equal to $\dim(H) - r(H)$, we see that H is a torus as desired. □

Example 4.13. The element $\mathbf{N} = \left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right)$ defines a point of $\mathcal{N}_{\mathrm{GL}_{2,k}}(k[t])$ not in $\mathcal{N}^{\sqcup}_{\mathrm{GL}_{2,k}}(k[t])$.

To state our geometric Jacobson–Morozov theorem, note that by Theorem 3.1 the map

$$\mathsf{JM}\colon \mathrm{Hom}(\mathrm{SL}_{2,k}, H)/H(k) \to \mathcal{N}(k)/H(k),$$

is a bijection. For each θ, writing $N = \mathsf{JM}(\theta)$, define JM_θ to be the map $\mathcal{O}_\theta \to \mathcal{O}_N$ which may be described as the quotient map $H/Z_H(\theta) \to H/Z_H(N)$.

Theorem 4.14 (Geometric Jacobson–Morozov)**.** *Suppose that H is split. The morphism $\mathsf{JM}\colon \underline{\mathrm{Hom}}(\mathrm{SL}_{2,k}, H) \to \mathcal{N}$ factorizes through $\mathcal{N}^{\sqcup}$, where it may be described as $\bigsqcup_\theta \mathsf{JM}_\theta$.*

4.3. **The relative Jacobson–Morozov theorem.** We now apply the geometric Jacobson–Morozov theorem to obtain a more concrete result on the level of A-points.

Theorem 4.15 (Relative Jacobson–Morozov)**.** *Let A be a k-algebra. Then, the map*

$$\mathsf{JM}\colon \mathrm{Hom}(\mathrm{SL}_{2,A}, H_A)/H(A) \to \mathcal{N}(A)/H(A)$$

is a bijection onto $\mathcal{N}^{\sqcup}(A)/H(A)$.

Proof. Assume first that $\mathrm{Spec}(A)$ is connected. By Theorem 4.14, it suffices to show that for each θ the map JM_θ induces a bijection $\mathcal{O}_\theta(A)/H(A) \to \mathcal{O}_N(A)/H(A)$. But, by Proposition 4.3 it suffices to show that the natural map $H^1_{\text{ét}}(\mathrm{Spec}(A), Z_H(\theta)) \to H^1_{\text{ét}}(\mathrm{Spec}(A), Z_H(N))$ is a bijection. But, this follows from Proposition 3.3 and [GP13, Lemma 4.14]. For the general case we reduce to the Noetherian case by standard approximation arguments, and then working on each component to the case when $\mathrm{Spec}(A)$ is connected. □

We now pursue the analogue of Theorem 3.2 in the relative setting.

Definition 4.16. Let A be a k-algebra and $\mathfrak{a}$ a Lie algebra over A. We call a triple of elements (e, h, f) in $\mathfrak{a}^3$ such that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h,$$

an $\mathfrak{sl}_2$*-triple* in $\mathfrak{a}$.

Denote by $\mathcal{T}(A)$ (or $\mathcal{T}_H(A)$ when we want to emphasize H) the set of $\mathfrak{sl}_2$-triples in $\mathfrak{h}_A$. Evidently $\mathcal{T}(A)$ carries a natural conjugation action by $H(A)$.

Theorem 4.17. *The following diagram is commutative and each arrow is a bijection*

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{SL}_{2,A}, H_A)/H(A) & \xrightarrow{\theta \longmapsto d\theta} & \mathrm{Hom}(\mathfrak{sl}_{2,A}, \mathfrak{h}_A)/H(A) \\ \downarrow{\scriptstyle \mathsf{JM}} & & \downarrow{\scriptstyle \nu \mapsto (\nu(e_0), \nu(h_0), \nu(f_0))} \\ \mathcal{N}^{\sqcup}(A)/H(A) & \xleftarrow{e \longleftarrow (e,h,f)} & \mathcal{T}(A)/H(A). \end{array}$$

Proof. By Theorem 4.15 the left vertical arrow is a bijection. The right vertical arrow is clearly a bijection, and the top horizontal arrow is a bijection by Proposition 2.6. We thus deduce that the bottom horizontal arrow is well-defined (i.e. takes values in $\mathcal{N}^{\sqcup}(A)$) and is bijective. □

4.4. **A relative version of Kostant's characterization of $\mathfrak{sl}_2$-triples.** This final subsection is dedicated to giving a proof of the following relative version of [Kos59, Corollary 3.5].

Proposition 4.18. *Let A be a k-algebra and $\mathfrak{a}$ a Lie subalgebra of $\mathfrak{h}_A$. Then, for a pair (e,h) in $\mathfrak{a}^2$, there exists an $\mathfrak{sl}_2$-triple of the form (e,h,f) in $\mathfrak{a}$ if and only if the following conditions hold:*

(1) $e \in \mathcal{N}^{\sqcup}(A)$,
(2) h is in the image of $\mathrm{ad}(e)\colon \mathfrak{a} \to \mathfrak{a}$,
(3) $[h,e] = 2e$.

Let us set

$$\mathfrak{h}_A^e := \ker\left(\mathrm{ad}(e)|\mathfrak{h}_A \to \mathfrak{h}_A\right), \qquad \mathfrak{a}^e := \ker\left(\mathrm{ad}(e)|\mathfrak{a} \to \mathfrak{a}\right).$$

If $\mathrm{ad}(e)(x)$ is zero then $\mathrm{ad}(e)(\mathrm{ad}(h)(x))$ is also zero. Thus, $\mathrm{ad}(h)$ stabilizes $\mathfrak{h}_A^e$ and $\mathfrak{a}^e$.

Lemma 4.19. *The A-linear map $\mathrm{ad}(h)+2\colon \mathfrak{a}^e \to \mathfrak{a}^e$ is an isomorphism.*

Proof. It suffices to show this result after passing to an etale cover $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Indeed, since $A \to B$ is faithfully flat we have that $(\mathfrak{a}^e)_B = \mathfrak{a}_B^e$, and moreover that $\ker(\mathrm{ad}(h)+2)$ and $\mathrm{coker}(\mathrm{ad}(h)+2)$ are trivial if and only if they are so after tensoring with B. Thus, we may assume without loss of generality that e is an element of $\mathcal{N}(k)$. With notation as in Lemma 4.20 below, the A-algebra map $A[T] \to \mathrm{End}_A(\mathfrak{a}^e)$ sending T to $\mathrm{ad}(h)$ factorizes through $A[T]/(p(T))$. But, by the Chinese remainder theorem $T+2$ is a unit in this ring. □

Lemma 4.20 (cf. [Kos59, Lemma 3.4])**.** *Suppose that e is an element of $\mathcal{N}(k)$. Let m be the smallest element such that $\mathrm{ad}(e)^{m+1}$ is trivial on $\mathfrak{h}$. Then, $p(\mathrm{ad}(h)|_{\mathfrak{h}_A^e}) = 0$ where*

$$p(T) = \prod_{i=0}^{m} (T-i).$$

Thus, a fortiori, we see that $p\left(\mathrm{ad}(h)|_{\mathfrak{a}^e}\right) = 0$.

Proof. For each $i = 0, \ldots, m+1$ let us set

$$\mathfrak{d}_i := (\mathrm{ad}(e)^i(\mathfrak{h}) \cap \mathfrak{h}^e) \otimes_k A.$$

Observe that

$$\mathfrak{h}_A^e = \mathfrak{d}_0 \supseteq \cdots \supseteq \mathfrak{d}_{m+1} = 0.$$

We claim then that $(\mathrm{ad}(h) - i)(\mathfrak{d}_i) \subseteq \mathfrak{d}_{i+1}$. Note that $\mathfrak{d}_i$ is generated as an A-algebra by elements of the form $\mathrm{ad}(e)^i(z)$ for z in $\mathfrak{h}$. The exact same algebra as in [Kos59, Lemma 3.4] then shows the desired containment, from where the claim is clear. □

Returning to the proof of Proposition 4.18, let us write $h = \mathrm{ad}(e)(f)$. Note that $[[h,f]+2f, e]$ vanishes and thus $[h,f]+2f$ is in $\mathfrak{a}^e$. By Lemma 4.19 we may write $[h,f]+2f = [h,g]+2g$ for some g in $\mathfrak{a}^e$. So then, if we take $f'' = f - g$ then

$$[h,e] = 2e, \qquad [h,f''] = [h,f] - [h,g] = -2f'', \qquad [e,f''] = [e,f] - [e,g] = h - 0 = h,$$

as desired.

5. Moduli spaces of Weil–Deligne parameters

To give a geometrization of the results of §3.2 it is useful to first develop a space intermediary between the moduli space of L-parameters (see §6) and the moduli space of Weil–Deligne parameters. We give such a space in this section which, in short, parameterizes Weil–Deligne parameters whose monodromy operator lies in $\mathcal{N}^{\sqcup}$.

5.1. **The moduli space of Weil–Deligne parameters.** We first recall the moduli space of Weil–Deligne parameters roughly following the presentation as in [Zhu20].

Initial definitions. We begin by defining the relative analogue of a Weil–Deligne parameter.

Definition 5.1. For a $\mathbb{Q}$-algebra A, we define a *Weil–Deligne parameter over* A to be a pair (φ, N) where

(WDP1) $\varphi\colon \mathcal{W}_{F,A} \to {}^{C}G_A$ is a morphism of group A-schemes such that $p_C \circ \varphi = (\|\cdot\|, \mathrm{id})$,

(WDP2) N is an element of $\widehat{\mathcal{N}}(A)$ such that $\mathrm{Ad}(\varphi(w))(N) = \|w\|N$ for all $w \in \mathcal{W}_F(A)$.

We denote the set of Weil–Deligne parameters over A by $\mathsf{WDP}_G(A)$ which clearly constitutes a presheaf on $\mathbb{Q}$-algebras. The presheaf WDP_G has a natural action by $\widehat{G}$ given by

$$g(\varphi, N)g^{-1} := (\mathrm{Int}(g) \circ \varphi, \mathrm{Ad}(g)(N)).$$

So, for a Weil–Deligne parameter (φ, N) we may consider the centralizer group presheaf $Z_{\widehat{G}}(\varphi, N)$.

We define the morphism $\check{\varphi}\colon \mathcal{W}_{F,A} \to \check{G}_A$ of schemes as the composition of φ with the projection to $\check{G}_A$. We denote by $\overline{\varphi}$ the homomorphism $\mathcal{W}_{F,A} \to (\widehat{G} \rtimes \underline{\Gamma_*})_A$ obtained by composing φ with the quotient map ${}^{C}G_A \to (\widehat{G} \rtimes \underline{\Gamma_*})_A$. Observe that while $\check{\varphi}$ may not be a homomorphism, it becomes so after restriction to $\mathcal{W}_{F^*,A}$. In particular, for any $w \in \mathcal{W}_F(A)$ the restriction of $\check{\varphi}$ to $\langle w^m \rangle$ is a homomorphism whenever $[F^* : F]$ divides m.

Let K be a finite extension of F^* Galois over F, and let us define for a $\mathbb{Q}$-algebra A the set

$$\mathsf{WDP}_G^K(A) := \left\{ (\varphi, N) \in \mathsf{WDP}_G(A) : \mathcal{I}_{K,A} \subseteq \ker(\check{\varphi}|_{\mathcal{W}_{F^*,A}}) \right\}.$$

We observe that WDP_G^K forms a $\widehat{G}$-stable subfunctor of WDP_G. In fact, one sees that there is an equality of functors $\mathsf{WDP}_G = \varinjlim \mathsf{WDP}_G^K$ as K travels over all such extensions.

We finally observe that WDP_G has a more familiar form over an extension k of $\mathbb{Q}$ containing an element c such that $c^2 = q$. More precisely, for a k-algebra A, we equip $\widehat{G}(A)$ with the discrete topology and put

$$\mathsf{WDP}'_{G,k}(A) := \left\{ (\varphi, N) : \begin{array}{ll} (1) & \varphi\colon W_F \to \widehat{G}(A) \rtimes W_F \text{ is a a continuous cross-section homomorphism,} \\ (2) & N \in \widehat{\mathcal{N}}(A) \text{ is such that } \mathrm{Ad}(\varphi(w))(N) = \|w\|N \text{ for all } w \in W_F \end{array} \right\}.$$

It is clear that WDP'_G is a functor on the category of k-algebras and comes equipped with a natural action of $\widehat{G}_k$. Let us also observe that if i_c is the map from §2.2 then there is a morphism $i_c^{\mathrm{WD}}\colon \mathsf{WDP}'_{G,k} \to \mathsf{WDP}_{G,k}$ which on A-points is given by sending (φ', N) to the unique element of $\mathsf{WDP}_G(A)$ of the form (φ, N) which is equal to $(i_c \circ \varphi', N)$ on A-points.

Proposition 5.2. *The morphism of functors $i_c^{\mathrm{WD}}\colon \mathsf{WDP}'_{G,k} \to \mathsf{WDP}_{G,k}$ is an isomorphism.*

Proof. This follows from the cartesian diagram

$$\begin{array}{ccc} {}^{L}G_k & \xrightarrow{\quad i_c \quad} & {}^{C}G_k \\ \downarrow & & \downarrow{\scriptstyle p_C} \\ \mathcal{W}_{F,k} & \xrightarrow{(\|\cdot\|,\mathrm{id})} & \mathbb{G}_{m,k} \times \mathcal{W}_{F,k} \end{array}$$

and that any morphism $\mathcal{W}_{F,A} \to \check{G}_A$ of schemes over A factors through $(\mathcal{W}_F/\mathcal{I}_K)_A$ for a finite extension K of F. □

Representability. We now establish the representability of the functor WDP_G. To this end, let us fix K a finite extension of F^* Galois over F. Note that for a $\mathbb{Q}$-algebra A and an element (φ, N) of $\mathsf{WDP}_G^K(A)$ we may define an element ϕ of $\underline{Z}^1(I_F/I_K, \widehat{G})(A)$ as follows. First observe that condition **(WDP1)** implies that $\varphi|_{\mathcal{I}_{F,A}}$ takes values in $\widehat{G}_A \rtimes \mathcal{I}_{F,A}$. Then, as (φ, N) is in $\mathsf{WDP}_G^K(A)$, the composition of $\varphi|_{\mathcal{I}_{F,A}}$ with the projection $\widehat{G}_A \rtimes \mathcal{I}_{F,A} \to \widehat{G}_A \rtimes (\mathcal{I}_F/\mathcal{I}_K)_A$ factorizes through a cross-section homomorphism $(\mathcal{I}_F/\mathcal{I}_K)_A \to \widehat{G}_A \rtimes (\mathcal{I}_F/\mathcal{I}_K)_A$. This gives an

element ϕ of $\underline{Z}^1(I_F/I_K, \widehat{G})(A)$ since $\mathcal{I}_F/\mathcal{I}_K \cong \underline{I_F/I_K}$. This association defines a morphism of presheaves $\mathsf{WDP}_G^K \to \underline{Z}^1(I_F/I_K, \widehat{G})$.

Let us now fix a lift w_0 of arithmetic Frobenius in W_F. Define a morphism of presheaves

$$j_{w_0} \colon \mathsf{WDP}_G^K \to \check{G} \times \underline{Z}^1(I_F/I_K, \widehat{G}) \times \widehat{\mathcal{N}}, \qquad (\varphi, N) \mapsto (\check{\varphi}(w_0), \phi, N).$$

On the other hand, we have a diagram

$$\mathcal{D}^{\mathrm{WD}} \colon \ \check{G} \times \underline{Z}^1(I_F/I_K, \widehat{G}) \times \widehat{\mathcal{N}} \rightrightarrows \underline{\mathrm{Hom}}(I_F/I_K, \widehat{G}) \times \mathbb{G}_{m,\mathbb{Q}} \times \widehat{\mathcal{N}}^{[I_F:I_K]+1}$$

given by

$$(g, f, M) \mapsto \Big(\mathrm{Int}(g, w_0) \circ f, p_{\mathbb{G}_m}(g), (\mathrm{Ad}(f(i))(M))_{i \in I_F/I_K}, \mathrm{Ad}(g, w_0)(M) \Big)$$

$$(g, f, M) \mapsto \Big(f \circ \mathrm{Int}(w_0), q, (M)_{i \in I_F/I_K}, qM \Big).$$

We then have the following explicit description of WDP_G^K.

Proposition 5.3. *The morphism j_{w_0} identifies WDP_G^K with the equalizer $\mathsf{Eq}(\mathcal{D}^{\mathrm{WD}})$. Thus, WDP_G^K is representable by a finite type affine $\mathbb{Q}$-scheme and j_{w_0} is a closed embedding.*

Observe that for an extension $K \subseteq K'$ of Galois extensions of F containing F^* there is a restriction morphism $\underline{Z}^1(I_F/I_{K'}, \widehat{G}) \to \underline{Z}^1(I_K/I_{K'}, \widehat{G})$. By Proposition 2.8 and Proposition 4.6 the subspace consisting of only the trivial homomorphism is a clopen subset of the target, and thus so is its preimage in $\underline{Z}^1(I_F/I_{K'}, \widehat{G})$, but this is precisely $\underline{Z}^1(I_F/I_K, \widehat{G})$. We deduce that the natural inclusion of functors $\mathsf{WDP}_G^K \to \mathsf{WDP}_G^{K'}$ is a clopen embedding. From the identification $\mathsf{WDP}_G = \varinjlim_K \mathsf{WDP}_G^K$ we deduce from Proposition 5.3 that WDP_G is representable by a scheme locally of finite type over $\mathbb{Q}$, all of whose connected components are affine.

The following non-trivial result will play an important technical role below.

Theorem 5.4 ([BG19, Corollary 2.3.7] and [Zhu20, Corollary 3.1.10])**.** *The schemes WDP_G^K are reduced for all K, and thus, a fortiori, WDP_G is reduced.*

5.2. **Semi-simplicity of parameters.** As in the Theorem 3.6 one requires Frobenius semi-simplicity conditions to get a Jacobson–Morozov result in the relative setting. Therefore, we now develop a sufficient notion of Frobenius semi-simplicity for a Weil–Deligne and L-parameter over a $\mathbb{Q}$-algebra A.

Definition 5.5. Let R be a $\mathbb{Q}$-algebra and H is a smooth group R-scheme such that H° is reductive. We then say that an element h of $H(R)$ is *semi-simple* if there exists some $m \geqslant 1$, an étale cover $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, and a torus T of H°_S such that h^m is in $T(S)$.

By [SGA3-1, Exposé VIB, Corollaire 4.4] H° is representable so the above makes sense. Moreover, by [Con14, Proposition B.3.4] we may assume that T is split in the above definition.

Proposition 5.6. *Let R be a $\mathbb{Q}$-algebra and H is a smooth group R-scheme such that H° is reductive, and let h be an element of $H(R)$. Then, the following statements are true.*

1. *If h is semi-simple, there exists an étale cover $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, an integer $m \geqslant 1$, and a split maximal torus T of H°_S such that h^m is in $T(S)$.*
2. *If Z is a closed subgroup R-scheme of $Z(H^\circ)$ which is flat over R, then h is semi-simple if and only if its image in $(H/Z)(R)$ is semi-simple.*

Proof. To show (1) let $\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ be an étale cover and T' a torus of $H^\circ_{S'}$ such that h^m is in $T'(S')$. Note that $Z_{H^\circ}(T')$ is a reductive group (combine [Con14, Lemma 2.2.4] and [Mil17, Corollary 17.59]). By [Con14, Corollary 3.2.7] there exists an étale cover $\mathrm{Spec}(S) \to \mathrm{Spec}(S')$ and a maximal torus T of $Z_{H^\circ}(T')_S$. Observe that T is also a maximal torus of H°_S. Indeed, it is evidently a torus, and its maximality can be checked over each point x of $\mathrm{Spec}(S)$ over which

it is clear. As T'_S is central in $Z_{H^\circ_{S'}}(T')_S$ it is clear that T contains T'_S and thus h^m is contained in $T(S)$. As we may pass to a futher étale extension to split T the claim follows.

Let $f\colon H^\circ \to H^\circ/Z$ be the tautological map. To prove (2) it is sufficient to note that for any R-algebra S one has that the map $T \mapsto T/Z$ and $T' \mapsto f^{-1}(T')$ are mutually inverse bijections between the maximal tori of H°_S and $(H^\circ/Z)_S$ by [Con14, Corollary 3.3.5]. □

Consider a representation $\rho\colon H \to \mathrm{GL}(M)$ where M is a finitely generated R-module. Let h be an element of $H(R)$ and I a finite subgroup of $H(R)$ that is stable under conjugation by h. For any R-algebra S and any λ in $S^\times$ let us set

$$M^I_S(h,\lambda) := \ker\left(\rho(h) - \lambda | M_S^{\rho(I)} \to M_S^{\rho(I)}\right).$$

Abbreviate $M^I_R(h,\lambda)$ to $M^I(h,\lambda)$, and further abbreviate to $M^I(\lambda)$ if h is clear from context. Finally, we omit I from the notation if I is trivial. Evidently $M^I_S(h,\lambda)\otimes_S S'$ is equal to $M^I_{S'}(h,\lambda)$ for any flat map of R-algebras $S \to S'$.

Proposition 5.7. *Assume that h is semi-simple. Then, there exists a unique decomposition*

$$M^I = \bigoplus_{\lambda\in R^\times} M^I(h,\lambda) \oplus M'$$

such that for any flat map $R \to S$ one has that

$$\bigoplus_{\lambda\in S^\times - R^\times} M^I_S(h,\lambda)$$

is a direct summand of M'_S, and such that this is an equality if for some $m \geqslant 1$:

(1) h^m is contained in a split torus of H°_S and commutes with I,
(2) S is a $\mathbb{Q}(\zeta_r)$-algebra, where $r := [\langle h\rangle : \langle h^m\rangle]$ and ζ_r is a primitive r^{th}-root of unity,
(3) and S contains an r^{th}-root of all λ such that $M(h^r,\lambda) \neq 0$.

Proof. Take an étale cover $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ and $m \geqslant 1$ such that h^m is contained in a split torus T of H°_S and h^m commutes with I. Then $h^r \in \langle h^m\rangle$ is contained in T and commutes with I. By [CGP15, Lemma A.8.8] one may decompose M_S into character spaces $M_S(\chi)$. One then observes that $M_S(h^r,\lambda)$ is precisely the direct sum of those character spaces $M_S(\chi)$ such that $\chi(h^r) = \lambda$. So, M_S admits a direct sum decomposition with respect to the spaces $M_S(h^r,\lambda)$.

As M_S is finitely generated, we know that $M_S(h^r,\lambda)$ is trivial for all but finitely many $\lambda_1,\ldots,\lambda_e$. In particular, we may further pass to the étale extension $S' := S[\lambda_1^{1/r},\ldots,\lambda_e^{1/r},\zeta_r]$. We extend the action of I on each nontrivial $M_{S'}(h^r,\lambda)$ by ρ to the action of the finite group $I \rtimes (\langle h\rangle/\langle h^m\rangle)$ letting h act $\lambda^{-1/r}\rho(h)$. As S' is a $\mathbb{Q}(\zeta_r)$-algebra, we have a decomposition of $M^I_{S'}(h^r,\lambda)$ into character spaces $M_{S'}(h^r,\lambda)[\nu]$ where ν travels over the characters $I \rtimes (\langle h\rangle/\langle h^m\rangle) \to \langle h\rangle/\langle h^m\rangle \to S'$. We then see that for each $\tau \in (S')^\times$ such that $\tau^r = \lambda$ the space $M^I_{S'}(h,\tau)$ admits a direct decomposition into the spaces $M_{S'}(h^r,\lambda)[\nu]$ as ν ranges over those characters with $\nu(h) = \lambda^{-1/r}\tau$.

One may then check that the module $\bigoplus_\tau M^I_{S'}(h,\tau)$ as τ ranges over those elements of $(S')^\times - R^\times$ is stabilized under the étale descent data associated to $M^I_{S'}$, and therefore (see [Sta21, Tag 023N]) descends to a submodule M' of M^I. One sees that M' is a complement of $\bigoplus_\lambda M^I(h,\lambda)$ as λ travels over the elements of $R^\times$, as this may be checked over the faithfully flat extension S'. One may then check that M' is independent of all choices, and satisfies the desired conditions. □

The following proposition will be helpful to define Frobenius semi-simple in a way that does not require the choice of an explicit arithmetic Frobenius lift.

Proposition 5.8. *Let $\varphi\colon \mathcal{W}_{F,A} \to {}^C G_A$ be a morphism of group schemes over a $\mathbb{Q}$-algebra A. Then there is a positive integer m divisible by $[F^* : F]$ such that the morphism $\mathcal{W}_{F,A} \to \check{G}_A$*

given by $w \mapsto \check{\varphi}(w^m)$ *admits a factorization*

$$\mathcal{W}_{F,A} \xrightarrow{d} \underline{\mathbb{Z}}_A \xrightarrow{\check{\varphi}_m} \check{G}_A$$

and $\check{\varphi}_m$ *takes values in* $Z_{\check{G}}(\varphi)$.

Proof. Take a finite extension K of F^* Galois over F such that $\check{\varphi}|_{\mathcal{I}_{K,A}}$ is trivial. Take a lift $w_0 \in W_F$ of arithmetic Frobenius and choose m_0 such that the image of $w_0^{m_0}$ in W_F/I_K is central. Let m be the order of $W_F/I_K\langle w_0^{m_0}\rangle$. Then for any $w \in W_F$, since w^m is trivial in $W_F/I_K\langle w_0^{m_0}\rangle$, we have that $w^m = iw_0^{d(w)m}$ for some $i \in I_K$. Hence, the images of w^m and $w_0^{md(w)}$ in W_{F^*}/I_K are the same. Since $\check{\varphi}|_{\mathcal{W}_{F^*,A}}$ factors through $(\mathcal{W}_{F^*}/\mathcal{I}_K)_A$, we have $\check{\varphi}(w^m) = \check{\varphi}(w_0^m)^{d(w)}$ for any point w of $\mathcal{W}_{F,A}$. Hence we have the factorization $\check{\varphi}_m \colon \underline{\mathbb{Z}}_A \to \check{G}_A$. The composition

$$\mathcal{W}_{F,A} \xrightarrow{\varphi} {}^C G \longrightarrow \check{G}_A \rtimes (\mathcal{W}_F/\mathcal{I}_K)_A$$

factors through $\varphi_K \colon (\mathcal{W}_F/\mathcal{I}_K)_A \to \check{G}_A \rtimes (\mathcal{W}_F/\mathcal{I}_K)_A$. To show that $\check{\varphi}_m$ factors through $Z_{\check{G}}(\varphi)$, it suffices to show $\check{\varphi}(w_0^m) \in Z_{\check{G}}(\varphi_K)$. Since the image of w_0^m in W_F/I_K is central, we have $\varphi_K(w_0^m) \in Z_{\check{G}_A \rtimes (\mathcal{W}_F/\mathcal{I}_K)_A}(\varphi_K)$. Since the image of $(1, w_0^m)$ in $\check{G}_A \rtimes (\mathcal{W}_F/\mathcal{I}_K)_A$ is central, we obtain $\check{\varphi}(w_0^m) \in Z_{\check{G}_A}(\varphi_K)$. □

To define the notion of Frobenius semi-simple parameters, it is useful to have the following analogue of Lemma 3.13.

Proposition 5.9. *Let* (φ, N) *be an element of* $\mathsf{WDP}_G(A)$. *Then, the following are equivalent:*

(1) for any (equiv. one) lift $w_0 \in W_F$ *of arithmetic Frobenius,* $\overline{\varphi}(w_0)$ *is semi-simple,*

(2) for some m *as in Proposition 5.8, the morphism* $\check{\varphi}_m$ *étale locally factorizes through a torus of* $\check{G}_A$.

Proof. By definition, (1) holds if and only if $\overline{\varphi}(w_0)$ has the property that $\overline{\varphi}(w_0)^m$ étale locally lies in a torus of $(\check{G} \rtimes \underline{\Gamma}_*)^\circ_A = \check{G}_A$ for some m as in Proposition 5.8. But, as an element of $\check{G}_A$, one easily sees that $\overline{\varphi}(w_0)^m$ is precisely $\check{\varphi}_m(1)$. As it is clear that (2) is equivalent to claim that étale locally on A there exists a torus containing $\check{\varphi}_m(1)$ the claim follows. □

Definition 5.10. For a $\mathbb{Q}$-algebra A, we call an element (φ, N) of $\mathsf{WDP}_G(A)$ *Frobenius semi-simple* if it satisfies any of the equivalent conditions of Proposition 5.9.

For each $\mathbb{Q}$-algebra A, let us denote by $\mathsf{WDP}_G^{\mathrm{ss}}(A)$ (resp. $\mathsf{WDP}_G^{K,\mathrm{ss}}(A)$) the subset of $\mathsf{WDP}_G(A)$ (resp. $\mathsf{WDP}_G^K(A)$) consisting of Frobenius semi-simple parameters. It is clear that this forms a $\widehat{G}$-stable subpresheaf[3] of WDP_G (resp. WDP_G^K). Note also that by Proposition 5.6, under the bijection of $\mathsf{WDP}_G(\mathbb{C})$ with $\Phi_G^{\mathrm{WD},\square}$ the set $\mathsf{WDP}_G^{\mathrm{ss}}(\mathbb{C})$ corresponds to $\Phi_G^{\mathrm{WD},\mathrm{ss},\square}$.

The following technical result will play an important role later in the paper.

Proposition 5.11. *If* A *is a reduced* $\mathbb{Q}$*-algebra and* (φ, N) *is an element of* $\mathsf{WDP}_G(A)$ *such that* $Z_{\widehat{G}}(\varphi, N)^\circ_x$ *is reductive of dimension* n *for all* x *in* $\mathrm{Spec}(A)$, *then* (φ, N) *is Frobenius semi-simple.*

Proof. Define $S(N)$ to be the closed subgroup scheme of $\check{G}_A$ cut out by the closed condition $gNg^{-1} = p_{\mathbb{G}_m}(g)N$. We have the equality $Z_{\widehat{G}}(\varphi, N) = \ker(p_{\mathbb{G}_m}|_{Z_{S(N)}(\varphi)})$. Note that for all x in $\mathrm{Spec}(A)$ one has a short exact sequence

$$1 \to Z_{\widehat{G}}(\varphi, N)_x \to Z_{S(N)}(\varphi)_x \to \mathbb{G}_{m,x} \to 1,$$

and as $Z_{\widehat{G}}(\varphi, N)^\circ_x$ is assumed to be reductive of dimension n for all x in $\mathrm{Spec}(A)$, that $Z_{S(N)}(\varphi)^\circ_x$ is reductive of dimension $n+1$, and thus $Z_{S(N)}(\varphi)^\circ$ is representable and smooth over A, and thus reductive over A, by [SGA3-1, Exposé VIB, Corollaire 4.4] and [Mil17, Theorem 3.23].

[3]Note that one does not expect this presheaf to be representable as the semi-simple elements in algebraic group form a constructible, but not locally closed, subset

We take m as Proposition 5.8. Then $\check{\varphi}_m$ factors through $Z_{S(N)}(\varphi)$. Further it factors through $Z(Z_{S(N)}(\varphi))$, since $\varphi(w^m)$ and $(1, w^m)$ commutes with $Z_{S(N)}(\varphi)$ for any point w of $\mathcal{W}_{F,A}$. Then there is an m' such that $\check{\varphi}_m^{m'} = \check{\varphi}_{mm'}$ factors through $Z(Z_{S(N)}(\varphi)^\circ)^\circ$. As $Z_{S(N)}(\varphi)^\circ$ is reductive, $Z(Z_{S(N)}(\varphi)^\circ)^\circ$ is a torus. Hence (φ, N) is Frobenius semi-simple. □

5.3. **The space $\mathsf{WDP}_G^{\sqcup}$.** In this section we study the moduli space of Weil–Deligne parameters (φ, N) where N lies in $\mathcal{N}^{\sqcup}$ and show that this moduli space has an exceedingly simple structure.

Definition 5.12. We denote by $\mathsf{WDP}_G^{K,\sqcup}$ (resp. $\mathsf{WDP}_G^{\sqcup}$) the space $\mathsf{WDP}_G^K \times_{\widehat{\mathcal{N}}} \widehat{\mathcal{N}}^{\sqcup}$ (resp. $\mathsf{WDP}_G \times_{\widehat{\mathcal{N}}} \widehat{\mathcal{N}}^{\sqcup} = \varinjlim_K \mathsf{WDP}_G^{K,\sqcup}$).

Now, let us fix a finite extension K of F^* Galois over F and a lift w_0 of arithmetic Frobenius. Then, by Proposition 5.3 we have an identification j_{w_0} of $\mathsf{WDP}_G^K(\overline{\mathbb{Q}})$ with

$$\left\{ (\gamma, \phi, N) \in \check{G}(\overline{\mathbb{Q}}) \times \underline{Z}^1(I_F/I_K, \widehat{G})(\overline{\mathbb{Q}}) \times \widehat{\mathcal{N}}(\overline{\mathbb{Q}}) : \begin{array}{ll} (1) & \mathrm{Int}(\gamma, w_0) \circ \phi = \phi \circ \mathrm{Int}(w_0), \\ (2) & p_{\mathbb{G}_m}(\gamma) = q, \\ (3) & \mathrm{Ad}(\phi(i))(N) = N \text{ for all } i \in I_F/I_K, \\ (4) & \mathrm{Ad}(\gamma, w_0)(N) = qN \end{array} \right\}.$$

Now, for (γ, ϕ, N) in $\mathsf{WDP}_G^K(\overline{\mathbb{Q}})$ let us define $Z_{\phi,N} := Z_{\widehat{G}}(\phi, N)$.

Definition 5.13. An element (γ', ϕ', N') in $\mathsf{WDP}_G^K(A)$, for a $\overline{\mathbb{Q}}$-algebra A, is *locally movable to* (γ, ϕ, N) if there exists an étale cover $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ and $(g, h) \in (\widehat{G} \times Z^\circ_{\phi,N})(A')$ such that $(\gamma', \phi', N') = g(h\gamma, \phi, N)g^{-1}$.

As this definition is clearly functorial, we observe that we may define a subpresheaf $U(\gamma, \phi, N)$ of $\mathsf{WDP}_{G,\overline{\mathbb{Q}}}^{K,\sqcup}$ whose A-points are given by

$$U(\gamma, \phi, N)(A) := \left\{ (\gamma', \phi', N') \in \mathsf{WDP}_{G,\overline{\mathbb{Q}}}^{K,\sqcup}(A) : (\gamma', \phi', N') \text{ is locally movable to } (\gamma, \phi, N) \right\}.$$

We then have the following.

Proposition 5.14. *The morphism of presheaves $U(\gamma, \phi, N) \to \mathsf{WDP}_{G,\overline{\mathbb{Q}}}^{K,\sqcup}$ is representable by an open immersion. Moreover, the $\overline{\mathbb{Q}}$-scheme $U(\gamma, \phi, N)$ is smooth and irreducible.*

Before we prove this proposition, we observe its major consequence. To this end, let us define an equivalence relation on $\mathsf{WDP}_G^K(\overline{\mathbb{Q}})$ by declaring that (γ, ϕ, N) is equivalent to (γ', ϕ', N') if there exists some $(g, h) \in (\widehat{G} \times Z_{\phi,N})(\overline{\mathbb{Q}})$ such that (γ', ϕ', N') is equal to $g(h\gamma, \phi, N)g^{-1}$. Let us denote an equivalence class under this relation by $[(\gamma, \phi, N)]$. Observe that as we do not require that h to actually lie in $Z^\circ_{\phi,N}(\overline{\mathbb{Q}})$ that $[(\gamma, \phi, N)]$ differs from $U(\gamma, \phi, N)(\overline{\mathbb{Q}})$. For each such equivalence class, let us choose an element (γ, ϕ, N). We consider $\pi_0(Z_{\phi,N})$ as a finite abstract group, and we define an equivalence relation on it by declaring that c is equivalent to $c_1 c \gamma c_1^{-1} \gamma^{-1}$ for any c_1 in $\pi_0(Z_{\phi,N})$. We denote by $[c]$ an equivalence class for this relation.

Remark 5.15. The group $\langle\gamma\rangle$ acts on $\pi_0(Z_{\phi,N})$ by $\gamma \cdot c = \gamma c \gamma^{-1}$. Note that $\langle\gamma\rangle \cong \mathbb{Z}$ since $p_{\mathbb{G}_m}(\gamma) = q$. Hence, the map $z \mapsto z(\gamma)$ for $z \in Z^1(\langle\gamma\rangle, \pi_0(Z_{\phi,N}))$ induces a bijection between $H^1(\langle\gamma\rangle, \pi_0(Z_{\phi,N}))$ and equivalence classes in $\pi_0(Z_{\phi,N})$.

We then have the following decomposition of $\mathsf{WDP}_{G,\overline{\mathbb{Q}}}^{K,\sqcup}$ into explicit connected components.

Theorem 5.16. *The choice of (γ, ϕ, N) in each class $[(\gamma, \phi, N)]$ of $\mathsf{WDP}_G^K(\overline{\mathbb{Q}})$ gives a scheme-theoretic decomposition*

$$\mathsf{WDP}_{G,\overline{\mathbb{Q}}}^{K,\sqcup} = \bigsqcup_{[(\gamma,\phi,N)]} \bigsqcup_{[c]} U(c\gamma, \phi, N).$$

Proof. From Proposition 5.14 we know that each $U(c\gamma,\phi,N)$ is an open subset of $\mathsf{WDP}^{K,\sqcup}_{G,\overline{\mathbb{Q}}}$. As $\mathsf{WDP}^{K,\sqcup}_{G,\overline{\mathbb{Q}}}$ is a finite type $\overline{\mathbb{Q}}$-scheme, it thus suffices to prove this claim at the level of $\overline{\mathbb{Q}}$-points. But, note that by Proposition 5.3, if (γ,ϕ,N) satisfies the conditions to be in $\mathsf{WDP}^K_G(\overline{\mathbb{Q}})$ then (γ',ϕ,N) does if and only if $\gamma'=h\gamma$ for h in $Z_{\phi,N}(\overline{\mathbb{Q}})$. Thus, we have a decomposition

$$\mathsf{WDP}^{K,\sqcup}_{G,\overline{\mathbb{Q}}}=\bigsqcup_{[(\gamma,\phi,N)]}\bigcup_{c\in\pi_0(Z_{\phi,N})}U(c\gamma,\phi,N).$$

Next observe that an element $(h\gamma,\phi,N)$ may be written in the form $g(h'\gamma,\phi,N)g^{-1}$ if and only if g is in $Z_{\phi,N}(\overline{\mathbb{Q}})$ and $h\gamma=gh'\gamma g^{-1}$ which implies that $h=gh'\gamma g^{-1}\gamma^{-1}$. With this, it is easy to see that

$$\bigcup_{c\in\pi_0(Z_{\phi,N})}U(c\gamma,\phi,N)=\bigsqcup_{[c]}U(c\gamma,\phi,N)$$

from where the desired equality follows. □

From this we deduce the following non-trivial result. Let us denote the set of equivalence classes for $\mathsf{WDP}^K_G(\overline{\mathbb{Q}})$ (resp. $\pi_0(Z_{\phi,N})$) by $[\mathsf{WDP}^K_G(\overline{\mathbb{Q}})]$ (resp. $[\pi_0(Z_{\phi,N})]$).

Corollary 5.17. *The $\mathbb{Q}$-scheme $\mathsf{WDP}^{K,\sqcup}_G$ is smooth, and there is a non-canonical $\Gamma_{\mathbb{Q}}$-equivariant bijection*

$$\pi_0\left(\mathsf{WDP}^{K,\sqcup}_{G,\overline{\mathbb{Q}}}\right)\xrightarrow{\sim}\left\{([(\gamma,\phi,N)],[c]):\begin{matrix}(1)&[(\gamma,\phi,N)]\in[\mathsf{WDP}^K_G(\overline{\mathbb{Q}})]\\(2)&[c]\in[\pi_0(Z_{\phi,N})]\end{matrix}\right\}$$

where the $\Gamma_{\mathbb{Q}}$ action on the target is inherited from $\mathsf{WDP}^{K,\sqcup}_G$ and $\widehat{G}$.

The proof of Proposition 5.14. Define the morphism $\pi_K\colon\mathsf{WDP}^{K,\sqcup}_G\to\underline{Z}^1(I_F/I_K,\widehat{G})\times\widehat{\mathcal{N}}^{\sqcup}$ by $\pi_K(\varphi,N)=(\phi,N)$. This morphism is $\widehat{G}$-equivariant when the target is endowed with the diagonal $\widehat{G}$-action. Now, by Proposition 2.8 there is a decomposition

$$\underline{Z}^1(I_F/I_K,\widehat{G})_{\overline{\mathbb{Q}}}\times\widehat{\mathcal{N}}^{\sqcup}_{\overline{\mathbb{Q}}}=\bigsqcup_{[(\phi_0,N_0)]\in\mathcal{J}}\mathcal{O}_{\phi_0}\times\mathcal{O}_{N_0}$$

where $\mathcal{J}$ is the set of $\widehat{G}(\overline{\mathbb{Q}})^2$ orbits of $(\underline{Z}^1(I_F/I_K,\widehat{G})\times\widehat{\mathcal{N}}^{\sqcup})(\overline{\mathbb{Q}})$. Observe though that if (φ,N) is in $\mathsf{WDP}^{K,\sqcup}_G(\overline{\mathbb{Q}})$ with $\pi_K(\varphi,N)=(\phi,N)$ then ϕ centralizes N. So, if we set $\mathcal{J}'$ to be the subset of $\mathcal{J}$ consisting of those $[(\phi_0,N_0)]$ with ϕ_0 centralizing N_0 then we may produce a factorization

$$\pi_K\colon\mathsf{WDP}^{K,\sqcup}_{G,\overline{\mathbb{Q}}}\longrightarrow\bigsqcup_{[(\phi_0,N_0)]\in\mathcal{J}'}\mathcal{O}_{\phi}\times\mathcal{O}_N$$

which is $\widehat{G}$-equivariant. For each $[(\phi_0,N_0)]$ in $\mathcal{J}'$ let us set $X(\phi_0,N_0):=\pi_K^{-1}(\mathcal{O}_{\phi_0}\times\mathcal{O}_{N_0})$, which is a clopen subset of $\mathsf{WDP}^{K,\sqcup}_{G,\overline{\mathbb{Q}}}$.

Set $L:=Z_{\widehat{G}}(\phi)$ which, by Lemma 2.5, is a closed subgroup scheme of $\widehat{G}_{\overline{\mathbb{Q}}}$ with reductive identity component. Let $\mathfrak{l}$ be the Lie algebra of L. Define $\mathcal{O}_N\cap\mathcal{N}_L:=\mathcal{O}_N\times_{\widehat{\mathcal{N}}}\mathcal{N}_L$. For each M in $(\mathcal{O}_N\cap\mathcal{N}_L)(\overline{\mathbb{Q}})$ we denote by $\mathcal{O}_{L,M}$ the locally closed L-orbit subscheme of $(\mathcal{O}_N\cap\mathcal{N}_L)_{\mathrm{red}}$.

Lemma 5.18. *There exists a finite set $\{N=N_1,N_2,\dots,N_m\}$ in $(\mathcal{O}_N\cap\mathcal{N}_L)(\overline{\mathbb{Q}})$ such that one has an equality of schemes $\mathcal{O}_N\cap\mathcal{N}_L=\bigsqcup_i\mathcal{O}_{L,N_i}$. In particular, $\mathcal{O}_N\cap\mathcal{N}_L$ is reduced.*

Proof. We first show that the claimed decomposition holds for $(\mathcal{O}_N\cap\mathcal{N}_L)_{\mathrm{red}}$. Now, there are only finitely many $L(\overline{\mathbb{Q}})$ orbits in $(\mathcal{O}_N\cap\mathcal{N}_L)(\overline{\mathbb{Q}})$ as there are only finitely many L°-orbits in $\mathcal{N}_L(\overline{\mathbb{Q}})$. Let $N=N_1,\dots,N_m$ represent these orbits. By Proposition 4.6 it suffices to show that each $\mathcal{O}_{L,N_i}$ is open or, as they form a set-theoretic partition of $(\mathcal{O}_N\cap\mathcal{N}_L)_{\mathrm{red}}$, that each is closed. Then, by the Noetherian valuative criterion for properness (see [Sta21, Tag 0208]) it suffices to show if R is a discrete valuation ring and $f\colon\mathrm{Spec}(R)\to(\mathcal{O}_N\cap\mathcal{N}_L)_{\mathrm{red}}$ is a morphism with $f(\eta)\in\mathcal{O}_{N_i,L}$ then $f(\mathrm{Spec}(R))\subseteq\mathcal{O}_{N_i,L}$. Assume not, and let $f\colon\mathrm{Spec}(R)\to(\mathcal{O}_N\cap\mathcal{N}_L)_{\mathrm{red}}$

be a morphism such that $f(\eta) \in \mathcal{O}_{L,N_i}(k(\eta))$ and $f(s) \in \mathcal{O}_{L,N_j}(k(s))$ with $i \neq j$. Note that f corresponds to an element $\mathbf{N}$ in $\mathcal{N}_L(R)$ which is, as an element of $\widehat{\mathcal{N}}(R)$, lies in $\mathcal{O}_N(R)$. Let us consider $Z_L(\mathbf{N})$. On the one hand, $Z_L(\mathbf{N})$ cannot be flat, as its generic fiber (resp. special fiber) is a twisted form of $Z_L(N_i)$ (resp. $Z_L(N_j)$) which has dimension $\dim(L) - \dim(\mathcal{O}_{L,N_i})$ (resp. $\dim(L) - \dim(\mathcal{O}_{L,N_j})$). Note though that as $f(s)$ lies in $\overline{\mathcal{O}_{L,N_i}}$, whose $\overline{\mathbb{Q}}$-points are unions of $\overline{\mathbb{Q}}$-points of orbits of smaller dimension (cf. [Mil17, Proposition 1.66]), $\dim(\mathcal{O}_{L,N_j})$ is strictly less than $\dim(\mathcal{O}_{N_i,L})$, and thus the fibers of $Z_L(\mathbf{N})$ have different dimensions, and so it cannot be flat over R (see [GW20, Corollary 14.95]). On the other hand, $Z_{\widehat{G}}(\mathbf{N})$ is flat as its étale locally isomorphic to $Z_{\widehat{G}}(N) = Z_{\widehat{G}}(N)_R$. But, by Lemma 2.5 this implies that $Z_{\widehat{G}}(\mathbf{N})^{\phi(I_F)} = Z_L(\mathbf{N})$ is flat, which is a contradiction.

As $(\mathcal{O}_N \cap \mathcal{N}_L)_{\mathrm{red}} \to \mathcal{O}_N \cap \mathcal{N}_L$ is a homeomorphism, there is a scheme-theoretic decomposition $\mathcal{O}_N \cap \mathcal{N}_L = \bigsqcup_i U_i$ where U_i is the open subscheme of $\mathcal{O}_N \cap \mathcal{N}_L$ with underlying space $\mathcal{O}_{L,N_i}$. As these schemes are Noetherian, to finish it suffices to show that for all i and all Noetherian $\overline{\mathbb{Q}}$-algebras A every morphism $\mathrm{Spec}(A) \to U_i$ factorizes through $\mathcal{O}_{L,N_i}$. As $\mathcal{O}_N = \mathcal{O}_{N_i}$ we may assume without loss of generality that $i = 1$, and so $N_i = N$. Let $\mathbf{N}$ be the element of $\mathfrak{l}_A$ coresponding to $\mathrm{Spec}(A) \to U_i$. We must then show that étale locally on A, $\mathbf{N}$ is conjugate to N. Let I denotes the nilradical of A, and write $A_0 = A/I$. As A is Noetherian, $I^m = (0)$ for some m, and thus by inducting we may assume that $I^2 = (0)$. Now, as A_0 is reduced the map $\mathrm{Spec}(A_0) \to U_i$ factorizes through $\mathcal{O}_{L,N}$ and thus $\mathbf{N}_{A_0}$ is étale locally conjugate to N. As the étale covers of A and A_0 are equivalent (see [Sta21, Tag 04DY]), and we are free to work étale locally on A, we may assume without loss of generality that $\mathrm{Ad}(l_0)(\mathbf{N}_{A_0}) = N$ for some l_0 in $L(A_0)$. As L is smooth, we may apply the infinitesimal lifting criterion to find a lift l in $L(A)$ of l_0. Replacing $\mathbf{N}$ by $\mathrm{Ad}(l)(\mathbf{N})$ we may assume without loss of generality that $\mathbf{N}_{A_0} = N$. Now, as $\underline{\mathrm{Transp}}_{\widehat{G}}(\mathbf{N}, N) \to \mathrm{Spec}(A)$ is a $Z_{\widehat{G}}(N)$-torsor, and thus smooth, we know by the infinitesimal lifting criterion that there exists some g in $\underline{\mathrm{Transp}}_{\widehat{G}}(\mathbf{N}, N)(A)$ lifting the identity. Using the notation of [DG70, II, §4, №3, 3.7], we may write $g = e^x$ for x in $I\widehat{\mathfrak{g}}_A$. Then, by [DG70, II, §4, №4, 4.2] we have

$$N = \mathrm{Ad}(g)(\mathbf{N}) = \mathbf{N} + \mathrm{ad}(x)(\mathbf{N}).$$

As N and $\mathbf{N}$ lie in $\mathfrak{l}_A$, they are invariant for the action of the finite group $\phi(I_F/I_K)$, and so if y denotes the average of x over the action of $\phi(I_F/I_K)$ then

$$N = \mathbf{N} + \mathrm{ad}(y)(\mathbf{N}).$$

But, by loc. cit. this right-hand side is equal to $\mathrm{Ad}(e^y)(\mathbf{N})$. By Lemma 2.5 we see that e^y lies in $L(A)$, from where the claim follows. □

Let us now denote by $(\gamma^{\mathrm{univ}}, \phi^{\mathrm{univ}}, N^{\mathrm{univ}})$ the universal object over $X(\phi, N)$. Consider the transporter scheme $\underline{\mathrm{Transp}}_{\widehat{G}}(\phi^{\mathrm{univ}}, \phi) \to \underline{Z}^1(I_F/I_K, \widehat{G})$ and set T to be the pullback to $X(\phi, N)$. Set $b\colon T \to X(\phi, N)$ to be the tautological map, which is smooth as T is visibly an L-torsor. Note that we have a morphism $a\colon T \to \mathcal{O}_N \cap \mathcal{N}_L$ given by $a(g) = \mathrm{Ad}(g)(N^{\mathrm{univ}})$ and observe then that we have a scheme-theoretic decomposition $T = \bigsqcup_i a^{-1}(\mathcal{O}_{L,N_i})$. But, for each i we also have a map $\kappa_i\colon a^{-1}(\mathcal{O}_{L,N_i}) \to \pi_0(Z_{\widehat{G}}(\phi, N))$ given by sending g to the component containing $\mathrm{Int}(g)(\gamma^{\mathrm{univ}})\gamma^{-1}$, and we define for each i and each $c \in \pi_0(Z_{\widehat{G}}(\phi, N))$ the open subscheme $U_{i,c} := \kappa_i^{-1}(c)$ of $a^{-1}(\mathcal{O}_{L,N_i})$. We then obtain a decomposition $T = \bigsqcup_{i,c} U_{i,c}$.

As $b\colon T \to X(\phi, N)$ is smooth, we see that $b(U_{1,\mathrm{id}})$ is an open subset of $X(\phi, N)$ whose A-points are precisely (by [EGA4-4, Corollaire 17.16.3.(ii)]) the set of A-points (γ', ϕ', N') of $X(\phi, N)$ which are étale locally in the image of b. It is simple to see that this implies that $U(\gamma, \phi, N) = b(U_{1,\mathrm{id}})$, which implies $U(\gamma, \phi, N)$ is representable by an open immersion.

Finally, to show that $U(\gamma, \phi, N)$ is smooth and irreducible consider the natural morphism $\widehat{G} \times Z^\circ_{\phi,N} \to U(\gamma, \phi, N)$. To simplify notation let us write $S = \widehat{G} \times Z^\circ_{\phi,N}$. Note that, by definition, $S \to U(\gamma, \phi, N)$ is surjective as étale sheaves and thus a fortiori surjective as schemes, and thus $U(\gamma, \phi, N)$ is irreducible. To see that $U(\gamma, \phi, N)$ is smooth, note that as $S \to U(\gamma, \phi, N)$ is surjective as étale sheaves there exists an étale cover $V \to U(\gamma, \phi, N)$ such that $p\colon S_V \to V$

admits a section. Note though that as $S_V \to S$ is étale and the target is reduced, so is the source (see [Sta21, Tag 025O]). But, as p has a section, this implies that V is reduced as the morphism of sheaves of rings $\mathcal{O}_V \to p_*\mathcal{O}_S$ has a section and thus is injective. This implies that $U(\gamma,\phi,N)$ is reduced by [Sta21, Tag 033F]. But, as we're in characteristic 0, this implies that $U(\gamma,\phi,N)$ is generically smooth over $\overline{\mathbb{Q}}$ (see [Sta21, Tag 056V]). But, as $S(\overline{\mathbb{Q}})$ acts $U(\gamma,\phi,N)$ by scheme automorphisms acting transitively on $U(\gamma,\phi,N)(\overline{\mathbb{Q}})$ we deduce that every point of $U(\gamma,\phi,N)(\overline{\mathbb{Q}})$ has regular local ring, and thus $U(\gamma,\phi,N)$ is smooth over $\overline{\mathbb{Q}}$ as desired (see [Sta21, Tag 0B8X]). This completes the proof of Proposition 5.14.

6. The moduli space of L-parameters and the Jacobson–Morozov morphism

In this section we define the moduli space LP_G^K of L-parameters for G, show it has favorable geometric properties, construct the Jacobson–Morozov morphism $\mathsf{LP}_G^K \to \mathsf{WDP}_G^{K,\sqcup}$, and show that an analogue of Theorem 3.6 holds for any $\mathbb{Q}$-algebra A.

6.1. The moduli space of L-parameters.

We begin with a slight modification of the Langlands group scheme $\mathcal{W}_F \times \mathrm{SL}_{2,\mathbb{Q}}$ better suited to arithmetic discussions over $\mathbb{Q}$.

Definition 6.1. We call the $\mathbb{Q}$-scheme representing the functor

$$\mathbf{Alg}_{\mathbb{Q}} \to \mathbf{Grp}, \qquad A \mapsto \{(w,g) \in \mathcal{W}_F(A) \times \mathrm{GL}_2(A) : \|w\| = \det(g)\}$$

the *twisted Langlands group scheme* and denote it $\mathcal{L}_F^{\mathrm{tw}}$.

To justify the naming of $\mathcal{L}_F^{\mathrm{tw}}$, note that if k is any extension of $\mathbb{Q}$ and c is any element of k such that $c^2 = q$, then the morphism

$$\eta_c \colon \mathcal{W}_{F,k} \times \mathrm{SL}_{2,k} \to \mathcal{L}_{F,k}^{\mathrm{tw}}, \qquad (w,g) \mapsto \left(w, g\begin{pmatrix} c^{-d(w)} & 0 \\ 0 & c^{-d(w)}\end{pmatrix}\right),$$

is an isomorphism. For future reference, we observe that we have a morphism

$$p_{\mathrm{tw}} \colon \mathcal{L}_F^{\mathrm{tw}} \to \mathbb{G}_{m,\mathbb{Q}} \times \mathcal{W}_F, \qquad (w,g) \mapsto (\|w\|, w).$$

Let us also observe that there is a natural embedding of group schemes $\mathrm{SL}_{2,\mathbb{Q}} \to \mathcal{L}_F^{\mathrm{tw}}$ given by sending g to $(1,g)$, as well as an embedding

$$\iota \colon \mathcal{W}_F \to \mathcal{L}_F^{\mathrm{tw}} \qquad w \mapsto \left(w, \begin{pmatrix} \|w\| & 0 \\ 0 & 1\end{pmatrix}\right).$$

With these embeddings, we shall implicitly think of $\mathrm{SL}_{2,\mathbb{Q}}$ and $\mathcal{I}_F$ as subfunctors of $\mathcal{L}_F^{\mathrm{tw}}$. Finally, we observe that the embedding of $\mathcal{W}_K$ into $\mathcal{W}_F$ for any finite extension K of F gives rise to an embedding of $\mathcal{L}_K^{\mathrm{tw}} \to \mathcal{L}_F^{\mathrm{tw}}$ which we implicitly use to think of $\mathcal{L}_K^{\mathrm{tw}}$ as a subgroup scheme of $\mathcal{L}_F^{\mathrm{tw}}$.

Definition 6.2. For a $\mathbb{Q}$-algebra A we define an *L-parameter over A* to be a homomorphism of group A-schemes $\psi \colon \mathcal{L}_{F,A}^{\mathrm{tw}} \to {}^C G_A$ such that $p_C \circ \psi = p_{\mathrm{tw}}$.

Denote by $\mathsf{LP}_G(A)$ the set of L-parameters over A, which is functorial in A. Note that LP_G has a natural conjugation action by $\widehat{G}$ and so one has the centralizer group presheaf $Z_{\widehat{G}}(\psi)$.

For an L-parameter ψ over A we define the morphism $\check{\psi} \colon \mathcal{L}_{F,A}^{\mathrm{tw}} \to \check{G}_A$ as the composition of ψ with the projection ${}^C G_A \to \check{G}_A$. We denote by $\overline{\psi}$ the homomorphism of group A-schemes $\mathcal{L}_{F,A}^{\mathrm{tw}} \to (\widehat{G} \rtimes \underline{\Gamma_*})_A$ obtained by composing ψ with the quotient homomorphism $\check{G}_A \to (\widehat{G} \rtimes \underline{\Gamma_*})_A$. Let us observe that while $\check{\psi}$ may not be a homomorphism, it becomes so after restriction to $\mathcal{L}_{F^*,A}^{\mathrm{tw}}$. Finally, by our assumptions on ψ the restriction to $\mathrm{SL}_{2,A}$ takes values in $\widehat{G}_A$ and we denote this resulting morphism $\mathrm{SL}_{2,A} \to \widehat{G}_A$ by θ (or θ_ψ when we want to emphasize ψ).

To relate this to more familiar objects, fix k to be an extension of $\mathbb{Q}$ containing an element c such that $c^2 = q$. For a k-algebra A, we endow $\widehat{G}(A)$ with the discrete topology and set

$$\mathsf{LP}'_{G,k}(A) := \left\{ W_F \times \mathrm{SL}_2(A) \xrightarrow{\psi} \widehat{G}(A) \rtimes W_F : \begin{array}{ll} (1) & \psi \text{ is a homomorphism over } W_F, \\ (2) & W_F \xrightarrow{\psi|_{W_F}} \widehat{G}(A) \rtimes W_F \to \widehat{G}(A) \text{ is continuous}, \\ (3) & \psi|_{\mathrm{SL}_2(A)} \colon \mathrm{SL}_2(A) \to \widehat{G}(A) \text{ is algebraic} \end{array} \right\}.$$

There is a morphism $i_c^{\mathrm{L}} \colon \mathsf{LP}'_{G,k} \to \mathsf{LP}_{G,k}$ which on A-points is given by sending ψ' to the element ψ of $\mathsf{LP}_G^K(A)$ that is equal to $i_c \circ \psi' \circ \eta_c^{-1}$ on A-points, where ψ is uniquely determined by Proposition 2.6. We can show the following proposition in the same way as Proposition 5.2.

Proposition 6.3. *The morphism $i_c^{\mathrm{L}} \colon \mathsf{LP}'_{G,k} \to \mathsf{LP}_{G,k}$ is an isomorphism.*

For a finite extension K of F^* Galois over F define

$$\mathsf{LP}_G^K(A) := \left\{ \psi \in \mathsf{LP}_G(A) : \mathcal{I}_K \subseteq \ker\left(\check{\psi}|_{\mathcal{L}^{\mathrm{tw}}_{F^*,A}} \right) \right\},$$

which clearly forms a subpresheaf of LP_G. We have the equality of presheaves $\mathsf{LP}_G = \varinjlim_K \mathsf{LP}_G^K$. As in the case of Weil–Deligne parameters, may associate to an L-parameter ψ in $\mathsf{LP}_G^K(A)$ an element ϕ of $\underline{Z}^1(I_F/I_K, \widehat{G})(A)$ and thus obtain a morphism of presheaves $\mathsf{LP}_G^K \to \underline{Z}^1(I_F/I_K, \widehat{G})$.

Fix a lift w_0 of arithmetic Frobenius in W_F and define a morphism of presheaves

$$j_{w_0} \colon \mathsf{LP}_G^K \to \check{G} \times \underline{Z}^1(I_F/I_K, \widehat{G}) \times \underline{\mathrm{Hom}}(\mathrm{SL}_{2,\mathbb{Q}}, \widehat{G}), \qquad \psi \mapsto \left(\check{\psi}\left(w_0, \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)\right), \phi, \theta \right).$$

On the other hand, we have a diagram

$$\mathcal{D}^L \colon \ \check{G} \times \underline{Z}^1(I_F/I_K, \widehat{G}) \times \underline{\mathrm{Hom}}(\mathrm{SL}_{2,\mathbb{Q}}, \widehat{G}) \rightrightarrows \underline{\mathrm{Hom}}(I_F/I_K, \widehat{G}) \times \mathbb{G}_{m,\mathbb{Q}} \times \underline{\mathrm{Hom}}(\mathrm{SL}_{2,\mathbb{Q}}, \widehat{G})^{[I_F:I_K]+1}$$

given by the two maps

$$(g, f, \nu) \mapsto \Big(\mathrm{Int}(g, w_0) \circ f, p_{\mathbb{G}_m}(g), (\mathrm{Int}(f(i)) \circ \nu)_{i \in I_F/I_K}, \mathrm{Int}(g, w_0) \circ \nu \Big)$$

$$(g, f, \nu) \mapsto \Big(f \circ \mathrm{Int}(w_0), q, (\nu)_{i \in I_F/I_K}, \nu \circ \mathrm{Int}\left(\left(w_0, \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right) \Big).$$

We then have the following explicit description of LP_G^K.

Proposition 6.4. *The morphism j_{w_0} gives an identification of LP_G^K with $\mathrm{Eq}(\mathcal{D}^L)$. In particular, LP_G^K is representable by a finite type affine $\mathbb{Q}$-scheme and j_{w_0} is a closed embedding.*

As already observed, for an extension $K \subseteq K'$ of finite extensions of F^* Galois over F, there is a restriction morphism $\underline{Z}^1(I_F/I_{K'}, \widehat{G}) \to \underline{Z}^1(I_K/I_{K'}, \widehat{G})$ which is a clopen embedding, and thus $\mathsf{LP}_G^K \to \mathsf{LP}_G^{K'}$ is also a clopen embedding. As we have the identification of presheaves $\mathsf{LP}_G = \varinjlim_K \mathsf{LP}_G^K$ we deduce from Proposition 5.3 that LP_G is representable by a scheme locally of finite type over $\mathbb{Q}$, all of whose connected components are affine.

6.2. **Decomposition into connected components.** We now establish the analogue of Theorem 5.16 for LP_G. Let us fix K a finite extension of F^* Galois over F, and a lift w_0 of arithmetic Frobenius. Then, by Proposition 6.4 we have an identification j_{w_0} of $\mathsf{LP}_G^K(\overline{\mathbb{Q}})$ with

$$\left\{ (\gamma, \phi, \theta) \in \check{G}(\overline{\mathbb{Q}}) \times \underline{Z}^1(I_F/I_K, \widehat{G})(\overline{\mathbb{Q}}) \times \underline{\mathrm{Hom}}(\mathrm{SL}_{2,\mathbb{Q}}, \widehat{G})(\overline{\mathbb{Q}}) : \begin{array}{ll} (1) & \mathrm{Int}(\gamma, w_0) \circ \phi = \phi \circ \mathrm{Int}(w_0), \\ (2) & p_{\mathbb{G}_m}(\gamma) = q, \\ (3) & \mathrm{Int}(\phi(i)) \circ \theta = \theta \text{ for all } i \in I_F/I_K, \\ (4) & \mathrm{Int}((\gamma, w_0)) \circ \theta = \theta \circ \mathrm{Int}\left(\left(w_0, \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right) \end{array} \right\}.$$

Now, for (γ, ϕ, θ) in $\mathsf{LP}_G^K(\overline{\mathbb{Q}})$ let us define $Z_{\phi,\theta}$ to be $Z_{\widehat{G}}(\phi, \theta)$. This is a linear algebraic group over $\overline{\mathbb{Q}}$ whose identity component is reductive. Let us then say that an element $(\gamma', \phi', \theta')$ in $\mathsf{LP}_G^K(A)$, for a $\overline{\mathbb{Q}}$-algebra A, is *locally movable to* (γ, ϕ, θ) if there exists an étale cover $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$

and $(g,h) \in (\widehat{G} \times Z^\circ_{\phi,\theta})(A')$ such that $(\gamma',\phi',\theta') = g(h\gamma,\phi,\theta)g^{-1}$. As this definition is clearly functorial, we obtain a subpresheaf of $\mathsf{LP}^K_{G,\overline{\mathbb{Q}}}$ as follows:

$$U(\gamma,\phi,\theta)(A) := \left\{(\gamma',\phi',\theta') \in \mathsf{LP}^K_{G,\overline{\mathbb{Q}}}(A) : (\gamma',\phi',\theta') \text{ is locally movable to } (\gamma,\phi,\theta)\right\}.$$

We then have the following, whose proof is identical to Proposition 5.14 except the analogue of Lemma 5.18 is simpler considering Proposition 4.10.

Proposition 6.5. *The morphism of presheaves $U(\gamma,\phi,\theta) \to \mathsf{LP}^K_{G,\overline{\mathbb{Q}}}$ is representable by an open immersion. Moreover, the $\overline{\mathbb{Q}}$-scheme $U(\gamma,\phi,\theta)$ is smooth and irreducible.*

Define an equivalence relation on $\mathsf{LP}^K_G(\overline{\mathbb{Q}})$ by declaring that (γ,ϕ,θ) is equivalent to (γ',ϕ',θ') if there exists some $(g,h) \in (\widehat{G} \times Z_{\phi,\theta})(\overline{\mathbb{Q}})$ such that $(\gamma',\phi',\theta') = g(h\gamma,\phi,\theta)g^{-1}$. Let us denote an equivalence class under this relation by $[(\gamma,\phi,\theta)]$. Observe that here we do not require h to lie in $Z^\circ_{\phi,\theta}(\overline{\mathbb{Q}})$, so that these equivalence classes differ from $U(\gamma,\phi,\theta)(\overline{\mathbb{Q}})$. For each such equivalence class, let us choose an element (γ,ϕ,θ). We consider $\pi_0(Z_{\phi,\theta})$ as a finite abstract group, and we define an equivalence relation on it by declaring that c is equivalent to $c_1 c \gamma c_1^{-1}\gamma^{-1}$ for any c_1 in $\pi_0(Z_{\phi,\theta})$. We denote by $[c]$ an equivalence class for this relation.

We then have the following decomposition of $\mathsf{LP}^K_{G,\overline{\mathbb{Q}}}$ into explicit connected components, whose proof is exactly the same as that of Theorem 5.16.

Theorem 6.6. *The choice of (γ,ϕ,θ) in each class $[(\gamma,\phi,\theta)]$ of $\mathsf{LP}^K_G(\overline{\mathbb{Q}})$ gives an identification*

$$\mathsf{LP}^K_{G,\overline{\mathbb{Q}}} = \bigsqcup_{[(\gamma,\phi,\theta)]} \bigsqcup_{[c]} U(c\gamma,\phi,\theta).$$

We derive from this two corollaries neither of which is a priori obvious.

Corollary 6.7. *For all (γ,ϕ,θ) in $\mathsf{LP}^K_G(\overline{\mathbb{Q}})$ the $\overline{\mathbb{Q}}$-scheme $U(\gamma,\phi,\theta)$ is affine.*

Denote the set of equivalence classes for $\mathsf{LP}^K_G(\overline{\mathbb{Q}})$ (resp. $\pi_0(Z_{\phi,\theta})$) by $[\mathsf{LP}^K_G(\overline{\mathbb{Q}})]$ (resp. $[\pi_0(Z_{\theta,N})]$).

Corollary 6.8. *The affine $\mathbb{Q}$-scheme LP^K_G is smooth, and there is a non-canonical $\Gamma_{\mathbb{Q}}$-equivariant bijection*

$$\pi_0\left(\mathsf{LP}^K_{G,\overline{\mathbb{Q}}}\right) \xrightarrow{\sim} \left\{([(\gamma,\phi,\theta)],[c]) : \begin{matrix}(1) & [(\gamma,\phi,\theta)] \in \left[\mathsf{LP}^K_G(\overline{\mathbb{Q}})\right] \\ (2) & [c] \in [\pi_0(Z_{\phi,\theta})]\end{matrix}\right\}$$

where the $\Gamma_{\mathbb{Q}}$ action on the target is inherited from LP^K_G and $\widehat{G}$.

6.3. **The Jacobson–Morozov morphism.** We now come to the definition of the Jacobson–Morozov map in the geometric setting.

Definition 6.9. The morphism $\mathsf{JM}\colon \mathsf{LP}_G \to \mathsf{WDP}_G$ given by sending ψ to $(\psi \circ \iota, d\theta_\psi(e_0))$ is called the *Jacobson–Morozov morphism.*

It is clear that JM is $\widehat{G}$-equivariant. By Theorem 4.14 it is also clear that JM factorizes uniquely through $\mathsf{WDP}^{\sqcup}_G$. Moreover, for any finite extension K of F^* Galois over F, one sees that $\mathsf{JM}^{-1}(\mathsf{WDP}^K_G)$ is precisely LP^K_G and so we get factorizations $\mathsf{LP}^K_G \to \mathsf{WDP}^K_G$ and $\mathsf{LP}^K_G \to \mathsf{WDP}^{K,\sqcup}_G$. We denote all these factorizations also by JM.

Observe that over $\overline{\mathbb{Q}}$ we may give a simpler description of the Jacobson–Morozov morphism on each connected component. Namely, let us fix (γ,ϕ,θ) in $\mathsf{LP}^K_G(\overline{\mathbb{Q}})$ as in the notation of §6.2. Then, first observe that $\mathsf{JM}(\gamma,\phi,\theta)$ is equal to (γ,ϕ,N) where $N = \mathsf{JM}(\theta)$. We may then observe that JM restricted to $U(\gamma,\phi,\theta)$ maps into $U(\gamma,\phi,N)$ and is the étale sheafification of the map which on A-points is the map

$$\left\{g(h\gamma,\phi,\theta)g^{-1} : (g,h) \in \widehat{G}(A) \times Z^\circ_{\phi,\theta}(A)\right\} \to \left\{g(h'\gamma,\phi,N)g^{-1} : (g,h') \in \widehat{G}(A) \times Z^\circ_{\phi,N}(A)\right\}$$

given by sending $g(h\gamma,\phi,\theta)g^{-1}$ to $g(h\gamma,\phi,N)g^{-1}$.

We also observe that if k is an extension of $\mathbb{Q}$ and c is an element of k such that $c^2 = q$ then under the isomorphisms described in Proposition 5.2 and Proposition 6.3 that the Jacobson–Morozov corresponds to the morphism $\mathsf{LP}'_{G,k} \to \mathsf{WDP}'_{G,k}$ sending ψ to the map on A-points of $(\psi \circ \iota', d\theta_\psi(e_0))$ where

$$\iota' \colon \mathcal{W}_{F,k} \to \mathcal{W}_{F,k} \times \mathrm{SL}_{2,k}, \qquad w \mapsto \left(w, \begin{pmatrix} c^{-d(w)} & 0 \\ 0 & c^{d(w)} \end{pmatrix}\right).$$

So, on the level of $\mathbb{C}$-points we see that our Jacobson–Morozov map agrees with that from §3.2.

We now move towards stating the analogue of Theorem 3.6 at the level of A-points. To begin, we must define the notion of semi-simplicity for L-parameters in the relative setting.

Proposition 6.10. *Let ψ be an L-parameter over a $\mathbb{Q}$-algebra A. Then there is a positive integer m divisible by $[F^* : F]$ such that the morphism*

$$\mathcal{W}_{F,A} \to \check{G}_A, \qquad w \mapsto \check{\psi}\left(w^{2m}, \begin{pmatrix} q^{-md(w)} & 0 \\ 0 & q^{-md(w)} \end{pmatrix}\right)$$

admits a factorization

$$\mathcal{W}_{F,A} \xrightarrow{d} \underline{\mathbb{Z}}_A \xrightarrow{\check{\psi}_m} \check{G}_A.$$

Proof. This is proved in the same way as Proposition 5.8. □

Definition 6.11. For A a $\mathbb{Q}$-algebra, we call an element ψ of $\mathsf{LP}_G(A)$ *Frobenius semi-simple* if there exists an integer m as in Proposition 6.10 such that $\check{\psi}_m$ factors through a subtorus of $\check{G}_A$ etale locally on A.

Let us denote by $\mathsf{LP}^{\mathrm{ss}}_G(A)$ (resp. $\mathsf{LP}^{K,\mathrm{ss}}_G(A)$) the subset of Frobenius semi-simple elements of $\mathsf{LP}_G(A)$ (resp. $\mathsf{LP}^K_G(A)$). This evidently forms a $\widehat{G}$-stable subfunctor of LP_G (resp. LP^K_G).

Remark 6.12. To understand the reasoning for this definition, observe that under the isomorphism in Proposition 6.3, this condition corresponds to an element ψ' of $\mathsf{LP}'_{G,k}(A)$ satisfying the property that the projection of $\psi'(w_0^{2m}, 1)$ to $\widehat{G}(A)$ is semi-simple for some m as in Proposition 6.10. In particular, this notion of semi-simple agrees with that from §3.2 for $\mathbb{C}$-points by Lemma 3.13.

We now prove the following surprisingly subtle semi-simplicity preservation property for the Jacobson–Morozov morphism.

Proposition 6.13. *Let A be a $\mathbb{Q}$-algebra and ψ an element of $\mathsf{LP}_G(A)$. Then, ψ is Frobenius semi-simple if and only if $\mathsf{JM}(\psi)$ is.*

Proof. Suppose that ψ is Frobenius semi-simple. As the conclusion is insensitive to passing to an étale extension and conjugating, we do so freely. Take m as in Proposition 6.10 and a split maximal torus T of $\check{G}_A$ such that $\check{\psi}_m$ factors through T. Note that the eigenspace $\check{\mathfrak{g}}_A(1)$ with respect to $\check{\psi}_m(1)$ is the Lie algebra of a Levi subgroup L of $\check{G}_A$ such that $\check{\psi}_m$ factors through $Z(L)$. Indeed, we may assume that $T = (T_0)_A$ for a maximal torus T_0 of $\check{G}$. Let L' be the Levi subgroup of $\check{G}$ generated by T_0 and the root groups for the roots α which annihilate $\check{\psi}_m(1)$. Then, we may take $L = L'_A$, where $\check{\psi}_m$ factors through $Z(L)$ by [Con14, Corollary 3.3.6].

Note that θ factorizes through L as by Proposition 2.6 it suffices to check this on the level of Lie algebras, from where it is clear. Let T_2 denote the standard diagonal subtorus of $\mathrm{SL}_{2,A}$. Since θ factorizes through L, by [Con14, Lemma 5.3.6] we may assume that the map $\theta|_{T_2}$ factorizes through a maximal torus T' of L. But, as $Z(L) \subseteq T'$ both $\theta|_{T_2}$ and $\check{\psi}_m$ factorize through T'. Hence, if we write $\mathsf{JM}(\psi) = (\varphi, N)$ then the morphism $\mathcal{W}_{F,A} \to \check{G}_A$ given by $w \mapsto \varphi(w^m)$ factors through T'. This implies that $\mathsf{JM}(\psi)$ is Frobenius semi-simple.

Conversely, suppose that $\mathsf{JM}(\psi) = (\varphi, N)$ is Frobenius semi-simple. Let m be any integer as Proposition 5.8. As above, we may build a reductive subgroup L_m of $\check{G}_A$ such that $\mathrm{Lie}(L_m)$ is identified with $\check{\mathfrak{g}}_A(1)$ with respect to $\check{\varphi}_m(1)$. We claim that the group L_{km} stabilizes for k

sufficiently large. Indeed, the roots of α of $\check{G}$ relative to T_0 that annihilate $\check{\varphi}_{km}(1) = \check{\varphi}_m(1)^k$ stabilize for k sufficiently large, from where the claim follows by the construction. Denote by L the group L_{km} for k sufficiently large, say for $k \geqslant k_0$. Let us write Z for the torus $Z(L)^\circ$ (see [Con14, Theorem 3.3.4]). Observe that as $\check{\varphi}_{km}$, for $k \geqslant k_0$, centralizes $\mathrm{Lie}(L)$ that $\check{\varphi}_{km}$ factors through $Z(L)$. So then, for some $k_1 \geqslant k_0$ we have that $\check{\varphi}_{k_1 m}$ factors through Z. We put $m_1 = k_1 m$. We will be done if we can show that $\theta|_{T_2}$ factorizes through the reductive group A-scheme $Z_{\check{G}}(Z)$ (see [Con14, Lemma 2.2.4] and [Mil17, Corollary 17.59]). Indeed, in this case by [Con14, Lemma 5.3.6], we know that after passing to an étale extension, $\theta|_{T_2}$ factorizes through a maximal torus T' of $Z_{\check{G}}(Z)$. Then $\theta|_{T_2}$ and $\check{\varphi}_{m_1}$ factor through T'. Hence

$$\mathcal{W}_{F,A} \to \check{G}_A, \qquad w \mapsto \check{\psi}\left(w^{2m_1}, \begin{pmatrix} q^{-m_1 d(w)} & 0 \\ 0 & q^{-m_1 d(w)} \end{pmatrix}\right)$$

factors through T'. This implies that ψ is Frobenius semi-simple.

Working etale locally, and by passing to a $\check{G}(A)$-conjugate, we may assume that Z is equal to Z'_A for a split subtorus Z' of $\check{G}$. Let R_0 be the set of nontrivial characters of Z' appearing in the adjoint action of Z' on $\check{\mathfrak{g}}_A$. Note that these characters are already defined over $\mathbb{Q}$. Consider the functor on $\mathbf{Alg}_{\mathbb{Q}}$ with

$$Y(B) := \left\{ z \in Z'(B) : \begin{array}{ll} (1) & \chi(z) \neq 1 \text{ for all } \chi \in R_0, \\ (2) & \chi(z) = q^{m_1} \text{ for all } \chi \in R_0 \text{ such that } \chi(\check{\varphi}_{m_1}(1)) = q^{m_1} \end{array} \right\}.$$

Clearly Y defines a locally closed subscheme of Z' which is non-empty as $\check{\varphi}_{m_1}(1)$ is an element of $Y(A)$. Take $y \in Y(F)$ for a finite extension F of $\mathbb{Q}$. By passing to an étale extension, we may assume that A contains F. We claim that inclusion $Z_{\check{G}}(Z) \subseteq Z_{\check{G}}(y)^\circ_A$ is an equality. As $Z_{\check{G}}(Z)$ is flat over $\mathrm{Spec}(A)$, we know from the fibral criterion for isomorphism (see [EGA4-4, Corollaire 17.9.5]), that it suffices to check this after base change to every point of $\mathrm{Spec}(A)$. But, as A is $\mathbb{Q}$-algebra, and $Z_{\check{G}}(Z)$ and $Z_{\check{G}}(y)^\circ_A$ are both connected, it then suffices to check they have the same Lie algebra (e.g. see [Mil17, Corollary 10.16]), but this is true by construction.

In the following, we use the notation $\check{\mathfrak{g}}_A(\lambda)$ for $\lambda \in A^\times$ with respect to $\check{\varphi}_{m_1}(1)$. By construction, we know that $\mathrm{Int}(y)$ acts on $\check{\mathfrak{g}}_A(q^{\pm m_1})$ by multiplication by $q^{\pm m_1}$. Moreover, the SL_2-triple (N, f, h) associated to θ by Theorem 4.17 satisfies $N \in \check{\mathfrak{g}}_A(q^{m_1})$, $f \in \check{\mathfrak{g}}_A(q^{-m_1})$ and $h \in \check{\mathfrak{g}}_A(1)$. Therefore, the $\mathfrak{sl}_2$-triple attached to $\mathrm{Int}(y) \circ \theta$ is $(q^{m_1} N, q^{-m_1} f, h)$. Thus, the $\mathfrak{sl}_2$-triple attached to $\mathrm{Int}(y) \circ \theta \circ \mu$ is (N, f, h) where

$$\mu \colon \mathrm{SL}_{2,A} \xrightarrow{\sim} \mathrm{SL}_{2,A}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & q^{-m_1} b \\ q^{m_1} c & d \end{pmatrix}.$$

By Theorem 4.17 $\mathrm{Int}(y) \circ \theta \circ \mu = \theta$, so $\theta|_{T_2}$ factorizes through $Z_{\check{G}}(y)^\circ_A = Z_{\check{G}}(Z)$ as desired. □

We end this section by proving a relative version of Proposition 3.7. Fix a $\mathbb{Q}$-algebra A and let N be an element of $\mathcal{N}^{\sqcup}(A)$. Let us denote by $\mathfrak{u}^N$ the A-submodule $\mathrm{im}(\mathrm{ad}(N)) \cap \mathrm{Ker}(\mathrm{ad}(N))$ of $\widehat{\mathfrak{g}}_A$, which we also treat as a subfunctor of $\widehat{\mathfrak{g}}_A$ in the obvious way. Note that $\mathfrak{u}^N$ is in fact a closed subscheme of $\widehat{\mathcal{N}}_A$ and for all A-algebras B there is an equality

$$\mathfrak{u}^N(B) = \mathrm{im}(\mathrm{ad}(N \otimes 1)) \cap \mathrm{Ker}(\mathrm{ad}(N \otimes 1)).$$

As these claims are étale local, we may assume that $N = gN_0g^{-1}$ for some N_0 in $\widehat{\mathcal{N}}(\mathbb{Q})$ and g in $\widehat{G}(A)$. Observe then that $\mathfrak{u}^N$ is equal to $g(\mathfrak{u}^{N_0})_A g^{-1}$ where $\mathfrak{u}^{N_0} \subseteq \widehat{\mathfrak{g}}$ is defined in the same way as $\mathfrak{u}^N$. As $\widehat{\mathcal{N}}_A$ is $\widehat{G}(A)$-equivariant it suffices to show that $\mathfrak{u}^{N_0}$ factorizes through $\widehat{\mathcal{N}}$ which may be checked on $\overline{\mathbb{Q}}$-points which is then clear. One similarly proves the claimed equality.

As $\mathfrak{u}^N$ is a closed subscheme of $\widehat{\mathcal{N}}_A$, we obtain a closed subscheme $U^N := \exp(\mathfrak{u}^N)$ of $\widehat{G}_A$. We claim that U^N is a closed subgroup scheme of $\widehat{G}_A$ flat over A. As this may be checked étale locally we are again reduced to checking that $\exp(\mathfrak{u}^{N_0})$ is a closed subgroup $\mathbb{Q}$-scheme of $\widehat{G}$ (automatically flat over $\mathbb{Q}$), but this is true by Proposition 2.3. For an element (φ, N) of $\mathsf{WDP}^{\sqcup}_G(A)$ we set

$$U^N(\varphi) := U^N \times_{\widehat{G}_A} Z_{\widehat{G}}(\varphi).$$

Concretely this means that for every A-algebra B one has an identification of $U^N(\varphi)(B)$ with $U^N(B) \cap Z_{\widehat{G}}(\varphi)(B)$ where this intersection is taken in $\widehat{G}(B)$.

Let us first establish the following relative version of Proposition 3.3, which follows easily (using the same reduction arguments as already used above) from Proposition 3.3

Lemma 6.14. *Let θ be an element of $\underline{\mathrm{Hom}}(\mathrm{SL}_{2,\mathbb{Q}}, \widehat{G})(A)$ and define $N = \mathsf{JM}(\theta)$. Then, $Z_{\widehat{G}}(N) = U^N \rtimes Z_{\widehat{G}}(\theta)$.*

Proposition 6.15. *Let A be a $\mathbb{Q}$-algebra, ψ is an element of $\mathsf{LP}_G(A)$, and set $(\varphi, N) = \mathsf{JM}(\psi)$. Then, $Z_{\widehat{G}}(\varphi, N) = U^N(\varphi) \rtimes Z_{\widehat{G}}(\psi)$.*

Proof. Let B be an A-algebra. Given Lemma 6.14 it clearly suffices to show that conjugation by an element in the image of φ stabilizes U^N, as the rest of the argument for Proposition 3.7 then goes through verbatim. Let $u = \exp(n)$ be an element of $U^N(B)$ and observe that $\mathrm{Int}(\varphi(w))(u)$ is equal to $\exp(\mathrm{Ad}(\varphi(w))(n))$, and so we are done as clearly $\mathrm{Ad}(\varphi(w))(n) \in \mathfrak{u}^N(B)$. □

6.4. **The relative Jacobson–Morozov theorem for parameters.** We now arrive at the relative analogue of Theorem 3.6. Let us set $\mathsf{WDP}_G^{\sqcup,\mathrm{ss}}$ to be the presheaf whose A-points consist of Frobenius semi-simple Weil–Deligne parameters (φ, N) such that N lies in $\mathcal{N}^{\sqcup}(A)$.

Theorem 6.16 (Relative Jacobson–Morozov theorem for parameters)**.** *The Jacobson–Morozov morphism $\mathsf{JM}\colon \mathsf{LP}_G^{\mathrm{ss}} \to \mathsf{WDP}_G^{\sqcup,\mathrm{ss}}$ is surjective, and induces an isomorphism of quotient presheaves*

$$\mathsf{JM}\colon \mathsf{LP}_G^{\mathrm{ss}}/\widehat{G} \xrightarrow{\sim} \mathsf{WDP}_G^{\sqcup,\mathrm{ss}}/\widehat{G}.$$

Let us fix a $\mathbb{Q}$-algebra A, an element (φ, N) of $\mathsf{WDP}_G^{\sqcup,\mathrm{ss}}(A)$, and an arithmetic Frobenius lift $w_0 \in \mathcal{W}_{F,A}$. In the notation from Proposition 5.7, with $\rho\colon (\check{G} \rtimes \Gamma_*)_A \to \mathrm{GL}(\widehat{\mathfrak{g}}_A)$ the adjoint action, $h = \overline{\varphi}(w_0)$, and $I = \phi(I_F/I_K)$, let $\mathfrak{h}$ and $\mathfrak{h}(\lambda)$ be $\widehat{\mathfrak{g}}_A^I$ and $\widehat{\mathfrak{g}}_A^{I,\overline{h}}(\lambda)$ respectively.

Proposition 6.17 (cf. [GR10, Lemma 2.1])**.** *There exists an $\mathfrak{sl}_2$-triple in $\widehat{\mathfrak{g}}_A$ of the form (N, f, h) where $N \in \mathfrak{h}(q)$, $f \in \mathfrak{h}(q^{-1})$, and $h \in \mathfrak{h}(1)$. Moreover, any two such $\mathfrak{sl}_2$-triples are conjugate by an element of $Z_{\widehat{G}}(\varphi, N)$ étale locally on A.*

Proof. By Theorem 4.17 there exists an $\mathfrak{sl}_2$-triple (N, h_{-1}, f_{-1}) in $\widehat{\mathfrak{g}}_A$. We take a finite extension K of F^* Galois over F such that $\mathcal{I}_{K,A} \subseteq \ker(\check{\varphi}|_{\mathcal{W}_{F^*,A}})$. Observe that N is in $\mathfrak{h}$ by definition and if we set h_0 to be the average over the action of $\phi(I_F/I_K)$, then h_0 is also in $\mathfrak{h}$ and (N, h_0) satisfies the conditions of Proposition 4.18 for $\mathfrak{h}$. Therefore there exists an $\mathfrak{sl}_2$-triple in $\mathfrak{h}$ of the form (N, h_0, f_0). Given this, the decomposition result from Proposition 5.7, and Proposition 4.18, the existence argument as in [GR10, Lemma 2.1] goes through without further comment.

To show the uniqueness part of the statement, let (N, h_1, f_1) be another $\mathfrak{sl}_2$-triple satisfying the same conditions. We shall pass to an étale extension freely in the following. By Proposition 4.10, we may assume that there exists a morphism $\theta\colon \mathrm{SL}_{2,\mathbb{Q}} \to \widehat{G}$ such that (N, h, f) is the associated $\mathfrak{sl}_2$-triple. Set $\mathfrak{m} := \mathfrak{h}^N \cap \mathfrak{h}(1)$, and for each $i \in \mathbb{N}$ set $\mathfrak{m}_i$ to be $\{x \in \mathfrak{m} : [h, x] = ix\}$. We can check that $\mathfrak{m} = \bigoplus_i \mathfrak{m}_i$ by using the adjoint action of $\theta|_{T_2}$ and Lemma 6.18 below, where T_2 is the diagonal subtorus of $\mathrm{SL}_{2,\mathbb{Q}}$. Let us now set $\mathfrak{u} := \bigoplus_{i>0} \mathfrak{m}_i$. Then $\mathfrak{u}$ is Lie subalgebra of $\widehat{\mathfrak{g}}_A$ contained in $\widehat{\mathcal{N}}(A)$ as it is contained in $\bigoplus_{i>0} \widehat{\mathfrak{g}}_{i,A}$, the base change to A of $\bigoplus_{i>0} \widehat{\mathfrak{g}}_i$ where $\widehat{\mathfrak{g}}_i = \{x \in \widehat{\mathfrak{g}} : [h, x] = ix\}$, and $\bigoplus_{i>0} \widehat{\mathfrak{g}}_i$ is quickly checked to be contained in $\widehat{\mathcal{N}}(\mathbb{Q})$. Consider $U := \exp(\mathfrak{u})$, which is a subgroup of $H(A)$ by (3) of Proposition 2.3.

We claim that $\{\mathrm{Ad}(u)(h) : u \in U\}$ is equal to $h + \mathfrak{u}$. To see this, we note that if we write $u = \exp(x)$ for $x \in \mathfrak{u}$ then by (2) of Proposition 2.3 $\mathrm{Ad}(u)(h)$ is equal to $\sum_{n\geq 0} \frac{1}{n!}\mathrm{ad}(x)^n(h)$. We need to show that for any $x_0 \in \mathfrak{u}$ there is $x \in \mathfrak{u}$ such that $x_0 = \sum_{n\geq 1} \frac{1}{n!}\mathrm{ad}(x)^n(h)$. We define a filtration $\mathrm{Fil}^i(\mathfrak{u}) = \bigoplus_{j\geqslant i} \mathfrak{m}_j$ for $i \geq 1$. It suffices to prove that there is $x_i \in \mathfrak{u}$ such that

$$x_0 \equiv \sum_{n\geq 1} \frac{1}{n!}\mathrm{ad}(x_i)^n(h) \mod \mathrm{Fil}^i(\mathfrak{u})$$

by induction on i. This is trivial for $i = 1$. We assume that it is proved for i. We take $x_i' \in \mathrm{Fil}^i(\mathfrak{u})$ such that $[x_i', h] = x_0 - \sum_{n\geq 1} \frac{1}{n!}\mathrm{ad}(x_i)^n(h)$. Then $x_{i+1} = x_i + x_i'$ is seen to satisfy

$$x_0 \equiv \sum_{n\geq 1} \frac{1}{n!}\mathrm{ad}(x_{i+1})^n(h) \mod \mathrm{Fil}^{i+1}(\mathfrak{u})$$

since $[\mathfrak{u}, \mathrm{Fil}^i(\mathfrak{u})] \subseteq \mathrm{Fil}^{i+1}(\mathfrak{u})$.

Note now that $y = h_1 - h = [N, f_1 - f]$ is in $\mathfrak{u}$. Indeed, by inspection $[N, y] = 0$ so that y is in $\mathfrak{h}^N$, but since h_1 and h are both in $\mathfrak{h}(1)$, so is their difference y. Note though that as $y = [N, f_1 - f]$ we have y is in $\mathfrak{u}$. Indeed, it again suffices to show that $\widehat{\mathfrak{g}}_A \cap [N, \widehat{\mathfrak{g}}_A]$ is equal to $\bigoplus_{i>0} \widehat{\mathfrak{g}}_{i,A}$ which, again, may be verified over $\mathbb{Q}$ in which case it is again classical (cf. [GR10, Proposition 2.2]). Thus, we know that there exists some u in U such that $\mathrm{Ad}(u)(h) = h+y = h_1$. One then verifies that $\mathrm{Ad}(u)(f) = f_1$ as in loc. cit. Finally, we now observe that the inclusion $U \subseteq Z_{\widehat{G}}(\varphi, N)(A)$ holds. Indeed, writing $u = \exp(x)$ we see that $\mathrm{Ad}(u)(N) = N$ since x is in $\mathfrak{h}^N$ and using the formula from (2) of Proposition 2.3. Similarly, as $\mathrm{Int}(\varphi(w))(\exp(x))$ is equal to $\exp(\mathrm{Ad}(\varphi(w))(x))$, this is just $\exp(x)$ as x is in $\mathfrak{h}(1)$. □

Lemma 6.18. *Let S be a scheme and H a smooth group S-scheme with Lie algebra $\mathfrak{h}$. Let $\rho\colon \mathbb{G}_{m,S} \to H$ be a morphism of group S-schemes. Set $h = d\rho(1)$, and for an integer i we set*

$$\mathfrak{h}_{\rho,i} = \{x \in \mathfrak{h} : \mathrm{Ad}(\rho(z))x = z^i x \text{ for all } z\}, \qquad \mathfrak{h}_{h,i} = \{x \in \mathfrak{h} : \mathrm{ad}(h)(x) = ix\}.$$

Then we have $\mathfrak{h}_{\rho,i} \subseteq \mathfrak{h}_{h,i}$. This is an equality if S is a $\mathbb{Q}$-scheme.

Proof. We have $d(\mathrm{Ad}\circ\rho)(1) = \mathrm{ad}(h)$ under the identification of the Lie algebra of $\mathrm{GL}(\mathfrak{h})$ with $\mathrm{End}(\mathfrak{h})$. By taking the weight decomposition of $\mathfrak{h}$ under $\mathrm{Ad}\circ\rho$ (cf. [CGP15, Lemma A.8.8]), we obtain the claim from the fact that the derivative of the i^{th}-power map $\mathbb{G}_{m,S} \to \mathbb{G}_{m,S}$ is the multiplication-by-i map. The last claim follows from $\mathfrak{h} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{h}_{\rho,i}$ and that $\mathfrak{h}_{h,i}$ for $i \in \mathbb{Z}$ are linearly independent if S is a $\mathbb{Q}$-scheme. □

To show the surjectivity claim in Theorem 6.16 let (N, f, h) be as in Proposition 6.17, and consider the morphism $\theta\colon \mathrm{SL}_{2,A} \to \widehat{G}_A$ associated by Theorem 4.15. We then consider the morphism of schemes

$$\psi\colon \mathcal{L}^{\mathrm{tw}}_{F,A} \to {}^C G_A, \quad (w, g) \mapsto \theta\left(g \begin{pmatrix} \|w\| & 0 \\ 0 & 1 \end{pmatrix}^{-1}\right)\varphi(w).$$

We claim that this a morphism of group A-schemes. To prove this, it suffices to show

$$\mathrm{Ad}(\varphi(w))(\theta(g)) = \theta\left(\mathrm{Ad}\left(\begin{pmatrix} \|w\| & 0 \\ 0 & 1 \end{pmatrix}\right)(g)\right)$$

for $w \in \mathcal{W}_{F,A}(B)$ and $g \in \mathrm{SL}_2(B)$, where B is any A-algebra. This follows from Proposition 2.6 and the construction of θ. One then easily check that ψ is an element of $\mathsf{LP}_G(A)$ such that $\mathsf{JM}(\psi) = (\varphi, N)$ as desired.

We now show that JM induces a bijection $\mathsf{LP}^{\mathrm{ss}}_G(A)/\widehat{G}(A) \xrightarrow{\sim} \mathsf{WDP}^{\sqcup,\mathrm{ss}}_G(A)/\widehat{G}(A)$, which now only requires the demonstration of injectivity. By the $\widehat{G}(A)$-equivariance of JM it suffices to show that if ψ_1 and ψ_2 are elements of $\mathsf{LP}^{\mathrm{ss}}_G(A)$ such that $\mathsf{JM}(\psi_1)$ and $\mathsf{JM}(\psi_2)$ both equal (φ, N), then ψ_1 and ψ_2 are $\widehat{G}(A)$-conjugate. Note that the $\mathfrak{sl}_2$-triples associated to θ_{ψ_i} for $i = 1, 2$ both satisfy the conditions of Proposition 6.17 for (φ, N). Therefore, étale locally on A the $\mathfrak{sl}_2$-triples associated to ψ and ψ' are conjugate in a way that centralizes (φ, N) and so ψ and ψ' are étale locally conjugate. From this we deduce that ψ_2 defines a class in $H^1_{\text{ét}}(\mathrm{Spec}(A), Z_{\widehat{G}}(\psi_1))$ given by $\underline{\mathrm{Transp}}_{\widehat{G}}(\psi_1, \psi_2)$. Note though that we have a natural map

$$H^1_{\text{ét}}(\mathrm{Spec}(A), Z_{\widehat{G}}(\psi_1)) \to H^1_{\text{ét}}(\mathrm{Spec}(A), Z_{\widehat{G}}(\varphi, N))$$

which maps $\underline{\mathrm{Transp}}_{\widehat{G}}(\psi_1, \psi_2)$ to the trivial element, and so $\underline{\mathrm{Transp}}(\psi_1, \psi_2)$ belongs to

$$\ker\left(H^1_{\text{ét}}(\mathrm{Spec}(A), Z_{\widehat{G}}(\psi_1)) \to H^1_{\text{ét}}(\mathrm{Spec}(A), Z_{\widehat{G}}(\varphi, N))\right),$$

and so we are done if this kernel is trivial. But, this follows from Proposition 6.15.

7. Geometric properties of the Jacobson–Morozov map

In this final section we use the material developed so far to prove that the Jacobson–Morozov morphism satisfies favorable geometric properties. Namely, we show that $\mathsf{JM}\colon \mathsf{LP}_G^K \to \mathsf{WDP}_G^{K,\sqcup}$ (resp. $\mathsf{JM}\colon \mathsf{LP}_G^K \to \mathsf{WDP}_G^K$) is birational (resp. weakly birational). We do this by exhibiting a more explicit space which embeds into all three moduli spaces weakly birationally. This is the geometric analogue of the reductive centralizer locus from §3.3. We then finally show that as a particular application of these ideas one may prove that the Jacobson–Morozov map is an isomorphism between the discrete loci in LP_G^K and WDP_G^K.

7.1. **Birationality properties.** To begin, note that as the morphism $\mathcal{N}^{\sqcup} \to \mathcal{N}$ is surjective and satisfies the conditions of Lemma 4.7, $\mathsf{WDP}_G^{K,\sqcup} \to \mathsf{WDP}_G^K$ is then also surjective and satisfies the same conditions. We therefore deduce from Lemma 4.7 the following.

Proposition 7.1. *The morphism $\mathsf{WDP}_G^{K,\sqcup} \to \mathsf{WDP}_G^K$ is weakly birational.*

We now give a more explicit effective version of this result. To start, we observe the following where we denote by $(\varphi^{\mathrm{univ}}, N^{\mathrm{univ}})$ the universal pair over WDP_G^K.

Proposition 7.2. *For each $n \geqslant 0$, the subset*

$$\mathsf{WDP}_G^{K,n} := \left\{ x \in \mathsf{WDP}_G^K : Z_{\widehat{G}}(\varphi^{\mathrm{univ}}, N^{\mathrm{univ}})^\circ_x \text{ is reductive of dimension } n + \dim(Z_0(\widehat{G})) \right\}$$

of WDP_G^K is locally closed, is open if $n = 0$, and is empty if $n > \dim(\widehat{G}/Z_0(\widehat{G}))$.

Proof. Consider the quotient $Q := Z_{\widehat{G}}(\varphi^{\mathrm{univ}}, N^{\mathrm{univ}})/Z_0(\widehat{G})_{\mathsf{WDP}_G^K}$. By [SGA3-1, Exposé VIB, Proposition 4.1], the function $f\colon \mathsf{WDP}_G^K \to \mathbb{N}$ given by $f(x) = \dim(Q_x)$ is upper semi-continuous. In particular the set $D_n = f^{-1}([0, n+1)) \cap f^{-1}([n, \infty)]$ of points where Q_x is of dimension n is locally closed, and as $D_0 = f^{-1}([0,1))$, D_0 is open. Let us endow D_n with the reduced substructure. Let us then note that by [SGA3-1, Exposé VIB, Corollaire 4.4] for all $n \geqslant 0$ the identity component functor $Q^\circ_{D_n}$ is representable and is smooth over D_n. Thus, by [Con14, Proposition 3.1.9], we deduce that the locus of x in D_n where Q°_x is reductive is open, and thus locally closed in WDP_G^K and open if $n = 0$. But, evidently this locus is equal to $\mathsf{WDP}_G^{K,n}$. □

Definition 7.3. We define the *reductive centralizer locus* in WDP_G^K to be the $\mathbb{Q}$-scheme $\mathsf{WDP}_G^{K,\mathrm{rc}} := \bigsqcup_n \mathsf{WDP}_G^{K,n}$ (where each $\mathsf{WDP}_G^{K,n}$ is given the reduced subscheme structure). We call the open subset $\mathsf{WDP}_G^{K,0}$ the *discrete locus* and denote it by $\mathsf{WDP}_G^{K,\mathrm{disc}}$.

Let us observe that by the proof of Proposition 7.2, if A is a reduced $\mathbb{Q}$-algebra and (φ, N) is a Weil–Deligne parameter over A such that the corresponding morphism $\mathrm{Spec}(A) \to \mathsf{WDP}_G^K$ factorizes through $\mathsf{WDP}_G^{K,\mathrm{rc}}$, then $Z_{\widehat{G}}(\varphi, N)^\circ$ is representable and reductive over A.

Now, while a priori unclear, we show now that the reducedness of WDP_G^K implies that N^{univ} pulled back to the reductive centralizer locus lies in $\mathcal{N}^{\sqcup}$. More precisely, we have the following.

Proposition 7.4. *The morphism $\mathsf{WDP}_G^{K,\mathrm{rc}} \to \mathsf{WDP}_G^K$ factorizes through $\mathsf{WDP}_G^{K,\sqcup}$.*

Indeed, as WDP_G^K is reduced by Theorem 5.4 this follows from the following proposition.

Proposition 7.5. *If A is a reduced $\mathbb{Q}$-algebra, and (φ, N) is an element of $\mathsf{WDP}_G(A)$ such that $Z_{\widehat{G}}(\varphi, N)^\circ_x$ is a reductive group scheme of dimension n for all x in $\mathrm{Spec}(A)$, then (φ, N) is an element of $\mathsf{WDP}_G^{\sqcup}(A)$.*

Proof. We break the argument into several steps to make the structure clear.

Step 1: It suffices to prove that if A is a strictly Henselian discrete valuation ring, then N is egc to some N_0 in $\widehat{\mathcal{N}}(\mathbb{Q})$. Indeed, we must show that the map $\mathrm{Spec}(A) \to \mathcal{N}$ induced by (φ, N)

factorizes through $\mathcal{N}^{\sqcup}$. By standard Noetherian approximation arguments we may assume that A is Noetherian. We may then assume that A is connected, in which case we must show that this morphism factorizes through some $\mathcal{O}_N$. As A is reduced, it suffices to show that $\mathrm{Spec}(A) \to \mathcal{N}$ factorizes through some $\mathcal{O}_N$ set-theoretically. As A is connected, any two points of $\mathrm{Spec}(A)$ may be connected by a finite chain of specialization and generalizations. This reduces us to showing that if x is a generalization of y in $\mathrm{Spec}(A)$ then these points map into a common $\mathcal{O}_N$. We are then reduced to the case of a discrete valuation ring by [Sta21, Tag 054F], and then trivially to the case of a strictly Henselian discrete valuation ring.

Step 2: We claim we may assume that (φ, N) is in $\mathsf{WDP}_G^{K,\mathrm{disc}}(A)$. Write η (resp. s) for the generic point (resp. special) of $\mathrm{Spec}(A)$. As $Z_{\widehat{G}}(\varphi, N)$ has constant fiber dimension, the same is true for $Z_{\widehat{G}^{\mathrm{der}}}(\varphi, N)$ and so again [SGA3-1, Exposé VIB, Corollaire 4.4] shows that $Z_{\widehat{G}^{\mathrm{der}}}(\varphi, N)^\circ$ is representable and reductive over A. As A is strictly Henselian, for any reductive group over A, all its tori are split, all its maximal tori are conjugate, and all its Borel subgroups are conjugate. Then, as ${}^C G$ is equal to ${}^L\widetilde{G}$ the arguments in [Bor79, Lemma 3.5] show that if T is a maximal torus of $Z_{\widehat{G}^{\mathrm{der}}}(\varphi, N)^\circ$ there exists some $g \in \check{G}(A)$ and a Levi subgroup H of G^* (where G^* is the quasi-split inner form of G) such that $gZ_{C_{G_A}}(T)g^{-1} = {}^C H_A$. Therefore $g^{-1}(\varphi, N)g$ factorizes through ${}^C H_A$. We claim then that (φ, N) is in $\mathsf{WDP}_H^{K,\mathrm{disc}}(A)$. By Proposition 5.11 $g^{-1}(\varphi_\eta, N_\eta)g$ and $g^{-1}(\varphi_s, N_s)g$ are Frobenius semi-simple. Moreover, the argument given in [Bor79, Proposition 3.6] shows that neither $g^{-1}(\varphi_\eta, N_\eta)g$ nor $g^{-1}(\varphi_s, N_s)g$ factorizes through a proper Levi (in the sense of loc. cit.) which, as they are both Frobenius semi-simple, implies by the usual arguments (cf. [Kot84, Lemma 10.3.1]) that they are discrete. As N is in $\mathcal{N}^{\sqcup}(A)$ if and only if $g^{-1}Ng$ is, the claimed reduction follows.

Step 3: We now show that we may assume $N_s \neq 0$. If both N_s and N_η are zero we're done, and so it suffices to show that if $N_\eta \neq 0$ then $N_s \neq 0$. To see this, assume otherwise. But the inequality $\dim Z_{\check{G}}(\varphi_\eta) \leqslant \dim Z_{\check{G}}(\varphi_s) = \dim Z_{\check{G}}(\varphi_s, N_s)$ holds by [SGA3-1, Exposé VIB, Proposition 4.1]. That said, $\dim Z_{\check{G}}(\varphi_\eta, N_\eta) < \dim Z_{\check{G}}(\varphi_\eta)$. Indeed, it suffices to note that if w_0 is any lift of arithmetic Frobenius then (as in Propoistion 5.8) for m sufficiently large $\check{\varphi}_\eta(w_0^m)$ defines a point of $Z_{\check{G}}(\varphi_\eta)$ but, as $N_\eta \neq 0$, does not define a point of $Z_{\check{G}}(\varphi_\eta, N_\eta)$ and thus $Z_{\check{G}}(\varphi_\eta, N_\eta)^\circ \subsetneq Z_{\check{G}}(\varphi_\eta)^\circ$ from where the claim follows. But, observe that $\dim(Z_{\check{G}}(\varphi_\eta, N_\eta))$ (resp. $\dim(Z_{\check{G}}(\varphi_\eta, N_\eta))$) is equal to $\dim(Z_{\widehat{G}}(\varphi_\eta, N_\eta)) + 1$ (resp. $\dim(Z_{\widehat{G}}(\varphi_s, N_s)) + 1$) and so we arrive at a contradiction.

Step 4: Replacing G with G^{der} we may assume that $Z_0(\widehat{G})$ is finite. Proposition 5.11 together with Theorem 6.16 imply that (φ_η, N_η) (resp. (φ_s, N_s)) comes from an L-parameter ψ_1 (resp. ψ_2). Write μ_i for the restriction of θ_{ψ_i} to the diagonal maximal torus. Fix w_0 to be an arithmetic Frobenius lift. By Frobenius semi-simplicity and the fact that A is strictly Henselian, there is, up to conjugacy, a positive integer m_0 divisible by $[F^* : F]$ such that $\check{\varphi}(w_0^{m_0})$ is contained in the A-points of a maximal torus T of $\check{G}_{\overline{\mathbb{Q}}}$. By the relationship between ψ_i and φ_i and the argument of [GR10, Lemma 3.1], we see that up to replacing m_0 by a power, we may further assume that $\check{\varphi}_\eta(w_0^{2m_0}) = \mu_1(q^{m_0})$ and $\check{\varphi}_s(w_0^{2m_0}) = \mu_2(q^{m_0})$. From this first equality it is simple to see that μ_1 factorizes through T_η, and thus there exists a unique lift μ_A of μ_1 to T_A where μ is a cocharacter of T. We note as $N_s \neq 0$, that μ_2 is characterized by the property that the image of μ_2 contains $\check{\varphi}_s(w_0^{2m_0})$ and $\mathrm{Ad}(\mu_2(q^{m_0}))(N_s) = q^{2m_0}N_s$. As $\check{G}_A$ and $\widehat{\mathfrak{g}}_A$ are separated over A, we have that the image of μ contains $\varphi(w_0^{2m_0})$ and $\mathrm{Ad}(\mu(q^{m_0}))(N) = q^{2m_0}N$. Hence, μ_s satisfies the above characterization of μ_2, so $\mu_s = \mu_2$. Let $P(\mu)$ be the parabolic subgroup of $\widehat{G}_{\overline{\mathbb{Q}}}$ associated to μ. Define $\widehat{\mathfrak{g}}_\eta(j)$ (resp. $\widehat{\mathfrak{g}}_s(i)$) using μ_η (resp. μ_s) as in [Car85, §5.7]. Then by [Car85, Proposition 5.7.3] N_η (resp. N_s) is in the unique open $P(\mu)_\eta$-orbit (resp. $P(\mu)_s$) of $\bigoplus_{i\geq 2}\widehat{\mathfrak{g}}_\eta(i)$ (resp. $\bigoplus_{i\geq 2}\widehat{\mathfrak{g}}_s(i)$). But, by the uniqueness of this open orbit, we then see that N_η and N_s are both conjugate to any $\overline{\mathbb{Q}}$-point of the unique open orbit of $P(\mu)$ on $\bigoplus_{i\geq 2}\widehat{\mathfrak{g}}(i)$, from where the conclusion follows. We are then done by Proposition 4.8. □

We next show the pleasant property that $\mathsf{WDP}_G^{K,\mathrm{rc}}$ actually has dense image in $\mathsf{WDP}_G^{K,\sqcup}$.

Lemma 7.6. *Let k be a field, X an irreducible finite type k-scheme equipped with an action of an algebraic k-group H, and Y an irreducible locally closed subscheme of X. Assume that the action morphism $\mu\colon H\times Y\to X$ is dominant. Then there is a dense open subset U of Y such that $\dim Z_H(y)\leqslant \dim(H)+\dim(Y)-\dim(X)$ for all $y\in U$.*

Proof. By [GW20, Corollary 14.116] there exists a dense open subset V of X with the property that $\dim \mu^{-1}(y)=\dim H+\dim Y-\dim X$ for all $y\in V$. As μ is H-equivariant when H is made to act on the first component of $H\times Y$, we may assume that V is H-stable. We put $U=V\cap Y$, which is non-empty as μ is dominant and V is H-stable. As $Z_H(y)\times\{y\}\subseteq\mu^{-1}(y)$ for $y\in U$, we obtain the claim. □

Proposition 7.7. *The set*

$$\{x\in \mathsf{WDP}_G^{K,\sqcup} : Z_{\widehat{G}}(\varphi^{\mathrm{univ}},N^{\mathrm{univ}})_x^\circ \text{ is a torus}\}$$

contains an open dense subset of $\mathsf{WDP}_G^{K,\sqcup}$.

Proof. Observe that this may be checked over $\overline{\mathbb{Q}}$, as the morphism $\mathrm{Spec}(\overline{\mathbb{Q}})\to\mathrm{Spec}(\mathbb{Q})$ is surjective and universally open (see [Sta21, Tag 0383]). Thus, from Theorem 5.16 it suffices to show that for each (γ,ϕ,N) in $\mathsf{WDP}_G^K(\overline{\mathbb{Q}})$ corresponding to (φ,N), one has that the set of points x in $U(\gamma,\phi,N)$ such that $Z_{\widehat{G}}(\varphi^{\mathrm{univ}},N^{\mathrm{univ}})_x^\circ$ is a torus contains a dense open subset.

Let H be the normalizer of ϕ in $(\widehat{G}\rtimes\underline{\Gamma_*})_{\overline{\mathbb{Q}}}$. Then $H^\circ=Z_{\widehat{G}}(\phi)^\circ$ which is a reductive group by Lemma 2.5. Consider the linear algebraic $\overline{\mathbb{Q}}$-group $S'_H(N)$ representing the functor

$$\mathbf{Alg}_{\overline{\mathbb{Q}}}\to\mathbf{Grp},\qquad A\mapsto\left\{(h,z)\in H(A)\times A^\times : \mathrm{Ad}(h)(N)=z^2N\right\},$$

which is clearly seen to be a closed subgroup scheme of $((\widehat{G}\times\mathbb{G}_m)\rtimes\underline{\Gamma_*})_{\overline{\mathbb{Q}}}$ by changing the order of the components. Let $S_H(N)$ be the image of $S'_H(N)$ in $(\check{G}\rtimes\underline{\Gamma_*})_{\overline{\mathbb{Q}}}$. Let s_0u_0 be the Jordan decomposition of $\overline{\varphi}(w_0)$ in $S_H(N)$. Then the image of u_0 in $\mathbb{G}_{m,\overline{\mathbb{Q}}}$ is trivial. Hence u_0 is an element of $Z^\circ_{\phi,N}$. Replacing γ by $u_0^{-1}\gamma$, we may assume that φ is Frobenius semi-simple from the beginning.

Let ψ be an element of $\mathsf{LP}_G^K(\overline{\mathbb{Q}})$ such that $\mathsf{JM}(\psi)=(\varphi,N)$ and write $\theta=\theta_\psi$. Let $U_H(N)$ be the unipotent radical of $Z_H(N)$. Then, as in Proposition 3.3, we have $Z_H(N)=U_H(N)\rtimes Z_H(\theta)$. We take a maximal quasi-torus T of $Z_H(\theta)$ in the sense of [HP18, Definition 8.6]. Set s_1 to be the image of $\left(\theta\left(\begin{pmatrix}q^{1/2}&0\\0&q^{-1/2}\end{pmatrix}\right),q^{1/2}\right)$ in $\check{G}(\overline{\mathbb{Q}})$. Then $Z^\circ_{\phi,N}\gamma s_1^{-1}\subseteq Z_H(N)$. So we can write $T\cap Z^\circ_{\phi,N}\gamma s_1^{-1}=t_1T^\circ$ for some $t_1\in T(\overline{\mathbb{Q}})$ by [HP18, Theorem 8.10 (d)]. Then, we have $Z^\circ_{\phi,N}\gamma=t_1Z^\circ_{\phi,N}s_1$.

We let T^{t_1} be the closed subgroup scheme of T of elements commuting with t_1. For t_0 in $(T^{t_1})^\circ(\overline{\mathbb{Q}})$, we consider the morphism

$$\Lambda_{t_0}\colon Z_H(N)^\circ\times(T^{t_1})^\circ\to Z_H(N)^\circ,\qquad (h,t)\mapsto(t_1t_0)^{-1}ht_1t_0ts_1h^{-1}s_1^{-1}.$$

This induces

$$\mathrm{Lie}(\Lambda_{t_0})\colon \mathrm{Lie}(Z_H(N)^\circ)\times\mathrm{Lie}((T^{t_1})^\circ)\to\mathrm{Lie}(Z_H(N)^\circ),\qquad (x,y)\mapsto\mathrm{ad}((t_1t_0)^{-1})x+y-\mathrm{ad}(s_1)x.$$

This is identified with the direct sum of

$$\mathrm{Lie}(\Lambda_{t_0})_1\colon \mathrm{Lie}(Z_H(\theta)^\circ)\times\mathrm{Lie}((T^{t_1})^\circ)\to\mathrm{Lie}(Z_H(\theta)^\circ),\qquad (x,y)\mapsto\mathrm{ad}((t_1t_0)^{-1})x+y-x,$$

$$\mathrm{Lie}(\Lambda_{t_0})_2\colon \mathrm{Lie}(U_H(N)^\circ)\to\mathrm{Lie}(U_H(N)^\circ),\qquad z\mapsto\mathrm{ad}((t_1t_0)^{-1})z-\mathrm{ad}(s_1)z.$$

In the proof of [HP18, Theorem 8.9 (c)], it is shown that the morphism

$$Z_H(\theta)^\circ\times t_1(T^{t_1})^\circ\to t_1Z_H(\theta)^\circ,\qquad (g,t)\mapsto gtg^{-1}$$

is dominant. Therefore, by Lemma 7.6 and the fact that $(T^{t_1})^\circ\subseteq Z_{Z_H(\theta)^\circ}(t_1t_0)^\circ$ for any $t_0\in(T^{t_1})^\circ$, there is an open dense subset $U_{t_1,1}\subseteq(T^{t_1})^\circ$ such that $Z_{Z_H(\theta)^\circ}(t_1t_0)^\circ=(T^{t_1})^\circ$ for $t_0\in U_{t_1,1}$. This implies that $\mathrm{Lie}(\Lambda_{t_0})_1$ is surjective for $t_0\in U_{t_1,1}$.

The eigenvalues of the diagonalizable $\mathrm{ad}(s_1)$ on $\mathrm{Lie}(U_H(N))$ are contained in $\{q^{i/2}\}_{1\leq i\leq n_0}$ for some n_0 by Proposition 3.3. Let m_1 be the order of t_1 in $\pi_0(T)$. Then there is a positive integer m such that the eigenvalues of the diagonalizable $\mathrm{ad}(t_1^{-1-mm_1})$ on $\mathrm{Lie}(U_H(N))$ are disjoint from $\{q^{i/2}\}_{1\leq i\leq n_0}$. Since $t_1^{-1-mm_1}$ and s_1 are commutative, $\mathrm{ad}(t_1^{-1-mm_1})$ and $\mathrm{ad}(s_1)$ are simultaneously diagonalizable. Hence we have the surjectivity of $\mathrm{Lie}(\Lambda_{t_1^{mm_1}})_2$. Since the surjectivity of $\mathrm{Lie}(\Lambda_{t_0})_2$ defines an open subset on $(T^{t_1})^\circ$, which we now know is non-empty, there is an open dense subset $U_{t_1,2}\subseteq (T^{t_1})^\circ$ such that $\mathrm{Lie}(\Lambda_{t_0})_2$ is surjective for $t_0\in U_{t_1,2}$.

We put $U_{t_1}=U_{t_1,1}\cap U_{t_1,2}$. Then, for $t_0\in U_{t_1}$, the map $\mathrm{Lie}(\Lambda_{t_0})$ is surjective, hence Λ_{t_0} is dominant. This implies that

$$Z_H(N)^\circ\times t_1(T^{t_1})^\circ s_1\to t_1Z_H(N)^\circ s_1,\qquad (g,t)\mapsto gtg^{-1}$$

is dominant. Further, for $t_0\in U_{t_1}$, the surjectivity of $\mathrm{Lie}(\Lambda_{t_0})$ implies that the kernel of

$$\mathrm{Lie}(Z_H(N)^\circ)\to \mathrm{Lie}(Z_H(N)^\circ),\qquad x\mapsto \mathrm{ad}((t_1t_0)^{-1})x-\mathrm{ad}(s_1)x$$

is equal to $\mathrm{Lie}((T^{t_1})^\circ)$. This means that for $t_0\in U_{t_1}$, we have $Z_{Z_H(N)}(t_1t_0s_1)^\circ=(T^{t_1})^\circ$. So we have toral centralizer for all points in the image of the dominant map

$$Z_H(N)^\circ\times t_1U_{t_1}s_1\to t_1Z_H(N)^\circ s_1,\qquad (g,t)\mapsto gtg^{-1},$$

whose target is equal to $Z^\circ_{\phi,N}\gamma$, and so the conclusion follows from Chevalley's theorem (see [GW20, Theorem 10.19]). □

From this, together with Proposition 7.1 and Lemma 4.7, we deduce that the two maps $\mathsf{WDP}_G^{K,\mathrm{rc}}\to\mathsf{WDP}_G^{K,\sqcup}$ and $\mathsf{WDP}_G^{K,\mathrm{rc}}\to\mathsf{WDP}_G^{K}$ are weakly birational. To connect this discussion to the Jacobson–Morozov map, we now show that JM is an isomorphism over $\mathsf{WDP}_G^{K,\mathrm{rc}}$.

Proposition 7.8. *The morphism* $\mathsf{JM}\colon \mathsf{JM}^{-1}(\mathsf{WDP}_G^{K,\mathrm{rc}})\to\mathsf{WDP}_G^{K,\mathrm{rc}}$ *is an isomorphism.*

Proof. Let A be a $\mathbb{Q}$-algebra. As JM is $\widehat{G}(A)$-equivariant, to show that this map is a bijection on A-points it suffices to prove that the map on A-points is a bijection upon quotienting both sides by $\widehat{G}(A)$, and that for all ψ in $\mathsf{JM}^{-1}(\mathsf{WDP}_G^{K,\mathrm{rc}}(A))$ the equality $Z_{\widehat{G}}(\psi)=Z_{\widehat{G}}(\varphi,N)$ holds where $(\varphi,N)=\mathsf{JM}(\psi)$. For the bijectivity on quotient sets, it suffices by Theorem 6.16 to show that every element of $\mathsf{WDP}_G^{K,\mathrm{rc}}(A)$ belongs to $\mathsf{WDP}_G^{K,\sqcup,\mathrm{ss}}(A)$. But, this follows from Proposition 5.11 and Proposition 7.4. Suppose now that ψ is an element of $\mathsf{JM}^{-1}(\mathsf{WDP}_G^{K,\mathrm{rc}}(A))$. To show that $Z_{\widehat{G}}(\psi)=Z_{\widehat{G}}(\varphi,N)$ it suffices by Proposition 6.15 to show that $U^N(\varphi)$ is trivial. Applying the fiberwise criterion for isomorphism (see [Con14, Lemma B.3.1]) to identity section of $U^N(\varphi)$ it suffices to show that $U^N(\varphi)_x$ is trivial for all x in $\mathrm{Spec}(A)$. But, as $U^N(\varphi)_x$ is unipotent it is contained in $Z(\varphi,N)^\circ_x$, and as it is also normal it must be trivial by our assumption that $Z(\varphi,N)^\circ_x$ is reductive. □

We deduce that $\mathsf{WDP}_G^{K,\mathrm{rc}}$ also admits a weakly birational monomorphism to LP_G^K. So, we now come to our main geometric result concerning the Jacobson–Morozov morphism.

Theorem 7.9. *The morphism* $\mathsf{JM}\colon\mathsf{LP}_G^K\to\mathsf{WDP}_G^{K,\sqcup}$ *(resp.* $\mathsf{JM}\colon\mathsf{LP}_G^K\to\mathsf{WDP}_G^{K}$*) is birational (resp. weakly birational).*

Proof. The weak birationality of both maps is clear from the above discussion, and therefore it suffices to show that the map $\mathsf{JM}\colon\mathsf{LP}_G^K\to\mathsf{WDP}_G^{K,\sqcup}$ induces a bijection on irreducible components. It clearly suffices to check this after base changing to $\overline{\mathbb{Q}}$. By Theorem 5.16 and Theorem 6.6 the connected components of $\mathsf{LP}^K_{G,\overline{\mathbb{Q}}}$ and $\mathsf{WDP}^{K,\sqcup}_{G,\overline{\mathbb{Q}}}$ are irreducible, so it suffices to show that the map $\mathsf{JM}\colon\pi_0(\mathsf{LP}^K_{G,\overline{\mathbb{Q}}})\to\pi_0(\mathsf{WDP}^{K,\sqcup}_{G,\overline{\mathbb{Q}}})$ is bijective. To do this we first show that the Jacobson–Morozov map induces a bijection $[\mathsf{LP}^K_G(\overline{\mathbb{Q}})]\to[\mathsf{WDP}^K_G(\overline{\mathbb{Q}})]$. By Proposition 7.7 and Proposition 5.11 every equivalence class of the target contains a Frobenius semi-simple element and thus surjectivity follows from Theorem 6.16. To show injectivity suppose that $(\gamma_i,\phi_i,\theta_i)$ for $i=1,2$ are elements of $\mathsf{LP}^K_G(\overline{\mathbb{Q}})$ such that (γ_i,ϕ_i,N_i) are equivalent in $\mathsf{WDP}^K_G(\overline{\mathbb{Q}})$. Without

loss of generality, we may assume that $\phi_1 = \phi_2 =: \phi$ and $N_1 = N_2 =: N$ and that $\gamma_2 = h\gamma_1$ with h in $Z_{\phi,N}(\overline{\mathbb{Q}})$. By Proposition 6.17 there exists z in $Z_{\phi,N}(\overline{\mathbb{Q}})$ such that $z\theta_1 z^{-1} = \theta_2$. Note then that $(\gamma_2, \phi, \theta_2) = z(s\gamma_1, \phi, \theta_1)z^{-1}$ where $s = z^{-1}\gamma_2 z\gamma_1^{-1}$. Writing $s = z^{-1}h\gamma_1 z\gamma_1^{-1}$ one sees from the fact that z^{-1} and h both centralize ϕ and γ_1 normalizes ϕ that s centralizes ϕ. On the other hand, one can just as easily check that as γ_1 centralizes θ_1 and γ_2 centralizes θ_2 that $s = z^{-1}\gamma_2 z\gamma_1^{-1}$ also centralizes θ_1. Therefore as $(\gamma_2, \phi, \theta_2) = z(s\gamma_1, \phi, \theta_1)z^{-1}$ we deduce that $(\gamma_2, \phi, \theta_2)$ and $(\gamma_1, \phi, \theta_1)$ are equivalent in $\mathsf{LP}_G^K(\overline{\mathbb{Q}})$ as desired. But, for (γ, ϕ, θ) with image (γ', ϕ, N) under the Jacobson–Morozov map, one has $\pi_0(Z_{\phi,N}) = \pi_0(Z_{\theta,N})$ as follows quickly from Proposition 6.15. These observations together with Corollary 5.17 and Corollary 6.8 give the desired conclusion. □

Let us finally note that as a possibly useful corollary of the above results, we also obtain the density of Frobenius semi-simple parameters in all three of these moduli spaces.

Corollary 7.10. *The subsets*

$$\mathsf{LP}_G^{\mathrm{ss}}(\overline{\mathbb{Q}}) \subseteq \mathsf{LP}_G, \qquad \mathsf{WDP}_G^{\sqcup,\mathrm{ss}}(\overline{\mathbb{Q}}) \subseteq \mathsf{WDP}_G^{\sqcup}, \qquad \mathsf{WDP}_G^{\mathrm{ss}}(\overline{\mathbb{Q}}) \subseteq \mathsf{WDP}_G$$

are dense.

7.2. **Isomorphism over the discrete locus.** In this final section we apply the material to give a geometric analogue of Corollary 3.16 or, in other words, we show that the Jacobson–Morozov morphism is an isomorphism over the discrete loci in LP_G^K and WDP_G^K.

We have defined the discrete locus $\mathsf{WDP}_G^{K,\mathrm{disc}}$ in Definition 7.3, and we now do so for LP_G^K.

Definition 7.11. Let ψ^{univ} be the universal L-parameter over LP_G^K. Then, the *discrete locus* in LP_G^K is the subset

$$\mathsf{LP}_G^{K,\mathrm{disc}} := \left\{ x \in \mathsf{LP}_G^K : Z_{\widehat{G}}(\psi^{\mathrm{univ}})_x / Z_0(\widehat{G})_x \to \mathrm{Spec}(k(x)) \text{ is finite} \right\}.$$

The same argument as in the proof of Proposition 7.2 shows that $\mathsf{LP}_G^{K,\mathrm{disc}}$ is an open subset of LP_G^K and we endow it with the open subscheme structure. The following relates the discrete loci in WDP_G^K and LP_G^K, giving a geometrization of Corollary 3.16.

Proposition 7.12. *The equality* $\mathsf{JM}^{-1}(\mathsf{WDP}_G^{K,\mathrm{disc}}) = \mathsf{LP}_G^{K,\mathrm{disc}}$ *holds.*

Proof. As these are both open subsets of the finite type affine $\mathbb{Q}$-scheme LP_G^K, it suffices to show that they have the same $\overline{\mathbb{Q}}$-points. In other words, we must show that for an element $\mathsf{LP}_G^K(\overline{\mathbb{Q}})$ one has that $Z_{\widehat{G}}(\psi)$ is finite (as a set) if and only if $Z_{\widehat{G}}(\mathsf{JM}(\psi))$ is finite. Choosing an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$ one then quickly deduces this from Proposition 3.15 and its proof. □

From this, and Proposition 7.8 we deduce the following.

Theorem 7.13. *The morphism* $\mathsf{JM} \colon \mathsf{LP}_G^{K,\mathrm{disc}} \to \mathsf{WDP}_G^{K,\mathrm{disc}}$ *is an isomorphism.*

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