Convolution morphisms and Kottwitz conjecture

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Abstract

We define etale cohomology of the moduli spaces of mixed characteristic local shtukas so that it gives smooth representations including the case where the relevant elements of the Kottwitz set are both non-basic. Then we relate the etale cohomology of different moduli spaces of mixed characteristic local shtukas using convolution morphisms, duality morphisms and twist morphisms. As an application, we show the Kottwitz conjecture in some new cases including the cases for all inner forms of GL_3 and minuscule cocharacters. We study also some non-minuscule cases and show that the Kottwitz conjecture is true for any inner form of GL_2 and any cocharacter if the Langlands parameter is cuspidal. On the other hand, we show that the Kottwitz conjecture does not hold as it is in non-minuscule cases if the Langlands parameter is not cuspidal. Further, we show that a generalization of the Harris–Viehmann conjecture for the moduli spaces of mixed characteristic local shtukas does not hold in Hodge–Newton irreducible cases.

Introduction

The Kottwitz conjecture says that etale cohomology of Rapoport–Zink spaces or more generally local Shimura varieties realize the local Langlands correspondence (*cf.* [Rap95, Conjecture 5.1], [RV14, Conjecture 7.4]). In [SW20], Scholze constructs local Shimura varieties as special cases of moduli spaces of mixed characteristic local shtukas. The Kottwitz conjecture makes sense also for the moduli spaces of mixed characteristic local shtukas. A weak version of the conjecture is studied by Hansen–Kaletha–Weinstein in [HKW22]. In the weak version, we ignore the action of the Weil groups and have an equality up to representations which have trace 0 on regular elliptic elements.

Let p be a prime number. Let G be a connected reductive group over a p-adic number field F. For $b, b' \in G(\breve{F})$ and a system $\mu_\bullet = (\mu_1, \ldots, \mu_m)$ of cocharacters of G, we define a moduli space $\mathrm{Sht}^{\mu_\bullet}_{b,b'}$ of mixed characteristic local shtukas. See §2 for the precise definition.

In this paper, we introduce convolution morphisms, duality morphisms and twist morphisms between moduli spaces of mixed characteristic local shtukas. The convolution morphism is related to a convolution morphism on affine Grassmannians. Using these morphisms and the convolution products in the geometric Satake equivalence for B^+_{dR}-Grassmannians, we relate the etale cohomology of different moduli spaces of mixed characteristic local shtukas. More concretely, we show the following:

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Theorem 1 (Corollary 5.2). *Assume that G is quasi-split and take a Borel pair $T \subset B$ of G. Let $\mu_\bullet = (\mu_1, \ldots, \mu_m)$ be a system of dominant cocharacters of T and $b_0, b_m \in G(\breve{F})$. Let E be a finite extension of F containing the fields of definition of μ_i for $1 \leq i \leq m$. We have*

$$\sum_{([b_i])_{1\leq i\leq m-1}\in I^{\mu_\bullet}_{b_0,b_m}} H_* \left(\prod_{i=1}^{m-1} G_{b_i}(F), \bigotimes_{1\leq i\leq m} H^*_{\mathrm{c}}(\mathrm{Sht}^{\mu_i}_{b_{i-1},b_i}) \otimes \bigotimes_{1\leq i\leq m-1} \delta_{b_i} \right) = \sum_{\lambda\in X_*(T)^+/\Gamma} V^\lambda_{\mu_\bullet} \otimes H^*_{\mathrm{c}}(\mathrm{Sht}^\lambda_{b_0,b_m})$$

as virtual representations of $G_{b_0}(F) \times G_{b_m}(F) \times W_E$, where $I^{\mu_\bullet}_{b_0,b_m}$ is a finite set defined in §5.

We note that even if b_0 and b_m are basic, non-basic elements appear in $I^{\mu_\bullet}_{b_0,b_m}$ and there are contributions from cohomology of non-basic moduli spaces of local shtukas. For a derived category version of the above statement, see Proposition 5.1.

As an application of Theorem 1 (or its derived category version) together with duality morphisms and twist morphisms, we show new cases of the Kottwitz conjecture for the moduli spaces of mixed characteristic local shtukas. In particular, we show the following:

Theorem 2 (Corollary 7.6). *Let G be an inner form of GL_3 over F. Let (G, b, μ) be a local shtuka datum such that μ is minuscule and b is basic. Let $\varphi \colon W_F \to {}^L\mathrm{GL}_3$ be a discrete local L-parameter. Let π and π_b be the irreducible smooth representations of $G(F)$ and $G_b(F)$ corresponding to φ via the local Langlands correspondence. Then we have*

$$\mathcal{H}^* \left(R\,\mathrm{Hom}_{G(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^\mu_{1,b}), \pi \right) \right) \simeq \pi_b \boxtimes (r_\mu \circ \varphi)$$

as representations of $G_b(F) \times W_F$.

It is remarkable that the proof of Theorem 2 requires moduli spaces of local shtukas for non-minuscule cocharacters, even though the statement involves only minuscule cocharacters: Using a derived category version of Theorem 1, we can calculate a sum of cohomology of moduli spaces of local shtukas for a minuscule cocharacter and a non-minuscule cocharacter. Then we separate them into each term using the duality isomorphism. We also note that it is essential to introduce convolution morphisms for moduli spaces of mixed characteristic local shtukas with multiple legs in §4, because we use it in the proof of a compatibility result, Proposition 6.2, which plays an important role in the proof of Theorem 2.

Theorem 1 is useful also for studying non-minuscule cases. We give inductive formulas that enable us to calculate the cohomology of moduli spaces of local shtukas for inner forms of GL_2. We can summarize the results in §8 as the following theorem:

Theorem 3. *Let G be an inner form of GL_2 over F. Let (G, b, μ) be a local shtuka datum. Let ρ be a discrete series representation of $G_b(F)$. We put*

$$H^\bullet_{\mathrm{c}}(\mathrm{Sht}^\mu_{1,b})[\rho] = \sum_{i,j\in\mathbb{Z}} (-1)^{i+j}\, \mathrm{Ext}^i_{G_b(F)} \left(R^j\Gamma_{\mathrm{c}}(\mathrm{Sht}^\mu_{1,b}), \rho \right).$$

Then we can calculate $H^\bullet_{\mathrm{c}}(\mathrm{Sht}^\mu_{1,b})[\rho]$ by inductive formulas. In particular, when b is basic, we see the following:

(1) The Kottwitz conjecture for $\mathrm{Sht}^{\mu}_{1,b}$ *holds if the L-parameter is cuspidal or* G *is not quasi-split.*

(2) The Kottwitz conjecture for $\mathrm{Sht}^{\mu}_{1,b}$ *does not hold in general if the L-parameter is not cuspidal and* G *is quasi-split.*

We note that in the first statement of Theorem 3, b can be non-basic and μ can be non-minusucle. Even if we are interested only in $H^{\bullet}_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{1,b})[\rho]$ for a basic b, the inductive formulas for the calculation of $H^{\bullet}_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{1,b})[\rho]$ involve moduli spaces of local shtukas for non-basic elements. Therefore it is important to study non-basic cases at the same time.

We note that Theorem 3 is compatible with the result in [HKW22], since the error term involves only representations which have trace 0 on regular elliptic elements. We remark also that this error term supports that the expectation [Far16, Remark 4.6] in the geometrization of the local Langalnds correspondence is true.

Further, we see that the Harris–Viemann conjecture for the moduli spaces of mixed characteristic local shtukas does not hold as it is in Example 8.10 and Remark 8.11. We note that Harris–Viemann conjecture for the moduli spaces of mixed characteristic local shtukas is proved in [GI16] and [Han21a] under the Hodge–Newton reducibility condition. On the other hand, the Hodge–Newton reducibility condition is not satisfied in Example 8.10.

In §1, we collect results on relative homologey and the geometric Satake correspondence. In §2, we give a definition of a moduli space of mixed characteristic local shtukas. The definition which we give here is slightly different from that in [SW20]. Our definition is suitable to construct convolution morphisms between moduli spaces of mixed characteristic local shtukas in §4. In §3, we construct a twist morphism between moduli spaces of mixed characteristic local shtukas, which has an origin in the twist of a vector bundle by a line bundle. In §5, we discuss a relation between cohomology of different moduli spaces of mixed characteristic local shtukas using convolution morphisms. In §6, we construct a duality morphism, which has an origin in the dual of a vector bundle. In §7, we give an application to the Kottwitz conjecture. In §8, we give some inductive formulas on cohomology and discuss more about the Kottwitz conjecture in non-minuscule cases.

After we put a former version of this paper on arXiv, a preprint [Han21b] by Hansen appeared, where a cohomology version of Theorem 2 is proved for cuspidal local L-parameters of GL_n using a result in [ALB21]. A merit of Theorem 2 is that it works for discrete local L-parameters.

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Notation

For a field F, let Γ_F denote the absolute Galois group of F. For a non-archimedean local field F, let $\breve{F}$ denote the completion of the maximal unramified extension of F. For an object X_Y over an object Y, its base change by the morphism $Y' \to Y$ is denoted by $X_{Y'}$.

1 Sheaves in ℓ-adic coefficients

1.1 Relative homology

Let p be a prime number. Let Λ be a solid $\widehat{\mathbb{Z}}^p$-algebra. For a small v-stack X, we define $D_\blacksquare(X,\Lambda)$ as [FS21, Definition VII.1.17]. There is a symmetric monoidal structure $-\overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda}-$ on $D_\blacksquare(X,\Lambda)$ constructed by [FS21, Proposition VII.2.2]. In the sequel, we simply write $\otimes^{\mathbb{L}}_{\Lambda}$ for $\overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda}$. For a morphism $f\colon X \to Y$ of small v-stacks, let

$$f_\natural\colon D_\blacksquare(X,\Lambda) \to D_\blacksquare(Y,\Lambda)$$

be a left adjoint to $f^*\colon D_\blacksquare(Y,\Lambda) \to D_\blacksquare(X,\Lambda)$ constructed by [FS21, Proposition VII.3.1].

The following lemma is already known (*cf.* the proof of [FS21, Proposition VII.6.3]).

Lemma 1.1. *Let $f\colon X \to Y$ be a quasi-compact, quasi-separated morphism of small v-stacks. Assume that $\Lambda = \varprojlim_{n\in I} \mathbb{Z}/n\mathbb{Z}$, where I is a filtered set of positive integers which are prime to p. Then we have*

$$f_\natural\Lambda \simeq \varprojlim_{n\in I} f_\natural(\mathbb{Z}/n\mathbb{Z}).$$

Proof. We recall a proof. We may assume that Y is a spatial diamond. Then Λ is a pseudo-coherent object on X (*cf.* [Kra20, Definition 7.2]) by the assumption on f. Since $f_\natural$ preserves pseudo-coherent objects, $f_\natural\Lambda$ is also a pseudo-coherent object. Since each cohomology sheaf of $f_\natural\Lambda$ is a finitely presented solid sheaf, we have

$$f_\natural\Lambda \simeq \varprojlim_{n\in I}(f_\natural\Lambda \otimes^{\mathbb{L}}_{\Lambda} \mathbb{Z}/n\mathbb{Z}) \simeq \varprojlim_{n\in I} f_\natural(\mathbb{Z}/n\mathbb{Z})$$

by [FS21, Theorem VII.1.3, Proposition VII.3.1]. □

Lemma 1.2. *Let $f\colon X \to Y$ be a morphism of small v-stacks. Let $\mathcal{F}$ be a solid $\widehat{\mathbb{Z}}^p$-sheaf on X. Let $\{U_i\}_{i\in I}$ be a filtered direct system of quasi-compact open substacks of X such that $X = \bigcup_{i\in I} U_i$. Let f_i and $\mathcal{F}_i$ be the restriction to U_i of f and $\mathcal{F}$ for $i \in I$. Then we have*

$$f_\natural\mathcal{F} \simeq \varinjlim_{i\in I} f_{i\natural}\mathcal{F}_i.$$

Proof. Let $j_i\colon U_i \to X$ be the inclusion for $i \in I$. Since $f_\natural$ commutes with a direct limit, it suffices to show $\mathcal{F} \simeq \varinjlim_{i\in I} j_{i\natural}\mathcal{F}_i$. By the projection formula, we may assume that $\mathcal{F} = \widehat{\mathbb{Z}}^p$. For any solid $\widehat{\mathbb{Z}}^p$-sheaf $\mathcal{G}$ on X, we have

$$\mathrm{Hom}(\varinjlim_{i\in I} j_{i\natural}\widehat{\mathbb{Z}}^p, \mathcal{G}) \simeq \varprojlim_{i\in I} \mathrm{Hom}(j_{i\natural}\widehat{\mathbb{Z}}^p, \mathcal{G}) \simeq \varprojlim_{i\in I} \mathcal{G}(U_i) \simeq \mathcal{G}(X) \simeq \mathrm{Hom}(\widehat{\mathbb{Z}}^p, \mathcal{G}).$$

Hence we obtain the claim. □

Lemma 1.3. *Let F be a non-archimedean field with residue characteristic p. Let d be a positive integer, and n a positive integer prime to p.*

(1) Let
$$f\colon \big(\mathrm{Spa}(\mathcal{O}_F[[x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}]])\times_{\mathrm{Spa}(\mathcal{O}_F)}\mathrm{Spa}(F)\big)^\diamond\to\mathrm{Spa}(F)^\diamond$$
be the natural morphism. Then we have $f_\natural\Lambda\simeq\Lambda$. We also have $f_!(\mathbb{Z}/n\mathbb{Z})\simeq(\mathbb{Z}/n\mathbb{Z})(-d)[-2d]$. Further, the geometric Frobenius morphism $x_i\mapsto x_i^p$ induces the multiplication by p^d on $f_!(\mathbb{Z}/n\mathbb{Z})$.

(2) Let
$$f\colon(\mathbb{A}_F^d)^\diamond\to\mathrm{Spa}(F)^\diamond$$
be the natural morphism. Then we have $f_\natural\Lambda\simeq\Lambda$. We also have $f_!(\mathbb{Z}/n\mathbb{Z})\simeq(\mathbb{Z}/n\mathbb{Z})(-d)[-2d]$. Further, the geometric Frobenius morphism $x_i\mapsto x_i^p$ induces the multiplication by p^d on $f_!(\mathbb{Z}/n\mathbb{Z})$.

Proof. We show the first claim of (1). We may assume that $\Lambda=\widehat{\mathbb{Z}}^p$ and F is algebraically closed of characteristic p. We write $\mathrm{Spa}(\mathcal{O}_F[[x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}]])\times_{\mathrm{Spa}(\mathcal{O}_F)}\mathrm{Spa}(F)$ as a union of affinoids isomorphic to $\mathrm{Spa}(F\langle x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}\rangle)$. By Lemma 1.2, it is reduced to show that $g_\natural\widehat{\mathbb{Z}}^p\simeq\widehat{\mathbb{Z}}^p$ for

$$g\colon\ \mathrm{Spa}(F\langle x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}\rangle)\to\mathrm{Spa}(F).$$

By Lemma 1.1 and [FS21, Proposition VII.5.2], the claim follows from that $g_!(\mathbb{Z}/n\mathbb{Z})\simeq(\mathbb{Z}/n\mathbb{Z})(-d)[-2d]$ for any integer n prime to p. The claim on $f_!(\mathbb{Z}/n\mathbb{Z})$ follows from the case for $g_!(\mathbb{Z}/n\mathbb{Z})$ in a similar way.

We can show the claim (2) similarly. □

Let ℓ be a prime number different from p.

Lemma 1.4. *Let G be a locally pro-p group. Let $\mathcal{H}(G)$ be the Hecke algebra of G with coefficients in Λ. Let $f\colon X\to Y$ be a morphism of small v-stacks which is a G-torsor. For a pro-p open subgroup K of G, let $f_K\colon X/K\to Y$ be the morphism induced by f. Let $g\colon Y\to Z$ be a morphism of small v-stacks. The morphisms f_K^* and $(g\circ f_K)_\natural$ induce*

$$\varinjlim_K(g\circ f_K)_\natural f_K^*\colon D_\blacksquare(Y,\Lambda)\to D_\blacksquare(Z,\mathcal{H}(G))$$

(1) For $A\in D_\blacksquare(Y,\Lambda)$, we have

$$(\varinjlim_K f_{K,\natural}f_K^*A)\otimes^{\mathbb{L}}_{\mathcal{H}(G)}\Lambda\cong A.$$

(2) Assume that $A\in D_\blacksquare(Y,\Lambda)$ is obtained from $V\in D^{\mathrm{b}}(G,\Lambda)$. Then we have

$$\big(\varinjlim_K(g\circ f_K)_\natural(\Lambda)\otimes_\Lambda V\big)\otimes^{\mathbb{L}}_{\mathcal{H}(G)}\Lambda\simeq g_\natural A.$$

Proof. (1) We have

$$(\varinjlim_K f_{K,\natural}f_K^*A)\otimes^{\mathbb{L}}_{\mathcal{H}(G)}\Lambda\cong((\varinjlim_K f_{K,\natural}\Lambda)\otimes^{\mathbb{L}}_\Lambda A)\otimes^{\mathbb{L}}_{\mathcal{H}(G)}\Lambda\cong((\varinjlim_K f_{K,\natural}\Lambda)\otimes^{\mathbb{L}}_{\mathcal{H}(G)}\Lambda)\otimes^{\mathbb{L}}_\Lambda A.$$

Hence it suffices to show that the natural morphism $(\varinjlim_K f_{K,\natural}\Lambda) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \Lambda \to \Lambda$ is an isomorphism. We can check this v-locally on Y by [FS21, Proposition VII.3.1 (iii)]. Hence the claim follows.

(2) The morphism $g_\natural$ induces

$$g_\natural \colon D_\blacksquare(Y, \mathcal{H}(G)) \to D_\blacksquare(Z, \mathcal{H}(G)).$$

By [FS21, Proposition VII.3.1 (i)], we have

$$\begin{aligned}\big(\varinjlim_K (g \circ f_K)_\natural(\Lambda) \otimes_\Lambda V\big) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \Lambda &\cong g_\natural\big(\varinjlim_K f_{K,\natural}(\Lambda) \otimes_\Lambda V\big) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \Lambda \\ &\cong g_\natural\Big(\big(\varinjlim_K f_{K,\natural}(\Lambda) \otimes_\Lambda V\big) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \Lambda\Big) \cong g_\natural\Big(\big(\varinjlim_K f_{K,\natural}(V)\big) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \Lambda\Big).\end{aligned}$$

Combined with (1), it remains to show

$$\varinjlim_K f_{K,\natural}(V) \cong \varinjlim_K f_{K,\natural} f_K^* A.$$

We can check that the morphism

$$\varinjlim_K f_{K,\natural}(V) \to \varinjlim_K f_{K,\natural} f_K^* A$$

induced by $V \twoheadrightarrow V^K \hookrightarrow f_K^* A$ is an isomorphism. □

Let Λ be a $\mathbb{Z}_\ell$-algebra. For an Artin v-stack X, let $D_{\mathrm{lis}}(X, \Lambda)$ be the category defined in [FS21, Definition VII.6.1].

Lemma 1.5. *Let $f \colon X \to Y$ be an ℓ-cohomologically smooth morphism of Artin v-stacks.*

(1) We have $f_\natural(D_{\mathrm{lis}}(X, \Lambda)) \subset D_{\mathrm{lis}}(Y, \Lambda)$.

(2) For $A \in D_\blacksquare(Y, \Lambda)$, we have $(f^* A)^{\mathrm{lis}} \cong f^*(A^{\mathrm{lis}})$.

Proof. The claim (1) follows from [FS21, Definition VII.6.1]. For $B \in D_\blacksquare(Y, \Lambda)$, we have

$$\begin{aligned}\mathrm{Hom}(B, (f^* A)^{\mathrm{lis}}) &\cong \mathrm{Hom}(B, f^* A) \cong \mathrm{Hom}(f_\natural(B), A) \\ &\cong \mathrm{Hom}(f_\natural(B), A^{\mathrm{lis}}) \cong \mathrm{Hom}(B, f^*(A^{\mathrm{lis}})),\end{aligned}$$

where we use (1) at the third isomorphism. Hence the claim (2) follows. □

Lemma 1.6. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a cartesian diagram of Artin v-stacks. Assume that g is ℓ-cohomologically smooth. Then we have

$$g^* Rf_{\mathrm{lis}\,*} A \cong Rf'_{\mathrm{lis}\,*} g'^* A$$

for $A \in D_{\mathrm{lis}}(X, \Lambda)$.

Proof. This follows from [FS21, Proposition VII.2.4] and Lemma 1.5. □

Lemma 1.7. *Let $f\colon X \to Y$ be an ℓ-cohomologically smooth morphism of Artin v-stacks. Let $A, B \in D_{\mathrm{lis}}(Y, \Lambda)$. Then we have $f^* R\mathscr{H}om_{\mathrm{lis}}(A, B) \cong R\mathscr{H}om_{\mathrm{lis}}(f^*A, f^*B)$.*

Proof. This follows from [FS21, Proposition VII.2.4] and Lemma 1.5. □

Lemma 1.8. *Let $f\colon X \to Y$ be a morphism of Artin v-stacks. Let $A \in D_{\mathrm{lis}}(X, \Lambda)$ and $B \in D_{\mathrm{lis}}(Y, \Lambda)$.*

(1) We have $R\mathscr{H}om_{\mathrm{lis}}(B, Rf_{\mathrm{lis}\,*}(A)) \cong Rf_{\mathrm{lis}\,*} R\mathscr{H}om_{\mathrm{lis}}(f^*B, A)$.

(2) If f is ℓ-cohomologically smooth, then we have

$$R\mathscr{H}om_{\mathrm{lis}}(f_\natural(A), B) \cong Rf_{\mathrm{lis}\,*} R\mathscr{H}om_{\mathrm{lis}}(A, f^*B).$$

Proof. (1) For $C \in D_{\mathrm{lis}}(Y, \Lambda)$, we can check

$$R\operatorname{Hom}(C, R\mathscr{H}om_{\mathrm{lis}}(B, Rf_{\mathrm{lis}\,*}(A))) \cong R\operatorname{Hom}(C, Rf_{\mathrm{lis}\,*} R\mathscr{H}om_{\mathrm{lis}}(f^*B, A))$$

by adjoint. The claim (2) is proved similarly. □

For an ℓ-cohomologically smooth morphism $f\colon X \to Y$, we put

$$D_f = (\varprojlim_n Rf^!(\mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_\ell} \Lambda$$

and

$$f_!(A) = f_\natural(A \otimes D_f^{-1})$$

for $A \in D_{\mathrm{lis}}(X, \Lambda)$. For an ℓ-cohomologically smooth morphism $f\colon X \to *$, we write D_X for D_f. For $f\colon X \to *$ and $A \in D_{\mathrm{lis}}(X, \Lambda)$, we put $R\Gamma_\natural(X, A) = f_\natural(A)$. For $f\colon X \to \operatorname{Spa} C$ and $A \in D_{\mathrm{lis}}(X, \Lambda)$ where C is an algebraically closed non-archimedean field of characteristic p, we put $R\Gamma_{\natural,C}(X, A) = f_\natural(A)$.

1.2 Geometric Satake equivalence

We recall the geometric Satake equivalence for B_{dR}^+-Grassmannians by Fargues–Scholze (*cf.* [FS21, VI, IX]).

Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. Let F be a finite extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$ with the residue field $\mathbb{F}_q$. For an algebraic field extension k of $\mathbb{F}_q$, let Perf_k denote the category of perfectoid spaces over k with v-topology in the sense of [Sch17, §8].

Let G be a connected reductive group over F. We define v-sheaves LG and L^+G over $\operatorname{Spd} F$ by sending $S = \operatorname{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$ with an untilt $S^\sharp = \operatorname{Spa}(R^\sharp, R^{\sharp,+})$ to $B_{\mathrm{dR}}(R^\sharp)$ and $B_{\mathrm{dR}}^+(R^\sharp)$, where $B_{\mathrm{dR}}(R^\sharp)$ and $B_{\mathrm{dR}}^+(R^\sharp)$ are defined as in [Far16, Definition 1.32]. We put $\mathrm{Gr}_G = LG/L^+G$ and

$$\mathcal{H}ck_G = [L^+G \backslash LG / L^+G].$$

For $A_1, A_2 \in D_\blacksquare(\mathcal{H}ck_G, \Lambda)$, let $A_1 \star A_2$ denote the convolution product of A_1 and A_2. Let Q be a finite quotient of W_F such that the action of W_F on $\widehat{G}$ factors through Q. Let

$$\mathcal{S}'\colon \operatorname{Rep}_\Lambda(\widehat{G} \rtimes Q) \longrightarrow D_\blacksquare(\mathcal{H}ck_G, \Lambda)$$

denote the functor that gives the geometric Satake equivalence (*cf.* [FS21, IX.2]). This functor is symmetric monoidal functor by the construction (*cf.* [FS21, Proposition VI.10.2]).

For $V_1, V_2 \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)$, let

$$c_{V_1,V_2} \colon \mathcal{S}'(V_1) \star \mathcal{S}'(V_2) \simeq \mathcal{S}'(V_2) \star \mathcal{S}'(V_1)$$

be the commutativity constraint uniquely characterized by

$$\begin{array}{ccc} \mathcal{S}'(V_1) \star \mathcal{S}'(V_2) & \xrightarrow{c_{V_1,V_2}} & \mathcal{S}'(V_2) \star \mathcal{S}'(V_1) \\ \downarrow & & \downarrow \\ \mathcal{S}'(V_1 \otimes V_2) & \xrightarrow{S'(\sigma_{V_1,V_2})} & \mathcal{S}'(V_2 \otimes V_1), \end{array}$$

where $\sigma_{V_1,V_2} \colon V_1 \otimes V_2 \to V_2 \otimes V_1$ is the isomorphism switching V_1 and V_2.

Assume that $\mu \in X_*(T)^+$. Let E_μ be the reflex field. Let $Q_\mu \subset Q$ be the image of W_{E_μ}. Let $r_{G,\mu}$ be the highest weight μ irreducible representation of $\widehat{G} \rtimes Q_\mu$. We simply write r_μ for $r_{G,\mu}$ if there is no confusion. We write V_μ for the representation space of r_μ. We put $\mathrm{IC}'_\mu = \mathcal{S}'(V_\mu)$, where $\mathcal{S}'$ is the one for G_{E_μ}. We use the same notation IC'_μ for the pullback of IC'_μ to other spaces.

2 Moduli of local shtukas

Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$. We put $W_{\mathcal{O}_F}(R^+) = W(R^+) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_F$. Take an topological nilpotent unit ϖ_R in R. Let $\mathcal{Y}_{(0,\infty)}(S)$ be the adic space defined by the condition $p \neq 0$ and $[\varpi_R] \neq 0$ in $\mathrm{Spa}(W_{\mathcal{O}_F}(R^+), W_{\mathcal{O}_F}(R^+))$. Then $\mathcal{Y}_{(0,\infty)}(S)$ has an action of the q-th power Frobenius element φ_S induced by the q-th power map on R. The quotient

$$X_S = \mathcal{Y}_{(0,\infty)}(S)/\varphi_S^{\mathbb{Z}}$$

is called the relative Fargues–Fontaine curve for S (*cf.* [SW20, Definition 15.2.6]). The construction glues together to give X_S for any $S \in \mathrm{Perf}_{\mathbb{F}_q}$.

We define a continuous map

$$\kappa_S \colon \mathcal{Y}_{(0,\infty)}(S) \longrightarrow (0, \infty)$$

by

$$\kappa_S(x) = \frac{\log|[\varpi_R]|_{\widetilde{x}}}{\log|p|_{\widetilde{x}}}$$

where $\widetilde{x}$ is the maximal generalization of $x \in \mathcal{Y}_{(0,\infty)}(S)$ and $|\cdot|_{\widetilde{x}}$ denotes the valuation corresponding to $\widetilde{x}$. For an interval I in $(0,\infty)$, let $\mathcal{Y}_I(S)$ denote the interior of $\kappa_S^{-1}(I)$. For $S \in \mathrm{Perf}_{\mathbb{F}_q}$, we put $\mathbb{B}(S) = \mathcal{O}(\mathcal{Y}_{(0,\infty)}(S))$. Then $\mathbb{B}$ is a v-sheaf by [FS21, Proposition II.2.1].

Let G be a connected reductive group over F. Let $b \in G(\breve{F})$. We define an algebraic group G_b over F by

$$G_b(R) = \{g \in G(\breve{F} \otimes_F R) \mid g(b\sigma \otimes 1) = (b\sigma \otimes 1)g\}$$

for any F-algebra R. We define a G-bundle $\mathscr{E}_{b,X_S}$ on X_S by

$$(G_{\breve{F}} \times_{\mathrm{Spa}(\breve{F})} \mathcal{Y}_{(0,\infty)}(S))/((b\sigma) \times \varphi_S)^{\mathbb{Z}}.$$

If $b' = g^{-1}b\sigma(g)$ for $b, b', g \in G(\breve{F})$, then the left multiplication by g^{-1} induces an isomorphim

$$t_g \colon \mathscr{E}_{b,X_S} \to \mathscr{E}_{b',X_S}. \tag{2.1}$$

We define a sheaf $\widetilde{J}_b$ on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ by

$$\widetilde{J}_b(S) = \mathrm{Aut}(\mathscr{E}_{b,X_S})$$

for $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. In the sequel, we simply write $\mathscr{E}_b$ for $\mathscr{E}_{b,X_S}$ if there is no confusion. We define $\widetilde{J}_b^{>0}$ as in [FS21, III.5]. Then we have $\widetilde{J}_b = \widetilde{J}_b^{>0} \rtimes \underline{G_b(F)}$ by [FS21, Proposition III.5.1]. If b is basic, we have $\widetilde{J}_b = \underline{G_b(F)}$.

Let $b, b' \in G(\breve{F})$. Let $\mu_1, \ldots, \mu_m$ be cocharacters of G. We put $\mu_\bullet = (\mu_1, \ldots, \mu_m)$. For $1 \leq i \leq m$, let E_i be the field of definition of μ_i.

Definition 2.1. *We define the presheaf* $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}$ *by sending* $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ *to the isomorphism classes of the following objects;*

- *an untilt* $S_i^\sharp$ *of* S *over* $\breve{E}_i$ *for* $1 \leq i \leq m$,
- *a* G*-torsor* $\mathcal{P}$ *on* $\mathcal{Y}_{(0,\infty)}(S)$ *with an isomorphism*

  $$\varphi_{\mathcal{P}} \colon (\varphi_S^* \mathcal{P})|_{\mathcal{Y}_{(0,\infty)}(S) \setminus \bigcup_{i=1}^m S_i^\sharp} \simeq \mathcal{P}|_{\mathcal{Y}_{(0,\infty)}(S) \setminus \bigcup_{i=1}^m S_i^\sharp}$$

  which is meromorphic along the Cartier divisor $\bigcup_{i=1}^m S_i^\sharp \subset \mathcal{Y}_{(0,\infty)}(S)$ *and the relative position of* $\varphi_S^*\mathcal{P}$ *and* $\mathcal{P}$ *at* $S_i^\sharp$ *is bounded by* $\sum_{j | S_j^\sharp = S_i^\sharp} \mu_j$ *at all geometric rank 1 points for all* $1 \leq i \leq m$,
- *an isomorphism*

  $$\iota_{[r,\infty)} \colon \mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)} \simeq G \times \mathcal{Y}_{[r,\infty)}(S)$$

  for large enough r *under which* $\varphi_{\mathcal{P}}$ *is identified with* $b \times \varphi_S$ *and an isomorphism*

  $$\iota_{(0,r']} \colon \mathcal{P}|_{\mathcal{Y}_{(0,r']}(S)} \simeq G \times \mathcal{Y}_{(0,r']}(S)$$

  for small enough r' *under which* $\varphi_{\mathcal{P}}$ *is identified with* $b' \times \varphi_S$.

If there is no confusion, we simply write $\mathrm{Sht}^{\mu_\bullet}_{b,b'}$ for $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}$. If $\mu_\bullet = (\mu)$, we simply write $\mathrm{Sht}^{\mu}_{G,b,b'}$ for $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}$. We use similar abbreviations also for other spaces.

We define the right action of $\widetilde{J}_b \times \widetilde{J}_{b'}$ on $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}$ by

$$(\iota_{[r,\infty)}, \iota_{(0,r']}) \mapsto (g^{-1} \circ \iota_{[r,\infty)}, g'^{-1} \circ \iota_{(0,r']})$$

for $(g, g') \in \widetilde{J}_b \times \widetilde{J}_{b'}$.

We define $\mathrm{Gr}^{\mathrm{tw}}_{G,\mathrm{Spd}\, E_1 \times \cdots \times \mathrm{Spd}\, E_m, \leq \mu_\bullet}$ as in [SW20, Definition 23.4.1]. It is a spacial diamond by [SW20, Proposition 23.4.2]. We have a morphism

$$\pi^{\mu_\bullet}_{G,b,b'} \colon \mathrm{Sht}^{\mu_\bullet}_{G,b,b'} \to \mathrm{Gr}^{\mathrm{tw}}_{G,\mathrm{Spd}\, \breve{E}_1 \times \cdots \times \mathrm{Spd}\, \breve{E}_m, \leq \mu_\bullet}$$

defined by forgetting $\iota_{(0,r']}$. The morphism $\pi^{\mu_\bullet}_{G,b,b'}$ is a $\widetilde{J}_{b'}$-torsor over a locally spatial subdiamond of $\mathrm{Gr}^{\mathrm{tw}}_{G,\mathrm{Spd}\, \breve{E}_1 \times \cdots \times \mathrm{Spd}\, \breve{E}_m, \leq \mu_\bullet}$ by [Sch17, Proposition 11.20]. Hence, $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}$ is a diamond by [Sch17, Proposition 11.6] and [Far16, 2.5, 2.6.2].

We have a natural inversing morphism

$$\mathrm{Sht}^{\mu_\bullet}_{G,b,b'} \to \mathrm{Sht}^{\mu_\bullet^{-1}}_{G,b',b} \tag{2.2}$$

compatible with the action of $\widetilde{J}_b \times \widetilde{J}_{b'}$.

Let $B(G)$ be the set of σ-conjugacy classes in $G(\breve{F})$. We write $B(G)_{\mathrm{bas}}$ for the set of the basic elements in $B(G)$. Let μ be a cocharacter of G. We define $B(G,\mu)$ as in [Kot97, 6.2].

Assume that G is quasi-split. We fix subgroups $A \subset T \subset B$ of G where A is a maximal split torus, T is a maximal torus and B is a Borel subgroup. We write $X_*(A)^+$ and $X_*(T)^+$ for the dominant cocharacter of A and T. For $b \in G(\breve{F})$, we define $\nu_b \in X_*(A)^+_{\mathbb{Q}}$ as in [Far16, 2.2.2] using the slope morphism constructed in [Kot85, 4.2]. Let $B(G,\mu,[b])$ be the set of acceptable neutral elements in $B(G)$ for $(\mu,[b])$ (*cf.* [GI16, Definition 4.3]).

Lemma 2.2. *Assume that b is basic. The map*

$$G(\breve{F}) \to G(\breve{F}) = G_b(\breve{F});\ g \mapsto gb^{-1}$$

induces bijections $B(G) \to B(G_b)$, $B(G)_{\mathrm{bas}} \to B(G_b)_{\mathrm{bas}}$ and $B(G,\mu,[b]) \to B(G_b,\mu)$.

Proof. The claim follows from the equality

$$(g'g\sigma(g')^{-1})b^{-1} = g'(gb^{-1})(b\sigma(g')b^{-1})^{-1}$$

for $g, g' \in G(\breve{F})$. □

Proposition 2.3. *Assume that b' is basic. We have a natural isomorphism*

$$\mathrm{Sht}^{\mu_\bullet}_{G,b,b'} \xrightarrow{\sim} \mathrm{Sht}^{\mu_\bullet}_{G_{b'},bb'^{-1},1}$$

which is compatible with the action of $\widetilde{J}_b \times \widetilde{J}_{b'}$.

Proof. We can view $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}$ as a moduli space of modifications of G-torsors on a Fargues–Fontaine curve. The category of G-torsor is equivalent to the category of $G_{b'}$-torsor on a Fargues–Fontaine curve as explained in the proof of [SW20, Corollary 23.2.3]. The claim follows from this equivalence. □

Remark 2.4. *Assume that b, b' are basic and $m = 1$. Then a weak version of Kottwitz conjecture for $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}$ holds by [HKW22, Theorem 1.0.4], Lemma 2.2 and Proposition 2.3.*

Remark 2.5. *Assume that b, b' are basic and $m = 1$. Under the isomorphism in Proposition 2.3, the inversing morphism (2.2) is identified with the Faltings–Fargues isomorphism proved in [SW20, Corollary 23.2.3].*

Lemma 2.6. *Assume that b' is basic. If $\mathrm{Sht}^{\mu}_{G,b,b'}$ is not empty, then we have $[b] \in B(G,\mu,[b'])$.*

Proof. By Proposition 2.3, we may assume that $b' = 1$ dropping the assumption that G is quasi-split. Then the claim follows from [CS17, Proposition 3.5.3]. □

We define a Weil descent datum of $\widetilde{J}_b$ by

$$\widetilde{J}_b \to \widetilde{J}_{\sigma(b)} = \sigma^*(\widetilde{J}_b);\ f \mapsto t_b \circ f \circ t_b^{-1},$$

where t_b is defined in (2.1). Let ρ_G denote the half-sum of the positive roots of G with respect to T and B. We put $N_b = \langle 2\rho_G, \nu_b \rangle$.

Lemma 2.7. *Let Λ be a solid $\widehat{\mathbb{Z}}^p$-algebra. Let $f \colon \widetilde{J}_b^{>0} \to *$ be the structure morphism. Then we have an isomorphism $f_\natural \Lambda \simeq \Lambda$ compatible with the actions of W_F.*

Proof. This is proved in the same way as [GI16, Lemma 4.17] using Lemma 1.3 and the definition of the Weil descent datum. □

Let $\delta_b \colon G_b(F) \to \Lambda^\times$ be the character obtained by the action of $G_b(F)$ on D_f, where $f \colon \widetilde{J}_b^{>0} \to *$.

3 Cohomology of moduli of local shtukas

Let $\mu_\bullet = (\mu_1, \ldots, \mu_m) \in (X_*(T)^+)^m$. Let E be the field of definition of $\mu_\bullet$. The space $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}$ is the moduli space of $(S^\sharp, \mathscr{E}_b \to \mathscr{E}_{b'})$, where $S^\sharp$ is an ultilt over $\breve{E}$ and $\mathscr{E}_b \to \mathscr{E}_{b'}$ is a modification bounded by $\mu_\bullet$ along the Cartier divisor defined by $S^\sharp$.

Let $\mathbb{C}_p^\flat$ denote the tilt of $\mathbb{C}_p$. The untilt $\mathbb{C}_p$ of $\mathbb{C}_p^\flat$ determines a morphism $\mathrm{Spa}\,\mathbb{C}_p^\flat \to \mathrm{Spd}\,\mathbb{Q}_p$. For the arithmetic Frobenius element $\sigma_E \in \mathrm{Gal}(E^{\mathrm{ur}}/E)$, we take m such that $\sigma_E|_{F^{\mathrm{ur}}} = \sigma^m$ and define a Weil descent datum of $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}$ by

$$\mathrm{Sht}^{\mu_\bullet}_{G,b,b'} \to \mathrm{Sht}^{\mu_\bullet}_{G,\sigma^m(b),\sigma^m(b')} = \sigma_E^*(\mathrm{Sht}^{\mu_\bullet}_{G,b,b'});$$
$$(S^\sharp, \mathscr{E}_b \xrightarrow{f} \mathscr{E}_{b'}) \mapsto (S^\sharp, \mathscr{E}_{\sigma^m(b)} \xrightarrow{t_{m,b}^{-1}} \mathscr{E}_b \xrightarrow{f} \mathscr{E}_{b'} \xrightarrow{t_{m,b'}} \mathscr{E}_{\sigma^m(b')})$$

where we put $t_{m,b} = t_{\sigma^{m-1}(b)} \circ \cdots \circ t_b \colon \mathscr{E}_b \to \mathscr{E}_{\sigma^m(b)}$ using (2.1).

We have fiber products

$$\begin{array}{ccccc}
\widetilde{\mathcal{M}}_b^\circ & \xrightarrow{\widetilde{j}_b} & \widetilde{\mathcal{M}}_b & \xleftarrow{\widetilde{i}_b} & * \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_b^\circ & \xrightarrow{j_b} & \mathcal{M}_b & \xleftarrow{i_b} & [*/\underline{G_b(F)}] \\
\downarrow & & \downarrow{\scriptstyle \pi_b} & & \downarrow{\scriptstyle h_b} \\
\mathrm{Bun}_G^{<b} & \xrightarrow{j^b} & \mathrm{Bun}_G & \xleftarrow{i^b} & \mathrm{Bun}_G^b
\end{array}$$

and morphisms $q_b \colon \mathcal{M}_b \to [*/\underline{G_b(F)}]$ and $\widetilde{q}_b \colon \widetilde{\mathcal{M}}_b \to *$ as [FS21, V.3]. Here i_b and $\widetilde{i}_b$ are sections of q_b and $\widetilde{q}_b$ respectively explained in [FS21, Proposition V.3.6]. Then i^b, j^b and π_b factor through

$$i'^b \colon \mathrm{Bun}_G^b \to \mathrm{Bun}_G^{\leq b}, \quad j'^b \colon \mathrm{Bun}_G^{<b} \to \mathrm{Bun}_G^{\leq b}, \quad \pi'_b \colon \mathcal{M}_b \to \mathrm{Bun}_G^{\leq b}.$$

Lemma 3.1. *The functors $h_{b,\natural}$ and $Rh_{b,*}$ are quasi-inverses of the equivalence*

$$h_b^* \colon D_{\mathrm{lis}}(\mathrm{Bun}_G^b, \Lambda) \to D_{\mathrm{lis}}([*/\underline{G_b(F)}], \Lambda)$$

of categories.

Proof. Since h_b^* is an equivalence of categories by [FS21, Proposition VII.7.1], its left adjoint $h_{b,\natural}$ and right adjoin $h_{b,*}$ give quasi-inverses. □

We define $\widetilde{i}_{b,!}\colon D_{\mathrm{lis}}(*,\Lambda) \to D_{\mathrm{lis}}(\widetilde{\mathcal{M}}_b,\Lambda)$ and $\widetilde{i}_b^!\colon D_{\mathrm{lis}}(\widetilde{\mathcal{M}}_b,\Lambda) \to D_{\mathrm{lis}}(*,\Lambda)$ by

$$\widetilde{i}_{b,!} = \mathrm{cone}(\widetilde{j}_{b,\natural}\widetilde{j}_b^* \to \mathrm{id}) \circ \widetilde{q}_b^*, \quad \widetilde{i}_b^! = R\widetilde{q}_{b,\mathrm{lis}\,*} \circ \mathrm{fib}(\mathrm{id} \to R\widetilde{j}_{b,\mathrm{lis}\,*}\widetilde{j}_b^*).$$

Then $\widetilde{i}_!^b$ is a left adjoint of $\widetilde{i}^{b,!}$. We define $i_{b,!}\colon D_{\mathrm{lis}}([*/\underline{G_b(F)}],\Lambda) \to D_{\mathrm{lis}}(\mathcal{M}_b,\Lambda)$ and $i_b^!\colon D_{\mathrm{lis}}(\mathcal{M}_b,\Lambda) \to D_{\mathrm{lis}}([*/\underline{G_b(F)}],\Lambda)$ by

$$i_{b,!} = \mathrm{cone}(j_{b,\natural}j_b^* \to \mathrm{id}) \circ q_b^*, \quad i_b^! = Rq_{b,\mathrm{lis}\,*} \circ \mathrm{fib}(\mathrm{id} \to Rj_{b,\mathrm{lis}\,*}j_b^*).$$

Then $i_!^b$ is a left adjoint of $i^{b,!}$. Further we define $i_!'^b\colon D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G^{\leq b},\Lambda)$ and $i'^{b,!}\colon D_{\mathrm{lis}}(\mathrm{Bun}_G^{\leq b},\Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda)$ by

$$i_!'^b = \pi'_{b,\natural} \circ i_{b,!} \circ h_b^*, \quad i'^{b,!} = Rh_{b,*} \circ i_b^! \circ \pi_b'^*.$$

Then $i_!'^b$ is a left adjoint of $i'^{b,!}$. We define $i_!^b\colon D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$ and $i^{b,!}\colon D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda) \to D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda)$ by

$$i_!^b = j_\natural^{\leq b} \circ i_!'^b, \quad i^{b,!} = i'^{b,!} \circ j^{\leq b,*}.$$

Then $i_!'^b$ is a left adjoint of $i'^{b,!}$.

Lemma 3.2. *For $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G^{\leq b},\Lambda)$, there is a distinguished triangle $A_1 \to A \to A_2 \to$ where $A_1 \in j_\natural^b D_{\mathrm{lis}}(\mathrm{Bun}_G^{<b},\Lambda)$ and $A_2 \in i_!^b D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda)$. Further, the full subcategories $j_\natural^b D_{\mathrm{lis}}(\mathrm{Bun}_G^{<b},\Lambda)$ and $i_!^b D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda)$ of $D_{\mathrm{lis}}(\mathrm{Bun}_G^{\leq b},\Lambda)$ are equivalent to $D_{\mathrm{lis}}(\mathrm{Bun}_G^{<b},\Lambda)$ and $D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda)$ by the restrictions respectively.*

Further similar claims hold for $\widetilde{\mathcal{M}}_b$ and $\mathcal{M}_b$.

Proof. The claim for $\mathrm{Bun}_G^{\leq b}$ is proved in the proof of [FS21, Proposition VII.7.3]. The claims for $\widetilde{\mathcal{M}}_b$ and $\mathcal{M}_b$ are proved in the same way. □

Lemma 3.3. *(1) We have isomorphisms*

$$\mathrm{cone}(\widetilde{j}_{b,\natural}\widetilde{j}_b^* \to \mathrm{id}) \cong \widetilde{i}_{b,!}\widetilde{i}_b^*, \quad R\widetilde{i}_{b,\mathrm{lis}\,*}\widetilde{i}_b^! \cong \mathrm{fib}(\mathrm{id} \to R\widetilde{j}_{b,\mathrm{lis}\,*}\widetilde{j}_b^*).$$

(2) We have isomorphisms

$$\mathrm{cone}(j_{b,\natural}j_b^* \to \mathrm{id}) \cong i_{b,!}i_b^*, \quad Ri_{b,\mathrm{lis}\,*}i_b^! \cong \mathrm{fib}(\mathrm{id} \to Rj_{b,\mathrm{lis}\,*}j_b^*).$$

(3) We have isomorphisms

$$\mathrm{cone}(j_\natural'^b j'^{b,*} \to \mathrm{id}) \cong i_!'^b i'^{b,*}, \quad Ri'^b_{\mathrm{lis}\,*}i'^{b,!} \cong \mathrm{fib}(\mathrm{id} \to Rj'^b_{\mathrm{lis}\,*}j'^{b,*}).$$

Proof. Let $A \in D_{\mathrm{lis}}(\mathcal{M}_b,\Lambda)$. By Lemma 3.2, there is $A_1 \in D_{\mathrm{lis}}(\mathcal{M}_b^\circ,\Lambda)$ and $A_2 \in D_{\mathrm{lis}}([*/\underline{G_b(F)}],\Lambda)$ such that $j_{b,\natural}A_1 \to A \to i_{b,!}A_2 \to$ is a distinguished triangle. By taking j_b^* and i_b^*, we have $A_1 \cong j_b^*A$ and $A_2 \cong i_b^*A$. Hence we obtain the first isomorphism in (2). The second isomorphism in (2) follows from the first one by taking the right adjoint.

The other claims are proved in the same way using Lemma 3.2. □

Lemma 3.4. *For $A \in D_{\mathrm{lis}}(\mathcal{M}_b, \Lambda)$ and $B \in D_{\mathrm{lis}}([*/\underline{G_b(F)}], \Lambda)$, we have an isomorphism $i_{b,!}(i_b^*(A) \otimes^{\mathbb{L}} B) \cong A \otimes^{\mathbb{L}} i_{b,!}(B)$.*

Proof. We have

$$A \otimes^{\mathbb{L}} i_{b,!}(B) \cong \mathrm{cone}(j_{b,\natural} j_b^*(A \otimes^{\mathbb{L}} q_b^* B) \to A \otimes^{\mathbb{L}} q_b^* B) \cong i_{b,!} i_b^*(A \otimes^{\mathbb{L}} q_b^* B) \cong i_{b,!}(i_b^*(A) \otimes^{\mathbb{L}} B),$$

where we use Lemma 3.3 (2) at the second isomorphism. □

Lemma 3.5. *We have $\mathrm{fib}(D_{q_b} \to Rj_{b,\mathrm{lis}*} j_b^* D_{q_b}) \cong Ri_{b,\mathrm{lis}*}\Lambda$.*

Proof. By the change of coefficient and the inverse limit, we may assume that Λ is torsion. Then we have $\mathrm{fib}(D_{q_b} \to Rj_{b,\mathrm{lis}*} j_b^* D_{q_b}) \cong Ri_{b,\mathrm{lis}*} i_b^! q_b^! \Lambda \cong Ri_{b,\mathrm{lis}*}\Lambda$. □

Lemma 3.6. *We have $\mathbb{D} \circ i_{b,!} = i_{b,*} \circ \mathbb{D}$ and $i_b^! \circ \mathbb{D} = \mathbb{D} \circ i_b^*$.*

Proof. Let $A \in D_{\mathrm{lis}}(\mathcal{M}_b, \Lambda)$. We have

$$(i_b^! \circ \mathbb{D})(A) = i_b^! R\mathscr{H}om_{\mathrm{lis}}(A, D_{q_b}) \cong Rq_{b,\mathrm{lis}*}(R\mathscr{H}om_{\mathrm{lis}}(A, \mathrm{fib}(D_{q_b} \to Rj_{b,\mathrm{lis}*} j_b^* D_{q_b}))).$$

By Lemma 3.5, this is isomorphic to

$$\begin{aligned} Rq_{b,\mathrm{lis}*}(R\mathscr{H}om_{\mathrm{lis}}(A, i_{b,\mathrm{lis}*}\Lambda)) &\cong Rq_{b,\mathrm{lis}*}(Ri_{b,\mathrm{lis}*} R\mathscr{H}om_{\mathrm{lis}}(i_b^* A, \Lambda)) \\ &\cong R\mathscr{H}om_{\mathrm{lis}}(i_b^* A, \Lambda)) = (\mathbb{D} \circ i_b^*)(A). \end{aligned}$$

Hence we have $i_b^! \circ \mathbb{D} = \mathbb{D} \circ i_b^*$. Another claim follows from this by adjoint. □

The following lemma is already known (*cf.* [FS21, IX.3]).

Lemma 3.7. *We have $\mathbb{D} \circ i_!^b = i_*^b \circ \mathbb{D}$ and $i^{b,!} \circ \mathbb{D} = \mathbb{D} \circ i^{b,*}$.*

Proof. Let $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$. We have

$$\begin{aligned} h_b^*((i^{b,!} \circ \mathbb{D})(A)) &\cong i_b^! \pi_b^* R\mathscr{H}om_{\mathrm{lis}}(A, D_{\mathrm{Bun}_G}) \cong i_b^! R\mathscr{H}om_{\mathrm{lis}}(\pi_b^* A, \pi_b^* D_{\mathrm{Bun}_G}) \\ &\cong i_b^! \mathbb{D}((\pi_b^* A) \otimes D_{\pi_b}) \cong \mathbb{D}(i_b^* \pi_b^* A) \otimes i_b^* D_{\pi_b}^{-1}, \end{aligned}$$

where we use Lemma 3.4 at the second isomorphism and Lemma 3.6 at the fourth isomorphism. On the other hand we have

$$\begin{aligned} h_b^*((i_*^b \circ \mathbb{D})(A)) \cong h_b^* R\mathscr{H}om_{\mathrm{lis}}(i^{b,*} A, D_{\mathrm{Bun}_G^b}) &\cong R\mathscr{H}om_{\mathrm{lis}}(h_b^* i^{b,*} A, h_b^* D_{\mathrm{Bun}_G^b}) \\ &\cong \mathbb{D}(i_b^* \pi_b^* A) \otimes D_{h_b}^{-1}, \end{aligned}$$

where we use Lemma 3.4 at the second isomorphism. Hence $i^{b,!} \circ \mathbb{D} = \mathbb{D} \circ i^{b,*}$ follows from [Sch17, Proposition 23.12]. Another claim follows from this by adjoint. □

Lemma 3.8. *We have $\widetilde{i}_b^! \cong \widetilde{i}_b^*(\mathrm{fib}(\mathrm{id} \to R\tilde{j}_{b,\mathrm{lis}*} \tilde{j}_b^*))$, $i_b^! \cong i_b^*(\mathrm{fib}(\mathrm{id} \to Rj_{b,\mathrm{lis}*} j_b^*))$ and $i'^{b,!} \cong i'^{b,*}(\mathrm{fib}(\mathrm{id} \to Rj'^{b}_{\mathrm{lis}*} j'^{b,*}))$.*

Proof. For $A \in D_{\mathrm{lis}}(\mathcal{M}_b, \Lambda)$ and $B \in D_{\mathrm{lis}}([*/\underline{G_b(F)}], \Lambda)$, we have

$$\mathrm{Hom}(i_{b,!}(B), A) \cong \mathrm{Hom}(i_{b,!}(B), \mathrm{fib}(A \to Rj_{b,\mathrm{lis}*} j_b^* A)) \cong \mathrm{Hom}(B, i_b^*(\mathrm{fib}(A \to Rj_{b,\mathrm{lis}*} j_b^* A)))$$

by Lemma 3.2. Hence we obtain the second claim. Other claims are proved similarly. □

Lemma 3.9. *We have* $\widetilde{i}_b^{!}\widetilde{i}_{b,!} \cong \mathrm{id}$*,* $i_b^{!} i_{b,!} \cong \mathrm{id}$ *and* $i'^{b,!} i'^{b}_{!} \cong \mathrm{id}$.

Proof. We can check these using Lemma 3.8. □

Lemma 3.10. *We have* $\widetilde{i}_{b,!} \cong R\widetilde{i}_{b,\mathrm{lis}\,*}$*,* $i_{b,!} \cong Ri_{b,\mathrm{lis}\,*}$ *and* $i'^{b}_{!} \cong Ri'^{b}_{\mathrm{lis}\,*}$.

Proof. By Lemma 3.3, Lemma 3.8 and Lemma 3.9, we have

$$i_b^* Ri_{b,\mathrm{lis}\,*} \cong i_b^* Ri_{b,\mathrm{lis}\,*} i_b^! i_{b,!} \cong i_b^* \operatorname{fib}(\mathrm{id} \to Rj_{b,\mathrm{lis}\,*} j_b^*) i_{b,!} \cong i_b^! i_{b,!} \cong \mathrm{id}\,.$$

Hence $i_{b,!} \cong i_{b,\mathrm{lis}\,*}$ follows from Lemma 3.2 using Lemma 1.6. Other claims are proved similarly. □

For a compact open subgroup K of $G_b(F)$, we consider the fiber products

$$\begin{array}{ccccc}
\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b',\mathbb{C}_p^\flat} & \xrightarrow{f_K} & \mathrm{Hck}^{\mu_\bullet}_{b'} & \xrightarrow{f_{b'}} & \operatorname{Spa}\mathbb{C}_p^\flat \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle t_{b'}} \\
 & & \mathrm{Hck}^{\mu_\bullet} & \xrightarrow{p_{2,X}} & \mathrm{Bun}_G \times \mathrm{Div}_X^m \\
 & & \downarrow{\scriptstyle p_1} & & \\
[*/K] \xrightarrow{h_K} & \mathrm{Bun}_G^b & \xrightarrow{i^b} & \mathrm{Bun}_G &
\end{array}$$

where h_K and $t_{b'}$ are the compositions

$$[*/K] \xrightarrow{h_{K,G_b(F)}} [*/G_b(F)] \xrightarrow{h_b} \mathrm{Bun}_G^b,$$
$$\operatorname{Spa}\mathbb{C}_p^\flat \longrightarrow \mathrm{Bun}_G^{b'} \times \mathrm{Div}_X^m \longrightarrow \mathrm{Bun}_G \times \mathrm{Div}_X^m$$

of the natural morphisms. Let $p_{1,b'} \colon \mathrm{Hck}^{\mu_\bullet}_{b'} \to \mathrm{Hck}^{\mu_\bullet} \xrightarrow{p_1} \mathrm{Bun}_G$. We put

$$f_{K,!}\Lambda = p_{1,b'}^* i^b_! h_{K,!}\Lambda.$$

Remark 3.11. *If b is basic,* f_K *is etale, in particular* ℓ*-cohomologically smooth. In this case, the above definition of* $f_{K,!}\Lambda$ *coincides with the general definition before.*

We put

$$R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'}) = f_{b',\natural}\big((f_{K,!}\Lambda) \otimes^{\mathbb{L}} \mathrm{IC}'_{\mu_\bullet}\big).$$

We can view

$$R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'}) \cong t_{b'}^* T_{\mu_\bullet}(i^b_! h_{K,!}\Lambda)$$

as an object of $D(G_b(F) \times W_E)$ by [FS21, Corollary IX.2.3]. For a compact open subgroup K' of $G_{b'}(F)$, we define $R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,b',K'})$ in the symmetric way. Since $\mathrm{IC}_{\mu_\bullet}$ and $\mathrm{IC}_{-\mu_\bullet}$ corresponds under the natural isomorphism $\mathrm{Sht}^{\mu_\bullet}_{G,b,b'} \simeq \mathrm{Sht}^{-\mu_\bullet}_{G,b',b}$, we have

$$R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,b',K'}) \cong t_b^* T_{-\mu_\bullet}(i^{b'}_! h_{K',!}\Lambda).$$

Remark 3.12. *If b is basic,* $R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'})$ *is identified with* $(f_{b'} \circ f_K)_\natural(\mathrm{IC}'_{\mu_\bullet})$*. We define* $R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'})$ *as above since we do not have a good definition of*

$$f_{K,!} \colon D_{\mathrm{lis}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b',\mathbb{C}_p^\flat}, \Lambda) \to D_{\mathrm{lis}}(\mathrm{Hck}^{\mu_\bullet}_{b'}, \Lambda)$$

for a general b.

We put
$$R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}) = \varinjlim_{K\subset G_b(F)} R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'}).$$

Lemma 3.13. *We have $q_{b,!}\circ i_{b,!} = \mathrm{id}$.*

Proof. Let $B \in D_{\mathrm{lis}}([*/\underline{G_b(F)}],\Lambda)$. Then we have $i_{b,!}(B) \cong \mathrm{cone}(j_{b,\natural}j_b^*\Lambda \to \Lambda) \otimes^{\mathbb{L}} q_b^* B$. Hence we have
$$(q_{b,!}\circ i_{b,!})(B) \cong q_{b,\natural}(\mathrm{cone}(j_{b,\natural}j_b^*\Lambda \to \Lambda)\otimes D_{q_b}^{-1}) \otimes^{\mathbb{L}} B.$$
It remains to show $q_{b,\natural}(\mathrm{cone}(j_{b,\natural}j_b^*\Lambda \to \Lambda)\otimes D_{q_b}^{-1}) \cong \Lambda$. It suffices to show this after taking a pullback via $\mathrm{Spa}\,\mathbb{C}_p^\flat \to [*/\underline{G_b(F)}]$ since the induced actions of $G_b(F)$ on the both sides are trivial. Let $j_U\colon U \to \widetilde{\mathcal{M}}_{b,\mathbb{C}_p^\flat}$ be a quasicompact open neighborhood of $\widetilde{i}_b(\mathrm{Spa}\,\mathbb{C}_p^\flat)$. We have
$$\widetilde{q}_{b,\natural}(\mathrm{cone}(\widetilde{j}_{b,\natural}\widetilde{j}_b^*\Lambda \to \Lambda)\otimes(\widetilde{q}_b^!\Lambda)^{-1}) \cong (\widetilde{q}_b\circ j_U)_\natural j_U^*(R\widetilde{i}_{b,\mathrm{lis}\,*}(\Lambda)\otimes D_{\widetilde{q}_b}^{-1}).$$
Then the question is reduced to the torsion case by by Lemma 1.1, since $\widetilde{q}_b\circ j_U$ is quasicompact, separated by [FS21, Proposition V.3.5]. In the torsion case, the claim follows from [FS21, Proposition VII.5.2] and $\mathrm{cone}(j_{b,\natural}j_b^*\Lambda \to \Lambda) \cong i_{b,!}(\Lambda)$. □

Lemma 3.14. *For $A \in D_{\mathrm{lis}}(\mathcal{M}_b,\Lambda)$ and $B \in D_{\mathrm{lis}}([*/\underline{G_b(F)}],\Lambda)$, we have an isomorphism $q_{b,\natural}(A\otimes^{\mathbb{L}} i_{b,!}B) \cong i_b^*(A\otimes D_{q_b})\otimes^{\mathbb{L}} B$.*

Proof. We have
$$\begin{aligned} A\otimes^{\mathbb{L}} i_{b,!}B &\cong \mathrm{cone}(j_{b,\natural}j_b^*A \to A)\otimes^{\mathbb{L}} q_b^*B \\ &\cong \mathrm{cone}(j_{b,\natural}j_b^*(A\otimes D_{q_b}) \to A\otimes D_{q_b})\otimes D_{q_b}^{-1}\otimes^{\mathbb{L}} q_b^*B \\ &\cong (i_{b,!}i_b^*(A\otimes D_{q_b}))\otimes D_{q_b}^{-1}\otimes^{\mathbb{L}} q_b^*B, \end{aligned}$$
where we use Lemma 3.3 (2) at the last isomorphism. Hence we have
$$\begin{aligned} q_{b,\natural}(A\otimes^{\mathbb{L}} i_{b,!}B) &\cong q_{b,\natural}((i_{b,!}i_b^*(A\otimes D_{q_b}))\otimes D_{q_b}^{-1})\otimes^{\mathbb{L}} B \\ &\cong q_{b,!}(i_{b,!}i_b^*(A\otimes D_{q_b}))\otimes^{\mathbb{L}} B \cong i_b^*(A\otimes D_{q_b})\otimes^{\mathbb{L}} B, \end{aligned}$$
where we use Lemma 3.13 at the last isomorphism. □

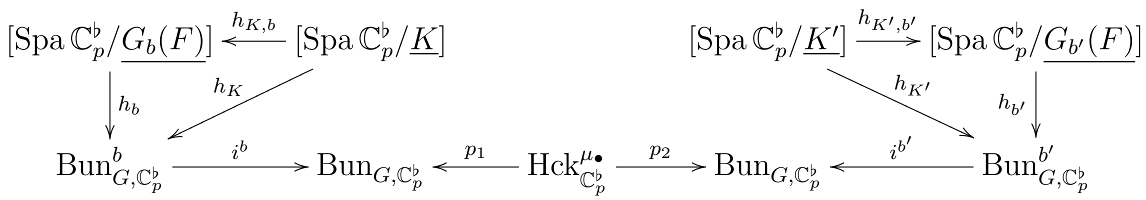
Lemma 3.15. *Let $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$ and $B \in D_{\mathrm{lis}}(\mathrm{Bun}_G^b,\Lambda)$. Then we have*
$$R\Gamma_\natural(\mathrm{Bun}_G, A\otimes^{\mathbb{L}} i_!^bB) \cong R\Gamma_\natural([*/\underline{G_b(F)}], h_b^*(i^{b,*}A\otimes^{\mathbb{L}} B)\otimes^{\mathbb{L}} i_b^*D_{q_b}).$$

Proof. We have
$$\begin{aligned} R\Gamma_\natural(\mathrm{Bun}_G, A\otimes^{\mathbb{L}} i_!^bB) &\cong R\Gamma_\natural(\mathcal{M}_b, \pi_b^*A\otimes^{\mathbb{L}} i_{b,!}h_b^*B) \\ &\cong R\Gamma_\natural([*/\underline{G_b(F)}], i_b^*(\pi_b^*A\otimes D_{q_b})\otimes^{\mathbb{L}} h_b^*B) \\ &\cong R\Gamma_\natural([*/\underline{G_b(F)}], h_b^*(i^{b,*}A\otimes^{\mathbb{L}} B)\otimes^{\mathbb{L}} i_b^*D_{q_b}), \end{aligned}$$
where we use Lemma 3.14 at the second isomorphism. □

Proposition 3.16. *We have a natural isomorphism*

$$R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'})_{K'} \cong R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,b',K'})_{K}.$$

Proof. We consider the following diagram:



We have

$$\begin{aligned}
&R\Gamma_{\natural,\mathbb{C}_p^\flat}(\mathrm{Hck}^{\mu_\bullet}_{\mathbb{C}_p^\flat}, p_1^* i^b_! h_{K,!}\Lambda \otimes^{\mathbb{L}} p_2^* i^{b'}_! h_{K',!}\Lambda \otimes^{\mathbb{L}} \mathrm{IC}'_{\mu_\bullet}) && (3.1)\\
&\cong R\Gamma_{\natural,\mathbb{C}_p^\flat}(\mathrm{Bun}_{G,\mathbb{C}_p^\flat}, T_{\mu_\bullet}(i^b_! h_{K,!}\Lambda) \otimes^{\mathbb{L}} i^{b'}_! h_{K',!}\Lambda)\\
&\cong R\Gamma_{\natural,\mathbb{C}_p^\flat}([\mathrm{Spa}\,\mathbb{C}_p^\flat/\underline{G_{b'}(F)}], h_{b'}^*(i^{b,*}T_{\mu_\bullet}(i^b_! h_{K,!}\Lambda) \otimes^{\mathbb{L}} h_{K',!}\Lambda) \otimes i_{b'}^* D_{q_{b'}}), && (3.2)
\end{aligned}$$

where we use Lemma 3.15 at the last isomorphism. We have

$$\begin{aligned}
h_{b'}^* h_{K',!}\Lambda \otimes i_{b'}^* D_{q_{b'}} &\cong h_{b'}^* h_{b',\natural}((h_{K',b',\natural}\Lambda) \otimes D_{h_{b'}}^{-1}) \otimes i_{b'}^* D_{q_{b'}}\\
&\cong h_{K',b',\natural} h_{K',b'}^*(D_{h_{b'}}^{-1} \otimes i_{b'}^* D_{q_{b'}}),
\end{aligned} \qquad (3.3)$$

where we use Lemma 3.1 at the last isomorphism. By [HI24, Proposition 3.15, (4.1)], we have $D_{h_{b'}}^{-1} \otimes i_{b'}^* D_{q_{b'}} \cong \Lambda$. Hence (3.2) is isomorphic to

$$R\Gamma_{\natural,\mathbb{C}_p^\flat}\big([\mathrm{Spa}\,\mathbb{C}_p^\flat/\underline{G_{b'}(F)}], h_{b'}^* i^{b',*} T_{\mu_\bullet}(i^b_! h_{K,!}\Lambda) \otimes^{\mathbb{L}} h_{K',b',\natural}\Lambda\big) \cong R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'})_{K'}$$

by (3.3). Since (3.1) is symmetric with respect to (b,K) and (b',K'), the claim follows. □

Corollary 3.17. *We have* $R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}) \cong R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{-\mu_\bullet}_{G,b',b})$.

Proof. This follows from Proposition 3.16. □

Proposition 3.18. *(1) If K is pro-p, then $R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'})$ is a compact object in $D(G_{b'}(F),\Lambda)$.*

(2) For $i \in \mathbb{Z}$, $H^i_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'})$ is finitely generated smooth $G_{b'}(F)$-representation.

(3) If ρ is an admissible representation of $G_{b'}$ over Λ, then $R\,\mathrm{Hom}_{G_{b'}}(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'}),\rho)$ is a perfect complex of Λ-modules.

(4) If $\Lambda = \overline{\mathbb{Q}}_\ell$ and ρ is a finite length representation of $G_{b'}(F)$ over $\overline{\mathbb{Q}}_\ell$, then

$$\varinjlim_{K \subset G_b(F)} R^i\,\mathrm{Hom}_{G_{b'}(F)}(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'}),\rho)$$

is finite length representation of $G_b(F)$ for $i \in \mathbb{Z}$.

Proof. We have

$$\varinjlim_{K\subset G_b(F)} R\operatorname{Hom}_{G_{b'}(F)}(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,K,b'}),\rho) \cong \varinjlim_{K\subset G_b(F)} R\operatorname{Hom}_{G_{b'}(F)}(t^*_{b'}T_{\mu_\bullet}(i^b_! h_{K,!}\Lambda),\rho)$$
$$\cong \varinjlim_{K\subset G_b(F)} R\operatorname{Hom}_{G_b(F)}(h_{K,G_b(F),!}\Lambda, h^!_b i^{b,!} T_{\mu^\vee_\bullet} Ri^{b'}_{\mathrm{lis}\,*} Rh_{b',*}[\rho])$$
$$\cong h^!_b i^{b,!} T_{\mu^\vee_\bullet} Ri^{b'}_{\mathrm{lis}\,*} Rh_{b',*}[\rho].$$

Then the claims are proved in the same way as [FS21, IX.3] using Lemma 3.7. □

We put

$$H^*_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}) = \sum_{i\in\mathbb{Z}} (-1)^i R^i\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{G,b,b'}).$$

4 Convolution morphism and twist morphism

4.1 Convolution morphism

Let $\Delta_{m,\operatorname{Spd} F}$ denote the diagonal subspace of $(\operatorname{Spd} F)^m$. For $1 \leq i < j \leq m$, let $\mathrm{pr}_{i,j}\colon (\operatorname{Spd} F)^m \to (\operatorname{Spd} F)^2$ denote the projection to the (i,j)-component. We put

$$U_m = (\operatorname{Spd} F)^m \setminus \bigcup_{1\leq i<j\leq m} \mathrm{pr}^{-1}_{i,j}\left(\bigcup_{n\in\mathbb{Z}\setminus\{0\}} (\varphi\times 1)^n(\Delta_{2,\operatorname{Spd} F})\right).$$

This is an open subspace of $(\operatorname{Spd} F)^m$ which contains $\Delta_{m,\operatorname{Spd} F}$.

Let $b_0,\dots,b_m \in G(\breve{F})$ and $\mu_\bullet = (\mu_1,\dots,\mu_m)$ where $\mu_i \in X_*(T)$ for $1 \leq i \leq m$. We put

$$\mathrm{Sht}^{\mu_\bullet}_{G,b_0,b_m,U_m} = \mathrm{Sht}^{\mu_\bullet}_{G,b_0,b_m} \times_{(\operatorname{Spd} F)^m} U_m.$$

We define the convolution morphism

$$m_{b_\bullet,\mu_\bullet,U_m}\colon (\mathrm{Sht}^{\mu_1}_{G,b_0,b_1}\times\cdots\times\mathrm{Sht}^{\mu_m}_{G,b_{m-1},b_m}) \times_{(\operatorname{Spd} F)^m} U_m \to \mathrm{Sht}^{\mu_\bullet}_{G,b_0,b_m,U_m}$$

over $\operatorname{Spd}\breve{E}_1\times\cdots\times\operatorname{Spd}\breve{E}_m$ as follows. Let $S=\operatorname{Spa}(R,R^+)\in\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ and

$$(S^\sharp_i, \mathcal{P}_i, \varphi_{\mathcal{P}_i}, \iota_{(0,r],i}, \iota_{[r',\infty],i})_{1\leq i\leq m}$$

be objects giving an S-valued point of

$$(\mathrm{Sht}^{\mu_1}_{G,b_0,b_1}\times\cdots\times\mathrm{Sht}^{\mu_m}_{G,b_{m-1},b_m}) \times_{(\operatorname{Spd} F)^m} U_m.$$

Define $\mathcal{P}$ by gluing $\mathcal{P}_1|_{\mathcal{Y}_{(0,r]}(S)}$ and $\mathcal{P}_m|_{\mathcal{Y}_{[r',\infty)}(S)}$ by the following modifications:

- Modifications occur only at $\bigcup_{i=1}^m \bigcup_{n\geq 0}\varphi^{-n}(S^\sharp_i)$.
- Take $1 \leq i_0 \leq m$. Put

  $$I_{i_0} = \{1\leq i\leq m \mid S^\sharp_i = S^\sharp_{i_0}\}.$$

  Define the modification at $S^\sharp_{i_0}$ by the composite of the modifications at $S^\sharp_{i_0}$ given by $\varphi_{\mathcal{P}_i}$ for all $i\in I_{i_0}$. For $n>0$, the modification at $\varphi^{-n}(S^\sharp_{i_0})$ is given by the pullback under φ^n of the modification at $S^\sharp_{i_0}$.

Then $\mathcal{P}$ is naturally equipped with an isomorphism

$$\varphi_{\mathcal{P}} \colon (\varphi_S^* \mathcal{P})|_{\text{“}S \times \operatorname{Spa} F\text{”} \setminus \bigcup_{i=1}^m S_i^\sharp} \simeq \mathcal{P}|_{\text{“}S \times \operatorname{Spa} F\text{”} \setminus \bigcup_{i=1}^m S_i^\sharp}.$$

Further, we have isomorphisms

$$\mathcal{P}|_{\mathcal{Y}_{(0,r]}(S)} = \mathcal{P}_1|_{\mathcal{Y}_{(0,r]}(S)} \xrightarrow{\iota_{(0,r],1}} G \times \mathcal{Y}_{(0,r]}(S),$$
$$\mathcal{P}|_{\mathcal{Y}_{[r',\infty)}(S)} = \mathcal{P}_m|_{\mathcal{Y}_{[r',\infty)}(S)} \xrightarrow{\iota_{[r',\infty),m}} G \times \mathcal{Y}_{[r',\infty)}(S).$$

These gives an S-valued point of $\mathrm{Sht}^{\mu_\bullet}_{G,b_0,b_m,U_m}$. Thus we obtain $m_{b_\bullet,\mu_\bullet,U_m}$.

We define

$$\mathrm{Gr}_{G,\operatorname{Spd} E_1 \times \cdots \times \operatorname{Spd} E_m, \leq \mu_\bullet}, \quad \widetilde{\mathrm{Gr}}_{G,\operatorname{Spd} E_1 \times \cdots \times \operatorname{Spd} E_m, \leq \mu_\bullet}$$

as in [SW20, Definition 20.4.4]. Then we have a convolution morphism

$$m_{\mu_\bullet} \colon \widetilde{\mathrm{Gr}}_{G,\operatorname{Spd} E_1 \times \cdots \times \operatorname{Spd} E_m, \leq \mu_\bullet} \longrightarrow \mathrm{Gr}_{G,\operatorname{Spd} E_1 \times \cdots \times \operatorname{Spd} E_m, \leq \mu_\bullet}$$

by [SW20, Proposition 20.4.5]. Note that

$$\mathrm{Gr}_{G,\operatorname{Spd} E_1 \times \cdots \times \operatorname{Spd} E_m, \leq \mu_\bullet} \times_{(\operatorname{Spd} F)^m} U_m \simeq \mathrm{Gr}^{\mathrm{tw}}_{G,\operatorname{Spd} E_1 \times \cdots \times \operatorname{Spd} E_m, \leq \mu_\bullet} \times_{(\operatorname{Spd} F)^m} U_m.$$

Then we have a morphism

$$\mathrm{Sht}^{\mu_1}_{G,b_0,b_1} \times \cdots \times \mathrm{Sht}^{\mu_m}_{G,b_{m-1},b_m} \longrightarrow \widetilde{\mathrm{Gr}}_{G,\operatorname{Spd} \breve{E}_1 \times \cdots \times \operatorname{Spd} \breve{E}_m, \leq \mu_\bullet}$$

by looking at a modification at each $S_i^\sharp$. Then we have the commutative diagram

$$\begin{array}{ccc}
(\mathrm{Sht}^{\mu_1}_{G,b_0,b_1} \times \cdots \times \mathrm{Sht}^{\mu_m}_{G,b_{m-1},b_m}) \times_{(\operatorname{Spd} F)^m} U_m & \xrightarrow{m_{b_\bullet,\mu_\bullet,U_m}} & \mathrm{Sht}^{\mu_\bullet}_{G,b_0,b_m,U_m} \\
\downarrow & & \downarrow \\
\widetilde{\mathrm{Gr}}_{G,\operatorname{Spd} \breve{E}_1 \times \cdots \times \operatorname{Spd} \breve{E}_m, \leq \mu_\bullet} \times_{(\operatorname{Spd} F)^m} U_m & \longrightarrow & \mathrm{Gr}_{G,\operatorname{Spd} \breve{E}_1 \times \cdots \times \operatorname{Spd} \breve{E}_m, \leq \mu_\bullet} \times_{(\operatorname{Spd} F)^m} U_m
\end{array}$$

where the bottom morphism is induced by $m_{\mu_\bullet}$.

4.2 Twist morphism

Let Z^0 be the identity component of the center of G. Let $a, a' \in Z^0(\breve{F})$ and $\lambda \in X_*(Z^0)$. Let E be a finite extension of F in $\mathbb{C}_p$ containing the fields of definition of μ and λ. We define the morphism

$$t^{\mu,\lambda}_{b,b',a,a'} \colon \mathrm{Sht}^{\mu}_{G,b,b',\operatorname{Spd} \breve{E}} \times_{\operatorname{Spd} \breve{E}} \mathrm{Sht}^{\lambda}_{Z^0,a,a',\operatorname{Spd} \breve{E}} \longrightarrow \mathrm{Sht}^{\mu-\lambda}_{G,ab,a'b',\operatorname{Spd} \breve{E}}$$

as follows. Let $(S^\sharp, \mathscr{E}_b \to \mathscr{E}_{b'})$ and $(S^\sharp, \mathscr{E}_a \to \mathscr{E}_{a'})$ be modifications defining points in $\mathrm{Sht}^{\mu}_{G,b,b'}$ and $\mathrm{Sht}^{\lambda}_{Z^0,a,a'}$. Then the diagonal arrow in the diagram

$$\begin{array}{ccc}
\mathscr{E}_b \times^{Z^0} \mathscr{E}_{a'} & \longrightarrow & \mathscr{E}_{b'} \times^{Z^0} \mathscr{E}_{a'} \\
\uparrow & \searrow & \uparrow \\
\mathscr{E}_b \times^{Z^0} \mathscr{E}_a & \longrightarrow & \mathscr{E}_{b'} \times^{Z^0} \mathscr{E}_a
\end{array}$$

defines the image of

$$\big((S^\sharp, \mathscr{E}_b \to \mathscr{E}_{b'}), (S^\sharp, \mathscr{E}_a \to \mathscr{E}_{a'})\big)$$

under $t^{\mu,\lambda}_{b,b',a,a'}$ in $\mathrm{Sht}^{\mu-\lambda}_{G,ab,a'b',\operatorname{Spd} \breve{E}}$. Note that we have equalities $G_b(F) = G_{ab}(F)$ and $G_{b'}(F) = G_{a'b'}(F)$.

Proposition 4.1. *We have*

$$\big(R\Gamma_c(\mathrm{Sht}^{\mu}_{G,b,b'}) \otimes^{\mathbb{L}} R\Gamma_c(\mathrm{Sht}^{\lambda}_{Z^0,a,a'})\big) \otimes^{\mathbb{L}}_{Z^0(F)} \overline{\mathbb{Q}}_\ell \simeq R\Gamma_c(\mathrm{Sht}^{\mu-\lambda}_{G,ab,a'b'})$$

in the derived category of representations of $G_b(F) \times G_{b'}(F) \times W_E$.

Proof. This follows from Lemma 1.4 and that $t^{\mu,\lambda}_{b,b',a,a'}$ is a $Z^0(F)$-torsor. □

5 Formula on cohomology

Let $b_0, \dots, b_m \in G(\breve{F})$ and $\mu_1, \dots, \mu_m \in X_*(T)^+$. Let E be a finite extension of F in $\mathbb{C}_p$ containing E_i for $1 \leq i \leq m$. Let

$$m_{b_\bullet,\mu_\bullet} \colon \mathrm{Sht}^{\mu_1}_{b_0,b_1,\mathrm{Spd}\,\breve{E}} \times_{\mathrm{Spd}\,\breve{E}} \cdots \times_{\mathrm{Spd}\,\breve{E}} \mathrm{Sht}^{\mu_m}_{b_{m-1},b_m,\mathrm{Spd}\,\breve{E}} \to \mathrm{Sht}^{|\mu_\bullet|}_{b_0,b_m,\mathrm{Spd}\,\breve{E}}$$

be the pullback of the convolution morphism $m_{b_\bullet,\mu_\bullet,U_m}$ defined in §4 under the morphism

$$\mathrm{Spd}\,\breve{E} = \Delta_{m,\mathrm{Spd}\,\breve{E}} \hookrightarrow (\mathrm{Spd}\,\breve{E})^m \longrightarrow \mathrm{Spd}\,\breve{E}_1 \times \cdots \times \mathrm{Spd}\,\breve{E}_m.$$

The morphism $m_{b_\bullet,\mu_\bullet}$ coincides with the morphism defined by the composition of modifications. This induces

$$\overline{m}_{b_\bullet,\mu_\bullet} \colon (\mathrm{Sht}^{\mu_1}_{b_0,b_1,\mathrm{Spd}\,\breve{E}} \times_{\mathrm{Spd}\,\breve{E}} \cdots \times_{\mathrm{Spd}\,\breve{E}} \mathrm{Sht}^{\mu_m}_{b_{m-1},b_m,\mathrm{Spd}\,\breve{E}})/(\widetilde{J}_{b_1} \times \cdots \times \widetilde{J}_{b_{m-1}}) \to \mathrm{Sht}^{|\mu_\bullet|}_{b_0,b_m,\mathrm{Spd}\,\breve{E}},$$

where $\widetilde{J}_{b_i}$ for $1 \leq i \leq m-1$ acts diagonally on the factor

$$\mathrm{Sht}^{\mu_i}_{b_{i-1},b_i,\mathrm{Spd}\,\breve{E}} \times_{\mathrm{Spd}\,\breve{E}} \mathrm{Sht}^{\mu_{i+1}}_{b_i,b_{i+1},\mathrm{Spd}\,\breve{E}}$$

and trivially on the other factors.

Let

$$\widetilde{\mathrm{Gr}}_{G,\mathrm{Spd}\,\breve{E},\leq\mu_\bullet} \xrightarrow{m_{\mu_\bullet}} \mathrm{Gr}_{G,\mathrm{Spd}\,\breve{E},\leq|\mu_\bullet|}$$

be the pullback of

$$m_{\mu_\bullet} \colon \widetilde{\mathrm{Gr}}_{G,\mathrm{Spd}\,E_1 \times \cdots \times \mathrm{Spd}\,E_m,\leq\mu_\bullet} \longrightarrow \mathrm{Gr}_{G,\mathrm{Spd}\,E_1 \times \cdots \times \mathrm{Spd}\,E_m,\leq\mu_\bullet}$$

under

$$\mathrm{Spd}\,\breve{E} = \Delta_{m,\mathrm{Spd}\,\breve{E}} \hookrightarrow (\mathrm{Spd}\,\breve{E})^m \longrightarrow \mathrm{Spd}\,E_1 \times \cdots \times \mathrm{Spd}\,E_m.$$

We define $m_{\mu_\bullet,b_0,b_m} \colon \mathrm{Sht}^{\mu_\bullet}_{b_0,b_m,\mathrm{Spd}\,\breve{E}} \to \mathrm{Sht}^{|\mu_\bullet|}_{b_0,b_m,\mathrm{Spd}\,\breve{E}}$ by the fiber product

$$\begin{array}{ccc} \mathrm{Sht}^{\mu_\bullet}_{b_0,b_m,\mathrm{Spd}\,\breve{E}} & \xrightarrow{m_{\mu_\bullet,b_0,b_m}} & \mathrm{Sht}^{|\mu_\bullet|}_{b_0,b_m,\mathrm{Spd}\,\breve{E}} \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{Gr}}_{G,\mathrm{Spd}\,\breve{E},\leq\mu_\bullet} & \xrightarrow{m_{\mu_\bullet}} & \mathrm{Gr}_{G,\mathrm{Spd}\,\breve{E},\leq|\mu_\bullet|}. \end{array}$$

Then $\mathrm{Sht}^{\mu_\bullet}_{b_0,b_m}$ is a moduli space of modifications

$$\mathscr{E}_{b_0} \xrightarrow{f_1} \mathscr{E}_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} \mathscr{E}_{m-1} \xrightarrow{f_m} \mathscr{E}_{b_m}$$

at $S^\sharp$ such that f_i is bounded by μ_i for $1 \le i \le m$. We define a subspace $\mathrm{Sht}_{b_0,b_m,\mathrm{Spd}\,\breve{E}}^{b_1,\dots,b_{m-1},\mu_\bullet} \subset \mathrm{Sht}_{b_0,b_m,\mathrm{Spd}\,\breve{E}}^{\mu_\bullet}$ as a moduli space of modifications

$$\mathscr{E}_{b_0} \xrightarrow{f_1} \mathscr{E}_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} \mathscr{E}_{m-1} \xrightarrow{f_m} \mathscr{E}_{b_m}$$

at $S^\sharp$ such that f_i is bounded by μ_i for $1 \le i \le m$ and $\mathscr{E}_i$ is isomorphic to $\mathscr{E}_{b_i}$ geometric fiberwisely for $1 \le i \le m-1$.

We put

$$I_{b_0,b_m}^{\mu_\bullet} = \{([b_1],\dots,[b_{m-1}]) \in B(G)^{m-1} \mid \mathrm{Sht}_{b_{i-1},b_i}^{\mu_i} \neq \emptyset \text{ for } 1 \le i \le m\}.$$

We take μ_{m+1} such that $[b_m] \in B(G,\mu_{m+1},[1])$. Then $I_{b_0,b_m}^{\mu_\bullet}$ is a finite set, since it is contained in $\prod_{1 \le i \le m-1} B(G, \sum_{j=i+1}^{m+1} \mu_j, [1])$ by Lemma 2.6. For $\lambda \in X_*(T)^+/\Gamma_F$, we put

$$V_{\mu_\bullet}^\lambda = \mathrm{Hom}_{{}^L G}(V_\lambda, \bigotimes_{1 \le i \le m} V_{\mu_i}).$$

For $([b_i])_{1 \le i \le m-1} \in I_{b_0,b_m}^{\mu_\bullet}$, we put $N_{b_\bullet} = \sum_{1 \le i \le m-1} N_{b_i}$. We write $\mathrm{Gr}_{G,\mathrm{Spd}\,E,\le\mu}^{(\infty)}$ for the inverse image of $\mathrm{Gr}_{G,\mathrm{Spd}\,E,\le\mu}$ under $LG_{\mathrm{Spd}\,E} \to \mathrm{Gr}_{G,\mathrm{Spd}\,E}$.

Proposition 5.1. *The sum*

$$\sum_{\lambda \in X_*(T)^+/\Gamma} V_{\mu_\bullet}^\lambda \otimes^{\mathbb{L}} R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_0,b_m}^\lambda)$$

is decomposed into

$$\left(\bigotimes_{1 \le i \le m} R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_{i-1},b_i}^{\mu_i}) \otimes^{\mathbb{L}} \bigotimes_{1 \le i \le m-1} \delta_{b_i} \right) \otimes^{\mathbb{L}}_{\prod_{i=1}^{m-1} G_{b_i}(F)} \Lambda[2N_{b_\bullet}]$$

for $([b_i])_{1 \le i \le m-1} \in I_{b_0,b_m}^{\mu_\bullet}$ by distinguished triangles in the derived category of representations of $G_{b_0}(F) \times G_{b_m}(F) \times W_E$.

Proof. Let $\mathrm{IC}_{\mu_\bullet}$ be the external twisted product of $\mathrm{IC}_{\mu_1},\dots,\mathrm{IC}_{\mu_m}$ on $\widetilde{\mathrm{Gr}}_{\mathrm{Spd}\,\breve{E},\le\mu_\bullet}$. By the construction of convolution product [FS21, VI.8] in geometric Satake equivalence and [FS21, Proposition VII.4.3], we have

$$(m_{\mu_\bullet})_\natural \mathrm{IC}_{\mu_\bullet} = \sum_{\lambda \in X_*(T)^+/\Gamma} V_{\mu_\bullet}^\lambda \otimes^{\mathbb{L}} \mathrm{IC}_\lambda.$$

Hence the sum

$$\sum_{\lambda \in X_*(T)^+/\Gamma} V_{\mu_\bullet}^\lambda \otimes^{\mathbb{L}} R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_0,b_m}^\lambda)$$

is isomorphic to $R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_0,b_m}^{\mu_\bullet}, \mathrm{IC}_{\mu_\bullet})$.

We put $\mu'_\bullet = (\mu_1,\dots,\mu_{m-2})$. Let $\{[b_{m-1}^j]\}_{1 \le j \le n}$ be the image of the projection $I_{b_0,b_m}^{\mu_\bullet} \to B(G)$ to the $(m-1)$-th component. It suffices to show that $R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_0,b_m}^{\mu_\bullet}, \mathrm{IC}_{\mu_\bullet})$ is decomposed into

$$\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_0,b_{m-1}^j}^{\mu'_\bullet}) \otimes^{\mathbb{L}} R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_{m-1}^j,b_m}^{\mu_m}) \otimes^{\mathbb{L}} \delta_{b_{m-1}^j} \right) \otimes^{\mathbb{L}}_{G_{b_{m-1}^j}(F)} \Lambda[2N_{b_{m-1}^j}] \tag{5.1}$$

for $1 \leq j \leq n$.

Let $K \subset G_{b_0}(F)$ be enough small compact open subgroup. Then

$$R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_\bullet}_{b_0,K,b_m}, \mathrm{IC}_{\mu_\bullet}) \cong t^*_{b_m} i^*_{b_m} T_{\mu_\bullet} i_{b_0,!}(h_{K,!}\Lambda) \cong t^*_{b_m} i^*_{b_m} T_{\mu_{m-1}} T_{\mu'_\bullet} i_{b_0,!}(h_{K,!}\Lambda)$$

is decomposed into

$$t^*_{b_m} i^*_{b_m} T_{\mu_{m-1}} i_{b^j_{m-1},!} i^*_{b^j_{m-1}} T_{\mu'_\bullet} i_{b_0,!}(h_{K,!}\Lambda)$$

for $1 \leq j \leq n$ by Lemma 3.3 (3). This is isomorphic to

$$t^*_{b_m} i^*_{b_m} T_{\mu_{m-1}} i_{b^j_{m-1},!}\big(\delta_{b^j_{m-1}}[2N_{b^j_{m-1}}] \otimes^{\mathbb{L}} h_{b^j_{m-1},!} h^*_{b^j_{m-1}} i^*_{b^j_{m-1}} T_{\mu'_\bullet} i_{b_0,!}(h_{K,!}\Lambda)\big) \tag{5.2}$$

by Lemma 2.7. By Lemma 1.4,

$$h^*_{b^j_{m-1}} i^*_{b^j_{m-1}} T_{\mu'_\bullet} i_{b_0,!}(h_{K,!}\Lambda) \cong \left(\Big(\varinjlim_{K' \subset G_{b^j_{m-1}}(F)} h_{K',\natural}\Lambda\Big) \otimes^{\mathbb{L}} R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu'_\bullet}_{b_0,K,b^j_{m-1}})\right) \otimes^{\mathbb{L}}_{\mathcal{H}(G_{b^j_{m-1}}(F))} \Lambda.$$

Hence (5.2) is isomorphic to

$$\Big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu'_\bullet}_{b_0,K,b^j_{m-1}}) \otimes^{\mathbb{L}} R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_m}_{b^j_{m-1},b_m}) \otimes^{\mathbb{L}} \delta_{b^j_{m-1}}\Big) \otimes^{\mathbb{L}}_{G_{b^j_{m-1}}(F)} \Lambda[2N_{b^j_{m-1}}]$$

since $t^*_{b_m} i^*_{b_m} T_{\mu_{m-1}} i_{b^j_{m-1},!}$ commutes with direct limits, tensors and changes of coefficients. Therefore we obtain the claim. □

Corollary 5.2. *We have*

$$\sum_{([b_i])_{1\leq i\leq m-1} \in I^{\mu_\bullet}_{b_0,b_m}} H_*\left(\prod_{i=1}^{m-1} G_{b_i}(F), \bigotimes_{1\leq i\leq m} H^*_{\mathrm{c}}(\mathrm{Sht}^{\mu_i}_{b_{i-1},b_i}) \otimes^{\mathbb{L}} \bigotimes_{1\leq i\leq m-1} \delta_{b_i}\right) = \sum_{\lambda \in X_*(T)^+/\Gamma} V^{\lambda}_{\mu_\bullet} \otimes^{\mathbb{L}} H^*_{\mathrm{c}}(\mathrm{Sht}^{\lambda}_{b_0,b_m})$$

as virtual representations of $G_{b_0}(F) \times G_{b_m}(F) \times W_E$.

Proof. This follows from Proposition 5.1 by taking cohomology. □

Lemma 5.3. *Assume that $m = 2$. Let π be a smooth representation of $G_{b_0}(F)$. Then we have*

$$\begin{aligned} &R\operatorname{Hom}_{G_{b_0}(F)}\Big((R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{b_0,b_1}) \otimes^{\mathbb{L}} R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_2}_{b_1,b_2}) \otimes^{\mathbb{L}} \delta_{b_1}) \otimes^{\mathbb{L}}_{G_{b_1}(F)} \Lambda, \pi\Big) \\ &\qquad \simeq R\operatorname{Hom}_{G_{b_1}(F)}\Big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_2}_{b_1,b_2}), R\operatorname{Hom}_{G_{b_0}(F)}\big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{b_0,b_1}), \pi\big) \otimes^{\mathbb{L}} \delta^{-1}_{b_1}\Big) \end{aligned}$$

in the derived category of representations of $G_{b_2}(F) \times W_E$ for $[b_1] \in I^{(\mu_1,\mu_2)}_{b_0,b_2}$.

Proof. We have

$$\begin{aligned} &R\operatorname{Hom}_{G_{b_0}(F)}\Big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{b_0,b_1}) \otimes R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_2}_{b_1,b_2}) \otimes^{\mathbb{L}} \delta_{b_1} \otimes^{\mathbb{L}}_{G_{b_1}(F)} \Lambda, \pi\Big) \\ &\qquad \simeq R\operatorname{Hom}_{G_{b_0}(F)\times G_{b_1}(F)}\big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{b_0,b_1}) \otimes R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_2}_{b_1,b_2}) \otimes^{\mathbb{L}} \delta_{b_1}, \Lambda \boxtimes \pi\big) \\ &\qquad \simeq R\operatorname{Hom}_{G_{b_0}(F)\times G_{b_1}(F)}\big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_2}_{b_1,b_2}) \otimes^{\mathbb{L}} \delta_{b_1}, \operatorname{Hom}\big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{b_0,b_1}), \pi\big)\big) \\ &\qquad \simeq R\operatorname{Hom}_{G_{b_1}(F)}\Big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_2}_{b_1,b_2}), R\operatorname{Hom}_{G_{b_0}(F)}\big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{b_0,b_1}), \pi\big) \otimes^{\mathbb{L}} \delta^{-1}_{b_1}\Big) \end{aligned}$$

in the derived category of representations of $G_{b_2}(F) \times W_E$. □

6 Duality morphism

Assume that 2 is invertible in Λ. We take a pinning $\mathcal{P} = (G, B, T, X_\alpha)$ of G. Then define a duality involution $\iota_{G,\mathcal{P}}$ on G as in [Pra19, Definition 1]. We simply write ι for $\iota_{G,\mathcal{P}}$. Note that $\mu = -\iota \circ \mu$ in $X_*(T)/W_G(T) \cong X_*(T)^+$. We define an anti-involution θ on G by $\theta(g) = \iota(g)^{-1}$. We define the duality morphism

$$\theta_{b,b'} \colon \mathrm{Sht}^{\mu}_{G,b,b'} \longrightarrow \mathrm{Sht}^{\mu}_{G,\iota(b'),\iota(b)}$$

by sending $f \colon \mathscr{E}_b \to \mathscr{E}_{b'}$ to $\iota(f)^{-1} \colon \mathscr{E}_{\iota(b')} \to \mathscr{E}_{\iota(b)}$. The above isomorphism is compatible with actions of $\widetilde{J}_b \times \widetilde{J}_{b'}$ and $\widetilde{J}_{\iota(b')} \times \widetilde{J}_{\iota(b)}$ under the isomorphism

$$\widetilde{J}_b \times \widetilde{J}_{b'} \longrightarrow \widetilde{J}_{\iota(b')} \times \widetilde{J}_{\iota(b)};\ (g, g') \mapsto (\iota(g'), \iota(g)).$$

Then $\theta_{b,\iota(b)}$ is an involution on $\mathrm{Sht}^{\mu}_{G,b,\iota(b)}$. On the other hand, θ induces a morphism $\theta \colon \mathcal{H}ck_G \to \mathcal{H}ck_G$. Let E be the field of definition of μ. We have a natural morphism

$$p^{\mu}_{b,b'} \colon \mathrm{Sht}^{\mu}_{G,b,b'} \longrightarrow \mathcal{H}ck_{G,\mathrm{Spd}\,\breve{E}}.$$

We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}^{\mu}_{G,b,b'} & \xrightarrow{\theta_{b,b'}} & \mathrm{Sht}^{\mu}_{G,\iota(b'),\iota(b)} \\ \downarrow{\scriptstyle p^{\mu}_{b,b'}} & & \downarrow{\scriptstyle p^{\mu}_{\iota(b'),\iota(b)}} \\ \mathcal{H}ck_{G,\mathrm{Spd}\,\breve{E}} & \xrightarrow{\theta} & \mathcal{H}ck_{G,\mathrm{Spd}\,\breve{E}}. \end{array}$$

We have $\mathcal{S}'(r_\mu \circ \mathrm{ad}(\widehat{\rho}(-1))) \cong \theta^* \mathrm{IC}'_\mu$ by [FS21, Proposition VI.12.1]. Hence $\widehat{\rho}(-1) \colon r_\mu \circ \mathrm{ad}(\widehat{\rho}(-1)) \to r_\mu$ induces $M_\mu \colon \theta^* \mathrm{IC}'_\mu \to \mathrm{IC}'_\mu$. Hence we obtain the isomorphism

$$R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{G,\iota(b'),\iota(b)}) \to R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{G,b,b'})$$

induced by $\theta_{b,b'}$.

Lemma 6.1. *The isomorphism*

$$R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{G,\iota(b'),\iota(b)}) \to R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{G,b,b'})$$

is compatible with actions of $\widetilde{J}_b \times \widetilde{J}_{b'}$ and $\widetilde{J}_{\iota(b')} \times \widetilde{J}_{\iota(b)}$ under the isomorphism

$$\widetilde{J}_b \times \widetilde{J}_{b'} \longrightarrow \widetilde{J}_{\iota(b')} \times \widetilde{J}_{\iota(b)};\ (g, g') \mapsto (\iota(g'), \iota(g)).$$

Proof. This follows from the definition. □

Further, we have an involution

$$\theta_b \colon \mathrm{Sht}^{\mu}_{G,b,1} \times \mathrm{Sht}^{\mu}_{G,1,\iota(b)} \longrightarrow \mathrm{Sht}^{\mu}_{G,b,1} \times \mathrm{Sht}^{\mu}_{G,1,\iota(b)};\ (x, x') \mapsto (\theta_{1,\iota(b)}(x'), \theta_{b,1}(x)).$$

We have a decomposition

$$V_\mu \otimes V_\mu = \mathrm{Sym}^2 V_\mu \oplus \bigwedge^2 V_\mu.$$

Let

$$\Psi_{b,\mu}\colon \Big(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{b,1})\otimes R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{1,\iota(b)})\Big)\otimes^{\mathbb{L}}_{G(F)}\Lambda \to \sum_{\lambda\in X_*(T)^+/\Gamma} V^{\lambda}_{\mu_\bullet}\otimes R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\lambda}_{b,\iota(b)})$$

be the morphism given by Proposition 5.1. Let $s_{b,\mu}$ be the involution on the source of $\Psi_{b,\mu}$ induced by θ_b and the multiplication by $(-1)^{\langle 2\widehat{\rho},\mu\rangle}$. On the other hand, let $t_{b,\mu}$ be the involution on the target of $\Psi_{b,\mu}$ induced by the permutation σ_{V_μ,V_μ} on $V_\mu\otimes V_\mu$ and $\theta_{b,\iota(b)}\colon \mathrm{Sht}^{\lambda}_{b,\iota(b)}\to \mathrm{Sht}^{\lambda}_{b,\iota(b)}$.

Proposition 6.2. *The morphism $\Psi_{b,\mu}$ is compatible with the involutions $s_{b,\mu}$ and $t_{b,\mu}$.*

Proof. By the characterization of the commutativity constraint, the equality

$$\mathrm{IC}'_\mu\star\mathrm{IC}'_\mu=\sum_{\lambda\in X_*(T)^+/\Gamma} V^{\lambda}_{\mu_\bullet}\otimes\mathrm{IC}'_\lambda$$

is compatible with the involutions c_{V_μ,V_μ} and σ_{V_μ,V_μ}. Hence the target of $\Psi_{b,\mu}$ is equal to $H^*_{\mathrm{c}}(\mathrm{Sht}^{2\mu}_{b,\iota(b)},\mathrm{IC}'_\mu\star\mathrm{IC}'_\mu)$ with the involution given by c_{V_μ,V_μ} and $\theta_{b,\iota(b)}$. Let $\sigma_{2,X}\colon(\mathrm{Div}^1_X)^2\to(\mathrm{Div}^1_X)^2$ and $\sigma_{2,G}\colon\mathcal{H}ck_G^{\{1,2\}}\to\mathcal{H}ck_G^{\{1,2\}}$ be the permutation of two Cartier divisors. Let $\mathrm{IC}'_\mu*\mathrm{IC}'_\mu$ be the fusion product on $\mathcal{H}ck_G^{\{1,2\}}$. Here we use the notation at the beginning of [FS21, VI.9]. Then we have the morphism

$$\widetilde{c}_{V_\mu,V_\mu}\colon\sigma^*_{2,G}(\mathrm{IC}'_\mu*\mathrm{IC}'_\mu)\to\mathrm{IC}'_\mu*\mathrm{IC}'_\mu$$

extending c_{V_μ,V_μ}.

The morphism θ induces $\theta^{\{1\},\{2\}}\colon\mathcal{H}ck_G^{\{1,2\};\{1\},\{2\}}\to\mathcal{H}ck_G^{\{1,2\};\{1\},\{2\}}$ switching two Cartier divisors. Here we use the notation in the proof of [FS21, Proposition VI.9.4]. Then we have a morphism

$$S_{\mu,\mu}\colon\theta^{\{1\},\{2\}*}(\mathrm{IC}'_\mu\boxtimes\mathrm{IC}'_\mu)\to\mathrm{IC}'_\mu\boxtimes\mathrm{IC}'_\mu$$

induced by M_μ and switching two factors of IC'_μ. The morphism θ induces $\theta^{\{1,2\}}\colon\mathcal{H}ck_G^{\{1,2\}}\to\mathcal{H}ck_G^{\{1,2\}}$ switching two Cartier divisors. Then we have

$$S'_{\mu,\mu}=m_\natural(S_{\mu,\mu})\colon\theta^{\{1,2\}*}(\mathrm{IC}'_\mu*\mathrm{IC}'_\mu)\to\mathrm{IC}'_\mu*\mathrm{IC}'_\mu\,.$$

Since $\theta^{\{1,2\}}\circ\sigma_{2,G}$ is the automorphism of $\mathcal{H}ck_G^{\{1,2\}}$ over $(\mathrm{Div}^1_X)^2$ induced by θ, we also have $M_{\mu,\mu}\colon(\theta^{\{1,2\}}\circ\sigma_{2,G})^*(\mathrm{IC}'_\mu*\mathrm{IC}'_\mu)\to\mathrm{IC}'_\mu*\mathrm{IC}'_\mu$ defined in the same way as M_μ.

Then it suffices to show that

$$\theta^{\{1,2\},*}(\mathrm{IC}'_\mu*\mathrm{IC}'_\mu)\xrightarrow{\sigma^*_{2,G}(M_{\mu,\mu})}\sigma^*_{2,G}(\mathrm{IC}'_\mu*\mathrm{IC}'_\mu))\xrightarrow{\widetilde{c}_{V_\mu,V_\mu}}\mathrm{IC}'_\mu*\mathrm{IC}'_\mu$$

and $S'_{\mu,\mu}$ are equal. It suffices to check this on $\mathcal{H}ck_G^{\{1,2\}}\times_{(\mathrm{Div}^1_X)^{\{1,2\}}}(\mathrm{Div}^1_X)^{\{1,2\};\{1\},\{2\}}$ by [FS21, Proposition VI.9.3]. This follows from the constructions of $\widetilde{c}_{V_\mu,V_\mu}$, $S'_{\mu,\mu}$ and $M_{\mu,\mu}$. □

7 Kottwitz conjecture

Definition 7.1. *Let $\varphi\colon W_F \to {}^L G$ be an ℓ-adic local L-parameter for G (cf. [Ima24, Definition 1.14]). We put*

$$S_\varphi = \{g \in \widehat{G}(\overline{\mathbb{Q}}_\ell) \mid g\varphi g^{-1} = \varphi\}.$$

We say that φ is discrete if $S_\varphi / Z(\widehat{G})^{\Gamma_F}$ is finite (cf. [Far16, Definition 4.1]).

Let $b, b' \in \mathrm{GL}_n(\breve{F})$ such that $[b] \in B(G, \mu, [b'])$. We put

$$H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}^{\mu}_{b,b'})[\pi] = \sum_{i,j \in \mathbb{Z}} (-1)^{i+j} \operatorname{Ext}^i_{G_b(F)} \left(R^j \Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{b,b'}), \pi \right)$$

for an irreducible smooth representation π of $G_b(F)$.

The following is a version of Kottwitz conjecture for moduli spaces of mixed characteristic local shtukas in GL_n-case (*cf.* [RV14, Conjecture 7.4]):

Conjecture 7.2. *Assume that b, b' are basic. Let $\varphi\colon W_F \to {}^L\mathrm{GL}_n$ be a discrete local L-parameter. Let π_b and $\pi_{b'}$ be the irreducible smooth representations of $G_b(F)$ and $G_{b'}(F)$ corresponding to φ via the local Langlands correspondence. Then we have*

$$H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}^{\mu}_{b,b'})[\pi_b] = \pi_{b'} \boxtimes (r_\mu \circ \varphi)$$

in $\mathrm{Groth}(G_{b'}(F) \times W_F)$.

For an object $\mathcal{C}$ in a derived category, we put $\mathcal{H}^*(\mathcal{C}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(\mathcal{C})$. The following conjecture is motivated by [Dat07, Théorème A].

Conjecture 7.3. *Assume that b, b' are basic. Let $\varphi\colon W_F \to {}^L\mathrm{GL}_n$ be a discrete local L-parameter. Let π_b and $\pi_{b'}$ be the irreducible smooth representations of $G_b(F)$ and $G_{b'}(F)$ corresponding to φ via the local Langlands correspondence. Then we have*

$$\mathcal{H}^* \left(R\operatorname{Hom}_{G_b(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu}_{b,b'}), \pi_b \right) \right) \simeq \pi_{b'} \boxtimes (r_\mu \circ \varphi)$$

as representations of $G_{b'}(F) \times W_F$.

Lemma 7.4. *Assume that b is basic. Let π_b and $\pi_{\iota(b)}$ be the irreducible smooth representations of $G_b(F)$ and $G_{\iota(b)}(F)$ corresponding via the local Jacquet–Langlands correspondence. Then the pullback of $\pi_{\iota(b)}$ under the isomorphism $\iota\colon G_b(F) \to G_{\iota(b)}(F)$ is isomorphic to π_b^*.*

Proof. By [Pra19, Corollary 1], we may assume that $\iota(g) = {}^t g^{-1}$. If $b = 1$, the calim follows from a theorem of Gelfand and Kazhdan (*cf.* [BZ76, 7.3. Theorem]). If regular elements $g \in \mathrm{GL}_n(F)$ and $g' \in G_b(F)$ have the same reduced characteristic polynomial, then $\iota(g) \in \mathrm{GL}_n(F)$ and $\iota(g') \in G_{\iota(b)}(F)$ are regular and have the same reduced characteristic polynomial. Hence the claim follows from the case where $b = 1$ and the characterization of the local Jacquet–Langlands correspondence. □

We put $\kappa(b) = v_F(\det(b))$. For $m_1, \ldots, m_n \in \mathbb{Z}$, let $(m_1, \ldots, m_n)$ denote the cocharacter of GL_n or its standard Levi subgroup defined by $z \mapsto \mathrm{diag}(z^{m_1}, \ldots, z^{m_n})$.

Theorem 7.5. *Conjecture 7.3 is true in the following cases:*

(1) $\kappa(b) \equiv \kappa(b') \mod n$ *and*

$$\mu = \frac{\kappa(b) - \kappa(b')}{n}(1, \ldots, 1).$$

(2) $\kappa(b) \equiv 0, 1,\ \kappa(b) \equiv \kappa(b') + 1 \mod n$ *and*

$$\mu = \frac{\kappa(b) - \kappa(b') - 1}{n}(1, \ldots, 1) + (1, 0, \ldots, 0).$$

(3) $\kappa(b) \equiv 0, -1,\ \kappa(b) \equiv \kappa(b') - 1 \mod n$ *and*

$$\mu = \frac{\kappa(b) - \kappa(b') + 1}{n}(1, \ldots, 1) + (0, \ldots, 0, -1).$$

(4) $\kappa(b) \equiv 1,\ \kappa(b') \equiv -1 \mod n$ *and*

$$\mu = \frac{\kappa(b) - \kappa(b') - 2}{n}(1, \ldots, 1) + \begin{cases} (2, 0, \ldots, 0), \\ (1, 1, 0, \ldots, 0). \end{cases}$$

(5) $\kappa(b) \equiv -1,\ \kappa(b') \equiv 1 \mod n$ *and*

$$\mu = \frac{\kappa(b) - \kappa(b') + 2}{n}(1, \ldots, 1) + \begin{cases} (0, \ldots, 0, -2), \\ (0, \ldots, 0, -1, -1). \end{cases}$$

Proof. By the inversing isomorphism (2.2), the claims (3) and (5) are reduced to the claims (2) and (4). By Proposition 4.1, we may assume that $\kappa(b) = \kappa(b') = 0$ in (1), $\kappa(b) = 0, -1$, $\kappa(b) = \kappa(b') + 1$ in (2) and $\kappa(b) = -1$, $\kappa(b') = 1$ in (4). Further, we may assume that $\kappa(b) = 0$ in (2) by Lemma 6.1 and Lemma 7.4. Then the claim (1) is trivial. The claim (2) follows from the proof of [Dat07, Thoérème A] taking care the degree in [Dat07, Thoérème 4.1.2].

We show the claim (4). We may assume that $b' = \iota(b)$. We put

$$\mu_1 = (1, 0, \ldots, 0),\ \mu_2 = (2, 0, \ldots, 0),\ \mu_{1,1} = (1, 1, 0, \ldots, 0).$$

Note that we have $I_{b,\iota(b)}^{(\mu_1,\mu_1)} = \{[1]\}$. Let π_1 be the irreducible smooth representations of $\mathrm{GL}_n(F)$ corresponding to φ via the local Langlands correspondence. By Proposition 5.1, Lemma 5.3, the claim (2) and [Dat07, Corollaire 4.2.1], we have

$$\begin{aligned}
&(V^{\mu_2}_{(\mu_1,\mu_1)})^* \otimes \mathcal{H}^* \left(R\operatorname{Hom}_{G_{\iota(b)}(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_2}_{\iota(b),b}), \pi_{\iota(b)} \right) \right) \\
&\quad + (V^{\mu_{1,1}}_{(\mu_1,\mu_1)})^* \otimes \mathcal{H}^* \left(R\operatorname{Hom}_{G_{\iota(b)}(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_{1,1}}_{\iota(b),b}), \pi_{\iota(b)} \right) \right) \\
&\qquad \simeq \mathcal{H}^* \left(R\operatorname{Hom}_{G_{\iota(b)}(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{\iota(b),1}) \otimes R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{1,b}) \otimes^{\mathbb{L}}_{\mathrm{GL}_n(F)} \overline{\mathbb{Q}}_\ell, \pi_{\iota(b)} \right) \right) \\
&\qquad \simeq \mathcal{H}^* \left(R\operatorname{Hom}_{\mathrm{GL}_n(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{1,b}), R\operatorname{Hom}_{G_{\iota(b)}(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{\iota(b),1}), \pi_{\iota(b)} \right) \right) \right) \\
&\qquad \simeq \mathcal{H}^* \left(R\operatorname{Hom}_{\mathrm{GL}_n(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{1,b}), \mathcal{H}^* \left(R\operatorname{Hom}_{G_{\iota(b)}(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{\iota(b),1}), \pi_{\iota(b)} \right) \right) \right) \right) \\
&\qquad \simeq \mathcal{H}^* \left(R\operatorname{Hom}_{\mathrm{GL}_n(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}^{\mu_1}_{1,b}), \pi_1 \boxtimes \varphi \right) \right) \\
&\qquad \simeq \pi_b \boxtimes (\varphi \otimes \varphi) \simeq \pi_b \boxtimes \left((r_{\mu_2} \circ \varphi) \oplus (r_{\mu_{1,1}} \circ \varphi) \right).
\end{aligned}$$

Using Proposition 6.2, we can separate the above equality to obtain the claim. □

Corollary 7.6. *Conjecture 7.3 is true if $n \leq 3$ and μ is minuscule.*

Proof. All the cases are contained in Theorem 7.5. □

8 Inductive formula

For a smooth representation π of $G(F)$ and the unipotent radical N of a parabolic subgroup of G, let π_N denote the Jacquet module of π with respect to N.

Assume that $G = \mathrm{GL}_2$. Let T be the diagonal torus and B be the upper triangle Borel subgroup of GL_2. Let N be the unipotent radical of B, and N^{op} be the the unipotent radical of the opposite Borel subgroup B^{op}. Let $\delta_B \colon T(F) \to \overline{\mathbb{Q}}_\ell^\times$ be the modulus character with respect to B. For $b = \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^l \end{pmatrix}$ with $m < l$, let $\delta_b \colon G_b(F) \to \overline{\mathbb{Q}}_\ell^\times$ be the character determined by δ_B and the natural isomorphism $G_b(F) \cong T(F)$.

Lemma 8.1. *Let $m \in \mathbb{Z}$. We put*

$$b = \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^m \end{pmatrix}, \quad b' = \begin{pmatrix} \varpi^{m-1} & 0 \\ 0 & \varpi^m \end{pmatrix}.$$

Let π be an admissible representation of $G(F)$. Then we have

$$R^\bullet \mathrm{Hom}_{G(F)} \left(R^\bullet \Gamma_{\mathrm{c}}(\mathrm{Sht}^{(1,0)}_{b,b'}), \pi \right) = -R^\bullet \mathrm{Hom}_{T(F)} \left(R^\bullet \Gamma_{\mathrm{c}}(\mathrm{Sht}^{(1,0)}_{T,b,b'}), \pi_{N^{\mathrm{op}}} \right) \left(\frac{1}{2} \right).$$

Proof. By [Cas82, A.11 Proposition, A.12 Theorem], [GI16, Theorem 4.25] (*cf.* [Han21a]) and [Ren10, III.2.7 Théorème, VI.9.6 Proposition], we have

$$\begin{aligned}
R^\bullet \mathrm{Hom}_{G(F)} \left(R^\bullet \Gamma_{\mathrm{c}}(\mathrm{Sht}^{(1,0)}_{b,b'}), \pi \right) &= R^\bullet \mathrm{Hom}_{G(F)} \left(\pi^*, R^\bullet \Gamma_{\mathrm{c}}(\mathrm{Sht}^{(1,0)}_{b,b'})^* \right) \\
&= R^\bullet \mathrm{Hom}_{G(F)} \left(\pi^*, - \left(\mathrm{Ind}_{B(F)}^{G(F)} R^\bullet \Gamma_{\mathrm{c}}(\mathrm{Sht}^{(1,0)}_{T,b,b'}) \otimes \delta_{b'}^{-1} \left(\frac{1}{2} \right) \right)^* \right) \\
&= -R^\bullet \mathrm{Hom}_{T(F)} \left((\pi^*)_N, \left(R^\bullet \Gamma_{\mathrm{c}}(\mathrm{Sht}^{(1,0)}_{T,b,b'}) \otimes \delta_B \right)^* \right) \otimes \delta_{b'} \left(-\frac{1}{2} \right) \\
&= -R^\bullet \mathrm{Hom}_{T(F)} \left(R^\bullet \Gamma_{\mathrm{c}}(\mathrm{Sht}^{(1,0)}_{T,b,b'}) \otimes \delta_B, \pi_{N^{\mathrm{op}}} \right) \otimes \delta_{b'} \left(-\frac{1}{2} \right) \\
&= -R^\bullet \mathrm{Hom}_{T(F)} \left(R^\bullet \Gamma_{\mathrm{c}}(\mathrm{Sht}^{(1,0)}_{T,b,b'}), \pi_{N^{\mathrm{op}}} \otimes \delta_B^{-1} \right) \otimes \delta_{b'} \left(-\frac{1}{2} \right) \\
&= -R^\bullet \mathrm{Hom}_{T(F)} \left(R^\bullet \Gamma_{\mathrm{c}}(\mathrm{Sht}^{(1,0)}_{T,b,b'}), \pi_{N^{\mathrm{op}}} \right) \left(\frac{1}{2} \right).
\end{aligned}$$

□

Proposition 8.2. *Let $\chi_1, \chi_2 \colon F^\times \to \overline{\mathbb{Q}}_\ell^\times$ be characters. Let $\varphi_{\chi_i} \colon W_F \to \overline{\mathbb{Q}}_\ell^\times$ be the character corresponding to χ_i. We put $\rho = \chi_1 \boxtimes \chi_2$ as representations of $T(F)$. Let $m \geq 0$ and $m/2 \geq l \geq 0$. We put*

$$b = \begin{pmatrix} \varpi^l & 0 \\ 0 & \varpi^{m-l} \end{pmatrix}, \quad b_1 = \begin{pmatrix} \varpi^{l-1} & 0 \\ 0 & \varpi^{m-l} \end{pmatrix}, \quad b_2 = \begin{pmatrix} \varpi^{l-1} & 0 \\ 0 & \varpi^{m-1-l} \end{pmatrix}.$$

(1) Assume $m \neq 2l$. We put

$$b_1' = \begin{pmatrix} \varpi^l & 0 \\ 0 & \varpi^{m-l-1} \end{pmatrix}.$$

If $l = 0$, then we have

$$H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b,1}^{(m,0)})[\rho] = (-1)^m (\mathrm{Ind}_{B(F)}^{G(F)} \rho) \boxtimes \varphi_{\chi_2}^m \left(\frac{m}{2}\right).$$

If $l \geq 1$, then we have

$$\begin{aligned} &H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b,1}^{(m,0)})[\rho] \\ &= -H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_1,1}^{(m-1,0)})[\rho] \otimes \varphi_{\chi_1}\left(-\frac{3}{2}\right) - H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_2,1}^{(m-2,0)})[\rho] \otimes \varphi_{\chi_1} \otimes \varphi_{\chi_2} \\ &\begin{cases} -H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_1',1}^{(m-1,0)})[\mathrm{Ind}_{B(F)}^{G(F)} \rho] \otimes \varphi_{\chi_2}\left(\frac{1}{2}\right) & \text{if } m = 2l+1 \\ -H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_1',1}^{(m-1,0)})[\rho] \otimes \varphi_{\chi_2}\left(\frac{1}{2}\right) & \text{if } m \geq 2l+2. \end{cases} \end{aligned}$$

(2) Assume $m = 2l$. If $l = 0$, then we have

$$H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b,1}^{(0,0)})[\mathrm{Ind}_{B(F)}^{G(F)} \rho] = (\mathrm{Ind}_{B(F)}^{G(F)} \rho) \boxtimes 1.$$

If $l \geq 1$, then we have

$$\begin{aligned} H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b,1}^{(m,0)})[\mathrm{Ind}_{B(F)}^{G(F)} \rho] = &- H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_1,1}^{(m-1,0)})[\rho] \otimes \varphi_{\chi_1}\left(-\frac{1}{2}\right) \\ &- H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_1,1}^{(m-1,0)})[\rho^w \otimes \delta_B^{-1}] \otimes \varphi_{\chi_2}\left(\frac{1}{2}\right) \\ &- H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_2,1}^{(m-2,0)})[\mathrm{Ind}_{B(F)}^{G(F)} \rho] \otimes \varphi_{\chi_1} \otimes \varphi_{\chi_2}. \end{aligned}$$

Proof. First we show the claim (1). If $l = 0$, we have

$$\begin{aligned} R^{\bullet}\mathrm{Hom}_{G_b(F)}&\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,1}^{(m,0)}), \rho\right) = R^{\bullet}\mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{1,b}^{(0,-m)}), \rho\right) \\ &= (-1)^m R^{\bullet}\mathrm{Hom}_{G_b(F)}\left(\mathrm{Ind}_{B(F)}^{G(F)} R^{\bullet}\Gamma_{\mathrm{c}}(\mathrm{Sht}_{T,1,b}^{(0,-m)}) \otimes \delta_b^{-1}, \rho\right)\left(-\frac{m}{2}\right) \\ &= (-1)^m \mathrm{Ind}_{B(F)}^{G(F)}\left(R^{\bullet}\mathrm{Hom}_{G_b(F)}\left(R^{\bullet}\Gamma_{\mathrm{c}}(\mathrm{Sht}_{T,b,1}^{(0,m)}) \otimes \delta_b^{-1}, \rho\right) \otimes \delta_B^{-1}\right)\left(-\frac{m}{2}\right) \\ &= (-1)^m \mathrm{Ind}_{B(F)}^{G(F)}\left(R^{\bullet}\mathrm{Hom}_{G_b(F)}\left(R^{\bullet}\Gamma_{\mathrm{c}}(\mathrm{Sht}_{T,b,1}^{(0,m)}), \rho \otimes \delta_b\right) \otimes \delta_B^{-1}\right)\left(-\frac{m}{2}\right) \\ &= (-1)^m \left(\mathrm{Ind}_{B(F)}^{G(F)} \rho\right) \boxtimes \varphi_{\chi_2}^m \left(\frac{m}{2}\right), \end{aligned}$$

where we use $\mathrm{Sht}_{1,b}^{(m-1,1)} = \emptyset$ and [GI16, Theorem 4.25] at the second equality. We assume that $l \geq 1$. By Proposition 5.1 and Lemma 5.3, the sum

$$R^{\bullet}\mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,1}^{(m,0)}), \rho\right) + R^{\bullet}\mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,1}^{(m-1,1)}), \rho\right)$$

is equal to the sum

$$\begin{aligned} &R^{\bullet}\mathrm{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}), R\,\mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1}^{(1,0)}), \rho\right) \otimes \delta_{b_1}^{-1}\right) \\ &\quad + R^{\bullet}\mathrm{Hom}_{G_{b_1'}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1',1}^{(m-1,0)}), R\,\mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1'}^{(1,0)}), \rho\right) \otimes \delta_{b_1'}^{-1}\right). \end{aligned}$$

Since the fiber of the natural morphism $\mathrm{Sht}_{b,b_1}^{(1,0)} \to \mathrm{Sht}_{T,b,b_1}^{(1,0)}$ is isomorphic to $\mathbb{B}^{\varphi=\varpi^{m+1-2l}}$, we have

$$R^\bullet \mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1}^{(1,0)}), \rho\right) = -R^\bullet \mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{T,b,b_1}^{(1,0)}) \otimes \delta_b^{-1}, \rho\right)\left(-\frac{1}{2}\right)$$
$$= -(\rho \otimes \delta_{b_1}) \boxtimes \varphi_{\chi_1}\left(-\frac{3}{2}\right).$$

Further, we have

$$R^\bullet \mathrm{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}), R\,\mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1}^{(1,0)}), \rho\right) \otimes \delta_{b_1}^{-1}\right)$$
$$= -R^\bullet \mathrm{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}), \rho\right) \boxtimes \varphi_{\chi_1}\left(-\frac{3}{2}\right).$$

If $m = 2l+1$, we have

$$R^\bullet \mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1'}^{(1,0)}), \rho\right) = -\left(\mathrm{Ind}_{B(F)}^{G(F)} \rho\right) \boxtimes \varphi_{\chi_2}\left(\frac{1}{2}\right)$$

by the claim in the case where $l = 0$.

If $m \geq 2l+2$, since the fiber of the natural morphism $\mathrm{Sht}_{b,b_1'}^{(1,0)} \to \mathrm{Sht}_{T,b,b_1'}^{(0,1)}$ is isomorphic to $\mathbb{B}^{\varphi=\varpi^{m-2l}}$, we have

$$R^\bullet \mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1'}^{(1,0)}), \rho\right) = -R^\bullet \mathrm{Hom}_{G_b(F)}\left(R^\bullet\Gamma_{\mathrm{c}}(\mathrm{Sht}_{T,b,b_1'}^{(0,1)}) \otimes \delta_b^{-1}, \rho\right)\left(-\frac{1}{2}\right)$$
$$= -(\rho \otimes \delta_B) \boxtimes \varphi_{\chi_2}\left(\frac{1}{2}\right).$$

Therefore

$$\begin{aligned}
&R^\bullet \mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,1}^{(m,0)}), \rho\right)\\
&= R^\bullet \mathrm{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}), R\,\mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1}^{(1,0)}), \rho\right) \otimes \delta_{b_1}^{-1}\right)\\
&\quad + R^\bullet \mathrm{Hom}_{G_{b_1'}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1',1}^{(m-1,0)}), R\,\mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1'}^{(1,0)}), \rho\right) \otimes \delta_{b_1'}^{-1}\right)\\
&\quad - R^\bullet \mathrm{Hom}_{G_b(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,1}^{(m-1,1)}), \rho\right)\\
&= -H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b_1,1}^{(m-1,0)})[\rho] \otimes \varphi_{\chi_1}\left(-\frac{1}{2}\right) - H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b,1}^{(m-2,0)})[\rho] \otimes \varphi_{\chi_1} \otimes \varphi_{\chi_2}\\
&\quad \begin{cases} -H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b_1',1}^{(m-1,0)})[\mathrm{Ind}_{B(F)}^{G(F)} \rho] \otimes \varphi_{\chi_2}\left(\frac{1}{2}\right) & \text{if } m = 2l+1,\\ -H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b_1',1}^{(m-1,0)})[\rho] \otimes \varphi_{\chi_2}\left(\frac{1}{2}\right) & \text{if } m \geq 2l+2. \end{cases}
\end{aligned}$$

Next we show the claim (2). The claim is trivial if $l = 0$. Assume that $l > 0$. We put $\pi = \mathrm{Ind}_{B(F)}^{G(F)} \rho$ and

$$b_1' = \begin{pmatrix} 0 & \varpi^{l-1} \\ \varpi^l & 0 \end{pmatrix}.$$

By Proposition 5.1 and Lemma 5.3, the sum

$$R^\bullet \mathrm{Hom}_{G(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,1}^{(m,0)}), \pi\right) + R^\bullet \mathrm{Hom}_{G(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,1}^{(m-1,1)}), \pi\right)$$

is equal to the sum

$$R^\bullet \operatorname{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}), R\operatorname{Hom}_{G(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1}^{(1,0)}),\pi\right)\otimes \delta_{b_1}^{-1}\right)$$
$$+ R^\bullet \operatorname{Hom}_{G_{b_1'}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1',1}^{(m-1,0)}), R\operatorname{Hom}_{G(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1'}^{(1,0)}),\pi\right)\otimes \delta_{b_1'}^{-1}\right).$$

We have

$$R^\bullet \operatorname{Hom}_{G(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1'}^{(1,0)}),\pi\right) = 0$$

by [Dat07, Théorème A].

By Lemma 8.1 and the geometric lemma (*cf.* [Ren10, VI.5.1 Théorème]), we have

$$\begin{aligned}
&R^\bullet \operatorname{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}), R\operatorname{Hom}_{G(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,b_1}^{(1,0)}),\pi\right)\otimes \delta_{b_1}^{-1}\right)\\
&= -R^\bullet \operatorname{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}), R^\bullet\operatorname{Hom}_{T(F)}\left(R^\bullet\Gamma_{\mathrm{c}}(\mathrm{Sht}_{T,b,b_1}^{(1,0)}),\pi_{N^{\mathrm{op}}}\right)\left(\frac{1}{2}\right)\otimes \delta_{b_1}^{-1}\right)\\
&= -R^\bullet \operatorname{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}), R^\bullet\operatorname{Hom}_{T(F)}\left(R^\bullet\Gamma_{\mathrm{c}}(\mathrm{Sht}_{T,b,b_1}^{(1,0)}),(\rho\otimes\delta_B)+\rho^w\right)\left(\frac{1}{2}\right)\otimes \delta_{b_1}^{-1}\right)\\
&= -R^\bullet \operatorname{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}),\rho\right)\otimes\varphi_{\chi_1}\left(-\frac{1}{2}\right)\\
&\quad - R^\bullet \operatorname{Hom}_{G_{b_1}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{(m-1,0)}),\rho^w\otimes\delta_B^{-1}\right)\otimes\varphi_{\chi_2}\left(\frac{1}{2}\right).
\end{aligned}$$

Hence

$$H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b,1}^{(m,0)})[\pi] = -H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b,1}^{(m-1,1)})[\pi] - H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b_1,1}^{(m-1,0)})[\rho]\otimes\varphi_{\chi_1}\left(-\frac{1}{2}\right)$$
$$- H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b_1,1}^{(m-1,0)})[\rho^w\otimes\delta_B^{-1}]\otimes\varphi_{\chi_2}\left(\frac{1}{2}\right).$$

Therefore we obtain the claim. □

By Proposition 8.2, we can calculate $H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b,1}^{(m,0)})[\rho]$ and $H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b,1}^{(m,0)})[\operatorname{Ind}_{B(F)}^{G(F)}\rho]$ in Proposition 8.2 inductively. We do not pursue the explicit formula here, but record the following corollary.

Corollary 8.3. *The* $\mathrm{GL}_2(F)$*-representations* $H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b,1}^{(m,0)})[\rho]$ *and* $H_{\mathrm{c}}^\bullet(\mathrm{Sht}_{b,1}^{(m,0)})[\operatorname{Ind}_{B(F)}^{G(F)}\rho]$ *in Proposition 8.2 are linear combinations of proper parabolic inductions.*

Proof. This follows from Proposition 8.2 by induction. □

Proposition 8.4. *We put*

$$b_1 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$$

and $b_m = b_1^m$ *for* $m \in \mathbb{Z}$*. For an odd integer* m*, we put*

$$b_m' = \begin{pmatrix} \varpi^{\frac{m-1}{2}} & 0 \\ 0 & \varpi^{\frac{m+1}{2}} \end{pmatrix}.$$

Assume that $m \geq 2$. If m is odd or φ is cuspidal, we have

$$H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_m,1}^{(m,0)})[\pi_{b_m}] = H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_{m-1},1}^{(m-1,0)})[\pi_{b_{m-1}}] \otimes \varphi - H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_{m-2},1}^{(m-2,0)})[\pi_{b_{m-2}}] \otimes (r_{(1,1)} \circ \varphi).$$

If m is even and φ is not cuspidal, we have

$$\begin{aligned} H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_m,1}^{(m,0)})[\pi_{b_m}] &= H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_{m-1},1}^{(m-1,0)})[\pi_{b_{m-1}}] \otimes \varphi - H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_{m-2},1}^{(m-2,0)})[\pi_{b_{m-2}}] \otimes (r_{(1,1)} \circ \varphi) \\ &\quad - H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b'_{m-1},1}^{(m-1,0)})[\chi \boxtimes \chi] \otimes \varphi_{\chi}\left(-\frac{1}{2}\right) \end{aligned}$$

where χ is a character of $F^{\times}$ such that $\pi_{b_m} \simeq \mathrm{St}_{\chi}$.

Proof. Assume that m is odd. By Proposition 5.1 and Lemma 5.3, the sum

$$R^{\bullet}\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,1}^{(m,0)}), \pi_{b_m}\right) + R^{\bullet}\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,1}^{(m-1,1)}), \pi_{b_m}\right)$$

is equal to

$$R^{\bullet}\mathrm{Hom}_{G_{b_{m-1}}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_{m-1},1}^{(m-1,0)}), R\,\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,b_{m-1}}^{(1,0)}), \pi_{b_m}\right)\right).$$

Hence the claim follows from Corollary 7.6.

Assume that m is even. By Proposition 5.1 and Lemma 5.3, the sum

$$R^{\bullet}\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,1}^{(m,0)}), \pi_{b_m}\right) + R^{\bullet}\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,1}^{(m-1,1)}), \pi_{b_m}\right)$$

is equal to the sum

$$\begin{aligned} &R^{\bullet}\mathrm{Hom}_{G_{b_{m-1}}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_{m-1},1}^{(m-1,0)}), R\,\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,b_{m-1}}^{(1,0)}), \pi_{b_m}\right)\right) \\ &\quad + R^{\bullet}\mathrm{Hom}_{G_{b'_{m-1}}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b'_{m-1},1}^{(m-1,0)}), R\,\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,b'_{m-1}}^{(1,0)}), \pi_{b_m}\right) \otimes \delta_B^{-1}\right). \end{aligned}$$

Hence, by Corollary 7.6, it suffices to show that

$$R\,\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,b'_{m-1}}^{(1,0)}), \pi_{b_m}\right) = \begin{cases} 0 & \text{if } \varphi \text{ is cuspidal,} \\ -((\chi \boxtimes \chi) \otimes \delta_B) \otimes \varphi_{\chi}\left(-\frac{1}{2}\right) & \text{if } \varphi \text{ is not cuspidal.} \end{cases}$$

By Lemma 8.1, we have

$$\begin{aligned} &R^{\bullet}\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,b'_{m-1}}^{(1,0)}), \pi_{b_m}\right) \\ &\quad = -R^{\bullet}\mathrm{Hom}_{T(F)}\left(R^{\bullet}\Gamma_{\mathrm{c}}(\mathrm{Sht}_{T,b_m,b'_{m-1}}^{(1,0)}), (\pi_{b_m})_{N^{\mathrm{op}}}\right)\left(\frac{1}{2}\right). \end{aligned}$$

Hence the claim follows from $(\mathrm{St}_{\chi})_{N^{\mathrm{op}}} \simeq (\chi \boxtimes \chi) \otimes \delta_B$. □

Proposition 8.5. *We put*

$$b_1 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$$

and $b_m = b_1^m$ for $m \in \mathbb{Z}$. For $m \geq 1$, we have

$$\begin{aligned} &R^{\bullet}\mathrm{Hom}_{G_{b_m}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,b_{-1},}^{(m+1,0)}), \pi_{b_m}\right) \\ &\quad = R^{\bullet}\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{1,b_{-1}}^{(1,0)}), R\,\mathrm{Hom}_{G_{b_m}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_m,1}^{(m,0)}), \pi_{b_m}\right)\right) \\ &\qquad - R^{\bullet}\mathrm{Hom}_{G_{b_{m-2}}(F)}\left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_{m-2},b_{-1}}^{(m-1,0)}), \pi_{b_{m-2}}\right) \otimes (r_{(1,1)} \circ \varphi). \end{aligned}$$

Proof. This follows from Proposition 5.1 and Lemma 5.3. □

Theorem 8.6. *Assume that $n = 2$. Then Conjecture 7.2 is true if $\kappa(b)$ is odd or φ is cuspidal.*

Proof. We put

$$b_1 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}.$$

To show the claim, we may assume that $\mu = (m, 0)$ for some $m \geq 0$ and b is 1 or b_1 by twisting.

Assume that φ is cuspidal. If $b = 1$, we can show the claim by induction using Proposition 8.4. If $b = b_1$, we can show the claim by induction using Proposition 8.5 and the case for $b = 1$.

It remains to treat the case where φ is not cuspidal and $b = b_1$. First, we can show that

$$H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b',1}^{(m,0)})[\pi_{b'}] - \pi_1 \boxtimes (r_{(m,0)} \circ \varphi)$$

is a linear combination of proper parabolic inductions as representations of $\mathrm{GL}_2(F)$ using Corollary 8.3 and Proposition 8.4. Hence, the claim follows from Proposition 8.5 and [Dat07, Théorème A]. □

On the other hand, the following example shows that Conjecture 7.2 is not true if μ is not minuscule and φ is not cuspidal.

Example 8.7. *Let $\mu = (2, 0)$ and b be a basic element such that $\kappa(b) = 2$. Assume that φ is not cuspidal and take a character χ of $F^{\times}$ such that $\pi_1 \simeq \mathrm{St}_{\chi}$. We put*

$$b_1 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}.$$

We note that

$$R^{\bullet} \operatorname{Hom}_{G_{b_1}(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b_1,1}^{\mu}), \pi_{b_1} \right) = \mathrm{St}_{\chi} \left(-\frac{1}{2} \right) - (\chi \circ \det) \left(\frac{1}{2} \right)$$

by [Dat07, Théorème 4.1.2]. Then we have

$$\begin{aligned} & R^{\bullet} \operatorname{Hom}_{G_b(F)} \left(R\Gamma_{\mathrm{c}}(\mathrm{Sht}_{b,1}^{\mu}), \pi_b \right) \\ & \quad = \pi_1 \boxtimes (r_{\mu} \circ \varphi) - \left(\operatorname{Ind}_{B(F)}^{\mathrm{GL}_2(F)} (\chi \boxtimes \chi) \right) \boxtimes (r_{(1,1)} \circ \varphi)(1) \end{aligned}$$

by Proposition 8.2 and Proposition 8.4.

Remark 8.8. *Example 8.7 is compatible with the main theorem of [HKW22], since the representation $\operatorname{Ind}_{B(F)}^{\mathrm{GL}_2(F)} (\chi \boxtimes \chi)$ has trace 0 on regular elliptic elements.*

Remark 8.9. *The error term in Example 8.7 supports that the expectation in [Far16, Remark 4.6] is true.*

Example 8.10. *Let $\chi_1, \chi_2 \colon F^{\times} \to \overline{\mathbb{Q}}_{\ell}^{\times}$ be characters. Let $\varphi_{\chi_i} \colon W_F \to \overline{\mathbb{Q}}_{\ell}^{\times}$ be the character corresponding to χ_i. We put $b = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^2 \end{pmatrix}$ and $\mu = (3, 0)$. We put $\rho = \chi_1 \boxtimes \chi_2$ as representations of $T(F)$. Then we have*

$$H^{\bullet}(\mathrm{Sht}_{G,b,1}^{\mu})[\rho] = -(\operatorname{Ind}_{B(F)}^{G(F)} \rho) \boxtimes \varphi_{\chi_1} \otimes \varphi_{\chi_2}^2 \left(-\frac{1}{2} \right).$$

Proof. We put

$$b_1 = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^2 \end{pmatrix}, \quad b_1' = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}.$$

By Proposition 8.2, we have

$$\begin{aligned}
&H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b,1}^{(3,0)})[\rho] \\
&= - H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_1,1}^{(2,0)})[\rho] \otimes \varphi_{\chi_1}\left(-\frac{3}{2}\right) - H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_2,1}^{(1,0)})[\rho] \otimes \varphi_{\chi_1} \otimes \varphi_{\chi_2} \\
&\quad - H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_1',1}^{(2,0)})[\mathrm{Ind}_{B(F)}^{G(F)} \rho] \otimes \varphi_{\chi_2}\left(\frac{1}{2}\right) \\
&= - \mathrm{Ind}_{B(F)}^{G(F)}(\rho) \boxtimes \varphi_{\chi_1} \otimes \varphi_{\chi_2}^2\left(-\frac{1}{2}\right) + (\mathrm{Ind}_{B(F)}^{G(F)} \rho) \boxtimes \varphi_{\chi_1} \otimes \varphi_{\chi_2}^2\left(\frac{1}{2}\right) \\
&\quad + H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_2,1}^{(1,0)})[\rho] \otimes \varphi_{\chi_1} \otimes \varphi_{\chi_2} + H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{b_2,1}^{(1,0)})[\rho^w \otimes \delta_B^{-1}] \otimes \varphi_{\chi_2}^2(1) \\
&\quad + H_{\mathrm{c}}^{\bullet}(\mathrm{Sht}_{1,1}^{(0,0)})[\mathrm{Ind}_{B(F)}^{G(F)} \rho] \otimes \varphi_{\chi_1} \otimes \varphi_{\chi_2}^2\left(\frac{1}{2}\right), \\
&= - \mathrm{Ind}_{B(F)}^{G(F)}(\rho) \boxtimes \varphi_{\chi_1} \otimes \varphi_{\chi_2}^2\left(-\frac{1}{2}\right)
\end{aligned}$$

using $\mathrm{Ind}_{B(F)}^{G(F)}(\rho^w \otimes \delta_B^{-1}) = \mathrm{Ind}_{B(F)}^{G(F)}(\rho)$ in $\mathrm{Groth}(G(F))$. □

Remark 8.11. *We use notation in Example 8.10. We define $I_{b,\mu,T}$ in the same way as [RV14, (31)]. Then we have $I_{b,\mu,T} = \emptyset$. Therefore Example 8.10 shows that the non-minuscule generalization of [RV14, Conjecture 8.5] does not hold as it is. We note that $([b], \mu)$ is not Hodge–Newton reducible (cf. [RV14, Definition 4.28]).*

References

[ALB21] J. Anschütz and A.-C. Le Bras, Averaging functors in Fargues' program for GL_n, 2021, arXiv:2104.04701.

[BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ, Representations of the group $GL(n, F)$, where F is a local non-Archimedean field, Uspehi Mat. Nauk 31 (1976), no. 3, 5–70.

[Cas82] W. Casselman, A new nonunitarity argument for p-adic representations, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1982), no. 3, 907–928.

[CS17] A. Caraiani and P. Scholze, On the generic part of the cohomology of compact unitary Shimura varieties, Ann. of Math. (2) 186 (2017), no. 3, 649–766.

[Dat07] J.-F. Dat, Théorie de Lubin-Tate non-abélienne et représentations elliptiques, Invent. Math. 169 (2007), no. 1, 75–152.

[Far16] L. Fargues, Geometrization of the local Langlands correspondence: An overview, 2016, arXiv:1602.00999.

[FS21] L. Fargues and P. Scholze, Geometrization of the local Langlands correspondence, 2021, arXiv:2102.13459.

[GI16] I. Gaisin and N. Imai, Non-semi-stable loci in Hecke stacks and Fargues' conjecture, 2016, arXiv:1608.07446.

[Han21a] D. Hansen, Moduli of local shtukas and Harris's conjecture, Tunis. J. Math. 3 (2021), no. 4, 749–799.

[Han21b] D. Hansen, On the supercuspidal cohomology of basic local Shimura varieties, 2021, preprint.

[HI24] L. Hamann and N. Imai, Dualizing complexes on the moduli of parabolic bundles, 2024, arXiv:2401.06342.

[HKW22] D. Hansen, T. Kaletha and J. Weinstein, On the Kottwitz conjecture for local shtuka spaces, Forum Math. Pi 10 (2022), Paper No. e13, 79.

[Ima24] N. Imai, Local Langlands correspondences in ℓ-adic coefficients, Manuscripta Math. 175 (2024), no. 1-2, 345–364.

[Kot85] R. E. Kottwitz, Isocrystals with additional structure, Compositio Math. 56 (1985), no. 2, 201–220.

[Kot97] R. E. Kottwitz, Isocrystals with additional structure. II, Compositio Math. 109 (1997), no. 3, 255–339.

[Kra20] H. Krause, Completing perfect complexes, Math. Z. 296 (2020), no. 3-4, 1387–1427, with appendices by Tobias Barthel, Bernhard Keller and Krause.

[Pra19] D. Prasad, Generalizing the MVW involution, and the contragredient, Trans. Amer. Math. Soc. 372 (2019), no. 1, 615–633.

[Rap95] M. Rapoport, Non-Archimedean period domains, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995 pp. 423–434.

[Ren10] D. Renard, Représentations des groupes réductifs p-adiques, vol. 17 of Cours Spécialisés, Société Mathématique de France, Paris, 2010.

[RV14] M. Rapoport and E. Viehmann, Towards a theory of local Shimura varieties, Münster J. Math. 7 (2014), no. 1, 273–326.

[Sch17] P. Scholze, Etale cohomology of diamonds, 2017, arXiv:1709.07343.

[SW20] P. Scholze and J. Weinstein, Berkeley lectures on p-adic geometry, vol. 207, Princeton, NJ: Princeton University Press, 2020.

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