AN INTEGRAL ANALOGUE OF FONTAINE'S CRYSTALLINE FUNCTOR

NAOKI IMAI[(1)], HIROKI KATO[(2)], AND ALEX YOUCIS[(3)]

Abstract. For a smooth formal scheme $\mathfrak{X}$ over the Witt vectors W of a perfect field k, we construct a functor $\mathbb{D}_{\mathrm{crys}}$ from the category of prismatic F-crystals $(\mathcal{E},\varphi_{\mathcal{E}})$ (or prismatic F-gauges) on $\mathfrak{X}$ to the category of filtered F-crystals on $\mathfrak{X}$. We show that $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ enjoys strong properties (e.g., strong divisibility in the sense of Faltings) when $(\mathcal{E},\varphi_{\mathcal{E}})$ is what we call *locally filtered free (lff)*. Most significantly, we show that $\mathbb{D}_{\mathrm{crys}}$ actually induces an equivalence between the category of prismatic F-gauges on $\mathfrak{X}$ with Hodge–Tate weights in $[0,p-2]$ and the category of Fontaine–Laffaille modules on $\mathfrak{X}$. Finally, we use our functor $\mathbb{D}_{\mathrm{crys}}$ to enhance the study of prismatic Dieduonné theory of p-divisible groups (as initiated by Anschütz–Le Bras) allowing one to recover the *filtered* crystalline Dieudonné crystal from the prismatic Dieudonné crystal. This in turn allows us to clarify the relationship between prismatic Dieudonné theory and the work of Kim on classifying p-divisible groups using Breuil–Kisin modules.

Contents

(1) Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
(2) Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France
(3) Department of Mathematics National University of Singapore, Level 4, Block S17, 10 Lower Kent Ridge Road, Singapore 119076
E-mail addresses: (1) naoki@ms.u-tokyo.ac.jp, (2) hiroki@ihes.fr, (3) alex.youcis@gmail.com.
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Introduction

Let k be a perfect extension of $\mathbb{F}_p$, and $\mathfrak{X}$ be a smooth formal scheme over $W := W(k)$ with generic fiber X. A question of central importance in p-adic Hodge theory is when a $\mathbb{Q}_p$-local system $\mathbb{V}$ on X should admit 'good reduction' relative to the model $\mathfrak{X}$. The operative property singled out by Fontaine in this regard is that $\mathbb{V}$ is *crystalline*. In this case, one may associate to $\mathbb{V}$ a filtered F-isocrystal $D_{\mathrm{crys}}(\mathbb{V})$ on $\mathfrak{X}$ which one may think of as the 'model' of $\mathbb{V}$ over $\mathfrak{X}$.

But, for arithmetic applications, for example to the study of Shimura varieties (e.g., see [IKY25]), it is important to have an *integral analogue* of this picture. Namely, if $\mathbb{L} \subseteq \mathbb{V}$ is a $\mathbb{Z}_p$-lattice, the collection of which (as $\mathbb{V}$ varies) we denote $\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X)$, then one would like to make sense of when $\mathbb{L}$ admits good reduction relative to $\mathfrak{X}$. The question of what sort of object models $\mathbb{L}$ is subtle, and there now exist several approaches.

Crystalline approach ([Fal89]): This utilizes the category $\mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$ of strongly divisible filtered F-crystals. This theory is best behaved when the filtration is supported in the Fontaine–Laffaille range $[0, p-2]$, where they are called *Fontaine–Laffaille modules* (cf. Proposition 2.5). In particular, in [Fal89] there is constructed a fully faithful functor

$$T_{\mathrm{crys}} \colon \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}}) \to \mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p,[0,p-2]}(X),$$

where the target consists of those $\mathbb{L}$ with Hodge–Tate weights in $[0, p-2]$.

Prismatic approach ([BS23]): This makes use of the category $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ of prismatic F-crystals on $\mathfrak{X}$. The relationship to local systems is given by the functor

$$T_{\text{ét}} \colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X),$$

(constructed in op. cit.), which was shown to be fully faithful in [GR24] or [DLMS24].

Syntomic approach ([Dri24], [Bha23]): This utilizes the category $\mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}})$ of prismatic F-gauges. By results in [GL23], this also admits a fully faithful functor

$$T_{\text{ét}} \colon \mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \to \mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X),$$

It is natural to ask what the precise relationship is between these three approaches. The latter two are directly related by a *forgetful functor*

$$\mathrm{R}_{\mathfrak{X}} \colon \mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \to \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}),$$

which is shown in [GL23] and [IKY25] to be fully faithful with essential image the subcategory $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$ of so-called locally filtered free (lff) prismatic F-crystals (see op. cit.).

The goal of this paper is to complete the comparisons of these three approaches by relating the crystalline approach to the prismatic and syntomic ones by defining *integral analogues* of D_{crys}

$$\mathbb{D}_{\mathrm{crys}} \colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}), \qquad \mathbb{D}_{\mathrm{crys}} \colon \mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \to \mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$$

(see §2.1 for the definition of $\mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$). In the rest of the introduction, we discuss the various good properties $\mathbb{D}_{\mathrm{crys}}$ enjoys, which not only provide the desired bridge between these various approaches, but help clarify the structure of the categories $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ and $\mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}})$.

But, we immediately state that the tightest possible connection holds, namely *equivalence*, in the Fontaine–Laffaille range, which clarifies the well-behavedness of Faltings's theory in this range. (See [DLMS24, Definition 3.16] and §3.1 for the definitions of the categories $\mathbf{Vect}^{\varphi}_{[0,p-2]}(\mathfrak{X}_{\Delta})$ and $\mathbf{Vect}_{[0,p-2]}(\mathfrak{X}^{\mathrm{syn}})$, respectively.)

Theorem A (see Theorem 3.1). *Suppose that $\mathfrak{X}$ is a smooth formal W-scheme and that $p > 2$. Then, the following diagram is commutative and all arrows are $\mathbb{Z}_p$-linear equivalences*

$$\begin{tikzcd}
\mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \arrow[d, hook] \arrow[drr, "\mathbb{D}_{\mathrm{crys}}"] & & \mathbf{Vect}_{[0,p-2]}(\mathfrak{X}^{\mathrm{syn}}) \arrow[ll, "\mathrm{R}_{\mathfrak{X}}"'] \arrow[d, "\mathbb{D}_{\mathrm{crys}}"] \\
\mathbf{Vect}^{\varphi}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \arrow[r, "T_{\text{ét}}"] & \mathbf{Loc}^{\mathrm{crys}}_{[0,p-2]}(X) & \mathbf{Vect}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}}) \arrow[l, "T_{\mathrm{crys}}"']
\end{tikzcd}$$

Remark 1. We comment on the relationship of Theorem 3.1 to other literature:

- In [Hok24], Hokaj obtains the equivalence of the arrow T_{crys} as in Theorem 3.1. Our proof is independent of his results, but our proof of the crucial Proposition 3.3 is inspired by (although simpler than) techniques in op. cit.
- In [Wür23], Würthen constructs an equivalence between $\mathbf{Vect}^{\varphi}_{[0,p-2]}(\mathfrak{X}_{\Delta})$ and a certain modified category of Fontaine–Laffaille modules. The relationship to our work is unclear.
- In [TVX24], for a perfect field k of chararacteristic p the authors construct an equivalence of ∞-categories (with notation as in op. cit):
$$\Phi_{\mathrm{Maz}}\colon \mathcal{D}_{\mathrm{qc},[0,p-2]}(W(k)^{\mathrm{syn}}) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-2]}(W(k)).$$
which is a derived analogue of $\mathbb{D}_{\mathrm{crys}}$ for $\mathfrak{X} = \mathrm{Spf}(W(k))$ in the Fontaine–Laffaille range. Moreover, the authors mention in op. cit. they intend to generalize their work to arbitrary smooth formal W-schemes.

We note that in all cases above these constructions do not address objects outside the Fontaine–Laffaille range, unlike our functor $\mathbb{D}_{\mathrm{crys}}$.

In the rest of the introduction, we discuss other aspects of the functor $\mathbb{D}_{\mathrm{crys}}$ and its applications to prismatic F-crystals, prismatic F-gauges, and the theory of p-divisible groups.

Construction of $\mathbb{D}_{\mathrm{crys}}$ and applications of prismatic F-crystals/gauges. Fix a smooth formal W-scheme $\mathfrak{X}$.[1] Our construction of the $\mathbb{Z}_p$-linear $\otimes$-functor, the *integral analogue of D_{crys}*,
$$\mathbb{D}_{\mathrm{crys}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{VectWF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$$
is achieved in a surprisingly pleasant way. Namely, we observe that for an object $(\mathcal{E},\varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ there is always a canonical *crystalline-de Rham comparison.*

Theorem B (see Theorem 1.19)**.** *Let $\mathfrak{X}$ be a smooth formal W-scheme, and $(\mathcal{E},\varphi_{\mathcal{E}})$ a prismatic F-crystal on $\mathfrak{X}$. Then there exists a canonical isomorphism*
$$\iota_{\mathfrak{X}}\colon \underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})|_{\mathfrak{X}_{\mathrm{Zar}}} \xrightarrow{\sim} \mathbb{D}_{\mathrm{dR}}(\mathcal{E},\varphi_{\mathcal{E}})$$
of vector bundles on $\mathfrak{X}$.

Here we are using the following notation:

- $\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$, the *crystalline realization*, denotes the F-crystal $\mathcal{E}^{\mathrm{crys}}$ associated to $\mathcal{E}|_{\mathfrak{X}_k}$,
- and $\mathbb{D}_{\mathrm{dR}}(\mathcal{E},\varphi_{\mathcal{E}})$, the *de Rham realization*, denotes the vector bundle on $\mathfrak{X}$ given by pullback along the de Rham point $\rho^{\Delta}_{\mathrm{dR},\mathfrak{X}}\colon \mathfrak{X} \to \mathfrak{X}^{\Delta}$ as in Definition 1.11 (see Proposition 1.15 for a more down-to-earth description).

Then, $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ has underlying F-crystal $\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ and obtains a filtration on its restriction to $\mathfrak{X}_{\mathrm{Zar}}$ via the crystalline-de Rham comparison, and the Nygaard filtration on $\phi^*\mathcal{E}$.

Remark 2. In the case when $\mathfrak{X} = \mathrm{Spf}(\mathbb{Z}_p)$ this is essentially contained in [BL22a] (see Proposition 3.6.6 of op. cit.). A more novel aspect of our study is a concrete reinterpretation of this comparison using Breuil and Breuil–Kisin prisms, see Construction 1.20 and Proposition 1.21, which is central to our study of $\mathbb{D}_{\mathrm{crys}}$.

Remark 3. In the case when $\mathfrak{X} = \mathrm{Spf}(W)$, one may view the crystalline-de Rham isomorphism as an integral refinement of [Kis06, §1.2.7] (cf. Remark 1.22).

As indicated by our choice of terminology, $\mathbb{D}_{\mathrm{crys}}$ forms a functorial lattice in D_{crys} in a way made precise by the following theorem.

Theorem C (see Theorem 2.10)**.** *There is a canonical identification*
$$\mathbb{D}_{\mathrm{crys}}[1/p] \xrightarrow{\sim} D_{\mathrm{crys}} \circ T_{\mathrm{\acute{e}t}}.$$

[1] Although most of these results apply for a base formal W-scheme $\mathfrak{X}$ in the sense of [IKY24, §1.1.5].

The functor $\mathbb{D}_{\mathrm{crys}}$ enjoys even more favorable properties when restricted to the full subcategory $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$ of lff prismatic F-crystals and, in fact, can be used to detect lffness.

Proposition A (see Proposition 2.13). *Suppose that $\mathfrak{X}$ is a smooth formal W-scheme. Then, a prismatic F-crystal $(\mathcal{E},\varphi_{\mathcal{E}})$ on $\mathfrak{X}$ is lff if and only if $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ belongs to $\mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$.*

Combining Proposition A with the forgetful functor

$$\mathrm{R}_{\mathfrak{X}}\colon \mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \xrightarrow{\sim} \mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}^{\mathrm{syn}})$$

(see [Bha23, Remark 6.3.4] and [IKY25, Proposition 1.28]), we obtain a functor

$$\mathbb{D}_{\mathrm{crys}}\circ \mathrm{R}_{\mathfrak{X}}\colon \mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \to \mathbf{Vect}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}}), \tag{1}$$

which we also denote $\mathbb{D}_{\mathrm{crys}}$. Using ideas of Faltings, we may show that (1) is exact. Using this, we obtain the following result of independent of interest (and of great importance in [IKY25]).

Proposition B (see Proposition 2.17). *The equivalence*

$$\mathrm{R}_{\mathfrak{X}}\colon \mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \xrightarrow{\sim} \mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$$

given by the forgetful functor is bi-exact.

We finally observe that our study of $\mathbb{D}_{\mathrm{crys}}$ gives rise to the following syntomic refinement of cohomological results obtained in [Fal89].

Proposition C (see Proposition 3.12). *Let $f\colon \mathscr{X}\to \mathrm{Spec}(W)$ be smooth and proper. Then, for an object $\mathcal{V}$ of $\mathbf{Vect}_{[0,a]}(\widehat{\mathscr{X}}^{\mathrm{syn}})$ and b in $\mathbb{N}$ with $a+b<p-2$, there is a canonical isomorphism*

$$H^b_{\mathrm{\acute{e}t}}\left(\mathscr{X}_{\overline{K}}, T_{\mathrm{\acute{e}t}}(\mathcal{V})^{\mathrm{alg}}/p^n\right) \xrightarrow{\sim} T_{\mathrm{\acute{e}t}}\left(R^b f_*\mathcal{V}/p^n\right),$$

of Galois representations, for any n in $\mathbb{N}\cup\{\infty\}$.

Applications to p-divisible groups. We finally describe the ways in which the functor $\mathbb{D}_{\mathrm{crys}}$ can be used to clarify the relationship between p-divisible groups and various types of Dieudonné theory when $p>2$. To this end, let us fix $\mathfrak{X}$ to be a base formal W-scheme (see [IKY24, §1.1.5]).

Denote by $\mathbf{BT}_p(\mathfrak{X})$ the category of p-divisible groups over $\mathfrak{X}$. There are then several approaches to use p-adic Hodge theory to classify objects of $\mathbf{BT}_p(\mathfrak{X})$ via various 'Dieudonné theories':

(1) the *filtered crystalline Dieudonné functor* of Grothendieck–Messing

$$\mathbb{D}\colon \mathbf{BT}_p(\mathfrak{X}) \to \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,1]}(\mathfrak{X}_{\mathrm{crys}}),$$

which is an (anti-)equivalence by results of Grothendieck, Messing, and de Jong (see [dJ95]),

(2) the *Breuil–Kisin–Kim Dieudonné functor* (see [Kim15])

$$\mathfrak{M}\colon \mathbf{BT}_p(\mathfrak{X}) \to \mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, \nabla^0)$$

when $\mathfrak{X}=\mathrm{Spf}(R)$ for a formally framed W-algebra R (see [IKY24, §1.1.5] for the definition of framed algebra, and Definition 4.11 for the definition of the target category), which is an (anti-)equivalence by [Kim15],

(3) the *prismatic Dieudonné functor* of Anschütz–Le Bras

$$\mathcal{M}_{\Delta}\colon \mathbf{BT}_p(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{X}_{\Delta}),$$

which is an (anti-)equivalence by [ALB23] (cf. Theorem 4.2).

The relationship between these various Dieudonné theories is important to establish the connection between historical results and modern methods (e.g., as is needed in the study of Shimura varieties in [IKY25]). That said, this relationship is far from clear given the different ways in which these functors are constructed.

In [ALB23, Theorem 4.44], it is demonstrated that there is a canonical identification

$$\underline{\mathbb{D}}(H) \simeq \underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{M}_{\Delta}(H)),$$

in $\mathbf{Vect}^\varphi(\mathfrak{X}_{\mathrm{crys}})$, for an object H of $\mathbf{BT}_p(\mathfrak{X})$. Here, recall, $\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{M}_{\Delta}(H))$ is the underlying F-crystal of $\mathbb{D}_{\mathrm{crys}}(\mathcal{M}_{\Delta}(H))$. We further denote by $\underline{\mathbb{D}}(H)$ is the underlying F-crystal of $\mathbb{D}(H)$. In particular, this identification ignores filtrations, which is equivalent (by Grothendieck–Messing theory) to only remembering the the special fiber H_k of the p-divisible group H. Thus, it is desirable to upgrade this to a filtered identification.

Theorem D (see Theorem 4.8)**.** *There is an identification of functors* $\mathbf{BT}_p(\mathfrak{X}) \to \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,1]}(\mathfrak{X}_{\mathrm{crys}})$

$$\mathbb{D} \simeq \mathbb{D}_{\mathrm{crys}} \circ \mathcal{M}_{\Delta}.$$

With respect to the Breuil–Kisin–Kim Dieudonné functor, when $R = W$, it is shown in [ALB23, Proposition 5.18] that there is an identification of functors $\mathbf{BT}_p(R) \to \mathbf{Vect}^\varphi_{[0,1]}(\mathfrak{S}_R)$

$$\mathfrak{M} \simeq \mathrm{ev}_{\mathfrak{S}_R} \circ \mathcal{M}_{\Delta},$$

where $\mathrm{ev}_{\mathfrak{S}_R}$ is the functor given by evaluation on the Breuil–Kisin prism $(\mathfrak{S}_R, (E))$. But for applications, it is desirable to have versions of this identification for arbitrary R, which then necessitates the consideration of connections.

Using Theorem D, and an intermediate study of Breuil's Dieudonné functor (see §4.3.1), we are able to make such an identification.

Proposition D (see Proposition 4.16)**.** *For any formally framed W-algebra R, there is a canonical identification of functors* $\mathbf{BT}_p(R) \to \mathbf{Vect}^\varphi_{[0,1]}(\mathfrak{S}_R, \nabla^0)$

$$\mathfrak{M} \simeq \mathrm{ev}^{\mathrm{K}}_{\mathfrak{S}_R} \circ \mathcal{M}_{\Delta}.$$

Here $\mathrm{ev}^{\mathrm{K}}_{\mathfrak{S}_R}(\mathcal{M}_{\Delta}(H))$ is the evaluation of $\mathcal{M}_{\Delta}(H)$ on $(\mathfrak{S}_R, (E))$ equipped with the connection on $\phi^*\mathcal{M}_{\Delta}(\mathfrak{S}_R, (E))/E$ coming from the identification of this with $\mathbb{D}_{\mathrm{crys}}(H)$ (via Theorem D).

As an application of this result and work of Kim, we obtain the following strengthening of a result of Ito (cf. [Ito23, Theorem 6.1.3]).

Corollary A (Corollary 4.18)**.** *Let R be a formally framed W-algebra. Then, there are equivalences of categories*

$$\mathbf{Vect}_{[0,1]}(R^{\mathrm{syn}}) \xrightarrow[\sim]{\mathrm{R}_{\mathfrak{X}}} \mathbf{Vect}^\varphi_{[0,1]}(R_{\Delta}) \xrightarrow[\sim]{\mathrm{ev}^{\mathrm{K}}_{\mathfrak{S}_R}} \mathbf{Vect}^\varphi_{[0,1]}(\mathfrak{S}_R, \nabla^0),$$

is an equivalence of categories. If $R = W[\![t_1, \ldots, t_d]\!]$ for some $d \geqslant 0$, then there are equivalences

$$\mathbf{Vect}_{[0,1]}(R^{\mathrm{syn}}) \xrightarrow[\sim]{\mathrm{R}_{\mathfrak{X}}} \mathbf{Vect}^\varphi_{[0,1]}(R_{\Delta}) \xrightarrow[\sim]{\mathrm{ev}_{\mathfrak{S}_R}} \mathbf{Vect}^\varphi_{[0,1]}(\mathfrak{S}_R).$$

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Notation and terminology. Throughout this article we make use of the following notation and terminology, which is largely standard (or self-evident), and so the reader is encouraged to only refer back when necessary.

Fix the following (we refer the reader to [IKY24, §1.1.5] for undefined terminology):

- k a perfect extension of $\mathbb{F}_p$,
- $W := W(k)$ and $K_0 := \mathrm{Frac}(W)$,
- K is a finite totally ramified extension of K_0,
- $\mathcal{O}_K$ the valuation ring of K,
- π a uniformizer of $\mathcal{O}_K$ and $k = \mathcal{O}_K/\pi$,
- $e = [K : K_0]$,
- $E \in W[u]$ the minimal polynomial for π,
- R a formally framed base $\mathcal{O}_K$-algebra,
- $(\mathfrak{S}_R, (E))$ the Breuil–Kisin prism,
- $(S_R, (p))$ the Breuil prism,
- $\mathbb{W}$ the p-typical Witt vector scheme,
- $F\colon \mathbb{W} \to \mathbb{W}$ the Frobenius,
- $V\colon \mathbb{W} \to \mathbb{W}$ the Verschiebung,
- $[-]\colon \mathbb{A}^1 \to \mathbb{W}$ the Teichmüller lift,
- $\mathfrak{X}$ a base formal $\mathcal{O}_K$-scheme,
- $X = \mathfrak{X}_\eta$ the rigid generic fiber.

We refer the reader to [IKY24, §2.3.1] for our notation and terminology concerning the crystalline site and (iso)crystals and F-(iso)crystals.

We further refer the reader to [IKY24, §1.1.3] for our notation and terminology concerning the quasi-syntomic and qrsp sites $\mathfrak{Y}_{\mathrm{qsyn}}$ and $\mathfrak{Y}_{\mathrm{qrsp}}$.

We lastly refer the reader to [IKY25, §1] for our notation and terminology concerning

- the formal stacks $\mathfrak{Y}^{\Delta}$ (with its Frobenius $F_{\mathfrak{Y}}\colon \mathfrak{Y}^{\Delta} \to \mathfrak{Y}^{\Delta}$), $\mathfrak{Y}^{\mathcal{N}}$, and $\mathfrak{Y}^{\mathrm{syn}}$ over $\mathbb{Z}_p$ and for the site $\mathfrak{Y}_{\Delta}$ (which we abbreviate to $S^{\Delta}, S^{\mathcal{N}}, S^{\mathrm{syn}}$, and S_{Δ} when $\mathfrak{Y} = \mathrm{Spf}(S)$),
- the categories $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$, $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$, $\mathbf{Vect}^{\varphi,\mathrm{an}}(\mathfrak{X}_{\Delta})$, and $\mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}})$, of prismatic F-crystals, locally filtered free (lff) prismatic F-crystals, analytic prismatic F-crystals and prismatic F-gauges, respectively.
- the étale realization functor $T_{\text{ét}}\colon \mathbf{Vect}^{\mathrm{an},\varphi}\varphi(\mathfrak{X}_{\Delta}) \to \mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X)$ and the induced étale realization functors on $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$, $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$, and $\mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}})$,
- $\mathrm{R}_{\mathfrak{X}}\colon \mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \xrightarrow{\sim} \mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$ the forgetful functor,
- notation for filtered rings and modules and, in particular, the Rees algebra $\mathrm{Rees}(A, \mathrm{Fil}^\bullet)$ and completed Rees stack $\widehat{\mathcal{R}}(A, \mathrm{Fil}^\bullet)$ of a filtered ring.

Finally, we freely use the notion of quasi-ideals $d\colon I \to A$ (which we sometimes write as $[I \to A]$ for clarity) and their quotients $\mathrm{Cone}(d)$ as in [Dri21].

1. The crystalline-de Rham comparison

The definition of $\mathbb{D}_{\mathrm{crys}}$ relies on a comparison isomorphism between the crystalline and de Rham realizations of a prismatic F-crystal that we formulate and prove in this subsection. Throughout we use notation and terminology from Notation and terminology without comment.

1.1. The crystalline realization functor.

We now expand on the notion of the *crystalline realization functor* $\underline{\mathbb{D}}_{\mathrm{crys}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ as discussed in [BS23, Example 4.12].

1.1.1. Prismatic F-crystals on quasi-syntomic $\mathbb{F}_p$-schemes.

Fix Z to be a quasi-syntomic k-scheme. As in [BS23, Example 4.7], there is an equivalence of categories

$$(-)^{\mathrm{crys}}\colon \mathbf{Vect}(Z_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathbf{Vect}(Z_{\mathrm{crys}}), \qquad \mathcal{F} \mapsto \mathcal{F}^{\mathrm{crys}},$$

defined as follows. Recall (e.g., see [IKY24, Example 1.8 and Example 2.15]) that for a qrsp k-algebra R, the categories R_{Δ} and $(R/W)_{\mathrm{crys}}$ have initial objects $(\mathrm{A}_{\mathrm{crys}}(R), (p), \widetilde{\mathrm{nat.}})$ and $\mathrm{A}_{\mathrm{crys}}(R) \twoheadrightarrow R$, respectively. Thus, by evaluation, we have functorial equivalences

$$\mathbf{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathbf{Vect}(\mathrm{A}_{\mathrm{crys}}(R)) \xleftarrow{\sim} \mathbf{Vect}(R_{\mathrm{crys}}).$$

Passing to the limit, and using [IKY24, Proposition 1.31 and Proposition 2.16], we deduce the existence of a diagram of equivalences

$$\mathbf{Vect}(Z_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \underset{R\in Z_{\mathrm{qrsp}}}{2\text{-}\lim}\, \mathbf{Vect}(R_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \underset{R\in Z_{\mathrm{qrsp}}}{2\text{-}\lim}\, \mathbf{Vect}(R_{\mathrm{crys}}) \xleftarrow{\sim} \mathbf{Vect}(Z_{\mathrm{crys}}).$$

Define $(-)^{\mathrm{crys}}$ to be the obvious equivalence derived from this diagram.

For an object $\mathcal{V}$ of $\mathbf{Vect}(Z_{\Delta}, \mathcal{O}_{\Delta})$, one has that $(\phi^*\mathcal{V})^{\mathrm{crys}}$ is naturally identified with $\phi^*(\mathcal{V}^{\mathrm{crys}})$. Indeed, by construction it suffices to observe that for a qrsp k-algebra R over Z

$$\phi^*(\mathcal{V}^{\mathrm{crys}})(\mathrm{A}_{\mathrm{crys}}(R) \twoheadrightarrow R) = \phi_R^*\mathcal{V}(\mathrm{A}_{\mathrm{crys}}(R) \twoheadrightarrow R) = \phi_R^*\mathcal{V}(\mathrm{A}_{\mathrm{crys}}(R), (p)) = (\phi^*\mathcal{V})(\mathrm{A}_{\mathrm{crys}}(R), (p)),$$

where the first equality follows from [IKY24, Remark 2.18] and the second and third by definition. From this observation, one may upgrade $(-)^{\mathrm{crys}}$ to an equivalence

$$(-)^{\mathrm{crys}}\colon \mathbf{Vect}^{\varphi}(Z_{\Delta}) \xrightarrow{\sim} \mathbf{Vect}^{\varphi}(Z_{\mathrm{crys}}),$$

which is functorial in Z in the obvious way.

1.1.2. **The crystalline realization functor.** From the above discussion, we obtain a natural functor $(-)^{\mathrm{crys}}\colon \mathbf{Vect}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}((\mathfrak{X}_k)_{\mathrm{crys}})$ given by sending $\mathcal{E}$ to $\mathcal{E}^{\mathrm{crys}} := (\mathcal{E}|_{(\mathfrak{X}_k)_{\Delta}})^{\mathrm{crys}}$. The crystal $\mathcal{E}^{\mathrm{crys}}$ enjoys a concrete description when evaluated on base $\mathcal{O}_K$-algebras.

For the notion of formal framing (usually denoted t, not w, in op. cit.) and the notations ϕ_w and $R_0^{(\phi_w)}$ used in the following, we refer the reader to [IKY24, §1.1.5].

Proposition 1.1. *Let $R = R_0 \otimes_W \mathcal{O}_K$ be a base $\mathcal{O}_K$-algebra and w a formal framing. For $\mathcal{F}$ in $\mathbf{Vect}((R_k)_{\Delta})$ there is a canonical isomorphism*

$$\vartheta_w = \vartheta_{R,w}\colon (\phi^*\mathcal{F})(R_0^{(\phi_w)}, (p)) \xrightarrow{\sim} \mathcal{F}^{\mathrm{crys}}(R_0).$$

If $\mathcal{F}$ carries a Frobenius structure, then ϑ_w is Frobenius-equivariant.

Suppose A is a quasi-syntomic p-adically complete ring. As $\mathrm{Spf}(A)$ has a cover in $\mathrm{Spf}(A)_{\mathrm{fl}}$ whose entire Čech cover is qrsp, we see by p-adic faithfully flat descent that the natural map

$$M \to \varprojlim_{B \in A_{\mathrm{qrsp}}} (M \otimes_A B), \tag{1.1.1}$$

is an isomorphism for any p-adically complete A-module M.

Proof of Proposition 1.1. For any object T of $(R_k)_{\mathrm{qrsp}}$, there exists a canonical identification of modules $\mathcal{F}(\mathrm{A}_{\mathrm{crys}}(T), (p)) \xrightarrow{\sim} \mathcal{F}^{\mathrm{crys}}(\mathrm{A}_{\mathrm{crys}}(T) \twoheadrightarrow T)$ by the definition of $(-)^{\mathrm{crys}}$. For each object S of R_{qrsp}, let $w_i^\flat$ in $S^\flat$ be compatible sequences of p^{th}-power roots of w_i in S/p. We then have maps $\beta_{w^\flat} = \beta\colon R_0 \to \mathrm{A}_{\mathrm{crys}}(S)$ as in [IKY24, §1.1.5]. The map β induces morphisms $(R_0 \twoheadrightarrow R_k) \to (\mathrm{A}_{\mathrm{crys}}(S) \twoheadrightarrow S_k)$ and $(R_0, (p), F_{R_k} \circ q) \to (\mathrm{A}_{\mathrm{crys}}(S), (p), \widetilde{\mathrm{nat.}})$ in $(R_k)_{\mathrm{crys}}$ and R_{Δ}, respectively, where $\widetilde{\mathrm{nat.}}$ is as in [IKY24, Example 1.8], and $q\colon R \to R/\pi = R_0/p$ is the natural map. Thus, from the crystal property and [IKY24, Remark 1.19], we obtain isomorphisms

$$\begin{aligned}(\phi^*\mathcal{F})(R_0, (p)) \otimes_{R_0} S &\xrightarrow{\sim} \mathcal{F}(\mathrm{A}_{\mathrm{crys}}(S), (p)) \otimes_{\mathrm{A}_{\mathrm{crys}}(S),\theta} S,\\ \mathcal{F}^{\mathrm{crys}}(R_0) \otimes_{R_0} S &\xrightarrow{\sim} \mathcal{F}^{\mathrm{crys}}(\mathrm{A}_{\mathrm{crys}}(S) \twoheadrightarrow S_k) \otimes_{\mathrm{A}_{\mathrm{crys}}(S),\theta} S,\end{aligned}$$

compatible in S. Then by the isomorphism in (1.1.1), we get canonical isomorphisms

$$\begin{aligned}(\phi^*\mathcal{F})(R_0, (p)) &\xrightarrow{\sim} \varprojlim_{S \in R_{\mathrm{qrsp}}} \mathcal{F}(\mathrm{A}_{\mathrm{crys}}(S), (p)) \otimes_{\mathrm{A}_{\mathrm{crys}}(S),\theta} S\\ &\xrightarrow{\sim} \varprojlim_{S \in R_{\mathrm{qrsp}}} \mathcal{F}^{\mathrm{crys}}(\mathrm{A}_{\mathrm{crys}}(S) \twoheadrightarrow S_k) \otimes_{\mathrm{A}_{\mathrm{crys}}(S),\theta} S\\ &\xrightarrow{\sim} \mathcal{F}^{\mathrm{crys}}(R_0),\end{aligned}$$

which proves the assertion. The second claim follows from the first, via the natural identification of $(\phi^*\mathcal{E})^{\mathrm{crys}}$ and $\phi^*(\mathcal{E}^{\mathrm{crys}})$ for an object $\mathcal{E}$ of $\mathbf{Vect}(\mathfrak{X}_{\Delta})$. ☐

We now define the *crystalline realization functor*

$$\underline{\mathbb{D}}_{\mathrm{crys}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}), \qquad (\mathcal{E}, \varphi_{\mathcal{E}}) \mapsto (\mathcal{E}^{\mathrm{crys}}, \varphi_{\mathcal{E}^{\mathrm{crys}}}).$$

While $\underline{\mathbb{D}}_{\mathrm{crys}}$ is far from full, it is faithful.

Proposition 1.2. *The following functors are faithful:*

$$(-)^{\mathrm{crys}}\colon \mathbf{Vect}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}(\mathfrak{X}_{\mathrm{crys}}), \qquad \underline{\mathbb{D}}_{\mathrm{crys}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}).$$

Proof. It suffices to prove the first claim. As passing to a cover is a faithful operation, we see by [IKY24, Proposition 1.11] that it is sufficient to show the following: if R is a perfectoid ring, then the base change functor $\mathbf{Vect}(\mathrm{A}_{\mathrm{inf}}(R)) \to \mathbf{Vect}(\mathrm{A}_{\mathrm{crys}}(R))$ given by $M \mapsto M \otimes_{\mathrm{A}_{\mathrm{inf}}(R)} \mathrm{A}_{\mathrm{crys}}(R)$ is faithful. As M is flat over $\mathrm{A}_{\mathrm{inf}}(R)$, and $\mathrm{A}_{\mathrm{inf}}(R) \to \mathrm{A}_{\mathrm{crys}}(R)$ is injective (see [IKY24, Example 1.9]), the map $M \to M \otimes_{\mathrm{A}_{\mathrm{inf}}(R)} \mathrm{A}_{\mathrm{crys}}(R)$ is injective, from where the claim follows. □

Lastly, we give a calculation of $\underline{\mathbb{D}}_{\mathrm{crys}}$ in terms of Breuil–Kisin modules.

Proposition 1.3. *Let $R = R_0 \otimes_W \mathcal{O}_K$ be a formally framed base $\mathcal{O}_K$-algebra. Then, there is a canonical Frobenius-equivariant isomorphism*

$$\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})(R_0) \xrightarrow{\sim} (\phi^*\mathcal{E})(\mathfrak{S}_R, (E))/u.$$

Proof. As E is an Eisenstein polynomial, the map $\mathfrak{S}_R \to R_0$ sending u to 0 defines a morphism $(\mathfrak{S}_R, (E)) \to (R_0, (p))$ in R_{Δ}. The desired isomorphism then follows from applying the crystal property in conjunction with Proposition 1.1. □

Remark 1.4. The Frobenius structure on $(\phi^*\mathcal{E})(\mathfrak{S}_R, (E))/u$ in Proposition 1.3 is taken in the sense of [IKY24, Remark 1.19] (either before or after quotienting by (u)).

Example 1.5. For $R = \mathcal{O}_K$ we abuse notation and define the exact $\mathbb{Z}_p$-linear $\otimes$-functor

$$\underline{\mathbb{D}}_{\mathrm{crys}} \colon \mathbf{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathrm{Gal}(\overline{K}/K)) \to \mathbf{Vect}^{\varphi}(k_{\mathrm{crys}}) = \mathbf{Vect}^{\varphi}(W, (p)),$$

to be $\underline{\mathbb{D}}_{\mathrm{crys}} \circ T_{\text{ét}}^{-1}$. Then, there is a natural identification between $\underline{\mathbb{D}}_{\mathrm{crys}}(\Lambda)$ and the Frobenius module $\phi^*\mathfrak{M}(\Lambda)/u$ over W, where $\mathfrak{M}$ is the functor from [Kis06]. Indeed, this follows from Proposition 1.3 and [BS23, Remark 7.11].

1.1.3. $\underline{\mathbb{D}}_{\mathrm{crys}}$ **in terms of stacks.** We now explain how the crystalline realization functor $\underline{\mathbb{D}}_{\mathrm{crys}}$ can be understood stack-theoretically.

To begin, recall that for a quasi-syntomic k-scheme Z there is constructed in [Bha23, Remark 2.5.12] the formal stack $(Z/W)^{\mathrm{crys}}$ over W associating to a p-nilpotent W-algebra S

$$(Z/W)^{\mathrm{crys}}(S) := \mathrm{Map}_k(\mathrm{Spec}(\mathbb{G}_a^{\mathrm{dR}}(S)), Z),$$

where $\mathbb{G}_a^{\mathrm{dR}}$ is as in [Bha23, Definition 2.5.1] and $\mathbb{G}_a^{\mathrm{dR}}(S)$ has k-structure as in [Bha23, Corollary 2.5.10]. The absolute Frobenius F_Z induces a morphism $F_Z \colon (Z/W)^{\mathrm{crys}} \to (Z/W)^{\mathrm{crys}}$ of formal stacks over W lying over the Frobenius map $\phi_W \colon W \to W$. In particular, it induces a morphism

$$F_Z \colon \phi_W^*(Z/W)^{\mathrm{crys}} \to (Z/W)^{\mathrm{crys}}$$

of formal stacks over W.

Lemma 1.6 ([Bha23, Corollary 2.6.8])**.** *There exists a natural Frobenius-equivariant isomorphism of formal stacks over W*

$$(Z/W)^{\mathrm{crys}} \xrightarrow{\sim} \phi_W^*(Z^{\Delta}). \tag{1.1.2}$$

Proof. By [Bha23, Corollary 2.6.8], we have an isomorphism of W-algebra schemes

$$\mathbb{G}_a^{\mathrm{dR}} \xrightarrow{\sim} F_*\mathbb{W}/p. \tag{1.1.3}$$

That said, for a p-nilpotent W-algebra S

$$\phi_W^*(Z^{\Delta})(S) = \mathrm{Map}_k(\mathrm{Spec}(\mathbb{W}((\phi_W)_*S)/p), Z),$$

where $(\phi_W)_*S$ is S viewed as a W-algebra via restriction along ϕ_W. But, it is easy to see that $F_*\mathbb{W}(S)/p$ is canonically isomorphic to $\mathbb{W}((\phi_W)_*(S))/p$ as a k-algebra. The claim follows given the identification of Z^{Δ} with $\mathrm{Map}_k(\mathbb{W}/p, Z)$ as explained in [Bha23, Example 5.1.12]. □

Observe that there is a natural functor

$$\Psi_Z \colon \mathbf{QCoh}((Z/W)^{\mathrm{crys}}) \to \mathbf{Crys}((Z/W)_{\mathrm{crys}}).$$

Namely, by definition of $\mathbb{G}_a^{\mathrm{dR}}$, given an object $(Z \leftarrow U \hookrightarrow T)$ of $(Z/W)_{\mathrm{crys}}$ one obtains a map

$$\rho_{(Z\leftarrow U\hookrightarrow T)}\colon T \to (Z/W)^{\mathrm{crys}}. \tag{1.1.4}$$

More precisely, if $T = \mathrm{Spec}(B)$ and $J = \ker(\mathcal{O}_T(T) \to \mathcal{O}_U(U))$, then we have a natural map $[J \to B] \to [\mathbb{G}_a^\sharp(B) \to B]$ of quasi-ideals. Set

$$\Psi_Z(\mathcal{F})(Z \leftarrow U \hookrightarrow T) := (\pi_{(Z\leftarrow U\hookrightarrow T)})^*(\mathcal{F})(T).$$

This is a crystal by the quasi-coherence of $\mathcal{F}$.

Proposition 1.7. *We have a 2-commutative Frobenius-equivariant equivalences*

$$\begin{tikzcd}
\mathbf{Vect}((Z/W)^{\mathrm{crys}}) \arrow[rr, "\Psi_Z"] & & \mathbf{Vect}((Z/W)_{\mathrm{crys}}) \\
& \mathbf{Vect}(\phi_W^*(Z_{\Delta})). \arrow[ul, "(1.1.2)"] \arrow[ur, swap, "(-)^{\mathrm{crys}}\circ(\phi_W^*)^{-1}"] &
\end{tikzcd}$$

Proof. To show that Ψ_Z is a Frobenius-equivariant equivalence, it suffices to show this diagram 2-commutes. It further suffices to assume $Z = \mathrm{Spf}(R)$ with R qrsp. But, for an object $\mathcal{F}$ of $\mathbf{Vect}(\phi_W^*(Z_{\Delta}))$, applying (1.1.2) and then Ψ_Z gives the value $\mathcal{F}(\mathrm{A}_{\mathrm{crys}}(R) \twoheadrightarrow R)$, with twisted W-structure map, on $\mathrm{A}_{\mathrm{crys}}(R) \twoheadrightarrow R$. But, this is the same as $((\phi_W^*)^{-1}(\mathcal{F}))^{\mathrm{crys}}(\mathrm{A}_{\mathrm{crys}}(R) \twoheadrightarrow R)$. ☐

Assume now that $\mathcal{O}_K = W$. Then, our base formal W-scheme $\mathfrak{X}$ determines a natural map

$$i_{\mathfrak{X}} := \rho_{(\mathfrak{X}_k \xleftarrow{\mathrm{id}} \mathfrak{X}_k \hookrightarrow \mathfrak{X})}\colon \mathfrak{X} \to (\mathfrak{X}_k/W)^{\mathrm{crys}},$$

with notation as in (1.1.4).

Definition 1.8. The *(relative) crystalline point* of $\mathfrak{X}$ is the morphism $\rho_{\mathrm{crys},\mathfrak{X}}\colon \mathfrak{X} \to \mathfrak{X}^{\Delta}$ of formal stacks over W obtained as the composition

$$\mathfrak{X} \xrightarrow{i_{\mathfrak{X}}} (\mathfrak{X}_k/W)^{\mathrm{crys}} \xrightarrow{\sim} \phi_W^*(\mathfrak{X}_k^{\Delta}) \xrightarrow{F_{\mathfrak{X}_k}} \mathfrak{X}_k^{\Delta} \to \mathfrak{X}^{\Delta}.$$

The 2-commutativity in Proposition 1.7 implies the following.

Proposition 1.9. *For any prismatic crystal $\mathcal{E}$ on $\mathfrak{X}$, there exists a natural identification*

$$\rho_{\mathrm{crys},\mathfrak{X}}^*(\mathcal{E}) \simeq \mathcal{E}^{\mathrm{crys}}|_{\mathfrak{X}_{\mathrm{Zar}}}.$$

Remark 1.10. Suppose that $\mathfrak{X} = \mathrm{Spf}(R)$ and that w is a formal framing. Consider the following diagram where $\rho_{(R^{(\phi_w)},(p))}$ is as in [BL22b, Construction 3.10]

$$\begin{tikzcd}
& R^{\Delta} \\
\mathrm{Spf}(R) \arrow[ur, "\rho_{\mathrm{crys},\mathfrak{X}}"] \arrow[r, swap, "\phi_w"] & \mathrm{Spf}(R). \arrow[u, swap, "\rho_{(R^{(\phi_w)},(p))}"]
\end{tikzcd}$$

This diagram naturally 2-commutes as for any p-nilpotent R-algebra S, both compositions result in the Cartier–Witt divisor $\mathbb{W}(S) \xrightarrow{p} \mathbb{W}(S)$ with the structure map corresponding to

$$R \to R/p \xrightarrow{F_{R/p}} R/p \to \mathbb{W}(S)/p.$$

This gives an isomorphism $\rho_{\mathrm{crys},\mathfrak{X}}^*(\mathcal{E}) \simeq (\phi^*\mathcal{E})(R^{(\phi_w)},(p))$ which combined with Proposition 1.9 recovers $\vartheta_{R,w}$ from Proposition 1.1.

1.2. **The de Rham realization functor.** We now discuss the notion of a *de Rham realization functor* $\mathbb{D}_{\mathrm{dR}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathbb{\Delta}}) \to \mathbf{Vect}(\mathfrak{X})$. We will take a stack-theoretic approach, but the reader should consult Proposition 1.15 for a more down-to-earth interpretation in coordinates. In the following there are no restrictions on $\mathcal{O}_K$.

Definition 1.11 (cf. [GM24, §6.7])**.** Let $\mathfrak{Y}$ be a quasi-syntomic p-adic formal scheme.

- The *(relative) Nygaard de Rham point* over $\mathfrak{Y}$ is the morphism of formal $\mathbb{A}^1/\mathbb{G}_m$-stacks $\rho^{\mathcal{N}}_{\mathrm{dR},\mathfrak{Y}}\colon \mathfrak{Y} \times (\mathbb{A}^1/\mathbb{G}_m) \to \mathfrak{Y}^{\mathcal{N}}$ over $\mathbb{Z}_p$ associating to a morphism $f\colon \mathrm{Spec}(S) \to \mathfrak{Y}$ and generalized Cartier divisor $\alpha\colon L \to \mathcal{O}_{\mathrm{Spec}(S)}$ the filtered Cartier–Witt divisor
$$F_*(\mathbb{W}) \oplus V(L)^{\sharp} \xrightarrow{d=(V,\alpha^{\sharp})} \mathbb{W}$$
where $(-)^{\sharp}$ is as in [Bha23, Variant 2.4.3], and where the map $\mathrm{Spec}(\mathrm{Cone}(d)) \to \mathfrak{Y}$ is induced by precomposition with f from the natural map
$$S = \mathbb{W}(S)/V(\mathbb{W}(S)) \to \mathrm{Cone}(d)(S).$$
- The *(relative) prismatic de Rham point* over $\mathfrak{Y}$ is the morphism $\rho^{\mathbb{\Delta}}_{\mathrm{dR},\mathfrak{Y}}\colon \mathfrak{Y} \to \mathfrak{Y}^{\mathbb{\Delta}}$ obtained by restricting to $\rho^{\mathcal{N}}_{\mathrm{dR},\mathfrak{Y}}$ to $\mathfrak{Y} = \mathfrak{Y} \times (\mathbb{G}_m/\mathbb{G}_m)$.
- The *(relative) syntomic de Rham point* over $\mathfrak{Y}$ is $\rho^{\mathrm{syn}}_{\mathrm{dR},\mathfrak{Y}} := j_{\mathcal{N}} \circ \rho^{\mathcal{N}}_{\mathrm{dR},\mathfrak{Y}}\colon \mathfrak{Y} \times (\mathbb{A}^1/\mathbb{G}_m) \to \mathfrak{Y}^{\mathrm{syn}}$.

For a perfect complex $\mathcal{V}$ over $\mathfrak{Y}^{\mathbb{\Delta}}$, let us write $\mathcal{V}^{\mathrm{dR}} := (\rho^{\mathbb{\Delta}}_{\mathrm{dR}})^*\mathcal{V}$. Given the identification of $\mathbf{Perf}(\mathfrak{Y} \times (\mathbb{A}^1/\mathbb{G}_m))$ with the ∞-category $\mathbf{PerfF}(\mathfrak{Y})$ of filtered perfect complexes over $\mathfrak{Y}$ (see [Bha23, Proposition 2.2.6]), we observe that for any perfect complex $\mathcal{V}$ on $\mathfrak{Y}^{\mathcal{N}}$ we have
$$(\rho^{\mathcal{N}}_{\mathrm{dR},\mathfrak{Y}})^*(\mathcal{V}) \simeq \left(\mathcal{V}^{\mathrm{dR}}, \mathrm{Fil}^{\bullet}(\mathcal{V}^{\mathrm{dR}})\right), \tag{1.2.1}$$
for some filtration $\mathrm{Fil}^{\bullet}\left(\mathcal{V}^{\mathrm{dR}}\right)$ of $\mathcal{V}^{\mathrm{dR}}$.

Definition 1.12. The *de Rham realization functor* (resp. *filtered de Rham realization functor*)
$$\mathbb{D}_{\mathrm{dR}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathbb{\Delta}}) \to \mathbf{Vect}(\mathfrak{X}), \qquad \left(\text{resp. } \mathbb{D}^+_{\mathrm{dR}}\colon \mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathbb{\Delta}}) \to \mathbf{PerfF}(\mathfrak{X})\right)$$
is defined to be $(\rho^{\mathbb{\Delta}}_{\mathrm{dR},\mathfrak{X}})^*$ (resp. $(\rho^{\mathrm{syn}}_{\mathrm{dR},\mathfrak{X}})^* \circ \Pi_{\mathfrak{X}}$, where $\Pi_{\mathfrak{X}}$ is as in [GL23, Theorem 2.31]).[2]

As $\mathrm{R}_{\mathfrak{X}}(\Pi_{\mathfrak{X}}(\mathcal{E}, \varphi_{\mathcal{E}})) \simeq (\mathcal{E}, \varphi_{\mathcal{E}})$ (by [GL23, Theorem 2.31 (1)]), one sees from (1.2.1) that one may write
$$\mathbb{D}^+_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}}) \simeq (\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}}), \mathrm{Fil}^{\bullet}_{\mathbb{D}_{\mathrm{dR}}}(\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}})))$$
for some filtration $\mathrm{Fil}^{\bullet}_{\mathbb{D}_{\mathrm{dR}}}(\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}}))$ on $\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}})$.

The following construction will help to understand $\mathbb{D}^+_{\mathrm{dR}}$ more concretely. Before we describe it, let us recall (e.g., see [IKY25, Definition 1.23]) that for a prismatic F-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})$ on $\mathfrak{X}$ the *Nygaard filtration* on $\phi^*\mathcal{E}$ is given as follows:
$$\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi^*\mathcal{E}) := \{x \in \phi^*\mathcal{E} : \varphi_{\mathcal{E}}(x) \in \mathcal{I}^{\bullet}_{\mathbb{\Delta}} \cdot \mathcal{E}\},$$
which produces a filtered module over $(\mathcal{O}_{\mathbb{\Delta}}, \mathcal{I}_{\mathbb{\Delta}})$. For any object (A, I) of $\mathfrak{X}_{\mathbb{\Delta}}$ there is an induced filtration $\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi_A^*\mathcal{E}(A, I))$, a filtered module over (A, I).

Remark 1.13. Let us remark that the terminology 'Nygaard filtration' is potentially confusing when $\mathcal{E}$ is cohomological, i.e., $\mathcal{E} = R^i f_*\mathcal{O}_{\mathbb{\Delta}}$ for some morphism $f\colon \mathfrak{Y} \to \mathfrak{X}$. Indeed, in this case one may equip $\mathcal{E}$ with the filtration given by pushing forward the Nygaard filtration on $\mathcal{O}_{\mathbb{\Delta}}$, which is also called the Nygaard filtration in [BS22]. These two filtration need not agree in general, even after Frobenius pullback, but always agree rationally and do agree integrally in certain cases (see [LL25, §7.2]).

[2]Observe that, by definition, $\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}}) = \mathcal{E}^{\mathrm{dR}}$.

Construction 1.14. Let $R = R_0 \otimes_W \mathcal{O}_K$, with formal framing w. Set

$$\Big(\mathcal{V}_w(\mathcal{E},\varphi_{\mathcal{E}}), \mathrm{Fil}^\bullet(\mathcal{V}_w(\mathcal{E},\varphi_{\mathcal{E}}))\Big) := \Big(\phi_w^*\mathcal{E}(\mathfrak{S}_R^{(\phi_w)},(E))/E, \overline{\mathrm{Fil}}^\bullet_{\mathrm{Nyg}}\Big)$$

where

$$\overline{\mathrm{Fil}}^\bullet_{\mathrm{Nyg}} := \mathrm{im}\left(\mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi_w^*\mathcal{E}(\mathfrak{S}_R^{(\phi_w)},(E))) \to \phi_w^*\mathcal{E}(\mathfrak{S}_R^{(\phi_w)},(E))/E\right).$$

This construction commutes with base change along étale maps of framed algebras $R \to R'$.

Proposition 1.15. *For an open subset $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ with $R = R_0 \otimes_W \mathcal{O}_K$ and formal framing w, there is a canonical isomorphism in the filtered derived category of quasi-coherent sheaves on $\mathfrak{Y}$,*

$$\psi_{R,w}\colon (\mathcal{V}_w(\mathcal{E},\varphi_{\mathcal{E}}), \mathrm{Fil}^\bullet(\mathcal{V}_w(\mathcal{E},\varphi_{\mathcal{E}}))) \xrightarrow{\sim} \mathbb{D}^+_{\mathrm{dR}}(\mathcal{E},\varphi_{\mathcal{E}})|_{\mathrm{Spf}(R)}. \tag{1.2.2}$$

To prove Proposition 1.15 we first set up some notation. First, observe we can interpret the Nygaard filtration using the filtered prismatization $\mathfrak{Y}^{\mathcal{N}}$ as follows. Consider, for any object (A, I) of $\mathfrak{Y}_{\mathbb{\Delta}}$ with structure map $s\colon \mathrm{Spf}(A/I) \to \mathfrak{Y}$, the map over $\mathbb{A}^1/\mathbb{G}_m$

$$\rho^+_{(A,I)}\colon \widehat{\mathcal{R}}(\mathrm{Fil}^\bullet_I(A)) \to \mathfrak{Y}^{\mathcal{N}},$$

defined as in [Bha23, Remark 5.5.19]. Note that over $\mathbb{G}_m/\mathbb{G}_m$, this recovers the composition

$$\mathrm{Spf}(A) \xrightarrow{\rho_{(A,I)}} \mathfrak{Y}^{\mathbb{\Delta}} \xrightarrow{F_{\mathfrak{Y}}} \mathfrak{Y}^{\mathbb{\Delta}}, \tag{1.2.3}$$

where $\rho_{(A,I)}$ is the map from [BL22b, Construction 3.10]. In fact, for a p-nilpotent ring S and map $f\colon \mathrm{Spec}(S) \to \mathrm{Spf}(A) \simeq \{t \neq 0\} \subseteq \widehat{\mathcal{R}}(\mathrm{Fil}^\bullet_I(A))$, the underlying Cartier–Witt divisor of $\rho^+_{(A,I)} \circ f$ (i.e., the image under the map $\mathbb{Z}_p^{\mathcal{N}} \to \mathbb{Z}_p^{\mathbb{\Delta}}$ defined in [Bha23, Construction 5.3.3]) of this filtered Cartier–Witt divisor is given by $\phi_A^* I \otimes_A \mathbb{W}(S) \to \mathbb{W}(S)$; note that there is a natural identification of $\mathbb{W}(S)$-modules $I \otimes_A F_*\mathbb{W}(S) \xrightarrow{\sim} F_*(\phi_A^* I \otimes_A \mathbb{W}(S))$.

Lemma 1.16. *The following diagram is 2-commutative*

$$\begin{tikzcd} \mathrm{Spf}(A/I) \times (\mathbb{A}^1/\mathbb{G}_m) \arrow[r, "\iota_{(A,I)}"] \arrow[d, "s\times\mathrm{id}"'] & \widehat{\mathcal{R}}(\mathrm{Fil}^\bullet_I(A)) \arrow[d, "\rho^+_{(A,I)}"] \\ \mathfrak{Y} \times (\mathbb{A}^1/\mathbb{G}_m) \arrow[r, "\rho^{\mathcal{N}}_{\mathrm{dR},\mathfrak{Y}}"'] & \mathfrak{Y}^{\mathcal{N}}, \end{tikzcd} \tag{1.2.4}$$

where $\iota_{(A,I)}$ corresponds to the natural map of graded rings $\mathrm{Rees}(\mathrm{Fil}^\bullet_I(A)) \to \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathrm{triv}}(A/I))$.

Proof. Let S be a p-nilpotent ring, $f\colon \mathrm{Spec}(S) \to \mathrm{Spf}(A/I)$ a morphism, and $\alpha\colon L \to \mathcal{O}_{\mathrm{Spec}(S)}$ a generalized Cartier divisor. We first construct an isomorphism between the two filtered Cartier–Witt divisors obtained by two composites from the diagram, i.e., an isomorphism of $\rho^+_{(A,I)} \circ \iota_{(A,I)}$ and $\rho^{\mathcal{N}}_{\mathrm{dR},\mathfrak{Y}} \circ (s \times \mathrm{id})$ after applying the map $\mathfrak{Y}^{\mathcal{N}}(S) \to \mathbb{Z}_p^{\mathcal{N}}(S)$.

The object of $\mathbb{Z}_p^{\mathcal{N}}(S)$ obtained from $\rho^{\mathcal{N}}_{\mathrm{dR},\mathfrak{Y}} \circ (s \times \mathrm{id})$ is given by

$$V(L)^\sharp \oplus F_*(\mathbb{W}) \xrightarrow{(\alpha^\sharp, V)} \mathbb{W}. \tag{1.2.5}$$

On the other hand, first applying $\iota_{(A,I)}$ gives us the S-point of $\widehat{\mathcal{R}}(\mathrm{Fil}^\bullet_I(A))$ corresponding to the map $A \to A/I \xrightarrow{f} S$ and the line bundle L on S with the factorization $I \otimes_A S \xrightarrow{0} L \xrightarrow{\alpha} S$. Applying the construction of $\rho^+_{(A,I)}$ then gives the filtered Cartier–Witt divisor obtained as follows. The underlying admissible $\mathbb{W}$-module scheme is obtained by pushing forward

$$0 \to I \otimes_A \mathbb{G}_a^\sharp \to I \otimes_A \mathbb{W} \to I \otimes_A F_*\mathbb{W} \to 0$$

along the zero map $I \otimes_A \mathbb{G}_a^\sharp \to V(L)^\sharp$. So, the underlying admissible $\mathbb{W}$-module scheme is the trivial extension $V(L)^\sharp \oplus (I \otimes_A F_*\mathbb{W})$. The structure of a filtered Cartier–Witt divisor $V(L)^\sharp \oplus (I \otimes_A F_*\mathbb{W}) \to \mathbb{W}$ is the one naturally induced from $\alpha^\sharp\colon V(L)^\sharp \to \mathbb{G}_a^\sharp$.

Applying [BL22a, Proposition 3.6.6] to the Cartier–Witt divisor $\rho_{(A,I)}\circ f = (I\otimes_A \mathbb{W}(S)\to\mathbb{W}(S))$ we obtain an isomorphism of Cartier–Witt divisors

$$(\phi_A^* I\otimes_A \mathbb{W}(S)\to\mathbb{W}(S))\xrightarrow{(\gamma,\mathrm{id})}(\mathbb{W}(S)\xrightarrow{p}\mathbb{W}(S)),$$

where γ is induced from the factorization $I\otimes_A\mathbb{W}(S)\xrightarrow{\beta}\mathbb{W}(S)\xrightarrow{V}\mathbb{W}(S)$; specifically, γ is the linearization of the F-semi-linear map β. This then induces an isomorphism of filtered Cartier–Witt divisors

$$\left(V(L)^\sharp\oplus(I\otimes F_*\mathbb{W})\to\mathbb{W}\right)\xrightarrow{(\mathrm{id}_{V(L)^\sharp}\oplus F_*(\gamma),\mathrm{id}_{\mathbb{W}})}\left(V(L)^\sharp\oplus F_*\mathbb{W}\xrightarrow{(\alpha^\sharp,V)}\mathbb{W}\right). \tag{1.2.6}$$

This is our desired identification of these two compositions in $\mathbb{Z}_p^{\mathcal{N}}(S)$.

We now show this isomorphism respects the $\mathfrak{Y}$-structures (i.e., lifts to an isomorphism in $\mathfrak{Y}^{\mathcal{N}}(S)$). From the natural map

$$A/I\to R\Gamma(\mathrm{Spec}(S),\mathbb{W}/(I\otimes_A\mathbb{W}))$$

and the given map $s\colon \mathrm{Spf}(A/I)\to\mathfrak{Y}$, the $\mathfrak{Y}$-structure on the source of (1.2.6) is induced from the map of quasi-ideals $[(0,\beta),\mathrm{id}]$:

$$\begin{tikzcd} I\otimes_A\mathbb{W} \arrow[r, "(0{,}\beta)"] \arrow[d] & V(L)^\sharp\oplus F_*\mathbb{W} \arrow[d, "(\alpha^\sharp{,}V)"] \\ \mathbb{W} \arrow[r, "\mathrm{id}"] & \mathbb{W}. \end{tikzcd}$$

Similarly, the $\mathfrak{Y}$-structure on the target of (1.2.6) is induced from

$$\begin{tikzcd} I\otimes_A\mathbb{W} \arrow[r, "(0{,}F)"] \arrow[d] & V(L)^\sharp\oplus F_*(\phi^*I\otimes_A\mathbb{W}) \arrow[d] \\ \mathbb{W} \arrow[r, "\mathrm{id}"] & \mathbb{W}. \end{tikzcd}$$

Here, by construction, the two maps β and $F_*(\gamma)\circ F$ coincide. Thus, the $\mathfrak{Y}$-structures agree, and hence we obtain the desired commutativity. □

Proposition 1.17. *For any open subset $\mathrm{Spf}(R)\subseteq\mathfrak{X}$ with $R=R_0\otimes_W\mathcal{O}_K$ and with formal framing w, there exists an isomorphism in $\mathbf{QCoh}(\widehat{\mathcal{R}}(\mathrm{Fil}_E^\bullet(\mathfrak{S}_R)))$*

$$\left(\rho^+_{(\mathfrak{S}_R^{(\phi_w)},E),\mathfrak{X}}\right)^*\Pi_{\mathfrak{X}}(\mathcal{E})\xrightarrow{\sim}\bigoplus_{i\in\mathbb{Z}}\mathrm{Fil}^{-i}_{\mathrm{Nyg}}(\phi_w^*\mathcal{E}(\mathfrak{S}_R^{(\phi_w)},(E))),$$

which is compatible with inclusions of affine open subsets of $\mathfrak{X}$ with compatible framings.[3]

Proof. We may assume that $\mathfrak{X}=\mathrm{Spf}(R)$. To define the isomorphism of underlying vector bundles

$$\left(\rho^+_{(\mathfrak{S}_R^{(\phi_w)},E)),\mathfrak{X}}\right)^*\Pi_{\mathfrak{X}}(\mathcal{E})\xrightarrow{\sim}\phi_w^*\mathcal{E}(\mathfrak{S}_R^{(\phi_w)},(E)),$$

it suffices to recall that $\rho^+_{(A,I)}$ restricted to the fiber of $\mathbb{G}_m/\mathbb{G}_m$ is precisely the map (1.2.3).

Consider the faithfully flat cover $R\to\widetilde{R}$ by a perfectoid ring $\widetilde{R}$ from [IKY24, Lemma 1.15]. Since the left hand side comes from a filtered module (i.e., is t-torsion-free) by the construction of $\Pi_{\mathfrak{X}}$, it suffices to verify the claim after base changing along the map $\widetilde{\alpha}_{\mathrm{inf},w^\flat}\colon\mathfrak{S}_R\to\mathrm{A}_{\mathrm{inf}}(\widetilde{R})$ from [IKY24, §1.1.5], i.e., it suffices to construct a natural isomorphism

$$\left(\rho^+_{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}),(\tilde{\xi})),\mathfrak{X}}\right)^*\Pi_{\mathfrak{X}}(\mathcal{E})\xrightarrow{\sim}\bigoplus_{i\in\mathbb{Z}}\mathrm{Fil}^{-i}_{\mathrm{Nyg}}(\phi^*\mathcal{E}(\mathrm{A}_{\mathrm{inf}}(\widetilde{R})),(\tilde{\xi})).$$

But, the functor $\Pi_{\mathfrak{X}}$ is precisely constructed so that this holds. □

[3]Here we implicitly use the equivalence from [LvO96, Chapter I, §4.3, Proposition 7] showing that we may explicitly identify t-torsion-free quasi-coherent sheaves on $\widehat{\mathcal{R}}(\mathrm{Fil}^\bullet_E(\mathfrak{S}_R^{(\phi_w)}))$ with filtered modules over $(\mathfrak{S}_R^{(\phi_w)},\mathrm{Fil}^\bullet_E)$.

Proof of Proposition 1.15. By Lemma 1.16, we have

$$\mathbb{D}^+_{\mathrm{dR}}(\mathcal{E},\varphi_{\mathcal{E}}) \xrightarrow{\sim} \iota^*_{(\mathfrak{S}_R^{(\phi_w)},(E))}(\rho^+_{(\mathfrak{S}_R^{(\phi_w)},(E))})^*\Pi_{\mathfrak{X}}(\mathcal{E}).$$

By Proposition 1.17, the right hand side is given by

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^{-i}_{\mathrm{Nyg}}(\phi_w^*\mathcal{E}(\mathfrak{S}_R^{(\phi_w)},(E))) \otimes^L_{\mathrm{Rees}(\mathrm{Fil}^\bullet_E(\mathfrak{S}_R))} R[t].$$

The proof then follows by Lemma 1.18 below. ☐

Lemma 1.18. *Let $(A,(d))$ be a bounded prism, and (M,φ_M) an object of $\mathbf{Vect}^\varphi(A,(d))$. Set $\mathrm{Fil}^\bullet := \mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*M)$ and $\overline{\mathrm{Fil}}^\bullet := \overline{\mathrm{Fil}}^\bullet_{\mathrm{Nyg}}(\phi^*M/(d))$. Then, the natural map*

$$\left(\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^{-i}\,t^i\right) \otimes^L_{\mathrm{Rees}(\mathrm{Fil}^\bullet_d(A))} A/I[t] \to \bigoplus_{i\in\mathbb{Z}} \overline{\mathrm{Fil}}^{-i} t^i \tag{1.2.7}$$

is an isomorphism of graded $A/I[t]$-modules.

Proof. By the proof of [IKY25, Lemma 1.7], the source of (1.2.7) is quasi-isomorphic to the complex

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^{-(i-1)}\,t^i \xrightarrow{\cdot d} \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^{-i}\,t^i,$$

placed in degree $[-1,0]$. Since the multiplication-by-d map is injective by the assumption that M is projective, the map in question, on the r^{th} graded piece is identified with the natural map

$$\mathrm{Fil}^r/d\,\mathrm{Fil}^{r-1} \to \overline{\mathrm{Fil}}^r,$$

which is surjective by the definition of $\overline{\mathrm{Fil}}^\bullet$. We can show the injectivity as follows. Let $x = dy$ be an element of $\mathrm{Fil}^r \cap d\phi^*M$ where here y belongs to ϕ^*M. Then, by the definition of the Nygaard filtration, we get that $d\varphi_M(y)$ is in d^rM, and hence that $\varphi_M(y)$ belongs to $d^{r-1}M$, that is, y is an element of Fil^{r-1}. Thus, the above natural map is an isomorphism from where the claim follows. ☐

1.3. **The crystalline-de Rham comparison.** We now describe the crystalline–de Rham comparison, which identifies the the vector bundles on $\mathfrak{X}$ given by $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})|_{\mathfrak{X}_{\mathrm{Zar}}}$ and $\mathbb{D}_{\mathrm{dR}}(\mathcal{E},\varphi_{\mathcal{E}})$ for a prismatic F-crysal $(\mathcal{E},\varphi_{\mathcal{E}})$. Throughout we will assume $\mathcal{O}_K = W$.

Theorem 1.19 (Crystalline–de Rham comparison)**.** *Let $\mathfrak{X}$ be a base formal W-scheme.*

(1) There is an identification between $\rho_{\mathrm{crys},\mathfrak{X}}$ and $\rho^{\Delta}_{\mathrm{dR},\mathfrak{X}}$ as objects of $\mathrm{Map}(\mathfrak{X},\mathfrak{X}^{\Delta})$.

(2) For an object $\mathcal{E}$ of $\mathbf{Vect}(\mathfrak{X}_\Delta)$, there is a canonical isomorphism in $\mathbf{Vect}(\mathfrak{X})$:

$$\iota_{\mathfrak{X}}\colon \mathcal{E}^{\mathrm{crys}} \xrightarrow{\sim} \mathcal{E}^{\mathrm{dR}}.$$

In particular, for an object $(\mathcal{E},\varphi_{\mathcal{E}})$ of $\mathbf{Vect}(\mathfrak{X}_\Delta)$, there is a canonical isomorphism

$$\iota_{\mathfrak{X}}\colon \mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})|_{\mathfrak{X}_{\mathrm{Zar}}} \xrightarrow{\sim} \mathbb{D}_{\mathrm{dR}}(\mathcal{E},\varphi_{\mathcal{E}}). \tag{1.3.1}$$

Proof. Assertion (2) is an immediate consequence of assertion (1) together with Proposition 1.9.

To prove assertion (1), let B be a p-nilpotent ring and $s\colon \mathrm{Spec}(B)\to\mathfrak{X}$ be a B-point of $\mathfrak{X}$. Recall that the underlying Cartier–Witt divisors of both $\rho^{\Delta}_{\mathrm{dR},\mathfrak{X}}(s)$ and $\rho_{\mathrm{crys},\mathfrak{X}}(s)$ is $\mathbb{W}(B)\xrightarrow{p}\mathbb{W}(B)$, and that their $\mathfrak{X}$-structures given respectively by

$$\mathrm{Spec}(\mathbb{W}(B)/p) \xrightarrow{F} \mathrm{Spec}(\mathbb{W}(B)/V) \xrightarrow{s} \mathfrak{X},$$
$$\mathrm{Spec}(\mathbb{W}(B)/p) \xrightarrow{\sim} \mathrm{Spec}(\mathbb{G}_a^{\mathrm{dR}}(B)) \to \mathrm{Spec}(B) \xrightarrow{s} \mathfrak{X},$$

where the map $\mathbb{W}(B)/p \xleftarrow{\sim} \mathbb{G}_a^{\mathrm{dR}}(B)$ is from [Bha23, Corollary 2.6.8] (note that we are not keeping track of the W-structure, and hence we can ignore the twist of W-module structure appearing in loc. cit.). If C denote the animated ring $\mathrm{Cone}(\mathbb{G}_a^\sharp(B)\oplus\mathbb{W}(B)\xrightarrow{(\mathrm{can},V)}\mathbb{W}(B))$, this isomorphism is defined to be the composite of the two isomorphisms

$$\mathbb{G}_a^{\mathrm{dR}}(B) \xleftarrow{\sim} C \xrightarrow{\sim} \mathbb{W}(B)/p.$$

Here, the left arrow is given by the restriction map $\mathbb{W}(B) \to B$ and the first projection onto $\mathbb{G}_a^\sharp$. The right arrow is given by the Frobenius map $F\colon \mathbb{W}(B) \to \mathbb{W}(B)$ and the second projection onto $\mathbb{W}(B)$ (see loc. cit. for details). So, it suffices to show that the following diagram commutes:

$$\begin{tikzcd} B \arrow[d] & \mathbb{W}(B)/V(\mathbb{W}(B)) \arrow[l, "\sim"'] \arrow[d, "\mathrm{can}"] \\ \mathbb{G}_a^{\mathrm{dR}}(B) & C. \arrow[l, "\sim"'] \end{tikzcd} \tag{1.3.2}$$

But, this is obtained from a commutative diagram of underlying quasi-ideals, and thus a commutative diagram of quotients. Thus, we obtain a functorial identification of the two objects $\rho_{\mathrm{crys},\mathfrak{X}}(s)$ and $\rho^{\Delta}_{\mathrm{dR},\mathfrak{X}}(s)$, and hence an isomorphism $\rho_{\mathrm{crys},\mathfrak{X}} \xrightarrow{\sim} \rho^{\Delta}_{\mathrm{dR},\mathfrak{X}}$. ☐

We end by giving a more down-to-earth version of the crystalline–de Rham comparison over a W-algebra R with formal framing w. To do so, we first make a construction using the Breuil prism $(S_R^{(\phi_w)}, (p))$ as in [IKY24, §1.1.5]. In what follows, we drop the decoration ϕ_w on Breuil(–Kisin) rings when only the ring structure is important.

Let $\rho_{(S_R^{(\phi_w)},(p))}\colon \mathrm{Spf}(S_R) \to R^{\Delta}$ be as in [BL22b, Construction 3.10] and denote by $\mathrm{sp}_{\mathrm{dR}}$ the map on formal spectra induced by the composition $S_R \twoheadrightarrow S_R/\operatorname{Fil}^1_{\mathrm{PD}} \xrightarrow{\sim} R$ (i.e., the map $u \mapsto \pi$).

Construction 1.20. Let $\mathcal{E}$ be an object of $\mathbf{Vect}(R_{\Delta})$. Using [IKY24, Diagram (1.1.2) and Lemma 1.14] and the crystal property, we obtain isomorphisms:

$$\phi_w^*\mathcal{E}(R^{(\phi_w)}, (p)) \otimes_R S_R \xrightarrow{\sim} \mathcal{E}(S_R^{(\phi_w)}, (p)) \xleftarrow{\sim} \phi_w^*\mathcal{E}(\mathfrak{S}_R^{(\phi_w)}, (E)) \otimes_{\mathfrak{S}_R} S_R. \tag{1.3.3}$$

Reducing this isomorphism along $\mathrm{sp}_{\mathrm{dR}}^*\colon S_R \to R$, we obtain an isomorphism of R-modules

$$\phi_w^*\mathcal{E}(R^{(\phi_w)}, (p)) \xrightarrow{\sim} \phi_w^*\mathcal{E}((\mathfrak{S}_R^{(\phi_w)}, (E))/E. \tag{1.3.4}$$

Since the left-hand (resp. right-hand) side is identified with $\mathcal{E}^{\mathrm{crys}}$ (resp. $\mathcal{E}^{\mathrm{dR}}$) by Proposition 1.1 (resp. Proposition 1.15), we obtain an isomorphism

$$\mathcal{E}^{\mathrm{crys}} \xrightarrow{\sim} \mathcal{E}^{\mathrm{dR}}. \tag{1.3.5}$$

Proposition 1.21. *Let R be a base W-algebra with formal framing w. Set $\mathfrak{X} = \mathrm{Spf}(R)$.*

(1) The identification from Theorem 1.19 naturally factors as

$$\rho_{\mathrm{crys},\mathfrak{X}} \xrightarrow{\sim} \rho_{(S_R^{(\phi_w)},(p))} \circ \mathrm{sp}_{\mathrm{dR}} \xleftarrow{\sim} \rho^{\Delta}_{\mathrm{dR},\mathfrak{X}}.$$

(2) The isomorphism in (1.3.4) can be identified with the $\iota_{\mathfrak{X}}$ from Theorem 1.19.

Proof. To prove assertion (1), first consider the following diagram:

$$\begin{tikzcd} & \mathrm{Spf}(\mathfrak{S}_R) \arrow[r, "\phi_w"] & \mathrm{Spf}(\mathfrak{S}_R) \arrow[dr, "\rho_{(\mathfrak{S}_R^{(\phi_w)},(E))}"] & \\ \mathrm{Spf}(R) \arrow[ur, "\mathrm{sp}_{\mathrm{dR},\mathfrak{S}}"] \arrow[r, "\mathrm{sp}_{\mathrm{dR}}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Spf}(S_R) \arrow[u] \arrow[d] \arrow[rr, "\rho_{(S_R^{(\phi_w)},(p))}"] & & \mathfrak{X}^{\Delta} \\ & \mathrm{Spf}(R) \arrow[r, "\phi_w"] & \mathrm{Spf}(R). \arrow[ur, "\rho_{(R^{(\phi_w)},(p))}"'] & \end{tikzcd}$$

Here, the map $\mathrm{sp}_{\mathrm{dR},\mathfrak{S}}$ comes from the natural isomorphism $\mathfrak{S}_R/E \xrightarrow{\sim} R$, the two vertical maps are given by the natural inclusions $\mathfrak{S}_R \to S_R$ and $R \to S_R$, and the three arrows mapping to $\mathfrak{X}^{\Delta}$ are obtained as in [BL22b, Construction 3.10]. The left two triangles are obviously commutative and the right two trapezoids are 2-commutative with obvious identifications of the compositions.

Let $\rho'_{\mathrm{dR},\mathfrak{X}}$ (resp. $\rho'_{\mathrm{crys},\mathfrak{X}}$) denote the upper (lower) composite maps $\mathrm{Spf}(R) \to \mathfrak{X}^{\Delta}$. By the commutativity of the above diagram, we obtain isomorphisms

$$\rho'_{\mathrm{crys},\mathfrak{X}} \xrightarrow{\sim} \rho_{(S_R^{(\phi_w)},(p))} \circ \mathrm{sp}_{\mathrm{dR}} \xleftarrow{\sim} \rho'_{\mathrm{dR},\mathfrak{X}}.$$

By Proposition 1.15 (resp. Remark 1.10), we have an isomorphism $\rho'_{\mathrm{dR},\mathfrak{X}} \xrightarrow{\sim} \rho^{\Delta}_{\mathrm{dR},\mathfrak{X}}$ (resp. $\rho'_{\mathrm{crys},\mathfrak{X}} \xrightarrow{\sim} \rho_{\mathrm{crys},\mathfrak{X}}$). Thus, we have obtained the diagram of isomorphisms

$$\begin{tikzcd} \rho_{\mathrm{crys},\mathfrak{X}} \arrow[rr] \arrow[d] & & \rho^{\Delta}_{\mathrm{dR},\mathfrak{X}} \arrow[d] \\ \rho'_{\mathrm{crys},\mathfrak{X}} \arrow[r] & \rho_{(S_R^{(\phi_w)},(p))} \circ \mathrm{sp}_{\mathrm{dR}} & \rho'_{\mathrm{dR},\mathfrak{X}}. \arrow[l] \end{tikzcd} \tag{1.3.6}$$

This diagram commutes as each isomorphism is defined by the identity map on $\mathbb{W}$ (and a canonical identification of its quasi-ideals), which proves assertion (1).

To prove assertion (2), we consider the following commutative diagram induced by (1.3.6)

$$\begin{tikzcd} \mathcal{E}^{\mathrm{crys}} \arrow[rr] \arrow[d] & & \mathcal{E}^{\mathrm{dR}} \arrow[d] \\ \phi_w^*\mathcal{E}(R^{(\phi_w)},(p)) \arrow[r] & \mathcal{E}(S_R^{(\phi_w)},(p)) \otimes_{S_R,\mathrm{sp}^*_{\mathrm{dR}}} R & \phi_w^*\mathcal{E}(\mathfrak{S}_R^{(\phi_w)},(E))/E. \arrow[l] \end{tikzcd}$$

The composition of the two lower horizontal isomorphism is the isomorphism in (1.3.4). Moreover, the left (resp. right) horizontal isomorphism is the identification from Proposition 1.1 (resp. Proposition 1.15) by Remark 1.10 (resp. by definition). This proves assertion (2). ☐

Remark 1.22. When $R = W$, the isomorphism in (1.3.3) is compatible with that from [Gao19, Proposition 3.6] (and after inverting p, that from [Kis06, Lemma 1.2.6]). More precisely, it is equal to the isomorphism $1 \otimes s$ from [Gao19, Proposition 3.6] (resp. the base change of the isomorphism ξ from [Kis06, Lemma 1.2.6] along the map $\phi\colon \mathcal{O} \to S_R[1/p]$, where $\mathcal{O}$ is as in loc. cit). This follows from the uniqueness property discussed in loc. cit.

Remark 1.23. Our assumption that $K = K_0$ was necessary for Construction 1.20 so that the arrow labeled $(*)$ in [IKY24, Diagram (1.1.2)] was a morphism in R_{Δ}. But, using [IKY24, Lemma 1.14], one may adjust this for arbitrary $\mathcal{O}_K$ giving an isomorphism of R-modules

$$(\phi^e)^*\mathcal{E}(\mathfrak{S}_R,(E))/E \xrightarrow{\sim} (\phi^e)^*\mathcal{E}(\mathfrak{S}_R,(E))/u.$$

In fact, such an isomorphism should hold, by the same method of proof, with $(\mathcal{E},\varphi_{\mathcal{E}})$ replaced by an object of $\mathbf{D}^{\varphi}_{\mathrm{perf}}(\mathfrak{X}_{\Delta})$, where the quotients should now be considered in the derived sense. One interesting implication of this would be the existence of canonically matched lattices under

$$Rf_*(\Omega^{\bullet}_{\mathfrak{X}/\mathfrak{Y}}) \otimes^L_{\mathcal{O}_K} K \xrightarrow{\sim} Rf_*(\mathcal{O}_{(\mathfrak{X}_k/W)_{\mathrm{crys}}})|_{\mathfrak{Y}_{\mathrm{Zar}}} \otimes^L_W K,$$

the isomorphism of Berthelot–Ogus (see [BO83, Theorem 2.4]), where $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a smooth proper morphism of formal $\mathcal{O}_K$-schemes, where $\mathfrak{Y}$ is smooth. This idea is explored in [AY25].

2. The functor $\mathbb{D}_{\mathrm{crys}}$

In this section we use the crystalline-de Rham comparision (see Theorem 1.19) to give the definition of our integral analogue $\mathbb{D}_{\mathrm{crys}}$ and establish basic properties about it. Throughout we use notation and terminology from Notation and terminology without comment.

2.1. Various categories of filtered Frobenius crystals.

We begin by explicating several categories of F-crystals with filtration that will be important in the sequel.

2.1.1. **Naive filtered F-crystals.** A *naive filtered F-crystal* on $\mathfrak{X}$ is a triple $(\mathcal{F},\varphi_{\mathcal{F}},\mathrm{Fil}^{\bullet}_{\mathcal{F}})$ with $(\mathcal{F},\varphi_{\mathcal{F}})$ an object of $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ and $\mathrm{Fil}^{\bullet}_{\mathcal{F}}$ a filtration by $\mathcal{O}_{\mathfrak{X}}$-submodules of $\mathcal{F}_{\mathfrak{X}}$. Morphisms of naive filtered F-crystals are morphisms of F-crystals respecting filtrations. Denote the category of naive filtered F-crystals by $\mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$, which has a $\mathbb{Z}_p$-linear $\otimes$-structure where

$$\mathrm{Fil}^k_{\mathcal{F}_1\otimes\mathcal{F}_2} = \sum_{i+j=k} \mathrm{Fil}^i_{\mathcal{F}_1} \otimes \mathrm{Fil}^j_{\mathcal{F}_2}.$$

It has an exact structure where a sequence is exact if its associated sequence in $\mathbf{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ is exact, and for all i the sequence of $\mathcal{O}_{\mathfrak{X}}$-modules on $\mathfrak{X}$ given by the i^{th}-graded piece is exact. We say that a naive filtered F-crystal has *level* in $[0,a]$ if $\mathrm{Fil}^0_{\mathcal{F}} = \mathcal{F}_{\mathfrak{X}}$ and $\mathrm{Fil}^{a+1}_{\mathcal{F}} = 0$.

2.1.2. **Filtered F-crystals.** We now examine several refinements of the notion of a naive filtered F-crystal that will play an important role in our discussion below.

Definition 2.1. For a base formal W-scheme $\mathfrak{X}$, we say an object $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ of $\mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$:

(1) is a *weakly filtered F-crystal* if $\mathrm{Fil}^{\bullet}_{\mathcal{F}}$ satisfies Griffiths transversality with respect to $\nabla_{\mathcal{F}}$ after inverting p, and the filtration $\mathrm{Fil}^{\bullet}_{\mathcal{F}}[1/p] \subset \mathcal{F}_{\mathfrak{X}}[1/p]$ is locally split,[4]
(2) is *graded p-torsion-free (gtf)* if $\mathrm{Gr}^r(\mathrm{Fil}^{\bullet}_{\mathcal{F}})$ is a p-torsion-free $\mathcal{O}_{\mathfrak{X}}$-module for all r,
(3) is a *filtered F-crystal* if $\mathrm{Fil}^{\bullet}_{\mathcal{F}}$ satisfies Griffiths transversality with respect to $\nabla_{\mathcal{F}}$, and the filtration $\mathrm{Fil}^{\bullet}_{\mathcal{F}} \subset \mathcal{F}_{\mathfrak{X}}$ is locally split,
(4) is *strongly divisible* if for any affine open $\mathrm{Spf}(R) \subset \mathfrak{X}$ and any formal framing w, the equality $\varphi_{\mathcal{F}}\left(\sum_{r\in\mathbb{Z}} p^{-r}\phi_w^* \mathrm{Fil}^r_{\mathcal{F}}(R)\right) = \mathcal{F}_{\mathfrak{X}}(R)$ holds.

Remark 2.2. We use different terms than [Lov17]: a filtered F-crystal (resp. strongly divisible filtered F-crystal) here is a weak filtered F-crystal (resp. filtered F-crystal) there.

We give notation to the full subcategories of $\mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ defined by these objects:

- $\mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ is the full subcategory consisting of weakly filtered F-crystals,
- $\mathbf{VectNF}^{\mathrm{gtf}}(\mathfrak{X}_{\mathrm{crys}})$ is the full subcategory consisting of gtf naive filtered F-crystals,
- $\mathbf{VectF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ is the full subcategory consisting of filtered F-crystals,
- $\mathbf{VectNF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$ is the full subcategory of strongly divisible naive filtered F-crystals.

We obtain further full subcategories of $\mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ by intersection, which are denoted by the obvious symbols. Furthermore, the subscript $[0,a]$ will denote the intersection with $\mathbf{VectNF}^{\varphi}_{[0,a]}(\mathfrak{X}_{\mathrm{crys}})$. Observe that $\mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ and $\mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$ are stable under tensor products and duals and so inherit an exact $\mathbb{Z}_p$-linear $\otimes$-structure from $\mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$.

Lastly, we note that if $\mathfrak{X} \to \mathrm{Spf}(W)$ is smooth, there is an exact $\mathbb{Z}_p$-linear $\otimes$-functor $\mathbf{VectWF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}) \to \mathbf{IsocF}^{\varphi}(\mathfrak{X})$ sending $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ to the filtered F-isocrystal $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})[1/p]$ which is defined to be $(\mathcal{F}[1/p], \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}}[1/p]) = (\mathcal{F}[1/p], \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{F})$.

2.1.3. **The Faltings morphism.** To study the relationship between our various conditions on a naive filtered F-crystal, it is useful to recall the following construction of Faltings.

Fix a naive filtered F-crystal $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ on $\mathrm{Spf}(R)$, where R is a base W-algebra with formal framing w. Consider the following module as in [Fal89, II.c), p. 30]:

$$\widetilde{\mathcal{F}}_{\mathfrak{X}} := \mathrm{colim}\left(\cdots \to \mathrm{Fil}^{r+1}_{\mathcal{F}} \xleftarrow{\cdot p} \mathrm{Fil}^{r+1}_{\mathcal{F}} \to \mathrm{Fil}^{r}_{\mathcal{F}} \xleftarrow{\cdot p} \mathrm{Fil}^{r}_{\mathcal{F}} \to \mathrm{Fil}^{r-1}_{\mathcal{F}} \leftarrow \cdots\right). \tag{2.1.1}$$

The maps $\mathrm{Fil}^r_{\mathcal{F}} \to \mathcal{F}_{\mathfrak{X}}[1/p]$ sending x to $p^{-r}x$ induce a natural map $\widetilde{\mathcal{F}}_{\mathfrak{X}} \to \mathcal{F}_{\mathfrak{X}}[1/p]$ whose image is the sum $\sum_{r\in\mathbb{Z}} p^{-r}\mathrm{Fil}^r_{\mathcal{F}}$. We then have the following *Faltings morphism*

$$\phi_w^*\widetilde{\mathcal{F}}_{\mathfrak{X}} \to \phi_w^*\mathcal{F}_{\mathfrak{X}}[1/p] \xrightarrow{\varphi_{\mathcal{F}}} \mathcal{F}_{\mathfrak{X}}[1/p]. \tag{2.1.2}$$

Observe that if $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ is gtf then the map $\widetilde{\mathcal{F}}_{\mathfrak{X}} \to \mathcal{F}_{\mathfrak{X}}[1/p]$ is injective. Thus, as ϕ_w is flat, we see that if $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ is gtf and strongly divisible, then the Faltings morphism induces an siomorphism $\phi_w^*\widetilde{\mathcal{F}}_{\mathfrak{X}} \xrightarrow{\sim} \mathcal{F}_{\mathfrak{X}}$.

Observe that

$$\widetilde{\mathcal{F}}_{\mathfrak{X}}/p \cong \mathrm{colim}\left(\cdots \to \mathrm{Fil}^{r+1}_{\mathcal{F}}/p \xleftarrow{0} \mathrm{Fil}^{r+1}_{\mathcal{F}}/p \to \mathrm{Fil}^{r}_{\mathcal{F}}/p \xleftarrow{0} \mathrm{Fil}^{r}_{\mathcal{F}}/p \to \mathrm{Fil}^{r-1}_{\mathcal{F}}/p \leftarrow \cdots\right),$$

and so is isomorphic to $\bigoplus_r \mathrm{Gr}^r(\mathrm{Fil}^{\bullet}_{\mathcal{F}})/p$. Leveraging this, we show the following.

[4]Recall that a filtered R-module $\mathrm{Fil}^{\bullet} \subseteq M$ is *locally split* if $\mathrm{Gr}^r(\mathrm{Fil}^{\bullet})$ is a finite projective R-module for all r.

Proposition 2.3 (cf. [Fal89, Theorem 2.1], [LMP24, Theorem 2.9]). *Let $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}})$ be a gtf strongly divisible naive filtered F-crystal. Then, the filtration $\mathrm{Fil}^\bullet_{\mathcal{F}} \subseteq \mathcal{F}_{\mathfrak{X}}$ is locally split. If $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}})$ is further a weakly filtered F-crystal, then it is a strongly divisible filtered F-crystal.*

Proof. We may assume that $\mathfrak{X} = \mathrm{Spf}(R)$, where R is a base $\mathcal{O}_K$-algebra with formal framing w.

Let us first verify that the filtration $\mathrm{Fil}^\bullet_{\mathcal{F}}$ is locally split. In other words, we must show that $\mathrm{Gr}^r(\mathrm{Fil}^\bullet_{\mathcal{F}})$ is a locally free R-module for all r. As $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}})$ is gtf and strongly divisible, the Faltings morphism is an isomorphism. This implies that $\phi_w^* \widetilde{\mathcal{F}}(R)$ is a projective R-module. But, as observed above, this implies that $\phi_w^* \mathrm{Gr}^r(\mathrm{Fil}^r_{\mathcal{F}}(R))/p$ is a projective R/p-module for all r, and since ϕ_w is faithfully flat this implies that $\mathrm{Gr}^r(\mathrm{Fil}^\bullet_{\mathcal{F}}(R))/p$ is a locally free R/p-module for all r. This implies that $\mathrm{Gr}^r(\mathrm{Fil}^\bullet_{\mathcal{F}}(R))$ is a projective R-module by the following simple lemma.

Lemma 2.4. *Let A be a Noetherian p-adically complete ring and Q a finitely generated p-torsion-free A-module such that Q/p is a projective A/p-module. Then, Q is a projective A-module.*

Proof. Take a short exact sequence $0 \to K \xrightarrow{\iota} A^n \to Q \to 0$. As Q is p-torsion-free, reducing modulo p gives an exact sequence $0 \to K/p \xrightarrow{\bar{\iota}} (A/p)^n \to Q/p \to 0$. As Q/p is projective, there exists a retraction $\bar{\rho}\colon (A/p)^n \to K/p$ to $\bar{\iota}$. Consider the composition $A^n \to (A/p)^n \to K/p$, and lift it to $\rho\colon A^n \to K$. Note that $\rho \circ \iota$ is the identity modulo p. As K is p-adically complete, this implies that $\rho\circ\iota$ is an automorphism, with inverse $\sum_{k\geqslant 0}(-1)^k((\rho\circ\iota) - \mathrm{id})^k$. So then, $(\rho\circ\iota)^{-1}\circ\rho$ is a retraction to ι. Thus, Q is a direct summand of a free module, so projective. ☐

Suppose further that $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}})$ is a weakly filtered F-crystal. To show that it is a strongly divisible F-crystal, it remains to show that $\mathrm{Fil}^\bullet_{\mathcal{F}}$ satisfies Griffiths transversality with respect to $\nabla_{\mathcal{F}}$. But, let us observe that by assumption, for each r we have that

$$\nabla_{\mathcal{F}}(\mathrm{Fil}^r_{\mathcal{F}}) \subseteq (\mathcal{F}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \cap (\mathrm{Fil}^{r-1}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K})[1/p].$$

But, the right-hand is just $\mathrm{Fil}^{r-1}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K}$. Indeed, this follows from the observation that

$$(\mathcal{F}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K})/(\mathrm{Fil}^{r-1}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \cong (\mathcal{F}_{\mathfrak{X}}/\mathrm{Fil}^{r-1}_{\mathcal{F}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathcal{O}_K}$$

is p-torsion-free, being the tensor product of two vector bundles on $\mathfrak{X}$. ☐

We may now explain the relationship between the subcategories of naive filtered F-crystals considered above, and the categories $\mathbf{MF}^{\nabla}_{[0,a]}(R)$ of *Fontaine–Laffaille modules* as in [Fal89, II.d)], when $p \neq 2$ and $0 \leqslant a \leqslant p-2$.[5]

Proposition 2.5. *Assume that $p \neq 2$. Let R be a base W-algebra. Then, for any a in $[0, p-2]$, the functor*

$$\mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,a]}(R_{\mathrm{crys}}) \to \mathbf{MF}^{\nabla}_{[0,a]}(R), \qquad (\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}}) \mapsto (\mathcal{F}(\mathrm{id}\colon R \to R), \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_{\mathcal{F}}, \nabla_{\mathcal{F}}),$$

is an equivalence of categories.

Proof. This follows from Proposition 2.6 by adding Frobenius structures to both side and checking that the notions of divisibility match. ☐

2.1.4. **The category $\mathbf{VectF}^{\nabla}(\mathfrak{X})$.** We next discuss some results of Tsuji which allows one to understand the filtrations of some naive filtered F-crystals via the crystalline site.

Consider a pair $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})$, where $\mathcal{E}$ is an object of $\mathbf{Vect}(\mathfrak{X}_{\mathrm{crys}})$, and $\mathrm{Fil}^\bullet_{\mathcal{E}} \subseteq \mathcal{E}$ is a filtration by locally quasi-coherent (see [SP, Tag 07IS]) $\mathcal{O}_{\mathrm{crys}}$-submodules. We call $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})$ a *filtered crystal* if

1. $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})(x)$ is a filtered module over $(A, \mathrm{Fil}^\bullet_{\mathrm{PD}})$ for all $x = (i\colon A \twoheadrightarrow B, \gamma)$ in $(\mathfrak{X}/W)_{\mathrm{crys}}$,
2. for a morphism $x = (i\colon A \to B, \gamma) \to (i'\colon A' \to B', \gamma') = y$ in $(\mathfrak{X}/W)_{\mathrm{crys}}$, the natural map $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})(x) \otimes_{(A, \mathrm{Fil}^\bullet_{\mathrm{PD}}(A))} (A', \mathrm{Fil}^\bullet_{\mathrm{PD}}(A')) \to (\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})(y)$ is an isomorphism.

[5] In loc. cit. one considers categories $\mathbf{MF}^{\nabla}_{[0,a]}(R, \Phi)$ for some Frobenius lift Φ on R. But, as a is in $[0, p-2]$, this category is independent of Φ by [Fal89, Theorem 2.3].

We will mostly be interested in the case when $(\mathcal{E}, \mathrm{Fil}^\bullet_{\mathcal{E}})$ is locally free over $(\mathcal{O}_{\mathrm{crys}}, \mathrm{Fil}^\bullet_{\mathrm{PD}})$ in the sense of [IKY25, Definition 1.1], and we denote the category of such by $\mathbf{VectF}(\mathfrak{X}_{\mathrm{crys}})$.

Now, define $\mathbf{VectF}^\nabla(\mathfrak{X})$ to consist of triples $(\mathcal{V}, \nabla_{\mathcal{V}}, \mathrm{Fil}^\bullet_{\mathcal{V}})$ where $(\mathcal{V}, \nabla_{\mathcal{V}})$ is an object $\mathbf{Vect}^\nabla(\mathfrak{X})$ and $\mathrm{Fil}^\bullet_{\mathcal{V}}$ is a locally split filtration on $\mathcal{V}$ which satisfying Griffiths transversality.

Proposition 2.6 (cf. [Tsu20, Theorem 29])**.** *The functor*

$$\mathbf{VectF}(\mathfrak{X}_{\mathrm{crys}}) \to \mathbf{VectF}^\nabla(\mathfrak{X}), \qquad \mathcal{E} \mapsto (\mathcal{E}|_{\mathfrak{X}_{\mathrm{Zar}}}, \nabla_{\mathcal{E}}, (\mathrm{Fil}^\bullet_{\mathcal{E}})|_{\mathfrak{X}_{\mathrm{Zar}}}),$$

is an equivalence.

Remark 2.7. By definition, $\mathcal{E}|_{\mathfrak{X}_{\mathrm{Zar}}}(\mathfrak{U}) = \mathcal{F}(\mathrm{id}\colon \mathfrak{U} \to \mathfrak{U}, \gamma)$. But while $\mathcal{E}(\mathrm{id}\colon \mathfrak{U} \to \mathfrak{U}, \gamma)$ agrees with $\mathcal{E}(\mathfrak{U}_0 \hookrightarrow \mathfrak{U}, \gamma)$, this is not true for $\mathrm{Fil}^\bullet_{\mathcal{E}}$ as it is a *filtered* crystal and the PD filtrations for $(\mathrm{id}\colon \mathfrak{U} \to \mathfrak{U}, \gamma)$ and $(\mathfrak{U}_0 \hookrightarrow \mathfrak{U}, \gamma)$.

Note that, by definition, we have

$$\mathbf{VectF}^\varphi(\mathfrak{X}_{\mathrm{crys}}) = \mathbf{Vect}^\varphi(\mathfrak{X}_{\mathrm{crys}}) \times_{\mathbf{Vect}^\nabla(\mathfrak{X})} \mathbf{VectF}^\nabla(\mathfrak{X}).$$

Thus, by Proposition 2.6, we obtain an equivalence of categories

$$\mathbf{VectF}^\varphi(\mathfrak{X}_{\mathrm{crys}}) \xrightarrow{\sim} \mathbf{Vect}^\varphi(\mathfrak{X}_{\mathrm{crys}}) \times_{\mathbf{Vect}(\mathfrak{X}_{\mathrm{crys}})} \mathbf{VectF}(\mathfrak{X}_{\mathrm{crys}}).$$

For this reason, for an object $(\mathcal{F}, \mathrm{Fil}^\bullet_{\mathcal{F}})$ of $\mathbf{Vect}^\varphi(\mathfrak{X}_{\mathrm{crys}})$ we may consider the evaluation of $\mathrm{Fil}^\bullet_{\mathcal{F}}$ at objects of $(\mathfrak{X}/W)_{\mathrm{crys}}$, by which we mean the evaluation of associated filtered crystal.

2.2. **The functor $\mathbb{D}_{\mathrm{crys}}$.** We now apply the crystalline-de Rham comparison theorem to define our integral analogue $\mathbb{D}_{\mathrm{crys}}$ of the functor D_{crys} and show that it really forms a lattice in D_{crys}. Throughout we assume that $\mathcal{O}_K = W$.

2.2.1. **The definition.** We now come to the definition of the naive filtered F-crystal $\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$.

Definition 2.8. Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^\varphi(\mathfrak{X}_{\Delta})$. Define $\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$ to have underlying F-crystal $\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$ and have the filtration induced by $\iota_{\mathfrak{X}}$ from Theorem 1.19:

$$\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}}) \supseteq \mathrm{Fil}^\bullet_{\mathbb{D}_{\mathrm{crys}}}(\mathcal{E}, \varphi_{\mathcal{E}}) := \iota_{\mathfrak{X}}^{-1}\Big(\mathrm{Fil}^\bullet_{\mathbb{D}_{\mathrm{dR}}}(\mathbb{D}_{\mathrm{dR}}(\mathcal{E}, \varphi_{\mathcal{E}}))\Big).$$

When $(\mathcal{E}, \varphi_{\mathcal{E}})$ is clear from context, we often omit it from the notation, writing $\mathrm{Fil}^\bullet_{\mathbb{D}_{\mathrm{crys}}}$ instead.

Proposition 2.9. *The functor*

$$\mathbb{D}_{\mathrm{crys}}\colon \mathbf{Vect}^\varphi(\mathfrak{X}_{\Delta}) \to \mathbf{VectNF}^\varphi(\mathfrak{X}_{\mathrm{crys}}),$$

is a $\mathbb{Z}_p$-linear $\otimes$-functor, which preserves duals, and maps $\mathbf{Vect}^\varphi_{[0,a]}(\mathfrak{X}_{\Delta})$ into $\mathbf{VectNF}^\varphi_{[0,a]}(\mathfrak{X}_{\mathrm{crys}})$.

2.2.2. **Comparison to D_{crys} and Griffiths transversality.** We now verify (when $\mathfrak{X} \to \mathrm{Spf}(W)$ is smooth) that $\mathbb{D}_{\mathrm{crys}}$ really does form a functorial filtered F-crystal lattice in D_{crys}.

Theorem 2.10. *If $\mathfrak{X} \to \mathrm{Spf}(W)$ is smooth, then $\mathbb{D}_{\mathrm{crys}}$ takes values in $\mathbf{VectWF}^\varphi(\mathfrak{X}_{\mathrm{crys}})$, and the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{Vect}^\varphi(\mathfrak{X}_{\Delta}) \arrow[r, "\mathbb{D}_{\mathrm{crys}}"] \arrow[d, "T_{\text{ét}}"'] & \mathbf{VectWF}^\varphi(\mathfrak{X}_{\mathrm{crys}}) \arrow[d, "(-)[1/p]"] \\
\mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(X) \arrow[r, "D_{\mathrm{crys}}"] & \mathbf{IsocF}^\varphi(\mathfrak{X}_{\mathrm{crys}}).
\end{tikzcd}$$

As D_{crys} takes values in $\mathbf{IsocF}^\varphi(\mathfrak{X})$ it suffices to prove the commutativity.

Before we prove Theorem 2.10 we set some notational shorthand. First, shorten the notation for $D_{\mathrm{crys}} \circ T_{\text{ét}}$ to just D_{crys}. Let R be a framed small W-algebra. Then, with notation as in [IKY24, §1.1.5], we further set

- $\mathrm{A}_{\mathrm{inf}} := \mathrm{A}_{\mathrm{inf}}(\widetilde{R})$,
- $\mathrm{A}_{\mathrm{crys}} := \mathrm{A}_{\mathrm{crys}}(\widetilde{R})$,
- $\widetilde{\mathrm{A}}_{\mathrm{crys}} := \mathrm{A}_{\mathrm{inf}}[\{\tilde{\xi}^n/n!\}]^\wedge_p$,
- $\mathrm{B}^+_{\mathrm{dR}} := \mathrm{B}^+_{\mathrm{dR}}(\widetilde{R})$,
- $\widetilde{\mathrm{B}}^+_{\mathrm{dR}} := \mathrm{A}_{\mathrm{inf}}[1/p]^\wedge_{(\tilde{\xi})}$.

These rings are arranged in the following diagram

$$\begin{tikzcd}
& \mathrm{A}_{\mathrm{crys}} \arrow[r, "\phi"] \arrow[dl, hook] \arrow[dr, "\widetilde{\psi}"'] \arrow[rrr, bend left, "\psi"] & \widetilde{\mathrm{A}}_{\mathrm{crys}} \arrow[r, hook] \arrow[d, hook] & \mathrm{A}_{\mathrm{crys}} \arrow[r, hook] & \mathrm{B}^+_{\mathrm{dR}} \\
\mathrm{B}^+_{\mathrm{dR}} \arrow[rr, "\phi_{\mathrm{dR}}"'] & & \widetilde{\mathrm{B}}^+_{\mathrm{dR}}.
\end{tikzcd}$$

Here we use the following notation:

- $\widetilde{\mathrm{A}}_{\mathrm{crys}} \hookrightarrow \mathrm{A}_{\mathrm{crys}}$ is the natural inclusion,
- $\phi_{\mathrm{dR}} \colon \mathrm{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \widetilde{\mathrm{B}}^+_{\mathrm{dR}}$ and $\phi \colon \mathrm{A}_{\mathrm{crys}} \xrightarrow{\sim} \widetilde{\mathrm{A}}_{\mathrm{crys}}$ are the maps induced by $\phi \colon \mathrm{A}_{\mathrm{inf}} \to \mathrm{A}_{\mathrm{inf}}$,
- and the maps ψ and $\widetilde{\psi}$ are defined uniquely to make the diagram commute.

Set $\widetilde{\mathrm{B}}_{\mathrm{dR}} := \mathrm{A}_{\mathrm{inf}}[1/p]^\wedge_{(\tilde{\xi})}[1/\tilde{\xi}]$. Then, a *$\varphi$-module* over $\mathrm{B}^+_{\mathrm{dR}}$ is a triple $(M, \widetilde{M}, \varphi_M)$ with M (resp. $\widetilde{M}$) a finitely generated projective $\mathrm{B}^+_{\mathrm{dR}}$-module (resp. $\widetilde{\mathrm{B}}^+_{\mathrm{dR}}$-module) and φ_M a $\widetilde{\mathrm{B}}_{\mathrm{dR}}$-linear isomorphism $(\phi^*_{\mathrm{dR}} M)[1/\tilde{\xi}] \to \widetilde{M}[1/\tilde{\xi}]$. Let us denote the category of φ-modules over $\mathrm{B}^+_{\mathrm{dR}}$ by $\mathbf{Vect}^\varphi(\mathrm{B}^+_{\mathrm{dR}})$.

We now observe that there is a natural functor

$$\mathcal{M} \colon \mathbf{IsocF}^\varphi(R) \to \mathbf{Vect}^\varphi(\mathrm{B}^+_{\mathrm{dR}}), \quad (\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_F) \mapsto (\mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}}, \psi} \mathrm{B}^+_{\mathrm{dR}}, \widetilde{\mathrm{Fil}}^0_F, \varphi_{\mathcal{F}} \otimes 1).$$

Here we define the filtration $\widetilde{\mathrm{Fil}}^\bullet_F$ on $\mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}}, \tilde{\psi}} \widetilde{\mathrm{B}}_{\mathrm{dR}}$ as follows:

$$\widetilde{\mathrm{Fil}}^r_F = \sum_{i+j=r} \mathrm{Fil}^i_F(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}}, \tilde{\psi}} \tilde{\xi}^j \widetilde{\mathrm{B}}^+_{\mathrm{dR}}. \tag{2.2.1}$$

Note that the Frobenius map $\varphi_{\mathcal{F}} \otimes 1$ is sensible as $\widetilde{\mathrm{Fil}}^0_F$ is a lattice in $\mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \check{R}) \otimes_{\mathrm{A}_{\mathrm{crys}}, \tilde{\psi}} \widetilde{\mathrm{B}}_{\mathrm{dR}}$. On the other hand, consider the functor

$$\mathrm{Nyg}_{\mathrm{dR}} \colon \mathbf{Vect}^\varphi(\mathrm{B}^+_{\mathrm{dR}}) \to \mathbf{MF}(\widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \mathrm{Fil}^\bullet_{\tilde{\xi}}), \quad (M, \widetilde{M}, \varphi_M) \mapsto (\phi^*_{\mathrm{dR}} M, \mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*_{\mathrm{dR}} M)),$$

where $\mathrm{Fil}^r_{\mathrm{Nyg}}(\phi^*_{\mathrm{dR}} M) := \phi^*_{\mathrm{dR}} M \cap \varphi_M^{-1}(\tilde{\xi}^r \widetilde{M})$. Then we have the following result.

Lemma 2.11. *The following diagram commutes.*

$$\begin{tikzcd}
\mathbf{Vect}^\varphi(R_{\Delta}) \arrow[r, "D_{\mathrm{crys}}"] \arrow[d, "\mathrm{ev}_{\mathrm{A}_{\mathrm{inf}}}"'] & \mathbf{IsocF}^\varphi(R) \arrow[r] \arrow[d, "\mathcal{M}"] & \mathbf{MF}(\mathrm{A}_{\mathrm{crys}}[1/p], \mathrm{Fil}^\bullet_{\xi}) \arrow[d] \\
\mathbf{Vect}^\varphi(\mathrm{A}_{\mathrm{inf}}) \arrow[r, "-\otimes_{\mathrm{A}_{\mathrm{inf}}} \mathrm{B}^+_{\mathrm{dR}}"] & \mathbf{Vect}^\varphi(\mathrm{B}^+_{\mathrm{dR}}) \arrow[r, "\mathrm{Nyg}_{\mathrm{dR}}"] & \mathbf{MF}(\widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \mathrm{Fil}^\bullet_{\tilde{\xi}}).
\end{tikzcd}$$

The top-right arrow is the evaluation of $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_F)$ at $\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}$ and the right vertical arrow is scalar extension along the map of filtered rings $\widetilde{\psi} \colon (\mathrm{A}_{\mathrm{crys}}[1/p], \mathrm{Fil}^\bullet_{\xi}) \to (\widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \mathrm{Fil}^\bullet_{\tilde{\xi}})$.

Proof of Lemma 2.11. The left square commutes by the proof of [GR24, Theorem 4.8] (cf. the proof of [IKY24, Lemma 2.26]). Let $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_F)$ be an object of $\mathbf{IsocF}^\varphi(R)$. As the category $\mathbf{MF}(\widetilde{B}^+_{\mathrm{dR}}, \mathrm{Fil}^\bullet_{\tilde{\xi}})$ satisfies descent (cf. [SW20, Corollary 17.1.9]), to show the commutativity of the right square, we are free to localize on $\mathrm{Spf}(R)$. But, note that as the graded pieces of the filtration on F are locally free, we may localize on $\mathrm{Spf}(R)$ to assume that they are free.[6] This

[6]Indeed, suppose that $\mathcal{E}$ is a vector bundle on $\mathrm{Spa}(R[1/p])$. Then, $\mathcal{E}(R[1/p])$ is a projective $R[1/p]$-module (see [Ked19, Theorem 1.4.2], and [Kie67]). Thus, there exists an open cover $\mathrm{Spec}(R[1/f_i p])$ of $\mathrm{Spec}(R[1/p])$ such that $\mathcal{E}(R[1/p]) \otimes_{R[1/p]} R[1/p f_i]$ is trivial, where f_i is a collection of elements of R. So, replacing $\mathrm{Spf}(R)$ by $\mathrm{Spf}(R[1/f_i]^\wedge_p)$, one may Zariski localize on $\mathrm{Spf}(R)$ to assume that $\mathcal{E}$ is free.

implies that the evaluation of $(\mathcal{F}, \mathrm{Fil}^\bullet_F)$ at any object $A \twoheadrightarrow R'$ of $(R/W)_{\mathrm{crys}}$ admitting a map to $\mathrm{id}_R \colon R \to R$ is free as a filtered module over $(A[1/p], \mathrm{Fil}^\bullet_{\mathrm{PD}}[1/p])$.

Fix a filtered basis $(e_\nu, r_\nu)_{\nu=1}^n$ of $\mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R})[1/p]$, so $\mathcal{M}(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^\bullet_F) = (M, \widetilde{M}, \varphi_M)$ where:

$$M = \mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}}, \psi} \mathrm{B}^+_{\mathrm{dR}}, \quad \widetilde{M} = \sum_{\nu=1}^n \tilde{\xi}^{-r_\nu} \widetilde{\mathrm{B}}^+_{\mathrm{dR}} \cdot e_\nu \subset \bigoplus_{\nu=1}^n \widetilde{\mathrm{B}}_{\mathrm{dR}} \cdot e_\nu,$$

and φ_M is the scalar extension of $\varphi_{\mathcal{F}}$. Thus, the Nygaard filtration on $\phi^*_{\mathrm{dR}} M$ is

$$\mathrm{Fil}^r_{\mathrm{Nyg}}(\phi^*_{\mathrm{dR}} M) = \phi^*_{\mathrm{dR}} M \cap \varphi_M^{-1} \left(\sum_{\nu=1}^n \tilde{\xi}^{r - r_\nu} \widetilde{\mathrm{B}}^+_{\mathrm{dR}} \cdot e_\nu \right).$$

On the other hand, if the object $(M', \mathrm{Fil}^\bullet_{M'})$ denotes the image under the other composition in the right-hand square, then it may be described as follows:

$$M' = \mathcal{F}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \widetilde{R}) \otimes_{\mathrm{A}_{\mathrm{crys}}, \widetilde{\psi}} \widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \qquad \mathrm{Fil}^r_{M'} = \sum_{\nu=1}^n \mathrm{Fil}^{r - r_\nu}_{\tilde{\xi}} \cdot e_\nu.$$

Thus, φ_M induces an isomorphism $(\phi^*_{\mathrm{dR}} M, \mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*_{\mathrm{dR}} M)) \xrightarrow{\sim} (M', \mathrm{Fil}^\bullet_{M'})$. $\square$

Proof of Theorem 2.10. Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^\varphi(\mathfrak{X}_{\Delta})$. We will show that there is a canonical isomorphism $\underline{\mathbb{D}}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})[1/p] \xrightarrow{\sim} \underline{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})$ in $\mathbf{Isoc}^\varphi(\mathfrak{X})$ respecting filtrations. Note that by the proof of [GR24, Theorem 4.8], the underlying F-isocrystals are identified, and so it suffices to show that the filtrations are matched. So, we may assume that $\mathfrak{X} = \mathrm{Spf}(R)$ where R is a framed small W-algebra. For an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathbf{Vect}^\varphi(R_{\Delta})$, let $\mathbb{D}_{\mathrm{crys},R}[1/p](\mathcal{E}, \varphi_{\mathcal{E}})$ (resp. $\underline{\mathbb{D}}_{\mathrm{crys},R}[1/p](\mathcal{E}, \varphi_{\mathcal{E}})$) denote the filtered $R[1/p]$-module $\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})(R)[1/p]$ (resp. its underlying $R[1/p]$-module). Define $D_{\mathrm{crys},R}(\mathcal{E}, \varphi_{\mathcal{E}})$ and $\underline{D}_{\mathrm{crys},R}(\mathcal{E}, \varphi_{\mathcal{E}})$ similarly.

Define the functor

$$\mathrm{Nyg}_{\mathfrak{S}_R} \colon \mathbf{Vect}^\varphi(\mathfrak{S}_R, (E)) \to \mathbf{MF}(\mathfrak{S}_R, \mathrm{Fil}^\bullet_E), \quad (\mathfrak{M}, \varphi_{\mathfrak{M}}) \mapsto (\phi^* \mathfrak{M}, \mathrm{Fil}^\bullet_{\mathrm{Nyg}}).$$

Then we have the following diagram.

$$\begin{tikzcd}
\mathbf{Vect}^\varphi(R_{\Delta}) \arrow[rr, "\mathbb{D}_{\mathrm{crys},R}[1/p]"] \arrow[d, "\mathrm{ev}_{\mathfrak{S}_R}"] & & \mathbf{MF}(R[1/p], \mathrm{Fil}^\bullet_{\mathrm{triv}}) \arrow[d, equal] \\
\mathbf{Vect}^\varphi(\mathfrak{S}_R) \arrow[r, "\mathrm{Nyg}_{\mathfrak{S}_R}"] \arrow[d, "(-)\otimes_{\mathfrak{S}_R} \mathrm{B}^+_{\mathrm{dR}}"] & \mathbf{MF}(\mathfrak{S}_R, \mathrm{Fil}^\bullet_E) \arrow[r, "(-)\otimes_{\mathfrak{S}_R} R[1/p]"] \arrow[d, "(-)\otimes_{\mathfrak{S}_R} \widetilde{\mathrm{B}}^+_{\mathrm{dR}}"] & \mathbf{MF}(R[1/p], \mathrm{Fil}^\bullet_{\mathrm{triv}}) \arrow[d, "(-)\otimes_R \widetilde{R}"] \\
\mathbf{Vect}^\varphi(\mathrm{B}^+_{\mathrm{dR}}) \arrow[r, "\mathrm{Nyg}_{\mathrm{dR}}"] & \mathbf{MF}(\widetilde{\mathrm{B}}^+_{\mathrm{dR}}, \mathrm{Fil}^\bullet_{\tilde{\xi}}) \arrow[r, "\bmod \tilde{\xi}"] & \mathbf{MF}(\widetilde{R}[1/p], \mathrm{Fil}^\bullet_{\mathrm{triv}}),
\end{tikzcd}$$

with the maps $\mathfrak{S}_R \to \mathrm{B}^+_{\mathrm{dR}}$ and $\mathfrak{S}_R \to \widetilde{B}^+_{\mathrm{dR}})$ given by the compositions $\mathfrak{S}_R \xrightarrow{\alpha_{\mathrm{crys}}} \mathrm{A}_{\mathrm{crys}} \xrightarrow{\psi} \mathrm{B}^+_{\mathrm{dR}}$ and $\mathfrak{S}_R \xrightarrow{\alpha_{\mathrm{crys}}} \mathrm{A}_{\mathrm{crys}} \xrightarrow{\widetilde{\psi}} \widetilde{\mathrm{B}}^+_{\mathrm{dR}}$, respectively.

The upper rectangle and the right lower square commute by definition. Noting that $\mathfrak{S}_R \to \mathrm{B}^+_{\mathrm{dR}}$ is flat as $\mathfrak{S}_R$ is Noetherian and it is E-adically flat (indeed, the image of E, which is ξ is a nonzerodivisor and the map is flat mod E by [IKY24, Lemma 1.15]), we get that the left lower square commutes as well. Thus, we have a canonical identification

$$\mathbb{D}_{\mathrm{crys},R}[1/p](\mathcal{E}, \varphi_{\mathcal{E}}) \otimes_R \widetilde{R} \cong \mathrm{Nyg}_{\mathrm{dR}} \left(\mathcal{E}(\mathfrak{S}_R, (E)) \otimes_{\mathfrak{S}_R} \mathrm{B}^+_{\mathrm{dR}} \right) \otimes_{\widetilde{\mathrm{B}}^+_{\mathrm{dR}}} \widetilde{R}[1/p]$$

On the other hand, by Lemma 2.11, we also have a canonical identification

$$D_{\mathrm{crys},R}(\mathcal{E}, \varphi_{\mathcal{E}}) \otimes_R \widetilde{R} \cong \mathrm{Nyg}_{\mathrm{dR}} \left(\mathcal{E}(\mathfrak{S}_R, (E)) \otimes_{\mathfrak{S}_R} \mathrm{B}^+_{\mathrm{dR}} \right) \otimes_{\widetilde{\mathrm{B}}^+_{\mathrm{dR}}} \widetilde{R}[1/p].$$

These identifications induce an isomorphism $\mathbb{D}_{\mathrm{crys},R}[1/p](\mathcal{E})\otimes_R \widetilde{R} \xrightarrow{\sim} \underline{D}_{\mathrm{crys},R}(\mathcal{E})\otimes_R \widetilde{R}$ which agrees with that from [GR24, Theorem 4.8]. Thus, $\mathbb{D}_{\mathrm{crys},R}[1/p](\mathcal{E}) \xrightarrow{\sim} \underline{D}_{\mathrm{crys},R}(\mathcal{E})$ preserves filtrations after base change along the faithfully flat map $R[1/p] \to \widetilde{R}[1/p]$, and so preserves filtrations. $\square$

Example 2.12. Similar to Example 1.5, when $\mathfrak{X} = \mathrm{Spf}(W)$, we define

$$\mathbb{D}_{\mathrm{crys}} \colon \mathbf{Rep}^{\mathrm{crys}}(\mathrm{Gal}(\overline{K}_0/K_0)) \to \mathbf{VectWF}^{\varphi,\mathrm{div}}(W), \qquad \Lambda \mapsto \mathbb{D}_{\mathrm{crys}}(T_{\text{ét}}^{-1}(\Lambda)).$$

By Theorem 2.10, we have an identification of filtered F-isocrystals $\mathbb{D}_{\mathrm{crys}}(\Lambda)[1/p] \xrightarrow{\sim} D_{\mathrm{crys}}(\Lambda)$. This agrees with the composition

$$\underline{\mathbb{D}}_{\mathrm{crys}}(\Lambda)[1/p] \xrightarrow{\sim} (\phi^*(\mathfrak{M}(\Lambda))/u)[1/p] \xrightarrow{\sim} (\mathfrak{M}(\Lambda)/(u))[1/p] \xrightarrow{\sim} D_{\mathrm{crys}}(\Lambda),$$

where the first isomorphism is that in Example 1.5, the second is from the Frobenius structure, and the last is from the definition of $\mathfrak{M}$ (see [Kis10, Theorem (1.2.1) (1)]).

2.3. **Relationship to locally filtered free prismatic F-crystals.** Recall (see [IKY25, Definition 1.24]), that a prismatic F-crystal $(\mathcal{E},\varphi_{\mathcal{E}})$ on $\mathfrak{X}_{\Delta}$ is called *locally filtered free (lff)* if $(\phi^*\mathcal{E}, \mathrm{Fil}^\bullet_{\mathrm{Nyg}})$ is locally filtered free over $(\mathcal{O}_{\Delta}, \mathrm{Fil}^\bullet_{\mathcal{J}_{\Delta}})$ in the sense of [IKY25, Definition 1.1]. In this section we discuss the interplay between the functor $\mathbb{D}_{\mathrm{crys}}$ and the notion of being lff.

2.3.1. **Lffness in terms of $\mathbb{D}_{\mathrm{crys}}$.** We begin by observing that $\mathbb{D}_{\mathrm{crys}}$ not only enjoys nicer properties on lff prismatic F-crystals but, in fact, can detect lffness.

Proposition 2.13. *Let $\mathfrak{X}$ be a base formal W-scheme, and $(\mathcal{E},\varphi_{\mathcal{E}})$ a prismatic F-crystal on $\mathfrak{X}$. Let $\{\mathrm{Spf}(R_i)\}$ an open cover with each R_i a formally framed base W-algebra, and for each i let $\mathfrak{M}_i := \mathcal{E}(\mathfrak{S}_{R_i},(E))$. Then, the following conditions are equivalent:*

- *(1) the prismatic F-crystal $(\mathcal{E},\varphi_{\mathcal{E}})$ is locally filtered free,*
- *(2) the filtration $\mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*\mathfrak{M}_i)$ is locally free over $(\mathfrak{S}_{R_i}, \mathrm{Fil}^\bullet_E)$,*
- *(3) the filtration $\mathrm{Fil}^\bullet_{\mathbb{D}_{\mathrm{crys}}} \subseteq \mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is locally free over $(\mathcal{O}_{\mathfrak{X}}, \mathrm{Fil}^\bullet_{\mathrm{triv}})$,*
- *(4) and $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is an object of $\mathbf{VectNF}^{\varphi,\mathrm{div},\mathrm{gtf}}(\mathfrak{X}_{\mathrm{crys}})$.*

If $\mathfrak{X} \to \mathrm{Spf}(W)$ is smooth, then these conditions are further equivalent to

- *(5) $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is an object of $\mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}})$,*

We begin by establishing the following filtered-freeness lifting lemma.

Lemma 2.14. *Let $(A,(d))$ be a bounded prism, and (M,φ_M) an object of $\mathbf{Vect}^{\varphi}(A,(d))$ with M free over A. Set $\mathrm{Fil}^\bullet := \mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*M)$ and $\overline{\mathrm{Fil}}^\bullet := \mathrm{Fil}^\bullet_{\mathrm{Nyg}}(\phi^*(M)/(d))$. Let $(f_\nu, r_\nu)_{\nu=1}^n$ be a filtered basis of $(\phi^*(M)/(d), \overline{\mathrm{Fil}}^\bullet)$ over $(A/(d), \mathrm{Fil}^\bullet_{\mathrm{triv}})$. Choose e_ν in Fil^{r_ν} such that $\overline{e_\nu} = f_\nu$ (which exist as f_ν is in $\overline{\mathrm{Fil}}^{r_\nu}$). Then, the following is true:*

- *(1) $(e_\nu, r_\nu)_{\nu=1}^n$ is a filtered basis of $(\phi^*M, \mathrm{Fil}^\bullet)$ over $(A, \mathrm{Fil}^\bullet_d)$,*
- *(2) $\varphi_M(e_\nu)$ is in $d^{r_\nu}M$, and $(\frac{\varphi_M(e_\nu)}{d^{r_\nu}})_\nu$ is a basis of M.*

Proof. To see that (e_ν) is a basis of ϕ^*M we observe that the map $A^n \to \phi^*M$ sending $(a_1,\ldots,a_n)$ to $\sum_{\nu=1}^n a_\nu e_\nu$ is surjective modulo d by assumption, and so surjective by Nakayama's lemma as A is d-adically complete. It is then an isomorphism as the source and target are finite projective A-modules of the same rank.

To prove the first assertion we consider the natural map $(A, \mathrm{Fil}^\bullet_d) \to (A/(d), \mathrm{Fil}^\bullet_{\mathrm{triv}})$ of filtered rings and the corresponding map of Rees algebras $\mathrm{Rees}(\mathrm{Fil}^\bullet_d) \to \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathrm{triv}}) \simeq A/(d)[t]$. We observe that, by Lemma 1.18, the filtration $\overline{\mathrm{Fil}}^\bullet$ on $\phi^*M/(d)$ defines via the Rees construction the $\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathrm{triv}})$-module $\mathrm{Rees}(\mathrm{Fil}^\bullet)\otimes_{\mathrm{Rees}(\mathrm{Fil}^\bullet_d)} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathrm{triv}})$, more precisely, that the natural map

$$\mathrm{Rees}(\mathrm{Fil}^\bullet)\otimes_{\mathrm{Rees}(\mathrm{Fil}^\bullet_d)} \mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathrm{triv}}) \to \mathrm{Rees}(\overline{\mathrm{Fil}}^\bullet) \tag{2.3.1}$$

is an isomorphism. Thus, we can apply [IKY25, Remark 1.9] to obtain the first assertion.

We now prove the second assertion. We take an integer r large enough so that $r \geqslant r_\nu$ for any ν and that $\varphi_M^{-1}(d^rM) \subset \phi^*M$. In particular, φ_M induces $\mathrm{Fil}^r = \sum_\nu d^{r-r_\nu}A\cdot e_\nu \xrightarrow{\sim} d^rM$, which implies that $d^{r-r_\nu}\varphi(e_\nu)$ forms a basis of d^rM. Dividing by d^r we obtain assertion (2). $\square$

We next show that for 'filtered free' prismatic F-crystals $(\mathcal{E},\varphi_{\mathcal{E}})$, the strong divisibility condition on $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is automatic. To make this precise, fix a base W-algebra R. Define the category $\mathbf{Vect}^{\varphi}_{\mathrm{free}}(R_{\Delta})$ to be the full subcategory $\mathbf{Vect}^{\varphi}(R_{\Delta})$ consisting of those $(\mathcal{E},\varphi_{\mathcal{E}})$ with $(\mathbb{D}_{\mathrm{dR}}(\mathcal{E},\varphi_{\mathcal{E}}),\mathrm{Fil}^{\bullet}_{\mathbb{D}_{\mathrm{dR}}}(\mathcal{E},\varphi_{\mathcal{E}}))$ (equiv. $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$) filtered free over $(R,\mathrm{Fil}^{\bullet}_{\mathrm{triv}})$ in the sense of [IKY25, Definition 1.1].

Lemma 2.15. *Let R be a base W-algebra and $(\mathcal{E},\varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^{\varphi}_{\mathrm{free}}(R_{\Delta})$. Then, $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is strongly divisible.*

Proof. Write $M=\mathcal{E}(\mathfrak{S}_R,(E))$. By construction of $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$, we see that $\overline{\mathrm{Fil}}^{\bullet}_{\mathrm{Nyg}}(\phi^*M/(E))$ is free. Choose a filtered basis $(\overline{e_\nu},r_\nu)$ of $\phi^*M/(E)$ over $(R,\mathrm{Fil}^{\bullet}_{\mathrm{triv}})$. Then, we have the filtered basis of $(\iota^{-1}(\overline{e_\nu}),r_\nu)$ of $\phi_R^*\mathcal{E}(R,(p))$, where ι is the isomorphism as in (1.3.5). Unraveling the definitions, we must show that

$$\sum_{r\in\mathbb{Z}}\sum_{\nu=1}^{n}p^{-r}\,\mathrm{Fil}^{r-r_\nu}_{\mathrm{triv}}\,\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))=\phi_R^*\mathcal{E}(R,(p)).$$

But, observe that by Lemma 2.14 we have that $\varphi_{\mathcal{E}}(e_\nu)\in E^{r_\nu}M$, and that $\frac{\varphi_{\mathcal{E}}(e_\nu)}{E^{r_\nu}}$ is a basis of M. Pulling back along $\phi_{\mathfrak{S}_R}$ we see that $\phi_{\mathfrak{S}_R}^*(e_\nu)$ is a basis of $(\phi_{\mathfrak{S}_R}^2)^*M$, that $\varphi_{\mathcal{E}}(\phi_{\mathfrak{S}_R}^*(e_\nu))$ belongs to $\phi_{\mathfrak{S}_R}(E)^{r_\nu}\phi_{\mathfrak{S}_R}^*M$, and that $\frac{\varphi_{\mathcal{E}}(\phi_{\mathfrak{S}_R}^*(e_\nu))}{\phi_{\mathfrak{S}_R}(E)^{r_\nu}}$ is a basis of $\phi_{\mathfrak{S}_R}^*M$.

Now, recall that from the crystal property we have a diagram of isomorphisms

$$\begin{tikzcd}
(\phi_R^2)^*\mathcal{E}(R,(p))\otimes_R S_R \arrow[r, "\sim"] \arrow[d, dotted, "\varphi_{\mathcal{E}}"] & \phi_{S_R}^*\mathcal{E}(S_R,(p)) \arrow[d, dotted, "\varphi_{\mathcal{E}}"] & (\phi_{\mathfrak{S}_R}^2)^*M\otimes_{\mathfrak{S}_R}S_R \arrow[l, "\sim"'] \arrow[d, dotted, "\varphi_{\mathcal{E}}"] \\
\phi_R^*\mathcal{E}(R,(p))\otimes_R S_R \arrow[r, "\sim"'] & \mathcal{E}(S_R,(p)) & \phi_{\mathfrak{S}_R}^*M\otimes_{\mathfrak{S}_R}S_R. \arrow[l, "\sim"]
\end{tikzcd}$$

The dotted arrow indicates that these arrows only exist after inverting p. Moreover, the induced isomorphisms between the outer objects of the first (resp. second) row is precisely the isomorphism $\phi_R^*(\iota)$ (resp. ι). As $(\phi_{\mathfrak{S}_R}(E))=(p)$ in S_R, we deduce from this diagram and the contents of the previous paragraph that $\phi_R^*(\iota^{-1}(\overline{e_\nu}))$ is a basis of $(\phi_R^2)^*\mathcal{E}(R,(p))$, that $\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))$ belongs to $p^{r_\nu}\phi_R^*\mathcal{E}(R,(p))$, and that $\frac{\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))}{p^{r_\nu}}$ is a basis of $\phi^*\mathcal{E}(R,(p))$. Thus, we see that

$$\begin{aligned}
\sum_{r\in\mathbb{Z}}\sum_{\nu=1}^{n}p^{-r}\,\mathrm{Fil}^{r-r_\nu}_{\mathrm{triv}}\,\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu}))) &= \sum_{r\in\mathbb{Z}}\sum_{\nu=1}^{n}p^{r_\nu-r}\,\mathrm{Fil}^{r-r_\nu}_{\mathrm{triv}}\,\tfrac{\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))}{p^{r_\nu}}\\
&= \sum_{\nu=1}^{n}\sum_{r\in\mathbb{Z}}p^{r_\nu-r}\,\mathrm{Fil}^{r-r_\nu}_{\mathrm{triv}}\,\tfrac{\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))}{p^{r_\nu}}\\
&= \sum_{\nu=1}^{n}R\cdot\tfrac{\varphi_{\mathcal{E}}(\phi_R^*(\iota^{-1}(\overline{e_\nu})))}{p^{r_\nu}}\\
&= \phi_R^*\mathcal{E}(R,(p)),
\end{aligned}$$

as desired. □

We are now ready to prove Proposition 2.13.

Proof of Proposition 2.13. The equivalence of (1) and (2) follows from [IKY24, Proposition 1.16]. The equivalence of (2) and (3) follows from Lemma 2.14. To show that (3) implies (4), it suffices to consider a cover $\{\mathrm{Spf}(R_i)\}$ of $\mathfrak{X}$ where each R_i is a base W-algebra, and the restriction of $(\mathcal{E},\varphi_{\mathcal{E}})$ to $(R_i)_{\Delta}$ belongs to $\mathbf{Vect}^{\varphi}((R_i)_{\Delta})$. The desired implication then follows from Lemma 2.15. That (4) implies (3) is given by Proposition 2.3. Finally, when $\mathfrak{X}\to\mathrm{Spf}(W)$ is smooth, the equivalence of (4) and (5) follows by combining Proposition 2.3 and Theorem 2.10. □

2.3.2. **Exactness in the lff case.** We now make the observation that exactness of $\mathbb{D}_{\mathrm{crys}}$ holds when restricted to the category $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$ of lff prismatic F-crystals.

Proposition 2.16. *The functor*

$$\mathbb{D}_{\mathrm{crys}}\colon \mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta}) \to \mathbf{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}),$$

is exact.

Proof. For this we may reduce to the case when $\mathfrak{X} = \mathrm{Spf}(R)$, where R is a formally framed base W-algebra. Consider an exact sequence

$$0 \to (\mathcal{E}_1, \varphi_{\mathcal{E}_1}) \to (\mathcal{E}_2, \varphi_{\mathcal{E}_2}) \to (\mathcal{E}_3, \varphi_{\mathcal{E}_3}) \to 0$$

in $\mathbf{Vect}^{\varphi,\mathrm{lff}}(R_{\Delta})$. Set $\mathfrak{M}_i := \mathcal{E}_i(\mathfrak{S}_R, (E))$ and write $\mathrm{Fil}_i^{\bullet} \subseteq \phi^*\mathfrak{M}_i/u$ for $\iota_i^{-1}(\overline{\mathrm{Fil}}^{\bullet}_{\mathrm{Nyg}}(\phi^*\mathfrak{M}_i/E))$, with ι_i as in Construction 1.20. Then, as evaluation at $(\mathfrak{S}_R, (E))$ is exact (see [IKY24, Lemma 1.18]), and an exact sequence of vector bundles is universally exact, we see from Propositions 1.15 and 1.21 that it suffices to show that the sequence of filtered modules over $(R, \mathrm{Fil}^{\bullet}_{\mathrm{triv}})$

$$0 \to (\phi^*\mathfrak{M}_1/u, \mathrm{Fil}^{\bullet}_1) \to (\phi^*\mathfrak{M}_2/u, \mathrm{Fil}^{\bullet}_2) \to (\phi^*\mathfrak{M}_3/u, \mathrm{Fil}^{\bullet}_2) \to 0,$$

is short exact. As this sequence is short exact on the underlying module and each $(\phi^*\mathfrak{M}_i/u, \mathrm{Fil}^{\bullet}_i)$ is finitely supported, it suffices by [IKY25, Lemma 1.3] to show that the morphisms $(\phi^*\mathfrak{M}_1/u, \mathrm{Fil}^{\bullet}_1) \to (\phi^*\mathfrak{M}_2/u, \mathrm{Fil}^{\bullet}_2)$ and $(\phi^*\mathfrak{M}_2/u, \mathrm{Fil}^{\bullet}_2) \to (\phi^*\mathfrak{M}_3/u, \mathrm{Fil}^{\bullet}_3)$ are strict. But, as each $(\mathcal{E}_i, \varphi_{\mathcal{E}_i})$ is in $\mathbf{Vect}^{\varphi,\mathrm{lff}}(R_{\Delta})$, the Faltings morphism (2.1.2) is an isomorphism, and thus, the claim follows from [Fal89, Theorem 2.1 (3)] (cf. [LMP24, Theorem 2.9 (3)]). ☐

2.4. **Bi-exactness of forgetful functor.** In this final short subsection we observe that one can use Proposition 2.16 to prove that the forgetful functor $\mathrm{R}_{\mathfrak{X}}$ is bi-exact.

Proposition 2.17. *Let $\mathfrak{X}$ be a base formal $\mathcal{O}_K$-scheme. Then, the $\mathbb{Z}_p$-linear $\otimes$-equivalence*

$$\mathrm{R}_{\mathfrak{X}}\colon \mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \xrightarrow{\sim} \mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$$

is bi-exact.

Proof. The exactness of $\mathrm{R}_{\mathfrak{X}}$ is clear. Conversely, using [IKY25, Corollary 1.18], [IKY25, Proposition 1.5], and the method of proof in [IKY25, Proposition 1.28] (which shows that the Rees module over $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\Delta_R))$ is a descent of $\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi^*\mathcal{E}))$ along the map on Rees algebras induced by ϕ) we are thus reduced to showing that if

$$0 \to (\mathcal{E}_1, \varphi_{\mathcal{E}_1}) \to (\mathcal{E}_2, \varphi_{\mathcal{E}_2}) \to (\mathcal{E}_3, \varphi_{\mathcal{E}_3}) \to 0,$$

is an exact sequence in $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta})$ then for all r in $\mathbb{Z}$ we have that

$$0 \to \mathrm{Fil}^r(\phi^*\mathcal{E}_1) \to \mathrm{Fil}^r(\phi^*\mathcal{E}_2) \to \mathrm{Fil}^r(\phi^*\mathcal{E}_3) \to 0,$$

is exact. As this is a local condition, [IKY24, Proposition 1.16] reduces us to showing exactness after evaluation on the Breuil–Kisin prism associated to each open subset $\mathrm{Spf}(R) \subseteq \mathfrak{X}$, where R is a (formally framed) base W-algebra. But, this follows from Proposition 2.16 and its proof. ☐

3. Relationship to Fontaine–Laffaille theory

We now relate $\mathbb{D}_{\mathrm{crys}}$ to relative Fontaine–Laffaille theory as first developed in [Fal89], when $\mathfrak{X}$ is smooth over W. Throughout we use notation and terminology from Notation and terminology without comment but now assume further that p is odd.

3.1. **Statement of main result and reduction steps.** We are interested in understanding the following diagram of $\mathbb{Z}_p$-linear functors when $\mathfrak{X}$ is a smooth formal W-scheme:

$$\begin{tikzcd}
\mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \arrow[d, hook] \arrow[ddrr, "\mathbb{D}_{\mathrm{crys}}"] & & \mathbf{Vect}_{[0,p-2]}(\mathfrak{X}^{\mathrm{syn}}) \arrow[ll, "\mathrm{R}_{\mathfrak{X}}"'] \arrow[dd, "\mathbb{D}_{\mathrm{crys}}"] \\
\mathbf{Vect}^{\varphi}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \arrow[d, hook] & & \\
\mathbf{Vect}^{\varphi,\mathrm{an}}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \arrow[r, "T_{\text{ét}}"] & \mathbf{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p,[0,p-2]}(X) & \mathbf{Vect}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}}) \arrow[l, "T_{\mathrm{crys}}"']
\end{tikzcd} \tag{3.1.1}$$

Each of these functors, except the diagonal $\mathbb{D}_{\mathrm{crys}}$ and T_{crys}, has a self-evident definition or been defined in §2. We explain the definition of these remaining two functors below.

First, by Proposition 2.17 we have a bi-exact $\mathbb{Z}_p$-linear $\otimes$-equivalence

$$\mathrm{R}_{\mathfrak{X}}\colon \mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \xrightarrow{\sim} \mathbf{Vect}^{\varphi,\mathrm{lff}}(\mathfrak{X}_{\Delta}),$$

Composing this with the functor $\mathbb{D}_{\mathrm{crys}}$ and applying Proposition 2.13 gives us an exact $\mathbb{Z}_p$-linear $\otimes$-functor

$$\mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}}) \to \mathbf{VectF}^{\varphi,\mathrm{div}}(\mathfrak{X}_{\mathrm{crys}}), \tag{3.1.2}$$

which we abusively also denote $\mathbb{D}_{\mathrm{crys}}$. This gives rise to the diagonal arrow in (3.1.1), where we write $\mathbf{Vect}_{[a,b]}(\mathfrak{X}^{\mathrm{syn}})$ to be the full subcategory of $\mathbf{Vect}(\mathfrak{X}^{\mathrm{syn}})$ consisting of those $\mathcal{V}$ such that $\mathrm{R}_{\mathfrak{X}}(\mathcal{V})$ is an object of $\mathbf{Vect}^{\varphi}_{[a,b]}(\mathfrak{X}_{\Delta})$.[7]

Second, the functor

$$T^*_{\mathrm{crys}}\colon \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}}) \to \mathbf{Loc}_{\mathbb{Z}_p}(X),$$

constructed by Faltings in [Fal89, II.e), pp. 35–37] (see also [Tsu20, §4]), is described as follows. For an object $\mathcal{F} = (\mathcal{F}, \varphi, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ of $\mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}})$, the reductions $\mathcal{F}_m$ of $\mathcal{F}$ modulo p^m define a projective system of objects of the category $\mathfrak{MF}^{\nabla}(\mathfrak{X})$ defined in [Fal89, 2.c)–d), pp. 30–33]. With the notation of loc. cit., $T^*_{\mathrm{crys}}(\mathcal{F})$ is then given as the inverse limit $\varprojlim_m \mathbf{D}(\mathcal{F}_m)$. We then set $T_{\mathrm{crys}}(-) = T^*_{\mathrm{crys}}(-)^{\vee}$, where $(-)^{\vee}$ is the dual in $\mathbf{Loc}_{\mathbb{Z}_p}(X)$, which gives the last undefined arrow in (3.1.1).

Our main goal in this section is to prove the following.

Theorem 3.1. *Suppose that $\mathfrak{X}$ is a smooth W-scheme. Then,* (3.1.1) *is a 2-commutative diagram where every arrow is an equivalence.*

Many parts of Theorem 3.1 are either explicitly or implicitly contained in §2 or other references. Thus, we single out below the two remaining results to be demonstrated.

Proposition 3.2. *Let $\mathfrak{X}$ be a smooth formal W-scheme and set $T^*_{\text{ét}}(-) := T_{\text{ét}}(-)^{\vee}$. Then, the following diagram commutes:*

$$\begin{tikzcd}
& \mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \arrow[dl, "T^*_{\text{ét}}"'] \arrow[dr, "\mathbb{D}_{\mathrm{crys}}"] & \\
\mathbf{Loc}_{\mathbb{Z}_p}(\mathfrak{X}_{\eta}) & & \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(\mathfrak{X}_{\mathrm{crys}}). \arrow[ll, "T^*_{\mathrm{crys}}"]
\end{tikzcd}$$

Proposition 3.3. *Suppose that $\mathfrak{X}$ is a base formal W-scheme, and that $(\mathcal{E}, \varphi_{\mathcal{E}})$ is an effective prismatic F-crystal on $\mathfrak{X}$ of height at most $p-2$. Then, $(\mathcal{E}, \varphi_{\mathcal{E}})$ is locally filtered free.*

We claim that these two propositions are sufficient to prove Theorem 3.1.

[7] See [GM24, §8.4.1] for a stack-theoretic definition of these *Hodge–Tate weights*.

Proposition 3.4. *Propositions 3.2 and 3.3 imply Theorem 3.1*

Proof. Note $R_{\mathfrak{X}}$ is an equivalence by [IKY25, Proposition 1.28], the inclusion $\mathbf{Vect}^{\varphi}_{[0,p-2]}(\mathfrak{X}_{\Delta}) \hookrightarrow \mathbf{Vect}^{\varphi,\mathrm{an}}_{[0,p-2]}(\mathfrak{X}_{\Delta})$ is an equivalence by [DLMS24, Remark 3.38], $T_{\text{ét}}$ is an equivalence by [GR24, Theorem A], and T_{crys} is fully faithful by [Fal89, Theorem 2.6]. As we also know this diagram commutes by Proposition 3.2, it is simple to check we are reduced to Proposition 3.3. □

We spend the next few sections preparing for, and then proving, Propositions 3.2 and 3.3.

3.2. **Several functors of Tsuji.** A key to proving Proposition 3.2 are certain results in [Tsu20], which we now describe. Fix a small framed W-algebra R. Let us write $\Gamma_R = \pi_1^{\text{ét}}(R[1/p])$ (with respect to some base point), so that $\mathbf{Rep}^{\text{cont.}}_{\mathbb{Z}_p}(\Gamma_R) = \mathbf{Loc}_{\mathbb{Z}_p}(\mathrm{Spf}(R[1/p])_\eta)$ (cf. [IKY24, §2.1.4]).

To describe Tsuji's results, we first must describe certain subcategories of $\mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,p-2]}(R_{\Delta})$ and $\mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(R_{\mathrm{crys}})$. First, recall (see Proposition 2.5) that there is an equivalence of categories

$$\mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,p-2]}(R_{\mathrm{crys}}) \xrightarrow{\sim} \mathbf{MF}^{\nabla}_{[0,p-2]}(R), \qquad (\mathcal{F},\varphi_{\mathcal{F}},\mathrm{Fil}^\bullet_{\mathcal{F}}) \mapsto (\mathcal{F}_{\mathfrak{X}}(R),\varphi_{\mathcal{F}_{\mathfrak{X}}(R)},\nabla_{\mathcal{F}},\mathrm{Fil}^\bullet_{\mathcal{F}}).$$

Following [Tsu20, §4], we consider the full subcategory $\mathbf{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(R)$ consisting of those $(M,\varphi_M,\nabla_M,\mathrm{Fil}^\bullet_M)$ such that $\mathrm{Gr}^r(\mathrm{Fil}^\bullet_M)$ is free over R for every $r\in\mathbb{Z}$. Then, unraveling the definition of $\mathbb{D}_{\mathrm{crys}}$ we obtain a functor

$$\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta}) \to \mathbf{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(R),$$

which we denote again by $\mathbb{D}_{\mathrm{crys}}$.

Now, for notational simplicity, we again use abbreviations:

$$\mathrm{A}_{\mathrm{inf}} := \mathrm{A}_{\mathrm{inf}}(\check{R}), \quad \mathrm{A}_{\mathrm{crys}} := \mathrm{A}_{\mathrm{crys}}(\check{R}),$$

with notation as in [IKY24, §1.1.5]. We then further consider the categories

$$\mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R), \quad \mathbf{MF}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R), \quad \mathbf{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{crys}},\varphi,\Gamma_R),$$

from [Tsu20, Definition 51 and §8].[8] By [Tsu20, Equation (49) and Proposition 59], we have

$$\mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \xleftarrow{\sim} \mathbf{MF}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \xrightarrow{\sim} \mathbf{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{crys}},\varphi,\Gamma_R). \quad (3.2.1)$$

The first functor is that forgetting the filtration, and the second is defined by

$$(M,\mathrm{Fil}^\bullet_M,\varphi_M) \mapsto \Big((M,\mathrm{Fil}^\bullet_M)\otimes_{(\mathrm{A}_{\mathrm{inf}},\mathrm{Fil}^\bullet_{\tilde{\xi}})}(\mathrm{A}_{\mathrm{crys}},\mathrm{Fil}^\bullet_{\mathrm{PD}}),\varphi_M\otimes 1\Big)$$

with the semi-linearly extended action of Γ_R.

Construction 3.5 ([Tsu20, Equations (23) and (36)])**.** Define functors

$$T\mathrm{A}_{\mathrm{crys}}\colon \mathbf{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(R) \to \mathbf{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{crys}},\varphi,\Gamma_R),$$

$$T\mathrm{A}_{\mathrm{inf}}\colon \mathbf{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(R) \to \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R),$$

as follows. Let $(M,\varphi_M,\nabla_M,\mathrm{Fil}^\bullet_M)$ be an object of $\mathbf{MF}^{\nabla}_{[0,p-2]\mathrm{free}}(R)$, and $(\mathcal{F},\varphi_{\mathcal{F}},\mathrm{Fil}^\bullet_{\mathcal{F}})$ denote the corresponding object of $\mathbf{VectF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$. Then, $T\mathrm{A}_{\mathrm{crys}}(M,\varphi_M,\nabla_M,\mathrm{Fil}^\bullet_M)$ is defined as the evaluation $(\mathcal{F},\varphi_{\mathcal{F}},\mathrm{Fil}^\bullet_{\mathcal{F}})(\mathrm{A}_{\mathrm{crys}}\twoheadrightarrow\check{R})$, and $T\mathrm{A}_{\mathrm{inf}}$ is the result of translating $T\mathrm{A}_{\mathrm{crys}}$ through (3.2.1).

We make analogous constructions for prismatic F-crystals as follows.

Construction 3.6. We have a natural functor

$$\mathbb{D}\mathrm{A}_{\mathrm{inf}}\colon \mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta}) \to \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R), \quad (\mathcal{E},\varphi_{\mathcal{E}}) \mapsto (\mathcal{E}(\mathrm{A}_{\mathrm{inf}},(\tilde{\xi})),\varphi_{\mathcal{E}}),$$

where we equip $\mathcal{E}(\mathrm{A}_{\mathrm{inf}},(\tilde{\xi}))$ with the Γ_R-action induced by the Γ_R-action on $(\mathrm{A}_{\mathrm{inf}},(\tilde{\xi}))$. That $(\mathcal{E}(\mathrm{A}_{\mathrm{inf}},(\tilde{\xi})),\varphi_{\mathcal{E}})$ is an object of $M^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R)$ follows from Lemma 2.14.

[8] In [Tsu20, §2 and §8], the element $\tilde{\xi}$ of $\mathrm{A}_{\mathrm{inf}}$ is denoted by q.

Let $\mathbb{D}^{\mathrm{F}}\mathrm{A}_{\mathrm{inf}}$ and $\mathbb{D}\mathrm{A}_{\mathrm{crys}}$ denote the compositions

$$\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta}) \xrightarrow{\mathbb{D}\mathrm{A}_{\mathrm{inf}}} \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \xrightarrow{\sim} \mathbf{MF}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R),$$

$$\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta}) \xrightarrow{\mathbb{D}\mathrm{A}_{\mathrm{inf}}} \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \xrightarrow{\sim} \mathbf{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{crys}},\varphi,\Gamma_R),$$

respectively.

Note that for an object $(\mathcal{E},\varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta})$, the underlying φ-module of $\mathbb{D}\mathrm{A}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is given by $\mathcal{E}(\mathrm{A}_{\mathrm{crys}},(p))$. The action of Γ_R is induced by that on the object $(\mathrm{A}_{\mathrm{crys}},(p),\widetilde{\mathrm{nat.}})$ of R_{Δ}. The filtration on these objects is described as follows.

Lemma 3.7. *Let $(e_\nu,r_\nu)_{\nu=1}^n$ be a filtered basis of $(\phi^*\mathcal{E}(\mathfrak{S}_R,(E)),\mathrm{Fil}^\bullet_{\mathrm{Nyg}})$ over the filtered ring $(\mathfrak{S}_R,\mathrm{Fil}^\bullet_E)$. Then we have:*

(1) $(\alpha^*_{\mathrm{inf}}(e_\nu),r_\nu)_{\nu=1}^n$ is a flitered basis of $\mathbb{D}^{\mathrm{F}}\mathrm{A}_{\mathrm{inf}}(\mathcal{E},\varphi_{\mathcal{E}})$ over the filtered ring $(\mathrm{A}_{\mathrm{inf}},\mathrm{Fil}^\bullet_{\xi})$,
(2) $(\alpha^*_{\mathrm{crys}}(e_\nu),r_\nu)_{\nu=1}^n$ is a flitered basis of $\mathbb{D}\mathrm{A}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ over the filtered ring $(\mathrm{A}_{\mathrm{crys}},\mathrm{Fil}^\bullet_{\mathrm{PD}})$.

Proof. Claim (2) follows from the (1), and (1) follows from the description of the equivalence

$$\mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \xrightarrow{\sim} \mathbf{MF}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R)$$

given in the proof of [Tsu20, Lemma 46] combined with Lemma 2.14. □

3.3. **Proof of Proposition 3.2.** To begin, we consider the functor

$$T^*_{\mathrm{inf}}\colon \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \to \mathbf{Rep}^{\mathrm{cont.}}_{\mathbb{Z}_p}(\Gamma_R), \quad (M,\varphi_M)\mapsto \mathrm{Hom}((M,\varphi_M),(\mathrm{A}_{\mathrm{inf}},\phi)),$$

where Γ_R acts on $\mathrm{Hom}((M,\varphi_M),(\mathrm{A}_{\mathrm{inf}},\phi))$ via its action on (M,φ_M). By [Tsu20, Theorem 63 (2)], the composition $T^*_{\mathrm{inf}}\circ T\mathrm{A}_{\mathrm{inf}}$ is identified with T^*_{crys}. Thus, the proof of Proposition 3.2 is reduced to showing that the following diagram of categories commutes:

$$\begin{tikzcd}
& \mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta}) \arrow[dl, "T^*_{\text{ét}}"'] \arrow[d, "\mathbb{D}A_{\mathrm{inf}}"] \arrow[dr, "\mathbb{D}_{\mathrm{crys}}"] & \\
\mathbf{Rep}_{\mathbb{Z}_p}(\Gamma_R) & \mathbf{M}^{\tilde{\xi},\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{inf}},\varphi,\Gamma_R) \arrow[l, "T^*_{\mathrm{inf}}"] \arrow[d, "\wr"] & \mathbf{MF}^{\nabla}_{[0,p-2],\mathrm{free}}(R) \arrow[l, "T\mathrm{A}_{\mathrm{inf}}"] \arrow[dl, "T\mathrm{A}_{\mathrm{crys}}"] \\
& \mathbf{MF}^{p,\mathrm{cont}}_{[0,p-2],\mathrm{free}}(\mathrm{A}_{\mathrm{crys}},\varphi,\Gamma_R). &
\end{tikzcd} \tag{3.3.1}$$

Moreover, the lower-right triangle of (3.3.1) commutes by the definition of $T\mathrm{A}_{\mathrm{inf}}$.

Lemma 3.8. *The upper-right triangle of (3.3.1) commutes.*

Proof. It suffices to show that the right large triangle in (3.3.1) commutes. Moreover, note that by definition the composition of the two vertical arrows is precisely $\mathbb{D}\mathrm{A}_{\mathrm{crys}}$.

So, let $(\mathcal{E},\varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta})$. We first check that the underlying (φ,Γ_R)-modules of $(T\mathrm{A}_{\mathrm{crys}}\circ\mathbb{D}_{\mathrm{crys}})(\mathcal{E},\varphi_{\mathcal{E}})$ and $\mathbb{D}\mathrm{A}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ are canonically identified. The latter is by definition $\mathcal{E}(\mathrm{A}_{\mathrm{crys}},(p))$ with Γ_R-action induced by that on $(\mathrm{A}_{\mathrm{crys}},(p),\widetilde{\mathrm{nat.}})$ via its normal action on $\mathrm{A}_{\mathrm{crys}}$. To describe $(T\mathrm{A}_{\mathrm{crys}}\circ\mathbb{D}_{\mathrm{crys}})(\mathcal{E},\varphi_{\mathcal{E}})$, recall that the underlying F-crystal of $\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})$ is $(\mathcal{E}^{\mathrm{crys}},\varphi_{\mathcal{E}^{\mathrm{crys}}})$. So, by definition, the underlying (φ,Γ_R)-module of $(T\mathrm{A}_{\mathrm{crys}}\circ\mathbb{D}_{\mathrm{crys}})(\mathcal{E},\varphi_{\mathcal{E}})$ is $\mathcal{E}^{\mathrm{crys}}(\mathrm{A}_{\mathrm{crys}}\twoheadrightarrow\check{R}/p)$ with Γ_R-action induced by that on $\mathrm{A}_{\mathrm{crys}}\twoheadrightarrow\check{R}/p$ via its normal action on $\mathrm{A}_{\mathrm{crys}}$. But, these two coincide by the definition of $(-)^{\mathrm{crys}}$.

We now compare the filtrations. We put $\mathcal{M}:=\mathcal{E}(S_R,(p))$ which is naturally isomorphic to $\mathcal{E}^{\mathrm{crys}}(S_R\twoheadrightarrow R)$ (cf. Proposition 1.1). Let $\mathrm{Fil}^\bullet_1$ denote the filtration on $\mathcal{M}$ obtained from the Nygaard filtration on $\phi^*\mathcal{E}(\mathfrak{S},(E))$ by filtered scalar extension along $(\mathfrak{S}_R,\mathrm{Fil}^\bullet_E)\to(S_R,\mathrm{Fil}^\bullet_{\mathrm{PD}})$, where $\mathfrak{S}_R\to S_R$ is the natural inclusion. On the other hand, consider the filtration $\mathrm{Fil}^\bullet_2$ on $\mathcal{M}$ obtained from the filtration on $\mathbb{D}_{\mathrm{crys}}(\mathcal{E})(R)$ via the filtered map $(R,\mathrm{Fil}^\bullet_{\mathrm{triv}})\to(S_R,\mathrm{Fil}^\bullet_{\mathrm{PD}})$, where

$R \to S_R$ is the natural inclusion. Note that the filtration on $\mathbb{D}A_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}}) = \mathcal{E}(A_{\mathrm{crys}}, (p))$ (resp. $(T\mathrm{A}_{\mathrm{crys}} \circ \mathbb{D}_{\mathrm{crys}})(\mathcal{E}, \varphi_{\mathcal{E}}) = \mathcal{E}(\mathrm{A}_{\mathrm{crys}} \twoheadrightarrow \check{R}))$ is obtained from $\mathrm{Fil}_1^{\bullet}$ (resp. $\mathrm{Fil}_2^{\bullet}$) by scalar extension along the faithfully flat map $S_R \to \mathrm{A}_{\mathrm{crys}}$. Thus, it suffices to show the equality $\mathrm{Fil}_1^{\bullet} = \mathrm{Fil}_2^{\bullet}$.

Here we note that we have the equality $\mathrm{Fil}^r_{\mathrm{PD}} = \mathrm{Fil}^r_{\mathrm{PD}}[1/p] \cap S_R$, which follows from the fact that the graded quotients of $\mathrm{Fil}^{\bullet}_{\mathrm{PD}}$ is p-torsion free (see the sentence before [Tsu20, Lemma 8]). Then, using [Tsu20, Lemma 42.(1)], the problem is reduced to showing $\mathrm{Fil}_1^{\bullet}[1/p] = \mathrm{Fil}_2^{\bullet}[1/p]$.

We consider the filtration $\mathrm{F}^{\bullet}\mathcal{M}[1/p]$ (resp. $\mathrm{Fil}^{\bullet}\,\mathcal{D}$) from the paragraph before [DLMS24, Lemma 4.31] (for the p-adic representation $V = T_{\text{ét}}(\mathcal{E})[1/p]$). We claim that the $\mathrm{Fil}_1^{\bullet}[1/p]$ (resp. $\mathrm{Fil}_2^{\bullet}[1/p]$) coincides with the filtration $\mathrm{F}^{\bullet}\mathcal{M}[1/p]$ (resp. $\mathrm{Fil}^{\bullet}\,\mathcal{D}$).

To see $\mathrm{Fil}_1^{\bullet}[1/p] = \mathrm{F}^{\bullet}\mathcal{M}[1/p]$, we take a filtered basis $(e_\nu, r_\nu)_\nu$ of $(\phi^*\mathcal{E})(\mathfrak{S}, (E))$ over $(\mathfrak{S}, \mathrm{Fil}_E^{\bullet})$. We denote again by e_ν the image of e_ν under the map

$$\phi^*\mathcal{E}(\mathfrak{S}_R, (E)) \to \mathcal{E}(S_R, (p)) = \mathcal{M}$$

induced by the natural inclusion $\mathfrak{S}_R \hookrightarrow S_R$. Let $x = \sum_{\nu=1}^n a_\nu e_\nu$, with a_ν in $S_R[1/p]$, be an arbitrary element of $\mathcal{M}[1/p]$. Then x belongs to $\mathrm{F}^r\mathcal{M}[1/p]$ if and only if the element

$$\sum_\nu a_\nu \varphi_{\mathcal{M}}(e_\nu) = \sum_\nu a_\nu E^{r_\nu}\left(\frac{\varphi_{\mathcal{M}}(e_\nu)}{E^{r_\nu}}\right)$$

belongs to $E^r\mathcal{M}[1/p]$. By the second part of Lemma 2.14, this is equivalent to the claim that a_ν belongs to $E^{r-r_\nu}S_R[1/p] = \mathrm{Fil}^{r-r_\nu}_{\mathrm{PD}}\,S_R[1/p]$ for all ν. Thus, we have $\mathrm{Fil}_1^{\bullet}[1/p] = \mathrm{F}^{\bullet}\mathcal{M}[1/p]$.

We show $\mathrm{Fil}_2^{\bullet}[1/p] = \mathrm{Fil}^{\bullet}\,\mathcal{D}$ by induction on r. We take a filtered basis $(e'_\nu, r'_\nu)_\nu$ of $D_{\mathrm{crys}}(V)(R)$ over $(R, \mathrm{Fil}^{\bullet}_{\mathrm{triv}})$. Through the identification $D_{\mathrm{crys}}(V) \otimes_{R[1/p]} S_R[1/p] \xrightarrow{\sim} \mathcal{M}[1/p]$ induced by Theorem 2.10 and the crystal property, the collection $(e'_\nu \otimes 1, r'_\nu)$ gives a filtered basis of $(\mathcal{M}[1/p], \mathrm{Fil}_2^{\bullet}[1/p])$ over $(S_R[1/p], \mathrm{Fil}^{\bullet}_{\mathrm{PD}})$. Here we are implicitly using that the filtrations on $S_R[1/p]$ induced by $\mathrm{Fil}^{\bullet}_{\mathrm{PD}}$ and $\mathrm{Fil}^{\bullet}_E$ agree, and we use both notations below depending on which is convenient.

Now we assume the equality $\mathrm{Fil}_2^{r-1}[1/p] = \mathrm{Fil}^{r-1}\,\mathcal{D}$ and take an arbitrary element $x = \sum_\nu a_\nu e'_\nu$ (with a_ν in $S_R[1/p]$ for all ν) from $\mathrm{Fil}_2^{r-1}[1/p]$, so that we have $a_\nu \in \mathrm{Fil}^{r-1-r_\nu}_{\mathrm{PD}}$ for all ν. We use the notation as in [DLMS24, §4]. Noting that we have $N_u(x) = \sum_\nu N_u(a_\nu)e'_\nu$, we see that x is in $\mathrm{Fil}^r\,\mathcal{D}$ if and only if the following two conditions hold:

- $N_u(a_\nu)$ belongs to $\mathrm{Fil}_E^{r-1-r_\nu}$ for ν with $r-1 > r_\nu$,
- $a_\nu \in \mathrm{Fil}^1_E$ for ν with $r-1 = r_\nu$.

For ν with $r-1 > r_\nu$, writing the element a_ν of $\mathrm{Fil}_E^{r-1-r_\nu}$ as $a_\nu = \sum_{i \geqslant r-1-r_\nu} b_i E^{[i]}$, with b_i in $R[1/p]$ and where $E^{[i]}$ denotes $E^i/i!$, we obtain $N_u(a_\nu) = -u\sum_{i\geqslant r-1-r_\nu} b_i E^{[i-1]}$. Hence, the second condition is equivalent to the claim that $b_{r-1-r_\nu} = 0$, which happens if and only if a_ν belongs to $\mathrm{Fil}_E^{r-r_\nu}$. Thus, $\mathrm{Fil}_2^{\bullet}[1/p] = \mathrm{Fil}^{\bullet}\,\mathcal{D}$. Finally, the equality $\mathrm{Fil}_1^{\bullet}[1/p] = \mathrm{Fil}_2^{\bullet}[1/p]$ follows from [DLMS24, Lemma 4.31] (cf. [Bre97, Proposition 6.2.2.1]). □

Remark 3.9. The method of proof in Lemma 3.8 shows that if R is a base W-algebra, and $(\mathcal{E}, \varphi_{\mathcal{E}})$ an object of $\mathbf{Vect}^{\varphi,\mathrm{lff}}(R_{\Delta})$, then there is an identification of filtered Frobenius modules

$$\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})(\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \to \widetilde{R}) \xrightarrow{\sim} (\phi^*\mathcal{E}(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi)), \mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}) \otimes_{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\xi})} (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\mathrm{PD}}),$$

$$\mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})(S_R, (p)) \xrightarrow{\sim} (\phi^*\mathcal{E}(\mathfrak{S}_R, (E)), \mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}) \otimes_{(\mathfrak{S}_R, \mathrm{Fil}^{\bullet}_E)} (S_R, \mathrm{Fil}^{\bullet}_{\mathrm{PD}}).$$

Lemma 3.10. *The upper left triangle of the diagram* (3.3.1) *commutes.*

Proof. Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta})$. By definition, we have

$$(T^*_{\mathrm{inf}} \circ \mathbb{D}\mathrm{A}_{\mathrm{inf}})(\mathcal{E}, \varphi_{\mathcal{E}}) \cong \mathrm{Hom}((\mathcal{E}(\mathrm{A}_{\mathrm{inf}}, (\tilde{\xi})), \varphi_{\mathcal{E}}), (\mathrm{A}_{\mathrm{inf}}, \phi)).$$

On the other hand, by [IKY24, Example 2.14], we have

$$T^*_{\text{ét}}(\mathcal{E}, \varphi_{\mathcal{E}}) \cong \mathrm{Hom}((\mathcal{E}(\mathrm{A}_{\mathrm{inf}}, (\tilde{\xi})), \varphi_{\mathcal{E}}), (\mathrm{A}_{\mathrm{inf}}[1/\tilde{\xi}]^{\wedge}_p, \phi)).$$

The obvious map $(T^*_{\mathrm{inf}} \circ T\mathrm{A}_{\mathrm{inf}})(\mathcal{E}, \varphi_{\mathcal{E}}) \to T^*_{\text{ét}}(\mathcal{E}, \varphi_{\mathcal{E}})$ is an isomorphism. Indeed, it is an injective map between free $\mathbb{Z}_p$-modules of the same rank by [Tsu20, Proposition 66] and Lemma 3.8. So, the cokernel is killed by a power of p, but also embedds into $\mathrm{Hom}(\mathcal{E}(\mathrm{A}_{\mathrm{inf}}, (\tilde{\xi})), \mathrm{A}_{\mathrm{inf}}[1/\tilde{\xi}]^{\wedge}_p/\mathrm{A}_{\mathrm{inf}})$, which is p-torsion free as $\mathrm{A}_{\mathrm{inf}}[1/\tilde{\xi}]^{\wedge}_p/\mathrm{A}_{\mathrm{inf}}$ is, and so the cokernel is zero as desired. $\square$

With these observations, the proof of Proposition 3.2 is now an exercise in parts assembly.

Proof of Proposition 3.2. Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an object of $\mathbf{Vect}^{\varphi,\mathrm{lff}}_{[0,p-2]}(\mathfrak{X}_{\Delta})$. By taking an open covering $(\mathfrak{U}_i)_i$ by small affine opens of $\mathfrak{X}$ such that $\mathcal{E}_{\mathfrak{U}_{i,\Delta}}$ is in $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(\mathfrak{U}_{i,\Delta})$, the proof is reduced to constructing, for a small affine formal scheme smooth $\mathfrak{X} = \mathrm{Spf}(R)$ over W and an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathbf{Vect}^{\varphi}_{[0,p-2],\mathrm{free}}(R_{\Delta})$, an isomorphism

$$(T^*_{\mathrm{crys}} \circ \mathbb{D}_{\mathrm{crys}})(\mathcal{E}, \varphi_{\mathcal{E}}) \xrightarrow{\sim} T^*_{\mathrm{\acute{e}t}}(\mathcal{E}, \varphi_{\mathcal{E}})$$

functorial in $(\mathcal{E}, \varphi_{\mathcal{E}})$ and compatible with open immersions $\mathrm{Spf}(R') \to \mathrm{Spf}(R)$. Such an isomorphism is obtained using Lemmas 3.8 and 3.10, together with [Tsu20, Theorem 63.(2)], which is seen to satisfy the desired compatibility. $\square$

3.4. **Proof of Proposition 3.3.** We now proceed to the proof of Proposition 3.3. Our approach is inspired by techniques developed in [Hok24, §3.3], but is significantly simpler (e.g., doesn't require reduction to k_g as in loc. cit.).

Proof of Proposition 3.3. By Proposition 2.13 we may assume that $\mathfrak{X} = \mathrm{Spf}(R)$ for a formally framed W-algebra R. Moreover, by the same result it suffices to show that $\mathrm{Gr}^i(\mathrm{Fil}^{\bullet}_{\mathbb{D}_{\mathrm{crys}}})$ is a locally free R-module for all i. As $\mathbb{D}_{\mathrm{crys}}$ is compatible with flat base change, and one may check local freeness at $\widehat{R}_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{p}$ of R/p, we may further reduce to the power-series case, i.e., when $R = W[\![t_1, \ldots, t_d]\!]$ for some d.

Set $(\mathfrak{M}, \varphi_{\mathfrak{M}}) = \mathcal{E}(\mathfrak{S}_R, (E))$ and let $\overline{\mathrm{Fil}}^{\bullet} \subseteq \phi^*_{\mathfrak{S}_R}\mathfrak{M}/E$ be the image of the Nygaard filtration $\mathrm{Fil}^{\bullet} \subseteq \phi^*_{\mathfrak{S}_R}\mathfrak{M}$. We must show that $\mathrm{Gr}^i(\overline{\mathrm{Fil}}^{\bullet})$ is a free R-module for all i. The argument in [Hok24, Lemma 52] shows that the map $f_i \colon \phi^*_{\mathfrak{S}_R}\mathfrak{M} \to \mathfrak{M}/E^i$ induced by the relative Frobenius map $\varphi_{\mathfrak{M}}$ has cokernel which is a finite free R-module. Using this, we claim that if $\mathfrak{M}_0$ is the Breuil–Kisin module over W given by $\mathfrak{M} \otimes_{\mathfrak{S}_R} \mathfrak{S}_W$, with filtration $\mathrm{Fil}^{\bullet}_0 \subseteq \phi^*_{\mathfrak{S}_W}\mathfrak{M}_0$, then the natural map $\mathrm{Fil}^{\bullet} \otimes_{\mathfrak{S}_R} \mathfrak{S}_W \to \mathrm{Fil}^{\bullet}_0$ is an isomorphism. Indeed, for each i we have the short exact sequence

$$0 \to \phi^*_{\mathfrak{S}_R}\mathfrak{M}/\mathrm{Fil}^i \xrightarrow{f_i} \mathfrak{M}/E^i \to \mathrm{coker}(f_i) \to 0.$$

As $\mathrm{coker}(f_i)$ is a free R-module, we may tensor this along $R \to W$ to obtain the exact sequence

$$0 \to (\phi^*_{\mathfrak{S}_R}\mathfrak{M}/\mathrm{Fil}^i) \otimes_R W \to (\mathfrak{M}/E^i) \otimes_R W \to \mathrm{coker}(f_i) \otimes_R W \to 0.$$

The injectivity in the above exact sequence implies the claimed isomorphism $\mathrm{Fil}^{\bullet} \otimes_{\mathfrak{S}_R} \mathfrak{S}_W \cong \mathrm{Fil}^{\bullet}_0$. By [Gao19, Lemma 3.8], $\mathrm{Gr}^i(\overline{\mathrm{Fil}}^{\bullet}_0)$ is finite free over $W(k)$. Let r_i be the rank of $\mathrm{Gr}^i(\overline{\mathrm{Fil}}^{\bullet}_0)$ over $W(k)$. By Nakayama's lemma, we can lift $W(k)^{r_i} \cong \mathrm{Gr}^i(\overline{\mathrm{Fil}}^{\bullet}_0)$ to a surjection $R^{r_i} \twoheadrightarrow \mathrm{Gr}^i(\overline{\mathrm{Fil}}^{\bullet})$. Lifting those surjections to $R^{r_i} \twoheadrightarrow \overline{\mathrm{Fil}}^i$, we obtain a surjection $\bigoplus_{j \geq i} R^{r_j} \twoheadrightarrow \overline{\mathrm{Fil}}^i$. In particular, we have $\bigoplus_{j\geq 0} R^{r_j} \twoheadrightarrow \phi^*_{\mathfrak{S}_R}\mathfrak{M}/E$. Since the source and target of the last surjection have the same rank, it is an isomorphism. This implies the injectivity of $\bigoplus_{j\geq i} R^{r_j} \twoheadrightarrow \overline{\mathrm{Fil}}^i$. Hence this is an isomorphism, which implies $R^{r_i} \cong \mathrm{Gr}^i(\overline{\mathrm{Fil}}^{\bullet})$. $\square$

3.5. **Cohomological applications.** Finally, we record some cohomological applications of the discussion above. To this end, fix $\mathscr{X} \to \mathrm{Spec}(W)$ to be smooth and proper.

For $(\mathcal{E}, \varphi_{\mathcal{E}})$ in $\mathbf{Vect}^{\varphi,\mathrm{lff}}(\widehat{\mathscr{X}_{\Delta}})$ consider the object $T_{\mathrm{\acute{e}t}}(\mathcal{E}, \varphi_{\mathcal{E}})^{\mathrm{alg}}$ of $\mathbf{Loc}_{\mathbb{Z}_p}(\mathscr{X}_K)$ (see [IKY24, §2.1.3]). Then, for $i \geqslant 0$, we have a diagram

$$\begin{array}{ccc} D_{\mathrm{crys}}\left(H^i_{\mathrm{\acute{e}t}}(\mathscr{X}_{\overline{K}}, T_{\mathrm{\acute{e}t}}(\mathcal{E}, \varphi_{\mathcal{E}})^{\mathrm{alg}}[1/p])\right) & \xrightarrow{c_{(\mathcal{E},\varphi_{\mathcal{E}})}} & H^i\left((\mathscr{X}_k/W)_{\mathrm{crys}}, \mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})[1/p]\right) \\ \cup\vert & & \cup\vert \\ \mathbb{D}_{\mathrm{crys}}\left(H^i_{\mathrm{\acute{e}t}}(\mathscr{X}_{\overline{K}}, T_{\mathrm{\acute{e}t}}(\mathcal{E}, \varphi_{\mathcal{E}})^{\mathrm{alg}})\right) & & H^i\left((\mathscr{X}_k/W)_{\mathrm{crys}}, \mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})\right) \end{array} \tag{3.5.1}$$

where $c_{(\mathcal{E},\varphi_\mathcal{E})}$ is the isomorphism obtained by combining [TT19, Theorem 5.5], Theorem 2.10, and [Hub96, Theorem 3.7.2], and we are using notation as in Example 2.12.

From Proposition 3.2, Proposition 3.3, and [Fal89, Proposition 5.4] we deduce the following.

Corollary 3.11. *Let $\mathscr{X} \to \mathrm{Spec}(W)$ be smooth and proper. Then, for an object $(\mathcal{E},\varphi_\mathcal{E})$ of $\mathbf{Vect}^{\varphi}_{[0,a]}(\widehat{\mathscr{X}_{\Delta}})$ and b in $\mathbb{N}$ with $a+b<p-2$, the map $c_{(\mathcal{E},\varphi_\mathcal{E})}$ restricts to an isomorphism*

$$c_{(\mathcal{E},\varphi_\mathcal{E})}\colon \mathbb{D}_{\mathrm{crys}}\left(H^b(\mathscr{X}_{\overline{K}}, T_{\text{ét}}(\mathcal{E},\varphi_\mathcal{E})^{\mathrm{alg}})\right) \xrightarrow{\sim} H^b_{\text{ét}}\left((\mathscr{X}_k/W)_{\mathrm{crys}}, \mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_\mathcal{E})\right).$$

We can upgrade this to the following statement about prismatic F-gauges (which is a consequence of Corollary 3.11 when $n=\infty$ by Proposition 3.2).

Proposition 3.12. *Let $f\colon \mathscr{X} \to \mathrm{Spec}(W)$ be smooth and proper. Then, for an object $\mathcal{V}$ of $\mathbf{Vect}_{[0,a]}(\widehat{\mathscr{X}}^{\mathrm{syn}})$ and b in $\mathbb{N}$ with $a+b<p-2$, there is a canonical isomorphism*

$$H^b_{\text{ét}}\left(\mathscr{X}_{\overline{K}}, T_{\text{ét}}(\mathcal{V})^{\mathrm{alg}}/p^n\right) \xrightarrow{\sim} T_{\text{ét}}\left(R^b f_*\mathcal{V}/p^n\right),$$

of Galois representations, for any n in $\mathbb{N}\cup\{\infty\}$.

Proof. When $n=\infty$, let us observe that the right-hand side is

$$T_{\text{ét}}(R^b f_*\mathcal{V}) = T_{\text{ét}}(\mathrm{R}_{\mathrm{Spf}(W)}(R^b f_*\mathcal{V})) = T_{\text{ét}}(R^b f_*(\mathrm{R}_{\mathrm{Spf}(W)}(\mathcal{V}))),$$

where the first equality is by definition, and the second equality follows as $\mathrm{R}_{\mathfrak{X}}$ is obtained by pullback along an open embedding. The desired result for $n=\infty$ then follows from [GL23, Corollary 4.9] (see also [GR24, Theorem 6.1]).

To establish the claim for arbitrary n, we proceed as follows. For simplicity of notation, let T_n^b denote the left-hand side and T_n denote $R\Gamma_{\text{ét}}(\mathscr{X}_{\overline{K}}, T_{\text{ét}}(\mathcal{V})^{\mathrm{alg}}/p^n)$, omitting n when $n=\infty$. As $T_{\text{ét}}(\mathcal{V})$ is torsion free, we have $T_n^b = H^b(T/^{\mathbb{L}}p^n)$. On the other hand, by the exactness of $T_{\text{ét}}$, the right-hand side is identified with $T^b/p^n = H^b(T)/p^n$. By the Tor-spectral sequence, we have

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}_p}(H^{b+1}(T), \mathbb{Z}/p^n) \to H^b(T/^L p^n) \to H^b(T)/p^n \to 0.$$

But, $H^{b+1}(T)$ is p-torsionfree: combine [Fal89, Theorem 5.3] and [BMS18, Theorem 1.1, (ii)]. $\square$

4. Relationship to Dieudonné theory

In this final section we discuss how $\mathbb{D}_{\mathrm{crys}}$ can be used to unite various Dieudonné theories that appear in the literature. Throughout we use notation and terminology from Notation and terminology without comment but now assume further that p is odd. Furthermore, for a p-adically complete ring S we denote by $\mathbf{BT}_p(S)$ the category p-divisible groups over S.[9]

4.1. The prismatic Dieudonné functor.

Let S be a quasi-syntomic ring. Fix an object H of $\mathbf{BT}_p(S)$. We may then consider the sheaf[10]

$$H_{\overline{\mathcal{O}}_{\Delta}}\colon S_{\Delta} \to \mathbf{Grp}, \qquad (A,I) \mapsto H(A/I) = H(\overline{\mathcal{O}}_{\Delta}(A,I)).$$

We then have the following construction of Anschütz–Le Bras.

Definition 4.1 ([ALB23])**.** The *prismatic Dieudonné crystal* associated to H is

$$\mathcal{M}_{\Delta}(H) := \mathcal{E}xt^1_{\mathbf{Ab}(S_{\Delta})}(H_{\overline{\mathcal{O}}_{\Delta}}, \mathcal{O}_{\Delta}).$$

which has the structure of an $\mathcal{O}_{\Delta}$-module on S_{Δ} and a Frobenius morphism

$$\varphi_{\mathcal{M}_{\Delta}(H)}\colon \phi^*\mathcal{M}_{\Delta}(H) \to \mathcal{M}_{\Delta}(H),$$

inherited from those structures on $\mathcal{O}_{\Delta}$.

[9]See [dJ95, Lemma 2.4.4] for why this notation is not ambiguous.

[10]To see that this is a sheaf, we combine the following observations: $\overline{\mathcal{O}}_{\Delta}$ is a sheaf and H is finitely continuous (because $H = \varinjlim H[p^n]$, with each $H[p^n]$ representable, and finite limits commute with filtered colimits in **Set**).

The prismatic Dieudonné crystal is a complete invariant of H.

Theorem 4.2 ([ALB23]). *The functor $\mathcal{M}_{\Delta}$ defines a contravariant fully faithful embedding*

$$\mathcal{M}_{\Delta}\colon \mathbf{BT}_p(S) \to \mathbf{Vect}^{\varphi}_{[0,1]}(S_{\Delta}),$$

which is an anti-equivalence if S admits a quasi-syntomic cover $S \to \widetilde{S}$ with $\widetilde{S}$ perfectoid.[11]

Proof. By [ALB23, Theorem 4.74] (and [IKY24, Proposition 1.31]), the functor $\mathcal{M}_{\Delta}$ forms an equivalence between $\mathbf{BT}_p(S)$ and the full subcategory of $\mathbf{Vect}^{\varphi}_{[0,1]}(S_{\Delta})$ consisting of so-called admissible objects (see [ALB23, Definition 4.5]). Thus, to prove the second part of the claim, it suffices to show that the existence of such a cover $S \to \widetilde{S}$ implies any object of $\mathbf{Vect}^{\varphi}(S_{\Delta})$ is admissible. As admissibility is clearly a local condition on S_{qsyn} (see [ALB23, Proposition 4.9]), it suffices to prove the claim over $\widetilde{S}$. But, this is the content of [ALB23, Proposition 4.12]. □

Next, we record the compatibility of $\mathcal{M}_{\Delta}$ with the functors M^{SW} and M^{Lau} defined by [SW20, Theorem 17.5.2] and [Lau18, Theorem 9.8], respectively. More precisely, suppose that R is perfectoid. Then, there is a natural identification between $M^{\mathrm{SW}}(H)$ and $M^{\mathrm{Lau}}(H)$, by the unicity part of [SW20, Theorem 17.5.2], and we have the following result of Anschütz–Le Bras.

Proposition 4.3 (cf. [ALB23, Proposition 4.48]). *Let S be a perfectoid ring and H an object of $\mathbf{BT}_p(S)$. Then, we have canonical identifications:*

$$M^{\mathrm{SW}}(H)^* = M^{\mathrm{Lau}}(H) = \mathcal{M}_{\Delta}(H)(\mathrm{A}_{\mathrm{inf}}(S), (\tilde{\xi})).$$

4.2. **Crystalline Dieudonné theory.** We now recall the classical filtered crystalline Dieudonné theory of Grothendieck–Messing. Fix $\mathfrak{X} = \mathrm{Spf}(R)$ a formally framed base W-scheme.

Definition 4.4 (cf. [Kim15, Definition 3.1]). A *filtered Dieudonné crystal* on R is an object $(\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}})$ of $\mathbf{VectNF}^{\varphi}_{[0,1]}(\mathfrak{X}_{\mathrm{crys}})$ where:

- $(\mathcal{F}, \varphi_{\mathcal{F}})$ is an effective F-crystal,
- $\mathrm{Fil}^1_{\mathcal{F}}$ is a direct factor of $\mathcal{F}_{\mathfrak{X}}$,
- $\phi^*(\mathrm{Fil}^1_{\mathcal{F}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}_k}$ equal to the kernel of $\varphi_{\mathcal{F}}\colon \phi^*\mathcal{F}_{\mathfrak{X}_k} \to \mathcal{F}_{\mathfrak{X}_k}$,
- and there exists $V\colon \mathcal{F}_{\mathfrak{X}_k} \to \phi^*\mathcal{F}_{\mathfrak{X}_k}$ with $\varphi_{\mathcal{F}} \circ V = [p]_{\mathcal{F}_{\mathfrak{X}_k}}$ and $V \circ \varphi_{\mathcal{F}} = [p]_{\phi^*\mathcal{F}_{\mathfrak{X}_k}}$,

the category of which we denote $\mathbf{DieuF}(R)$.

Remark 4.5. One may check that $\mathbf{DieuF}(R) = \mathbf{VectF}^{\varphi,\mathrm{div}}_{[0,1]}(R_{\mathrm{crys}})$, and we only prefer the former notation/terminology when discussing p-divisible groups for historical reasons. In particular, by this equality and the discussion in §2.1, it makes sense to evaluate the filtration of filtered Dieudonné crystal on any object of $(R/W)_{\mathrm{crys}}$.

Let us now fix an object H of $\mathbf{BT}_p(R)$. We then consider the sheaf

$$H_{\overline{\mathcal{O}}_{\mathrm{crys}}}\colon (R/W)_{\mathrm{crys}} \to \mathbf{Grp}, \quad (i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma) \mapsto H(\mathfrak{U}) = H(\overline{\mathcal{O}}_{\mathrm{crys}}(i\colon \mathfrak{U} \hookrightarrow \mathfrak{T}, \gamma)),$$

(which is a sheaf as in Footnote 10). We may then form the sheaf

$$\underline{\mathbb{D}}(H) := \mathcal{E}xt^1_{\mathbf{Ab}((R/W)_{\mathrm{crys}})}(H_{\overline{\mathcal{O}}_{\mathrm{crys}}}, \mathcal{O}_{\mathrm{crys}}),$$

which comes with the structure of an $\mathcal{O}_{\mathrm{crys}}$-module from the second entry. By [BBM82, Théorème 3.3.3], the $\mathcal{O}_{\mathrm{crys}}$-module $\underline{\mathbb{D}}(H)$ is an object of $\mathbf{Vect}(R_{\mathrm{crys}})$. It is simple to check that

$$(\iota_{0,\infty})_*\underline{\mathbb{D}}(H_k) = \underline{\mathbb{D}}(H), \qquad \iota^*_{0,\infty}\underline{\mathbb{D}}(H) = \underline{\mathbb{D}}(H_k),$$

(with notation as in [IKY24, §2.3.1]). As in [BBM82, 1.3.5] we have a Frobenius morphism

$$\varphi\colon \phi^*\underline{\mathbb{D}}(H_k) \to \underline{\mathbb{D}}(H_k).$$

[11]When S is Noetherian and $\mathbb{Z}_p$-flat, this condition is equivalent to S being regular by combining [BIM19] and [ALB23, Proposition 5.8].

So, $\underline{\mathbb{D}}(H)$ is an object of $\mathbf{Vect}^{\varphi}(R_{\mathrm{crys}})$ called the *Dieudonné crystal* associated to H.

One defines a *Hodge filtration* (see [BBM82, Corollaire 3.3.5])

$$\mathrm{Fil}^1_{H,\mathrm{Hodge}} := \mathcal{E}xt^1_{\mathbf{Ab}((R/W)_{\mathrm{crys}})}(H_{\overline{\mathcal{O}}_{\mathrm{crys}}}, \mathcal{J}_{\mathrm{crys}})_{\mathfrak{X}} \subseteq \underline{\mathbb{D}}(H)_{\mathfrak{X}},$$

where $\mathcal{J}_{\mathrm{crys}} \subseteq \mathcal{O}_{\mathrm{crys}}$ is the PD ideal sheaf.

Definition 4.6. The functor

$$\mathbb{D}\colon \mathbf{BT}_p(R) \to \mathbf{DieuF}(R), \qquad H \mapsto \mathbb{D}(H) = (\underline{\mathbb{D}}(H), \mathrm{Fil}^1_{H,\mathrm{Hodge}}),$$

is called the *filtered Dieudonné crystal* functor.

The filtered Dieudonné crystal is also a complete invariant of H.

Theorem 4.7 (de Jong, cf. [Kim15, Theorem 3.17])**.** *The functor*

$$\mathbb{D}\colon \mathbf{BT}_p(R) \to \mathbf{DieuF}(R),$$

is an anti-equivalence of categories.

Our first main result is that $\mathbb{D}_{\mathrm{crys}}$ transforms $\mathcal{M}_{\Delta}$ into $\mathbb{D}$, improving upon [ALB23, Theorem 4.44] which proves the more naive statement obtained by ignoring filtrations.

Theorem 4.8. *There is a natural equivalence $\mathbb{D}_{\mathrm{crys}} \circ \mathcal{M}_{\Delta} \xrightarrow{\sim} \mathbb{D}$.*

Proof. We begin by observing that there is a tautological identification between $\underline{\mathbb{D}}_{\mathrm{crys}} \circ \mathcal{M}_{\Delta}$ and $\underline{\mathbb{D}}$ (see [ALB23, Theorem 4.44]). Thus, it suffices to check that the submodules $\mathrm{Fil}^1_{\mathbb{D}_{\mathrm{crys}}}$ and $\mathrm{Fil}^1_{H,\mathrm{Hodge}}$ agree. To check this, it suffices to pass to the faithfully flat cover $R \to \widetilde{R}$ (see [IKY24, Lemma 1.15]). But, observe that we have a commutative diagram of filtered rings

$$\begin{tikzcd}
(R, \mathrm{Fil}^{\bullet}_{\mathrm{triv}}) \arrow[r, "\beta"] \arrow[dr] & (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\mathrm{PD}}) \arrow[d] \\
& (\widetilde{R}, \mathrm{Fil}^{\bullet}_{\mathrm{triv}}),
\end{tikzcd}$$

so by the crystal property we need only check the equality after evaluation on $\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \twoheadrightarrow \widetilde{R}$.

We first observe that by Proposition 3.3 and Remark 3.9, for any object H of $\mathbf{BT}_p(R)$ there is a canonical identification of filtered $\mathrm{A}_{\mathrm{crys}}(\widetilde{R})$-modules

$$\mathbb{D}_{\mathrm{crys}}(\mathcal{M}_{\Delta}(H))(\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \twoheadrightarrow \widetilde{R}) \xrightarrow{\sim} \phi^*\mathcal{M}_{\Delta}(H)(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi)) \otimes_{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\xi})} (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\mathrm{PD}}).$$

On the other hand, by Lemma 4.9, the filtration on $\mathbb{D}(H)(\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \twoheadrightarrow \widetilde{R})$ is equal to the preimage of the filtration $\mathrm{Fil}^1 \subseteq \underline{\mathbb{D}}(H)(\check{R})$, defined by $\mathbb{D}(H)(\check{R})$, under the surjection

$$\Pi\colon \underline{\mathbb{D}}(H)(\mathrm{A}_{\mathrm{crys}}(\check{R}) \twoheadrightarrow \check{R}) \to \underline{\mathbb{D}}(H)(\check{R}).$$

Thus, it suffices to show the following equality of filtered $\mathrm{A}_{\mathrm{crys}}(\check{R})$-modules:

$$\phi^*\mathcal{M}_{\Delta}(H)(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi)) \otimes_{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\xi})} (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\mathrm{PD}}) = (\underline{\mathbb{D}}(H)(\mathrm{A}_{\mathrm{crys}}(\check{R}) \twoheadrightarrow \check{R}), \Pi^{-1}(\mathrm{Fil}^1)).$$

But combining Proposition 4.3 and the definition of M^{Lau} we have

$$\begin{aligned}
(\underline{\mathbb{D}}(H)(\mathrm{A}_{\mathrm{crys}}(\widetilde{R}) \to \widetilde{R}), \Pi^{-1}(\mathrm{Fil}^1)) &= \Phi^{\mathrm{cris}}_{\widetilde{R}}(H_{\widetilde{R}}) \\
&= \lambda^*(M^{\mathrm{Lau}}(H_{\widetilde{R}})) \\
&= \lambda^*(\phi^*\mathcal{M}_{\Delta}(H)(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi))) \\
&= \phi^*\mathcal{M}_{\Delta}(H)(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), (\xi)) \otimes_{(\mathrm{A}_{\mathrm{inf}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\xi})} (\mathrm{A}_{\mathrm{crys}}(\widetilde{R}), \mathrm{Fil}^{\bullet}_{\mathrm{PD}}),
\end{aligned}$$

where $\Phi^{\mathrm{cris}}_{\widetilde{R}}$ is as in [Lau18, Theorem 6.3] and λ^* is as in [Lau18, Proposition 9.3]. $\square$

Lemma 4.9. *Let $\mathcal{F}$ be an object of $\mathbf{VectF}_{[0,1]}(R_{\mathrm{crys}})$ and $(A \twoheadrightarrow R') \to (B \twoheadrightarrow R')$ be a morphism in $(R/W)_{\mathrm{crys}}$, with $A \to B$ surjective. Then $\mathrm{Fil}^1_{\mathcal{F}}(A \twoheadrightarrow R') \subseteq \mathcal{F}(A \twoheadrightarrow R')$ is the preimage of the submodule $\mathrm{Fil}^1_{\mathcal{F}}(B \twoheadrightarrow R') \subseteq \mathcal{F}(B \twoheadrightarrow R')$ via the surjection $\mathcal{F}(A \twoheadrightarrow R') \to \mathcal{F}(B \twoheadrightarrow R')$.*

Proof. By the crystal property, we may work Zariski locally on A. But, Zariski locally on A, there is a basis $(e_\nu)_{\nu=1}^n$ of $\mathcal{F}(A \twoheadrightarrow R')$ and a subset I of $\{1, \ldots, n\}$ such that the filtration $\mathrm{Fil}^1(A \twoheadrightarrow R')$ is given by $\sum_{\nu \in I} A \cdot e_\nu + \sum_{\nu \notin I} \mathrm{Fil}^1_{\mathrm{PD}}(A) \cdot e_\nu$. The claim then follows as $\mathrm{Fil}^1_{\mathrm{PD}}(A)$ is the preimage of $\mathrm{Fil}^1_{\mathrm{PD}}(B)$ under the surjection $A \to B$. $\square$

Remark 4.10. When R is small, Theorem 4.8 follows from Theorem 3.1. Indeed, it suffices to show that $T^*_{\mathrm{crys}}(\mathbb{D}_{\mathrm{crys}}(\mathcal{M}_{\Delta}(H)))$ is isomorphic to $T^*_{\mathrm{crys}}(\mathbb{D}(H))$. But, by Proposition 3.2 and [DLMS24, Proposition 3.35] the former is $T_p(H)$, which is also the latter by [Kim15, Corollary 5.3 and §5.4].

4.3. **Breuil–Kisin–Kim Dieudonné theory.** In this final section we use Theorem 4.8 to show that Anschütz–Le Bras's prismatic Dieudonné functor $\mathcal{M}_{\Delta}$ recovers the Breuil–Kisin–Kim Dieudonné functor $\mathfrak{M}$.

4.3.1. **The Breuil Dieudonné functor.** We begin by first recalling an intermediate construction between $\mathbb{D}_{\mathrm{crys}}$ and $\mathfrak{M}$ constructed by Breuil.

Definition 4.11 (cf. [Kim15, Definition 3.12])**.** A (*minuscule*) *Breuil-module* over R is a quadruple of data $(\mathcal{M}, \mathrm{Fil}^1_{\mathcal{M}}, \varphi_{\mathcal{M}}, \nabla^0_{\mathcal{M}})$ with:

- $\mathcal{M}$ a finite projective S_R-module,
- $\mathrm{Fil}^1_{\mathcal{M}} \subseteq \mathcal{M}$ an S_R-submodule with $\mathrm{Fil}^1_{\mathrm{PD}} \cdot \mathcal{M} \subseteq \mathrm{Fil}^1_{\mathcal{M}}$ and $\mathcal{M}/\mathrm{Fil}^1_{\mathcal{M}}$ projective over R,
- $\varphi_{\mathcal{M}} \colon \phi^*\mathcal{M} \to \mathcal{M}$ is an S_R-linear map with $\varphi_M(\phi^* \mathrm{Fil}^1_{\mathcal{M}}) = p\mathcal{M}$,
- $\nabla^0_{\mathcal{M}}$ is a topologically quasi-nilpotent integrable connection (cf. [dJ95, Remark 2.2.4]) on $\mathcal{M}_0 := \mathcal{M} \otimes_{S_R} R$, where $S_R \to R$ is induced by the map $\mathfrak{S}_R \to R$ sending u to 0, such that the induced Frobenius $\varphi_{\mathcal{M}_0}$ is horizontal.

These naturally form a category which we denote by $\mathbf{VectF}^{\varphi}_{[0,1]}(S_R, \nabla^0)$.

By [Kim15, Proposition 3.8 and Lemma 3.15], there is a fully faithful embedding

$$\mathbf{DieuF}(R) \to \mathbf{VectF}^{\varphi}_{[0,1]}(S_R, \nabla^0), \quad (\mathcal{F}, \varphi_{\mathcal{F}}, \mathrm{Fil}^{\bullet}_{\mathcal{F}}) \mapsto ((\mathcal{F}, \mathrm{Fil}^1_{\mathcal{F}}, \varphi_{\mathcal{F}})(S_R \twoheadrightarrow R), \nabla_{\mathcal{F}}). \tag{4.3.1}$$

This then suggests the following definition.

Definition 4.12 (cf. [Kim15, §3.4])**.** The functor

$$\mathcal{M}^{\mathrm{Br}} \colon \mathbf{BT}_p(R) \to \mathbf{VectF}^{\varphi}_{[0,1]}(S_R, \nabla^0), \qquad H \mapsto \mathcal{M}^{\mathrm{Br}}(H) := (\mathbb{D}(H)(S_R \twoheadrightarrow R), \nabla_{\underline{\mathbb{D}}(H)}),$$

is called the *Breuil Dieudonné functor*.

As (4.3.1) and $\mathbb{D}$ are both fully faithful, it follows that $\mathcal{M}^{\mathrm{Br}}$ is also fully faithful.

4.3.2. **The Breuil–Kisin–Kim Dieudonné functor.** We now recall the definition of the Breuil–Kisin–Kim Dieudonné functor.

Definition 4.13 (cf. [Kim15, Definition 6.1])**.** A (*minuscule*) *Kisin module* over R is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla^0_{\mathfrak{M}})$ where:

- $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is an object of the category $\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, (E))$
- $\nabla^0_{\mathfrak{M}}$ is topologically quasi-nilpotent integrable connection on $\mathfrak{M}_0 := \phi^*\mathfrak{M}/u$.

We denote the category of such objects by $\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, \nabla^0_{\mathfrak{M}})$.

There is a functor

$$\mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, \nabla^0) \to \mathbf{VectF}^{\varphi}_{[0,1]}(S_R, \nabla^0), \tag{4.3.2}$$

where $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \nabla^0_{\mathfrak{M}_0})$ is sent to the object whose underlying filtered S_R-module is

$$(\phi^*\mathfrak{M}, \mathrm{Fil}^\bullet_{\mathrm{Nyg}}) \otimes_{(\mathfrak{S}_R, \mathrm{Fil}^\bullet_E)} (S_R, \mathrm{Fil}^\bullet_{\mathrm{PD}})$$

(forgetting everything but the 1-part of the filtration), with Frobenius given by $\varphi_{\phi^*\mathfrak{M}} \otimes 1$, and with connection $\nabla^0_{\mathfrak{M}}$, which is sensible via the natural identification of R-modules

$$\phi^*\mathfrak{M} \otimes_{\mathfrak{S}_R} R \simeq \phi^*\mathfrak{M} \otimes_{\mathfrak{S}_R} S_R \otimes_{S_R} R.$$

This functor is fully faithful by [Kim15, Lemma 6.5].

Definition 4.14 ([Kim15])**.** By [Kim15, Corollary 6.7], the functor $\mathcal{M}^{\mathrm{Br}}$ factorizes through the fully faithful embedding (4.3.2). Thus, we obtain a functor

$$\mathfrak{M}\colon \mathbf{BT}_p(R) \to \mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, \nabla^0_{\mathfrak{M}}), \quad H \mapsto \mathfrak{M}(H) = (\underline{\mathfrak{M}}(H), \varphi_{\underline{\mathfrak{M}}(H)}, \nabla^0_{\mathbb{D}(H)})$$

called the *Breuil–Kisin–Kim Dieudonné functor.*

Remark 4.15. When $R = \mathcal{O}_K$, this agrees with the functor from [Kis06]. This is not explicitly explained in op. cit., and can be argued directly, but also follows by combining Proposition 4.16 and Remark 4.17 below.

To relate this to prismatic Dieudonné modules, let us begin by observing that there are functors

$$\begin{aligned} \mathrm{ev}_{\mathfrak{S}_R}\colon \mathbf{Vect}^{\varphi}_{[0,1]}(R_{\Delta}) \to \mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, (E)), \qquad & (\mathcal{E}, \varphi_{\mathcal{E}}) \mapsto (\mathcal{E}, \varphi_{\mathcal{E}})(\mathfrak{S}_R, (E)), \\ \mathrm{ev}^{\mathrm{K}}_{\mathfrak{S}_R}\colon \mathbf{Vect}^{\varphi}_{[0,1]}(R_{\Delta}) \to \mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, \nabla^0), \qquad & (\mathcal{E}, \varphi_{\mathcal{E}}) \mapsto (\mathrm{ev}_{\mathfrak{S}_R}(\mathcal{E}, \varphi_{\mathcal{E}}), \nabla_{\mathbb{D}_{\mathrm{crys}}(\mathcal{E}.\varphi_{\mathcal{E}})}). \end{aligned}$$

Here by $\nabla_{\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})}$ we abusively mean the pullback of $\nabla_{\mathbb{D}_{\mathrm{crys}}(\mathcal{E},\varphi_{\mathcal{E}})}$ under the isomorphism

$$\phi^*\mathcal{E}(\mathfrak{S}_R, (E))/u \xrightarrow{\sim} \mathbb{D}_{\mathrm{crys}}(\mathcal{E}, \varphi_{\mathcal{E}})(R),$$

from Proposition 1.3.

Proposition 4.16. *There is a natural identification* $\mathrm{ev}^{\mathrm{K}}_{\mathfrak{S}_R} \circ \mathcal{M}_{\Delta} \xrightarrow{\sim} \mathfrak{M}$.

Proof. Let H be an object of $\mathbf{BT}_p(R)$. Then, by the definition of $\mathfrak{M}$ it suffices to show that $\mathrm{ev}^{\mathrm{K}}_{\mathfrak{S}_R}(\mathcal{M}_{\Delta}(H))$ and $\mathfrak{M}(H)$ have image under (4.3.2) which are canonically identified. But, this follows by combining Remark 3.9, Theorem 4.8, and the definition of $\mathcal{M}^{\mathrm{Br}}(H)$. □

Remark 4.17. For $R = \mathcal{O}_K$ this was previously shown in [ALB23, Proposition 5.18].

Combining Theorem 4.2, [Kim15, Corollary 6.7 and Corollary 10.4], Proposition 4.16, [IKY25, Proposition 1.28], and Proposition 3.3, we deduce the following.

Corollary 4.18. *Let R be a formally framed W-algebra. Then, there are equivalences of categories*

$$\mathbf{Vect}_{[0,1]}(R^{\mathrm{syn}}) \xrightarrow[\sim]{\mathrm{R}_{\mathfrak{X}}} \mathbf{Vect}^{\varphi}_{[0,1]}(R_{\Delta}) \xrightarrow[\sim]{\mathrm{ev}^{\mathrm{K}}_{\mathfrak{S}_R}} \mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, \nabla^0),$$

is an equivalence of categories. If $R = W[\![t_1, \ldots, t_d]\!]$ for some $d \geqslant 0$, then there are equivalences

$$\mathbf{Vect}_{[0,1]}(R^{\mathrm{syn}}) \xrightarrow[\sim]{\mathrm{R}_{\mathfrak{X}}} \mathbf{Vect}^{\varphi}_{[0,1]}(R_{\Delta}) \xrightarrow[\sim]{\mathrm{ev}_{\mathfrak{S}_R}} \mathbf{Vect}^{\varphi}_{[0,1]}(\mathfrak{S}_R, (E)).$$

Remark 4.19. The second claim of Corollary 4.18 was previously established by Anschütz–Le Bras (see [ALB23, Theorem 5.12]) and Ito (see [Ito23, Proposition 7.1.1]).

References

[ALB23] J. Anschütz and A.-C. Le Bras, Prismatic Dieudonné Theory, Forum Math. Pi 11 (2023), Paper No. e2. 4, 5, 29, 30, 31, 33

[AY25] Abhinandan and A. Youcis, Twisted crystalline and de Rham cohomology, 2025, in preparation. 15

[BBM82] P. Berthelot, L. Breen and W. Messing, Théorie de Dieudonné cristalline. II, vol. 930 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1982. 30, 31

[Bha23] B. Bhatt, Prismatic F-gauges, 2023, unpublished course notes . 2, 4, 8, 10, 11, 13

[BIM19] B. Bhatt, S. B. Iyengar and L. Ma, Regular rings and perfect(oid) algebras, Comm. Algebra 47 (2019), no. 6, 2367–2383. 30

[BL22a] B. Bhatt and J. Lurie, Absolute prismatic cohomology, 2022, 2201.06120. 3, 12

[BL22b] B. Bhatt and J. Lurie, The prismatization of p-adic formal schemes, 2022, 2201.06124. 9, 11, 14

[BMS18] B. Bhatt, M. Morrow and P. Scholze, Integral p-adic Hodge theory, Publ. Math. Inst. Hautes Études Sci. 128 (2018), 219–397. 29

[BO83] P. Berthelot and A. Ogus, F-isocrystals and de Rham cohomology. I, Invent. Math. 72 (1983), no. 2, 159–199. 15

[Bre97] C. Breuil, Représentations p-adiques semi-stables et transversalité de Griffiths, Math. Ann. 307 (1997), no. 2, 191–224. 27

[BS22] B. Bhatt and P. Scholze, Prisms and prismatic cohomology, Ann. of Math. (2) 196 (2022), no. 3, 1135–1275. 10

[BS23] B. Bhatt and P. Scholze, Prismatic F-crystals and crystalline Galois representations, Camb. J. Math. 11 (2023), no. 2, 507–562. 2, 6, 8

[dJ95] A. J. de Jong, Crystalline Dieudonné module theory via formal and rigid geometry, Inst. Hautes Études Sci. Publ. Math. (1995), no. 82, 5–96 (1996). 4, 29, 32

[DLMS24] H. Du, T. Liu, Y. S. Moon and K. Shimizu, Completed prismatic F-crystals and crystalline Z_p-local systems, Compos. Math. 160 (2024), no. 5, 1101–1166. 2, 25, 27, 32

[Dri21] V. Drinfeld, On a notion of ring groupoid, 2021, 2104.07090. 6

[Dri24] V. Drinfeld, Prismatization, 2024, 2005.04746. 2

[Fal89] G. Faltings, Crystalline cohomology and p-adic Galois-representations, in Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), pp. 25–80, Johns Hopkins Univ. Press, Baltimore, MD, 1989. 2, 4, 16, 17, 23, 24, 25, 29

[Gao19] H. Gao, Fontaine-Laffaille modules and strongly divisible modules, Ann. Math. Qué. 43 (2019), no. 1, 145–159. 15, 28

[GL23] H. Guo and S. Li, Frobenius height of prismatic cohomology with coefficients, 2023, 2309.06663. 2, 10, 29

[GM24] Z. Gardner and K. Madapusi, An algebraicity conjecture of Drinfeld and the moduli of p-divisible groups, 2024, 2412.10226. 10, 24

[GR24] H. Guo and E. Reinecke, A prismatic approach to crystalline local systems, Invent. Math. 236 (2024), no. 1, 17–164. 2, 19, 20, 21, 25, 29

[Hok24] C. Hokaj, Fontaine-Laffaille Theory over Power Series Rings, 2024, 2407.21327. 3, 28

[Hub96] R. Huber, Étale cohomology of rigid analytic varieties and adic spaces, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996. 29

[IKY24] N. Imai, H. Kato and A. Youcis, A Tannakian framework for prismatic F-crystals, 2024, 2406.08259. 3, 4, 5, 6, 7, 8, 12, 14, 15, 18, 19, 20, 22, 23, 25, 27, 28, 30, 31

[IKY25] N. Imai, H. Kato and A. Youcis, The Prismatic Realization Functor for Shimura Varieties of Abelian Type, 2025, 2310.08472v5. 2, 4, 6, 10, 13, 18, 21, 22, 23, 25, 33

[Ito23] K. Ito, Prismatic G-display and descent theory, 2023, 2303.15814. 5, 33

[Ked19] K. S. Kedlaya, Sheaves, Shtukas, and Stacks, in Perfectoid Spaces, vol. 242 of Mathematical Surveys and Monographs, pp. 45–191, American Mathematical Society, Providence, RI, 2019. 19

[Kie67] R. Kiehl, Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie, Invent. Math. 2 (1967), 256–273. 19

[Kim15] W. Kim, The relative Breuil-Kisin classification of p-divisible groups and finite flat group schemes, Int. Math. Res. Not. IMRN (2015), no. 17, 8152–8232. 4, 30, 31, 32, 33

[Kis06] M. Kisin, Crystalline representations and F-crystals, in Algebraic geometry and number theory, vol. 253 of Progr. Math., pp. 459–496, Birkhäuser Boston, Boston, MA, 2006. 3, 8, 15, 33

[Kis10] M. Kisin, Integral models for Shimura varieties of abelian type, J. Amer. Math. Soc. 23 (2010), no. 4, 967–1012. 21

[Lau18] E. Lau, Dieudonné theory over semiperfect rings and perfectoid rings, Compos. Math. 154 (2018), no. 9, 1974–2004. 30, 31

[LL25] S. Li and T. Liu, Comparison of prismatic cohomology and derived de Rham cohomology, J. Eur. Math. Soc. (JEMS) 27 (2025), no. 1, 183–268. 10

[LMP24] T. Liu, Y. S. Moon and D. Patel, Relative Fontaine-Messing theory over power series rings, Int. Math. Res. Not. IMRN (2024), no. 5, 4384–4455. 17, 23

[Lov17] T. Lovering, Filtered F-crystals on Shimura varieties of abelian type, 2017, 1702.06611. 16

[LvO96] H. Li and F. van Oystaeyen, Zariskian filtrations, vol. 2 of K-Monographs in Mathematics, Kluwer Academic Publishers, Dordrecht, 1996. 12

[SP] Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu, 2023. 17

[SW20] P. Scholze and J. Weinstein, Berkeley lectures on p-adic geometry, vol. 207 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 2020. 19, 30

[Tsu20] T. Tsuji, Crystalline-Representations and-Representations with Frobenius, in p-adic Hodge Theory, pp. 161–319, Springer, 2020. 18, 24, 25, 26, 27, 28

[TT19] F. Tan and J. Tong, Crystalline comparison isomorphisms in p-adic Hodge theory: the absolutely unramified case, Algebra Number Theory 13 (2019), no. 7, 1509–1581. 29

[TVX24] G. Terentiuk, V. Vologodsky and Y. Xu, Prismatic F-Gauges and Fontaine–Laffaille modules, 2024, 2402.17755. 3

[Wür23] M. Würthen, Divided prismatic Frobenius crystals of small height and the category $\mathcal{MF}$, 2023, 2305.06081. 3