Convolution morphisms and Kottwitz conjecture

Naoki Imai

Abstract

We introduce convolution morphisms, duality morphisms and twist morphisms between moduli spaces of mixed characteristic local shtukas. Using these morphisms, we relate the etale cohomology of different moduli spaces of mixed characteristic local shtukas. As an application, we show the Kottwitz conjecture in many new cases including the cases for all inner forms of GL$_3$ and minuscule cocharacters. We study also some non-minuscule cases and show that the Kottwitz conjecture is true for any inner form of GL$_2$ and any cocharacter if the Langlands parameter is cuspidal. On the other hand, we show that the Kottwitz conjecture does not hold as it is in non-minuscule cases if the Langlands parameter is not cuspidal.

Introduction

The Kottwitz conjecture says that etale cohomology of Rapoport–Zink spaces or more generally local Shimura varieties realize the local Langlands correspondence (cf. [Rap95, Conjecture 5.1], [RV14, Conjecture 7.4]). In [SW17], Scholze constructs local Shimura varieties as special cases of moduli spaces of mixed characteristic local shtukas. The Kottwitz conjecture makes sense also for the moduli spaces of mixed characteristic local shtukas. A weak version of the conjecture is studied by Kaletha–Weinstein in [KW17]. In the weak version, we ignore the action of the Weil groups and have an equality up to representations which have trace 0 on regular elliptic elements.

In this paper, we introduce convolution morphisms, duality morphisms and twist morphisms between moduli spaces of mixed characteristic local shtukas. The convolution morphism is related to a convolution morphism on affine Grassmannians. Using these morphisms and the convolution products in the geometric Satake equivalence for $B^+_{dR}$-Grassmannians, we relate the etale cohomology of different moduli spaces of mixed characteristic local shtukas.

As an application, we show new cases of the Kottwitz conjecture for the moduli spaces of mixed characteristic local shtukas. In particular, we show that the Kottwitz conjecture is true for all inner forms of GL$_3$ and minuscule cocharacters. The method is useful also for studying non-minuscule cases. We give inductive formulas that enable us to calculate the cohomology of the moduli space of mixed characteristic local shtukas for a local shtuka datum $(G, b, \mu)$ when $G$ is any inner form of GL$_2$, $b$ is any elements including non-basic one and $\mu$ is any cocharacter. In particular, we show that the Kottwitz conjecture is true for any inner form of GL$_2$ and any cocharacter if the Langlands parameter is cuspidal. On the other hand, we show that the Kottwitz conjecture need

2010 Mathematics Subject Classification. Primary: 11F70; Secondary: 14G35.
a modification in a non-minuscule case if the Langlands parameter is not cuspidal. We note that this is compatible with the result in [KW17], since the modification involves only representations which have trace 0 on regular elliptic elements.

In Section 1, we give a definition of a moduli space of mixed characteristic local shtukas. The definition which we give here is slightly different from that in [SW17]. Our definition is suitable to construct convolution morphisms between moduli spaces of mixed characteristic local shtukas in Section 2. In Section 4, we construct a twist morphism between moduli spaces of mixed characteristic local shtukas, which has an origin in the twist of a vector bundle by a line bundle. In Section 5, we discuss a relation between cohomology of different moduli spaces of mixed characteristic local shtukas using convolution morphisms. In Section 6, we construct a duality morphism, which has an origin in the dual of a vector bundle. In Section 7, we give an application to the Kottwitz conjecture. In Section 8, we give some inductive formulas on cohomology and discuss more about the Kottwitz conjecture in non-minuscule cases.

Notation

For a field $F$, let $\Gamma_F$ denote the absolute Galois group of $F$. For a non-archimedean local field $F$, let $\hat{F}$ denote the completion of the maximal unramified extension of $F$. For an object $X_Y$ over an object $Y$, its base change by the morphism $Y' \to Y$ is denoted by $X_{Y'}$.

1 Moduli of local shtukas

Let $p$ be a prime number. Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. Let $F$ be a finite extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$ with the residue field $\mathbb{F}_q$. For an algebraic field extension $k$ of $\mathbb{F}_q$, let $\text{Perf}_k$ denote the category of perfectoid spaces over $k$ with $v$-topology in the sense of [Sch17, §8].

Let $S = \text{Spa}(R, R^+) \in \text{Perf}_{\mathbb{F}_q}$. We put $W_{\mathcal{O}_R}(R^+) = W(R^+) \otimes_{W(\mathbb{F}_p)} \mathcal{O}_F$. Take a topological nilpotent unit $\varpi_R$ in $R$. Let $\mathcal{Y}_{(0,\infty)}(S)$ be the adic space defined by the condition $p \neq 0$ and $[\varpi] \neq 0$ in $\text{Spa}(W_{\mathcal{O}_R}(R^+), W_{\mathcal{O}_R}(R^+))$. Then $\mathcal{Y}_{(0,\infty)}(S)$ has an action of the $q$-th power Frobenius element $\varphi_S$ induced by the $q$-th power map on $R$. The quotient

$$X_S = \mathcal{Y}_{(0,\infty)}(S)/\varphi_S^2$$

is called the relative Fargues–Fontaine curve for $S$ (cf. [SW17, Definition 15.2.6]). The construction glues together to give $X_S$ for any $S \in \text{Perf}_{\mathbb{F}_q}$.

We define a continuous map

$$\kappa_S: \mathcal{Y}_{(0,\infty)}(S) \to (0, \infty)$$

by

$$\kappa_S(x) = \frac{\log|[\varpi]|_\varpi}{\log|p|_\varpi}$$

where $\varpi$ is the maximal generalization of $x \in \mathcal{Y}_{(0,\infty)}(S)$ and $| \cdot |_\varpi$ denotes the valuation corresponding to $\varpi$. For an interval $I$ in $(0, \infty)$, let $\mathcal{Y}_I(S)$ denote the interior of $\kappa_S^{-1}(I)$.
Let $G$ be a connected reductive group over $F$. Let $b \in G(\bar{F})$. We can construct a $G$-bundle $\mathcal{E}_b,X_S$ on $X_S$ (cf. [GI16, §1]). We define a sheaf $\tilde{J}_b$ on $\text{Perf}_{\bar{\mathbb{F}}_q}$ by

$$\tilde{J}_b(S) = \text{Aut}(\mathcal{E}_b,X_S)$$

for $S \in \text{Perf}_{\bar{\mathbb{F}}_q}$. In the sequel, we simply write $\mathcal{E}_b$ for $\mathcal{E}_b,X_S$ if there is no confusion. If $b$ is basic, let $J_b$ denote the inner form of $G$ determined by $b$. Then we have $\tilde{J}_b = J_b(F)$ for basis $b$.

Let $b,b' \in G(\bar{F})$. Let $\mu_1,\ldots,\mu_m$ be cocharacters of $G$. We put $\mu_* = (\mu_1,\ldots,\mu_m)$. For $1 \leq i \leq m$, let $E_i$ be the field of definition of $\mu_i$.

**Definition 1.1.** We define the presheaf $\text{Sht}^{\mu_*}_{G,b,b'}$ by sending $S = \text{Spa}(R,R^+) \in \text{Perf}_{\bar{\mathbb{F}}_q}$ to the isomorphism classes of the following objects:

- an untilt $S^2_i$ of $S$ over $\tilde{E}_i$ for $1 \leq i \leq m$,
- a $G$-torsor $\mathcal{P}$ on $\mathcal{Y}_{(0,\infty)}(S)$ with an isomorphism

$$\varphi_\mathcal{P} : (\varphi^*_S \mathcal{P})|_{\mathcal{Y}_{(0,\infty)}(S) \cup \sum_{i=1}^m S^2_i} \simeq \mathcal{P}|_{\mathcal{Y}_{(0,\infty)}(S) \cup \sum_{i=1}^m S^2_i}$$

which is meromorphic along the Cartier divisor $\cup_{i=1}^m S^2_i \subset \mathcal{Y}_{(0,\infty)}(S)$ and the relative position of $\varphi^*_S \mathcal{P}$ and $\mathcal{P}$ at $S^2_i$ is bounded by $\sum_{j,l} S^2_j \cap S^2_l$ at all geometric rank 1 points for all $1 \leq i \leq m$,

- an isomorphism

$$\iota_{[r,\infty]} : \mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)} \simeq G \times \mathcal{Y}_{[r,\infty)}(S)$$

for large enough $r$ under which $\varphi_\mathcal{P}$ is identified with $b \times \varphi_S$ and an isomorphism

$$\iota_{(0,r]} : \mathcal{P}|_{\mathcal{Y}_{(0,r]}(S)} \simeq G \times \mathcal{Y}_{(0,r]}(S)$$

for small enough $r'$ under which $\varphi_\mathcal{P}$ is identified with $b' \times \varphi_S$.

If there is no confusion, we simply write $\text{Sht}^{\mu_*}_{b,b'}$ for $\text{Sht}^{\mu_*}_{G,b,b'}$. If $\mu_* = (\mu)$, we simply write $\text{Sht}^{\mu_*}_{G,b,b'}$ for $\text{Sht}^{\mu_*}_{G,b,b'}$. We use similar abbreviations also for other spaces.

We define the right action of $\tilde{J}_b \times \tilde{J}_{b'}$ on $\text{Sht}^{\mu_*}_{G,b,b'}$ by

$$(\iota_{[r,\infty]}, \iota_{(0,r]}) \mapsto (g^{-1} \circ \iota_{[r,\infty]}, g'^{-1} \circ \iota_{(0,r]})$$

for $(g,g') \in \tilde{J}_b \times \tilde{J}_{b'}$.

We define $\text{Gr}^{\text{tw}}_{G,\text{Spd} \bar{E}_1 \times \cdots \times \text{Spd} \bar{E}_m, \leq \mu_*}$ as in [SW17, Definition 23.4.1]. It is a spacial diamond by [SW17, Proposition 23.4.2]. We have a morphism

$$\pi^{\mu_*}_{G,b,b'} : \text{Sht}^{\mu_*}_{G,b,b'} \to \text{Gr}^{\text{tw}}_{G,\text{Spd} \bar{E}_1 \times \cdots \times \text{Spd} \bar{E}_m, \leq \mu_*}$$

defined by forgetting $\iota_{(0,r]}$. The morphism $\pi^{\mu_*}_{G,b,b'}$ is a $\tilde{J}_{b'}$-torsor over a locally spatial subdiamond of $\text{Gr}^{\text{tw}}_{G,\text{Spd} \bar{E}_1 \times \cdots \times \text{Spd} \bar{E}_m, \leq \mu_*}$ by [Sch17, Proposition 11.20]. Hence, $\text{Sht}^{\mu_*}_{G,b,b'}$ is a diamond by [Sch17, Proposition 11.6] and [Far16, 2.5, 2.6.2].

We have a natural inversing morphism

$$\text{Sht}^{\mu_*}_{G,b,b'} \to \text{Sht}^{\mu_*}_{G,b,b'}$$

(1.1)
compatible with the action of $\tilde{J}_b \times \tilde{J}_{b'}$.

Let $B(G)$ be the set of $\sigma$-conjugacy classes in $G(\tilde{F})$. We write $B(G)_{\text{bas}}$ for the set of the basic elements in $B(G)$. Let $\mu$ be a cocharacter of $G$. We define $B(G, \mu)$ as in [Kot97, 6.2].

Assume that $G$ is quasi-split. We fix subgroups $A \subset T \subset B$ of $G$ where $A$ is a maximal split torus, $T$ is a maximal torus and $B$ is a Borel subgroup. We write $X_*(A)^+$ and $X_*(T)^+$ for the dominant cocharacter of $A$ and $T$. For $b \in G(\tilde{F})$, we define $\nu_b \in X_*(A)_G^+$ as in [Far16, 2.2.2] using the slope morphism constructed in [Kot85, 4.2]. We define $B(G, \mu, [b])$ as in [GI16, Definition 4.3].

**Lemma 1.2.** Assume that $b$ is basic. The map

$$G(\tilde{F}) \to G(\tilde{F}) = J_b(\tilde{F}); \ g \mapsto gb^{-1}$$

induces bijections $B(G) \to B(J_b)$, $B(G)_{\text{bas}} \to B(J_b)_{\text{bas}}$ and $B(G, \mu, [b]) \to B(J_b, \mu)$.

**Proof.** The claim follows from the equality

$$(g'g\sigma(g')^{-1})b^{-1} = g'(gb^{-1})(b\sigma(g')b^{-1})^{-1}.$$ for $g, g' \in G(\tilde{F})$. \hfill $\square$

**Proposition 1.3.** Assume that $b'$ is basic. We have a natural isomorphism

$$\text{Sht}_{G,b,b'}^{\mu} \cong \text{Sht}_{J_{b'},b'^{-1},1}^{\mu}$$

which is compatible with the action of $\tilde{J}_b \times \tilde{J}_{b'}$.

**Proof.** We can view $\text{Sht}_{G,b,b'}^{\mu}$ as a moduli space of modifications of $G$-torsors on a Fargues–Fontaine curve. The category of $G$-torsor is equivalent to the category of $J_{b'}$-torsor on a Fargues–Fontaine curve as explained in the proof of [SW17, Corollary 23.2.3]. The claim follows from this equivalence. \hfill $\square$

**Remark 1.4.** Assume that $b, b'$ are basic and $m = 1$. Then a weak version of Kottwitz conjecture for $\text{Sht}_{G,b,b'}^{\mu}$ holds by [KW17, Theorem 1.0.4], Lemma 1.2 and Proposition 1.3.

**Remark 1.5.** Assume that $b, b'$ are basic and $m = 1$. Under the isomorphism in Proposition 1.3, the inversing morphism (1.1) is identified with the Faltings–Fargues isomorphism proved in [SW17, Corollary 23.2.3].

**Lemma 1.6.** Assume that $b'$ is basic. If $\text{Sht}_{G,b,b'}^{\mu}$ is not empty, then we have $[b] \in B(G, \mu, [b'])$.

**Proof.** By Proposition 1.3, we may assume that $b' = 1$ dropping the assumption that $G$ is quasi-split. Then the claim follows from [CS17, Proposition 3.5.3]. \hfill $\square$
2 Convolution morphism

Let $\Delta_{m, \text{Spd} F}$ denote the diagonal subspace of $(\text{Spd} F)^m$. For $1 \leq i < j \leq m$, let $\text{pr}_{i,j} : (\text{Spd} F)^m \to (\text{Spd} F)^2$ denote the projection to the $(i, j)$-component. We put

$$U_m = (\text{Spd} F)^m \setminus \bigcup_{1 \leq i < j \leq m} \text{pr}_{i,j}^{-1}\left( \bigcup_{n \in \mathbb{Z} \setminus \{0\}} (\varphi \times 1)^n(\Delta_{2, \text{Spd} F}) \right).$$

This is an open subspace of $(\text{Spd} F)^m$ which contains $\Delta_{m, \text{Spd} F}$.

Let $b_0, \ldots, b_m \in G(F)$ and $\mu_* = (\mu_1, \ldots, \mu_m)$ where $\mu_i \in X_*(T)$ for $1 \leq i \leq m$. We put

$$\text{Sh}^{\mu_*}_{G, b_0, b_m, U_m} = \text{Sh}^{\mu_*}_{G, b_0, b_m} \times (\text{Spd} F)^m U_m.$$

We define the convolution morphism

$$m_{b_0, \mu, U_m} : (\text{Sh}^{\mu_1}_{G, b_0, b_1} \times \cdots \times \text{Sh}^{\mu_m}_{G, b_{m-1}, b_m}) \times (\text{Spd} F)^m U_m \to \text{Sh}^{\mu_*}_{G, b_0, b_m, U_m}$$

over $\text{Spd} \tilde{E}_1 \times \cdots \times \text{Spd} \tilde{E}_m$ as follows. Let $S = \text{Spa}(R, R^+)$ in $\text{Perf}_{\mathfrak{q}_q}$ and

$$\left( S^\sharp_{1, i}, \mathcal{P}_i, \varphi_{\mathcal{P}_i}, \kappa_{(0, r], i}, \kappa_{[r', \infty], i} \right)_{1 \leq i \leq m}$$

be objects giving an $S$-valued point of

$$\left( \text{Sh}^{\mu_1}_{G, b_0, b_1} \times \cdots \times \text{Sh}^{\mu_m}_{G, b_{m-1}, b_m} \right) \times (\text{Spd} F)^m U_m.$$

Define $\mathcal{P}$ by gluing $\mathcal{P}_1|_{\mathcal{Y}_{(0, r]}(S)}$ and $\mathcal{P}_m|_{\mathcal{Y}_{[r', \infty)}(S)}$ by the following modifications:

- Modifications occur only at $\bigcup_{n=1}^m \varphi^{-n}(S^\sharp_{1, i}).$
- Take $1 \leq i_0 \leq m$. Put

$$I_{i_0} = \{ 1 \leq i \leq m \mid S^\sharp_{1, i} = S^\sharp_{1, i_0} \}.$$

Define the modification at $S^\sharp_{1, i_0}$ by the composite of the modifications at $S^\sharp_{1, i}$ given by $\varphi_{\mathcal{P}_i}$ for all $i \in I_{i_0}$. For $n > 0$, the modification at $\varphi^{-n}(S^\sharp_{1, i_0})$ is given by the pullback under $\varphi^n$ of the modification at $S^\sharp_{1, i_0}$.

Then $\mathcal{P}$ is naturally equipped with an isomorphism

$$\varphi_{\mathcal{P}} : (\varphi_{\mathcal{S}} \mathcal{P})|_{S \times \text{Spa} F^m \setminus \bigcup_{i=1}^m S^\sharp_{1, i}} \sim \mathcal{P}|_{S \times \text{Spa} F^m \setminus \bigcup_{i=1}^m S^\sharp_{1, i}}.$$

Further we have isomorphisms

$$\mathcal{P}|_{\mathcal{Y}_{(0, r]}(S)} = \mathcal{P}_1|_{\mathcal{Y}_{(0, r]}(S)} \overset{\mathcal{I}_{(0, r], 1}}{\longrightarrow} G \times \mathcal{Y}_{(0, r]}(S),$$

$$\mathcal{P}|_{\mathcal{Y}_{[r', \infty)}(S)} = \mathcal{P}_m|_{\mathcal{Y}_{[r', \infty)}(S)} \overset{\mathcal{I}_{[r', \infty), m}}{\longrightarrow} G \times \mathcal{Y}_{[r', \infty)}(S).$$

These gives an $S$-valued point of $\text{Sh}^{\mu_*}_{G, b_0, b_m, U_m}$. Thus we obtain $m_{b_0, \mu, U_m}$.

We define

$$\text{Gr}_{G, \text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu_*} : \widetilde{\text{Gr}}_{G, \text{Spd} E_1 \times \cdots \times \text{Spd} E_m, \leq \mu_*}.$$
as in [SW17, Definition 20.4.4]. Then we have a convolution morphism
\[
m_{\mu,\ast} : \Gr_{G,\Spd} E_1 \times \cdots \times \Spd E_{m, \leq \mu} \rightarrow \Gr_{G,\Spd} E_1 \times \cdots \times \Spd E_{m, \leq \mu}
\]
by [SW17, Proposition 20.4.5]. Note that
\[
\Gr_{G,\Spd} E_1 \times \cdots \times \Spd E_{m, \leq \mu} \times (\Spd F)^m U_m \cong \Gr_{G,\Spd} E_1 \times \cdots \times \Spd E_{m, \leq \mu} \times (\Spd F)^m U_m.
\]
Then we have a morphism
\[
\Sh_{G,b_0,b_1} \times \cdots \times \Sh_{G,b_{m-1},b_m} \rightarrow \Gr_{G,\Spd} E_1 \times \cdots \times \Spd E_m, \leq \mu
\]
by looking at a modification at each \(S^2_i\). Then we have the commutative diagram
\[
\begin{array}{ccc}
\Sh_{G,b_0,b_1} \times \cdots \times \Sh_{G,b_{m-1},b_m} \times (\Spd F)^m U_m & \xrightarrow{m_{\mu,\ast} U_m} & \Sh_{G,b_0,b_m,U_m} \\
\downarrow & & \downarrow \\
\Gr_{G,\Spd} E_1 \times \cdots \times \Spd E_m, \leq \mu \times (\Spd F)^m U_m & \xrightarrow{H_{\mu,\ast}} & \Gr_{G,\Spd} E_1 \times \cdots \times \Spd E_m, \leq \mu \times (\Spd F)^m U_m
\end{array}
\]
where the bottom morphism is induced by \(m_{\mu,\ast}\).

3 Geometric Satake equivalence

In the sequel, we assume the geometric Satake equivalence for \(B_{\dR}^+\) Grassmannians which is announced by Fargues–Scholze (cf. [FS]).

We define \(v\)-sheaves \(LG\) and \(L^+ G\) over \(\Spd \Q_p\) by sending \(S = \Spa(R, R^+) \in \Perf_q\) with an untill \(S^2 = \Spa(R^2, R^{2+})\) to \(B_{\dR}^+(R^2)\) and \(B_{\dR}^+(R^2)\), where \(B_{\dR}^+(R^2)\) and \(B_{\dR}^+(R^2)\) are defined as in [Far16, Definition 1.32]. We put \(\Gr_G = LG/L^+ G\).

Let \(\ell\) be a prime number different from \(p\). Let \(P_{L^+ G}(\Gr_G)\) be the category of \(L^+ G\)-equivariant \(\overline{Q}_\ell\)-perverse sheaf on \(\Gr_G\). For \(A_1, A_2 \in P_{L^+ G}(\Gr_G)\), let \(A_1 \ast A_2\) denote the convolution product of \(A_1\) and \(A_2\). Let
\[
H^* : P_{L^+ G}(\Gr_G) \rightarrow \Rep_{\overline{Q}_\ell}(L^+ G)
\]
denote the tensor functor that gives the geometric Satake equivalence. For \(A_1, A_2 \in P_{L^+ G}(\Gr_G)\), let
\[
c_{A_1, A_2} : A_1 \ast A_2 \cong A_2 \ast A_1
\]
be the commutativity constraint uniquely characterized by
\[
\begin{array}{c}
H^*(A_1 \ast A_2) \\
\downarrow \\
H^*(A_1) \otimes H^*(A_2) \xrightarrow{\sigma} H^*(A_2) \otimes H^*(A_1).
\end{array}
\]
Assume that \(\mu \in X_*(T)^+\). Let \(\Gamma_\mu \subset \Gamma_F\) be the stabilizer of \(\mu\). Let \(r_{G,\mu}^r\) be the highest weight \(\mu\) irreducible representation of \(\hat{G} \rtimes \Gamma_\mu\). We put
\[
r_{G,\mu} = \Ind_{\G^\mu \Gamma_\mu}^{L^+ G} r_{G,\mu}^r.
\]
We simply write $r_\mu$ for $r_{G,\mu}$ if there is no confusion. We write $V_\mu$ for the representation space of $r_\mu$.

Let $\text{IC}_\mu$ be the $L^+G$-equivariant perverse sheaf on $\text{Gr}_G$ corresponding to $r_\mu$ via the geometric Satake equivalence. We use the same notation $\text{IC}_\mu$ for the pullback of $\text{IC}_\mu$ to other spaces. Let $E$ be the field of definition of $\mu$. We write $\text{Gr}_G^{(\infty)}_{E, \leq \mu}$ for the inverse image of $\text{Gr}_{G, \text{Spd}}_{E, \leq \mu}$ under $LG_{\text{Spd}} E \to \text{Gr}_{G, \text{Spd}} E$.

4 Twist morphism

The space $\text{Sht}^\mu_{G,b,b'}$ is the moduli space of $(S^\sharp, \mathcal{E}_b \to \mathcal{E}_b')$, where $S^\sharp$ is an ultilt over $\tilde{E}$ and $\mathcal{E}_b \to \mathcal{E}_b'$ is a modification bounded by $\mu$ along the Cartier divisor defined by $S^\sharp$.

Let $\mathbb{C}_p'$ denote the tilt of $\mathbb{C}_p$. The ultilt $\mathbb{C}_p$ of $\mathbb{C}_p'$ determine a morphism $\text{Spa} \mathbb{C}_p' \to \text{Spd} \mathbb{Q}_p$. We put

$$R\Gamma_c(\text{Sht}^\mu_{G,b,b'}) = R\Gamma_c \left( \text{Sht}^\mu_{G,b,b',C_\mu}, \text{IC}_\mu \right).$$

We put

$$H^*_c(\text{Sht}^\mu_{G,b,b'}) = \sum_{i \in \mathbb{Z}} (-1)^i H^i_c \left( \text{Sht}^\mu_{G,b,b',C_\mu}, \text{IC}_\mu \right).$$

Note that $H^*_c(\text{Sht}^\mu_{G,b,b'}) = H^*_c(\text{Sht}^{-\mu}_{G,b,b'})$ since $\text{IC}_\mu$ and $\text{IC}_{-\mu}$ corresponds under the natural isomorphism $\text{Sht}^\mu_{G,b,b'} \simeq \text{Sht}^{-\mu}_{G,b,b'}$.

Let $U$ be the unipotent radical of $B$. We define a subsheaf $\tilde{\mathcal{J}}^U_b$ of $\tilde{\mathcal{J}}_b$ as in [GI16, §4]. By [GI16, Lemma 4.16], we have isomorphisms

$$H^*_c(\text{Sht}^\mu_{G,b,b'}) = H^*_c(\text{Sht}^\mu_{G,b,b'} / \tilde{\mathcal{J}}_b^U) = H^*_c(\text{Sht}^\mu_{G,b,b'} / \tilde{\mathcal{J}}_b^U)$$

and these have actions of $J_b(F) = (\tilde{\mathcal{J}}_b / \tilde{\mathcal{J}}_b^U)(\mathbb{C}_p')$ and $J_{b'}(F) = (\tilde{\mathcal{J}}_{b'} / \tilde{\mathcal{J}}_{b'}^U)(\mathbb{C}_p')$.

Let $Z^0$ be the identity component of the center of $G$. Let $a, a' \in Z^0(\tilde{F})$ and $\lambda \in X_*(Z^0)$. Let $E$ be a finite extension of $F$ in $\mathbb{C}_p$ containing the fields of definition of $\mu$ and $\lambda$. We define the morphism

$$t^\mu_{G,b,b',E,Spd} \times \text{Spd} E \times \text{Spd} E \times \text{Spd} E \times \text{Spd} E \times \text{Spd} E$$

as follows. Let $(S^\sharp, \mathcal{E}_b \to \mathcal{E}_b')$ and $(S^\sharp, \mathcal{E}_a \to \mathcal{E}_a')$ be modifications defining points in $\text{Sht}^\mu_{G,b,b'}$ and $\text{Sht}^\mu_{G,a,b'}$. Then the diagonal arrow in the diagram

$$\begin{array}{ccc}
\mathcal{E}_b & \to & \mathcal{E}_{b'} \\
\downarrow & & \downarrow \\
\mathcal{E}_a & \to & \mathcal{E}_{a'}
\end{array}$$

defines the image of

$$((S^\sharp, \mathcal{E}_b \to \mathcal{E}_b'), (S^\sharp, \mathcal{E}_a \to \mathcal{E}_a'))$$

under $t^\mu_{G,b,b',E,Spd}$. Note that we have equalities $J_b(F) = J_{ab}(F)$ and $J_{b'}(F) = J_{a'b'}(F)$. 

7
Proposition 4.1. We have
\[(\mathcal{R} \Gamma_c(Sht^\mu_{G,b,b'}) \otimes \mathcal{R} \Gamma_c(Sht_{Z^0,a,a'})) \otimes_{Z^0(F)} \mathbb{Q}_\ell \simeq \mathcal{R} \Gamma_c(Sht^\mu_{G,a,b,a'})\]
in the derived category of representations of $J_0(F) \times J_0(F) \times W_E$.

Proof. This follows from that $m^\mu_{b,b,a,a}$ is a $Z^0(F)$-torsor. \qed

5 Formula on cohomology

Let $b_0, \ldots, b_m \in G(\tilde{F})$ and $\mu_1, \ldots, \mu_m \in X_*(T)^+$. Let $E$ be a finite extension of $F$ in $\mathbb{C}_p$ containing $E_i$ for $1 \leq i \leq m$. Assume that $[b_{i-1}] \in B(G, [\mu_i])$ for $1 \leq i \leq m$. Let

$$m_{b_i, \mu_i} : Sht^\mu_{b_i,b_i,Spd E} \times_{Spd E} \cdots \times_{Spd E} Sht^{\mu_m}_{b_m-1,b_m,Spd E} \to Sht_{b_0,b_m,Spd E}$$

by the pullback of the convolution morphism $m_{b_i, \mu_i, U_m}$ defined in Section 2 under the morphism

$$Spd \tilde{E} = \Delta_{m,Spd E} \hookrightarrow (Spd \tilde{E})^m \to Spd \tilde{E}_1 \times \cdots \times Spd \tilde{E}_m.$$

The morphism $m_{b_i, \mu_i}$ coincides with the morphism defined by the composition of modifications. This induces

$$\overline{m}_{b_i, \mu_i} : (Sht^\mu_{b_0,b_1,Spd E} \times_{Spd E} \cdots \times_{Spd E} Sht^{\mu_m}_{b_m-1,b_m,Spd E})/(\tilde{J}_{b_1} \times \cdots \times \tilde{J}_{b_m-1}) \to Sht_{b_0,b_m,Spd E},$$

where $\tilde{J}_i$ for $1 \leq i \leq m - 1$ acts diagonally on the factor

$$Sht^{\mu_i}_{b_i-1,b_i,Spd E} \times_{Spd E} Sht^{\mu_{i+1}}_{b_i,b_i+1,Spd E}$$

and trivially on the other factors.

Let

$$\overline{\text{Gr}}_{G,Spd E, \leq \mu_i} \xrightarrow{m_{b_i}} \text{Gr}_{G,Spd E, \leq \mu_i}$$

be the pullback of

$$m_{\mu_i} : \overline{\text{Gr}}_{G,Spd E_1 \times \cdots \times Spd E_m, \leq \mu_i} \to \text{Gr}_{G,Spd E_1 \times \cdots \times Spd E_m, \leq \mu_i}.$$

under

$$Spd \tilde{E} = \Delta_{m,Spd E} \hookrightarrow (Spd \tilde{E})^m \to Spd E_1 \times \cdots \times Spd E_m.$$

We define $m_{\mu_i, b_0,b_m} : Sht^\mu_{b_0,b_m,Spd E} \to Sht^{\mu_i}_{b_0,b_0,Spd E}$ by the fiber product

$$Sht^{\mu_i}_{b_0,b_m,Spd E} \xrightarrow{m_{\mu_i, b_0,b_m}} Sht^{\mu_i}_{b_0,b_0,Spd E}$$

$$\overline{\text{Gr}}_{G,Spd E, \leq \mu_i} \xrightarrow{m_{\mu_i}} \text{Gr}_{G,Spd E, \leq \mu_i}.$$

Then $Sht^\mu_{b_0,b_m}$ is a moduli space of modifications

$$\mathcal{E}_{b_0} \xrightarrow{f_1} \mathcal{E}_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} \mathcal{E}_{m-1} \xrightarrow{f_m} \mathcal{E}_{b_m}.$$
at $S_{	au}$ such that $f_i$ is bounded by $\mu_i$ for $1 \leq i \leq m$. We define a subspace $\text{Sh}_{b_0, b_m, \text{Spd} E}^{\mu_1, \ldots, \mu_m} \subset \text{Sh}_{b_0, b_m, \text{Spd} E}^{\mu_{\bullet}}$ as a moduli space of modifications

$$E_{b_0} f_1 \to E_1 f_2 \to \cdots \to E_{m-1} f_m \to E_{b_m}$$

at $S_{\tau}$ such that $f_i$ is bounded by $\mu_i$ for $1 \leq i \leq m$ and $E_i$ is isomorphic to $E_{b_i}$ geometric fiberwisely for $1 \leq i \leq m - 1$.

We put

$$I_{b_0, b_m}^{\mu_{\bullet}} = \{([b_1], \ldots, [b_{m-1}]) \in B(G)^{m-1} \mid \text{Sh}_{b_i, b_{i+1}}^{\mu_i} \neq \emptyset \text{ for } 1 \leq i \leq m - 1\}.$$ 

Then $I_{b_0, b_m}^{\mu_{\bullet}}$ is a finite set, since it is contained in $\prod_{1 \leq i \leq m-1} B(G, \sum_{j=i+1}^m \mu_j, [b_m])$ by Lemma 1.6. For $\lambda \in X_*(T)^+/T_E$, we put

$$V_{\mu_{\bullet}}^\lambda = \text{Hom}_{L}(V_{\lambda}, \bigotimes_{1 \leq i \leq m} V_{\mu_i}).$$

Let $\rho_U$ denote the half-sum of the positive roots of $G$ with respect to $T$ and $B$. We put $N_{U,b} = (2\rho_U, \nu_b)$.

**Proposition 5.1.** The sum

$$\sum_{\lambda \in X_*(T)^+/T_E} V_{\mu_{\bullet}}^\lambda \otimes R\Gamma_c(\text{Sh}_{b_0, b_m}^\lambda)$$

is decomposed into

$$\left(\bigotimes_{1 \leq i \leq m} R\Gamma_c(\text{Sh}_{b_i, b_{i+1}}^{\mu_i})[2N_{U,b}]\right) \otimes_{\prod_{i=1}^{m-1} J_{b_i}(F)} \mathbb{Q}_\ell$$

for $([b_1], \ldots, [b_{m-1}] \in I_{b_0, b_m}^{\mu_{\bullet}}$ by distinguished triangles in the derived category of representations of $J_{b_0}(F) \times J_{b_m}(F) \times W_E$.

**Proof.** Let $\text{IC}_{\mu_{\bullet}}$ be the external twisted product of $\text{IC}_{\mu_1}, \ldots, \text{IC}_{\mu_m}$ on $\text{Gr}_{\text{Spd} E, \underline{\mu}_{\bullet}}$. By the geometric Satake equivalence, we have

$$(m_{\mu_{\bullet}})_! \text{IC}_{\mu_{\bullet}} = \sum_{\lambda \in X_*(T)^+/T_E} V_{\mu_{\bullet}}^\lambda \otimes \text{IC}_{\lambda}.$$ 

Hence the sum

$$\sum_{\lambda \in X_*(T)^+/T_E} V_{\mu_{\bullet}}^\lambda \otimes R\Gamma_c(\text{Sh}_{b_0, b_m}^\lambda)$$

is isomorphic to $R\Gamma_c(\text{Sh}_{b_0, b_m}^{\mu_{\bullet}}, \text{IC}_{\mu_{\bullet}})$ by the proper base change theorem. Further, $\text{Sh}_{b_0, b_m}^{\mu_{\bullet}}$ has a stratification by $\text{Sh}_{b_i, b_{i+1}}^{\mu_i}$ for $([b_i])_{1 \leq i \leq m-1} \in I_{b_0, b_m}^{\mu_{\bullet}}$ by Lemma 1.6. Hence $R\Gamma_c(\text{Sh}_{b_0, b_m}^{\mu_{\bullet}}, \text{IC}_{\mu_{\bullet}})$ is decomposed into

$$R\Gamma_c(\text{Sh}_{b_0, b_m, C_p}^{b_1, \ldots, b_{m-1} \mu_{\bullet}}, \text{IC}_{\mu_{\bullet}}).$$

9
for \( ([b_i])_{1 \leq i \leq m-1} \in I_{b_0, b_m}^\bullet \) by distinguished triangles. The morphism \( \overline{\mathcal{M}}_{b_+, \mu^*} \) induces an isomorphism

\[
\left( \mathcal{Sht}^{\mu_1}_{b_0, b_1, \text{Spd} E} \times \text{Spd} E \cdots \times \text{Spd} E \mathcal{Sht}^{\mu_m}_{b_{m-1}, b_m, \text{Spd} E} \right) / (\widetilde{J}_{b_1} \times \cdots \times \widetilde{J}_{b_{m-1}}) \xrightarrow{\sim} \mathcal{Sht}^{b_1, \ldots, b_{m-1}, \mu^*}_{b_0, b_m, \text{Spd} E}.
\]

Hence we have

\[
R \Gamma_c(\mathcal{Sht}^{b_1, \ldots, b_{m-1}, \mu^*}_{b_0, b_m, C_p}, IC_{\mu^*}) \\
\simeq \left( R \Gamma_c(\mathcal{Sht}^{\mu_1}_{b_0, b_1, C_p} \times C_p \cdots \times C_p \mathcal{Sht}^{\mu_m}_{b_{m-1}, b_m, C_p}, IC_{\mu^*}) \left[ 2 \sum_{i=1}^{m-1} N_{U_i} \right] \right) \otimes \prod_{i=1}^{m-2} J_{b_i} (F) \overline{\mathbb{Q}}_\ell
\]

by [GI16, Lemma 4.17]. By the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Sht}^{\mu_1}_{b_0, b_1, \text{Spd} E} \times \text{Spd} E \cdots \times \text{Spd} E \mathcal{Sht}^{\mu_m}_{b_{m-1}, b_m, \text{Spd} E} & \xrightarrow{\sim} & \mathcal{Sht}^{\mu^*}_{b_0, b_m, \text{Spd} E} \\
\downarrow & & \downarrow \\
\text{Gr}_{\text{Spd} E, \leq \mu_1} \times \text{Spd} E \text{Gr}^{(\infty)}_{\text{Spd} E, \leq \mu_2} \times \text{Spd} E \cdots \times \text{Spd} E \text{Gr}^{(\infty)}_{\text{Spd} E, \leq \mu_m} & \xrightarrow{\sim} & \text{Gr}_{\text{Spd} E, \leq \mu^*} \\
\downarrow & & \downarrow \\
\text{Gr}_{\text{Spd} E, \leq \mu_1} \times \text{Spd} E \text{Gr}_{\text{Spd} E, \leq \mu_2} \times \text{Spd} E \cdots \times \text{Spd} E \text{Gr}_{\text{Spd} E, \leq \mu_m}
\end{array}
\]

the pullback of \( IC_{\mu^*} \) to

\[
\mathcal{Sht}^{\mu_1}_{b_0, b_1, \text{Spd} E} \times \text{Spd} E \cdots \times \text{Spd} E \mathcal{Sht}^{\mu_m}_{b_{m-1}, b_m, \text{Spd} E}
\]

is equal to the pullback of \( IC_{\mu_1} \boxtimes \cdots \boxtimes IC_{\mu_m} \). Hence we have

\[
R \Gamma_c \left( \mathcal{Sht}^{\mu_1}_{b_0, b_1, C_p} \times C_p \cdots \times C_p \mathcal{Sht}^{\mu_m}_{b_{m-1}, b_m, C_p}, IC_{\mu^*} \right) \simeq \bigotimes_{1 \leq i \leq m} R \Gamma_c (\mathcal{Sht}^{\mu_i}_{b_0, b_{i+1}}).
\]

Therefore we obtain the claim. \( \square \)

**Corollary 5.2.** We have

\[
\sum_{([b_i])_{1 \leq i \leq m-1} \in I_{b_0, b_m}^\bullet} H_* \left( \prod_{i=1}^{m-1} J_{b_i} (F), \bigotimes_{1 \leq i \leq m} H_*^c (\mathcal{Sht}^{\mu_i}_{b_0, b_{i+1}}) \right) = \sum_{\lambda \in X_+(T) / \Gamma} V_{\mu^*}^\lambda \otimes H_*^c (\mathcal{Sht}^\lambda_{b_0, b_m})
\]

as virtual representations of \( J_{b_0} (F) \times J_{b_m} (F) \times W_E \).

**Proof.** This follows from Proposition 5.1 by taking cohomology. \( \square \)

**Lemma 5.3.** Assume that \( m = 2 \). Let \( \pi \) be a smooth representation of \( J_{b_0} (F) \). Then we have

\[
R \text{Hom}_{J_{b_0} (F)} \left( \left( R \Gamma_c (\mathcal{Sht}^{\mu_1}_{b_0, b_1}) \otimes R \Gamma_c (\mathcal{Sht}^{\mu_2}_{b_1, b_2}) \right) \otimes_{J_{b_1} (F)} \overline{\mathbb{Q}}_\ell, \pi \right) \\
\simeq R \text{Hom}_{J_{b_1} (F)} \left( R \Gamma_c (\mathcal{Sht}^{\mu_2}_{b_1, b_2}), R \text{Hom}_{J_{b_0} (F)} \left( R \Gamma_c (\mathcal{Sht}^{\mu_1}_{b_0, b_1}), \pi \right) \right)
\]

in the derived category of representations of \( J_{b_2} (F) \times W_E \) for \([b_i] \in I^{(\mu_1, \mu_2)}_{b_0, b_2}\).
Proof. We have
\[
R \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}_{b_0,b_1}^\mu) \otimes R\Gamma_c(\text{Sh}_{b_1,b_2}^\mu) \otimes \mathbb{Q}_\ell, \pi \right)
\]
\[
\simeq R \text{Hom}_{J_b(F) \times J_b(F)} \left( R\Gamma_c(\text{Sh}_{b_0,b_1}^\mu) \otimes R\Gamma_c(\text{Sh}_{b_1,b_2}^\mu), \pi \otimes \mathbb{Q}_\ell \right)
\]
\[
\simeq R \text{Hom}_{J_b(F) \times J_b(F)} \left( R\Gamma_c(\text{Sh}_{b_0,b_1}^\mu), \Hom \left( R\Gamma_c(\text{Sh}_{b_0,b_1}^\mu), \pi \right) \right)
\]
\[
\simeq R \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}_{b_1,b_2}^\mu), R \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}_{b_0,b_1}^\mu), \pi \right) \right)
\]
in the derived category of representations of \( J_{b_2}(F) \times W_E \).

6 Duality morphism

We take a pinning \( \mathcal{P} = (G, B, T, X_0) \) of \( G \). Then define a duality involution \( \iota_{G, \mathcal{P}} \) on \( G \) as in [Pra19, Definition 1]. We simply write \( \iota \) for \( \iota_{G, \mathcal{P}} \). Note that \( \mu = -\iota \circ \mu \) in \( X_s(T)/W_G(T) \cong X_s(T)^+ \). We define an anti-involution \( \theta \) on \( G \) by \( \theta(g) = \iota(g)^{-1} \). We define the duality morphism
\[
\theta_{b,b'} : \text{Sh}_{G,b,b'}^\mu \rightarrow \text{Sh}_{G,\iota(b'),\iota(b)}^\mu
\]
by sending \( f : \mathcal{E}_b \rightarrow \mathcal{E}_{b'} \) to \( \iota(f)^{-1} : \mathcal{E}_{\iota(b')} \rightarrow \mathcal{E}_{\iota(b)} \). The above isomorphism is compatible with actions of \( \tilde{J}_b \times \tilde{J}_{b'} \) and \( \tilde{J}_{\iota(b')} \times \tilde{J}_{\iota(b)} \) under the isomorphism
\[
\tilde{J}_b \times \tilde{J}_{b'} \rightarrow \tilde{J}_{\iota(b')} \times \tilde{J}_{\iota(b)} ; \ (g, g') \mapsto (\iota(g'), \iota(g)).
\]
Then \( \theta_{b,\iota(b)} \) is an involution on \( \text{Sh}_{G,b,\iota(b)}^\mu \). We put \( \text{Gr}_{G}^{\text{op}} = L^+ G \backslash LG \). Then \( \theta \) induces a morphism \( \theta : \text{Gr}_{G}^{\text{op}} \rightarrow \text{Gr}_{G} \). Let \( E \) be the field of definition of \( \mu \). We have a morphism
\[
\pi_{b,b'}^{\mu,\text{op}} : \text{Sh}_{G,b,b'}^\mu \rightarrow \text{Gr}_{G,\text{Spd}E}^{\text{op}}
\]
obtained by forgetting the trivialization of \( \mathcal{E}_b \). We have the commutative diagram
\[
\begin{array}{ccc}
\text{Sh}_{G,b,b'}^\mu & \xrightarrow{\theta_{b,b'}} & \text{Sh}_{G,\iota(b'),\iota(b)}^\mu \\
\pi_{b,b'}^{\mu,\text{op}} & & \pi_{\iota(b'),\iota(b)}^{\mu,\text{op}} \\
\text{Gr}_{G,\text{Spd}E}^{\text{op}} & \xrightarrow{\theta} & \text{Gr}_{G,\text{Spd}E}^{\text{op}}.
\end{array}
\]
We have a canonical isomorphism \( N_{\mu} : \theta^* \text{IC}_\mu \rightarrow \text{IC}_{\mu}^{\text{op}} \) as in [Zhu17, (2.4.2)]. Further we have a canonical isomorphism \( (\pi_{b,b'}^{\mu,\text{op}})^* \text{IC}_{\mu}^{\text{op}} \rightarrow (\pi_{b,b'}^{\mu})^* \text{IC}_\mu \) as in [Zhu17, Lemma 2.24]. Hence we obtain the isomorphism
\[
R\Gamma_c(\text{Sh}_{G,\iota(b'),\iota(b)}^\mu) \rightarrow R\Gamma_c(\text{Sh}_{G,b,b'}^\mu)
\]
induced by \( \theta_{b,b'} \).

Lemma 6.1. The isomorphism
\[
R\Gamma_c(\text{Sh}_{G,\iota(b'),\iota(b)}^\mu) \rightarrow R\Gamma_c(\text{Sh}_{G,b,b'}^\mu)
\]
is compatible with actions of \( \tilde{J}_b \times \tilde{J}_{b'} \) and \( \tilde{J}_{\iota(b')} \times \tilde{J}_{\iota(b)} \) under the isomorphism
\[
\tilde{J}_b \times \tilde{J}_{b'} \rightarrow \tilde{J}_{\iota(b')} \times \tilde{J}_{\iota(b)} ; \ (g, g') \mapsto (\iota(g'), \iota(g)).
\]
Proof. This follows from the definition. 

Further, we have an involution
\[ \theta_b : \text{Sh}_G^{\mu} \times \text{Sh}_{G,1,c(\ell)}^{\mu} \to \text{Sh}_G^{\mu} \times \text{Sh}_{G,1,c(\ell)}^{\mu}; (x, x') \mapsto (\theta_{1,1}(x), \theta_{b,1}(x)). \]
We have a decomposition
\[ V_\mu \otimes V_\mu = \text{Sym}^2 V_\mu \oplus \bigwedge^2 V_\mu. \]
Let
\[ \Psi_{b,\mu} : \left( R\Gamma_c(\text{Sh}_G^{\mu}, 1) \otimes R\Gamma_c(\text{Sh}_{G,1,c(\ell)}^{\mu}) \right) \otimes_{G(F)} \overline{\mathbb{Q}_\ell} \to \sum_{\lambda \in X_s(T)/T} V_{\mu^*}^{\lambda} \otimes R\Gamma_c(\text{Sh}_{b,\lambda}^{\mu}) \]
be the morphism given by Proposition 5.1. Then \( \theta_b \) induces an involution on the source of \( \Psi_{b,\mu} \). On the other hand, the permutation \( \sigma \) on \( V_\mu \otimes V_\mu \) induces an involution on the target of \( \Psi_{b,\mu} \).

**Proposition 6.2.** The morphism \( \Psi_{b,\mu} \) is compatible with the involutions induced by \( \theta_b \) and \( \sigma \).

Proof. By the characterization of the commutativity constraint, the equality
\[ \text{IC}_\mu \star \text{IC}_\mu = \sum_{\lambda \in X_s(T)/T} V_{\mu^*}^{\lambda} \otimes \text{IC}_\lambda \]
is compatible with the involutions \( c_{\text{IC}_\mu, \text{IC}_\mu} \) and \( \sigma \). Hence the target of \( \Psi_{b,\mu} \) is equal to \( H^*_c(\text{Sh}_{b,\mu}^{\mu}, \text{IC}_\mu \star \text{IC}_\mu) \) with the involution given by \( c_{\text{IC}_\mu, \text{IC}_\mu} \). We define the morphisms \( i_1 \) and \( j_1 \) by the cartesian diagrams
\[
\begin{array}{ccc}
\text{Sh}_b^{\mu} \times \text{Spd} \bar{E} & \xrightarrow{i_1} & (\text{Sh}_b^{\mu} \times \text{Sh}_{1,c(\ell)}^{\mu})_U \leftarrow j_1 \newline \text{Spd} \bar{E} = \Delta_{2,\text{Spd} \bar{E}} & \xrightarrow{\gamma} & \text{Spd} \bar{E} \times \text{Spd} \bar{E} \leftarrow \gamma(\text{Spd} \bar{E} \times \text{Spd} \bar{E}) \setminus \Delta_{2,\text{Spd} \bar{E}}.
\end{array}
\]
Further, we define the morphisms \( i_2 \) and \( j_2 \) by the cartesian diagrams
\[
\begin{array}{ccc}
\text{Sh}_{b,\mu}^{\mu} \Delta_2 & \xrightarrow{i_2} & \text{Sh}_{b,\mu}^{\mu} \Delta_2 \leftarrow j_2 \newline \text{Spd} \bar{E} = \Delta_{2,\text{Spd} \bar{E}} & \xrightarrow{\gamma} & \text{Spd} \bar{E} \times \text{Spd} \bar{E} \leftarrow \gamma(\text{Spd} \bar{E} \times \text{Spd} \bar{E}) \setminus \Delta_{2,\text{Spd} \bar{E}}.
\end{array}
\]
Then we have the following commutative diagram
\[
\begin{array}{ccc}
R\Gamma_c((\text{Sh}_b^{\mu} \times \text{Sh}_{1,c(\ell)}^{\mu})_U, i_{1,1}j_{1,1}(\text{IC}_\mu \otimes \text{IC}_\mu)) \otimes_{G(F)} \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \left(R\Gamma_c(\text{Sh}_b^{\mu}, 1) \otimes R\Gamma_c(\text{Sh}_{1,c(\ell)}^{\mu}) \right) \otimes_{G(F)} \overline{\mathbb{Q}_\ell} \\
R\Gamma_c((\text{Sh}_{b,\mu}^{(\mu,\mu)} \otimes j_{2,2}j_{2,1}(\text{IC}_\mu \otimes \text{IC}_\mu)) \xrightarrow{\sim} R\Gamma_c(\text{Sh}_{b,\mu}^{(\mu,\mu)}, \text{IC}_\mu \star \text{IC}_\mu)
\end{array}
\]
which is compatible with involutions. Therefore we obtain the claim. \( \square \)
7 Kottwitz conjecture

Let \( b, b' \in \text{GL}_n(F) \) such that \([b] \in B(G, \mu, [b'])\). We put

\[
H^\bullet(\text{Sht}_{b, b'})[\pi] = \sum_{i,j \in \mathbb{Z}} (-1)^{i+j} \text{Ext}^i_{J_{b}(F)}(R^j \Gamma(\text{Sht}_{b, b'}^\mu), \pi)
\]

for an irreducible smooth representation \( \pi \) of \( J_b(F) \).

The following is a version of Kottwitz conjecture for moduli spaces of mixed characteristic local shtukas in \( \text{GL}_n \)-case (cf. [RV14, Conjecture 7.4]):

**Conjecture 7.1.** Assume that \( b, b' \) are basic. Let \( \varphi: W_F \to L \text{GL}_n \) be a discrete local Langlands parameter. Let \( \pi_b \) and \( \pi_{b'} \) be the irreducible smooth representations of \( J_b(F) \) and \( J_{b'}(F) \) corresponding to \( \varphi \) via the local Langlands correspondence. Then we have

\[
H^\bullet(\text{Sht}_{b, b'})[\pi_b] = \pi_{b'} \boxtimes (r_\mu \circ \varphi)
\]

in \( \text{Groth}(J_{b'}(F) \times W_F) \).

The following conjecture is motivated by [Dat07, Théorème A].

**Conjecture 7.2.** Assume that \( b, b' \) are basic. Let \( \varphi: W_F \to L \text{GL}_n \) be a discrete local Langlands parameter. Let \( \pi_b \) and \( \pi_{b'} \) be the irreducible smooth representations of \( J_b(F) \) and \( J_{b'}(F) \) corresponding to \( \varphi \) via the local Langlands correspondence. Then we have

\[
R \text{Hom}_{J_{b}(F)}(R^\bullet \Gamma(\text{Sht}_{b, b'}^\mu), \pi_b) \simeq \pi_{b'} \boxtimes (r_\mu \circ \varphi)
\]

as representations of \( J_{b'}(F) \times W_F \).

**Remark 7.3.** We have

\[
H^\bullet(\text{Sht}_{b, b'})[\pi] = \sum_{i \in \mathbb{Z}} (-1)^i \text{Ext}^i_{J_{b}(F)}(R^\bullet \Gamma(\text{Sht}_{b, b'}^\mu), \pi).
\]

Hence Conjecture 7.2 is stronger than Conjecture 7.1.

**Lemma 7.4.** Assume that \( b, b' \) are basic. Let \( \pi_b \) and \( \pi_{\iota(b)} \) be the irreducible smooth representations of \( J_b(F) \) and \( J_{\iota(b)}(F) \) corresponding via the local Jacquet–Langlands correspondence. Then the pullback of \( \pi_{\iota(b)} \) under the isomorphism \( \iota: J_b(F) \to J_{\iota(b)}(F) \) is isomorphic to \( \pi_{b'}^\iota \).

**Proof.** By [Pra19, Corollary 1], we may assume that \( \iota(g) = g^{-1} \). If \( b = 1 \), the claim follows from a theorem of Gelfand and Kazhdan (cf. [BZ76, 7.3. Theorem]). If regular elements \( g \in \text{GL}_n(F) \) and \( g' \in J_b(F) \) have the same reduced characteristic polynomial, then \( \iota(g) \in \text{GL}_n(F) \) and \( \iota(g') \in J_{\iota(b)}(F) \) are regular and have the same reduced characteristic polynomial. Hence the claim follows from the case where \( b = 1 \) and the characterization of the local Jacquet–Langlands correspondence.

We put \( \kappa(b) = v_F(\det(b)) \). For \( m_1, \ldots, m_n \in \mathbb{Z} \), let \( (m_1, \ldots, m_n) \) denote the cocharacter of \( \text{GL}_n \) or its standard Levi subgroup defined by \( z \mapsto \text{diag}(z^{m_1}, \ldots, z^{m_n}) \).

**Theorem 7.5.** Conjecture 7.2 is true in the following cases:
Lemma 5.3 and the claim (2), we have

Note that we have 

The claim (1) follows from the proof of [Dat07, Théorème A] taking care the degree in (2) and (4). By Proposition 4.1, we may assume that $(a_1, a_2)$ corresponding to $(F_{a_1}, b)$ and $(F_{a_2}, b')$ are contained in Theorem 7.5.

Proof. By the inversing isomorphism (1.1), the claims (3) and (5) are reduced to the claims (2) and (4). By Proposition 4.1, we may assume that $\kappa(b) = \kappa(b') = 0$ in (1), $\kappa(b) = 0, -1, \kappa(b') = \kappa(b) + 1$ in (2) and $\kappa(b) = -1, \kappa(b') = 1$ in (4). Further we may assume that $\kappa(b) = 0$ in (2) by Lemma 6.1 and Lemma 7.4. Then the claim (1) is trivial. The claim (1) follows from the proof of [Dat07, Théorème A] taking care the degree in [Dat07, Théorème 4.1.2].

We show the claim (4). We may assume that $b' = \nu(b)$. We put

Note that we have $f_{\nu_1, \nu_1} = \{1\}$. Let $\pi_1$ be the irreducible smooth representations of $GL_n(F)$ corresponding to $\varphi$ via the local Langlands correspondence. By Proposition 5.1, Lemma 5.3 and the claim (2), we have

$$
(V_{\mu_1, \mu_1}^{\mu_2})^* R \text{Hom}_{J_b(F)} \left( R \Gamma_c(\text{Sh}_{b,1}(F), \pi_b) \right) + (V_{\mu_1, \mu_1}^{\mu_1,1})^* R \text{Hom}_{J_b(F)} \left( R \Gamma_c(\text{Sh}_{b,1}^{\mu_1,1}(F), \pi_b) \right)
$$

Using Proposition 6.2, we can separate the above equality to obtain the claim.

Corollary 7.6. Conjecture 7.2 is true if $n \leq 3$ and $\mu$ is minuscule.

Proof. All the cases are contained in Theorem 7.5.
8 Inductive formula

For a smooth representation $\pi$ of $G(F)$ and the unipotent radical $N$ of a parabolic subgroup of $G$, let $\pi_N$ denote the Jacquet module of $\pi$ with respect to $N$.

Assume that $G = GL_2$. Let $B$ be the upper triangle Borel subgroup of $GL_2$. Let $N$ be the unipotent radical of $B$, and $N^{op}$ be the the unipotent radical of the opposite Borel subgroup $B^{op}$.

**Lemma 8.1.** Let $m \in \mathbb{Z}$. We put

$$b = \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^m \end{pmatrix}, \quad b' = \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^{m-1} \end{pmatrix}.$$ 

Let $\pi$ be an admissible representation of $G(F)$. Then we have

$$R^* \text{Hom}_{G(F)} \left( R^* \Gamma_c(\text{Sh}(1,0)), \pi \right) = -R^* \text{Hom}_{T(F)} \left( R^* \Gamma_c(\text{Sh}_{T,b'}(0,1)), \pi_{N^{op}} \otimes \delta_B^{-1} \right) \left( -\frac{1}{2} \right).$$

**Proof.** By [Cas82, A.11 Proposition, A.12 Theorem], [GI16, Theorem 4.25] (cf. [Han16]) and [Ren10, VI.9.6 Proposition], we have

$$R^* \text{Hom}_{G(F)} \left( R^* \Gamma_c(\text{Sh}_{b,b'}(1,0)), \pi \right) = R^* \text{Hom}_{G(F)} \left( \pi^*, R^* \Gamma_c(\text{Sh}_{b,b'}(1,0))^* \right) = R^* \text{Hom}_{G(F)} \left( \pi^*, R^* \Gamma_c(\text{Sh}_{b,b'}(0,1)^*) \right) = R^* \text{Hom}_{T(F)} \left( (\pi^*)_N, -R^* \Gamma_c(\text{Sh}_{T,b'}(0,1)^*) \otimes \delta_B^{-1} \right) \left( -\frac{1}{2} \right)$$

$$= R^* \text{Hom}_{T(F)} \left( (\pi^*)_N \otimes \delta_B, R^* \Gamma_c(\text{Sh}_{T,b'}(0,1)^*) \right) \left( -\frac{1}{2} \right)$$

$$= -R^* \text{Hom}_{T(F)} \left( R^* \Gamma_c(\text{Sh}_{T,b'}(0,1)), (\pi)_N \otimes \delta_B^{-1} \right) \left( -\frac{1}{2} \right)$$

$$= -R^* \text{Hom}_{T(F)} \left( R^* \Gamma_c(\text{Sh}_{T,b'}(0,1)), (\pi)_N \otimes \delta_B^{-1} \right) \left( -\frac{1}{2} \right).$$

\[ \square \]

**Proposition 8.2.** Let $\chi_1, \chi_2: F^\times \to \overline{\mathbb{Q}}^\times_\ell$ be characters. Let $\varphi_{\chi_1}: W_F \to \overline{\mathbb{Q}}^\times_\ell$ be the character corresponding to $\chi_1$. We put $\rho = \chi_1 \boxplus \chi_2$ as representations of $T(F)$. Let $m \geq 0$ and $m/2 \geq l \geq 0$. We put

$$b = \begin{pmatrix} \varpi^{m-l} & 0 \\ 0 & \varpi^l \end{pmatrix}, \quad b_1 = \begin{pmatrix} \varpi^{m-l} & 0 \\ 0 & \varpi^{l-1} \end{pmatrix}, \quad b_2 = \begin{pmatrix} \varpi^{m-l-1} & 0 \\ 0 & \varpi^{l-1} \end{pmatrix}.$$

(1) Assume $m \neq 2l$. We put

$$b'_l = \begin{pmatrix} \varpi^{m-l-1} & 0 \\ 0 & \varpi^l \end{pmatrix}.$$

If $l = 0$, then we have

$$H^*(\text{Sh}_{G,b_1}(m,0))|_{\rho} = (-1)^m \text{Ind}_{B(F)}^G(\rho \otimes \delta_B^{-1}) \boxplus \varphi_{\chi_1}^m \left( -\frac{m}{2} \right).$$
If \( l \geq 1 \), then we have

\[
H^\bullet(\text{Sh}^{(m,0)}_{G,b,1})[\rho] = -H^\bullet(\text{Sh}^{(m-1,0)}_{G,b,1})[\rho] \otimes \varphi_{x_2} \left( m + \frac{1}{2} - 2l \right) - H^\bullet(\text{Sh}^{(m-2,0)}_{G,b,1})[\rho] \otimes \varphi_{x_1} \otimes \varphi_{x_2}
\]

\[
\begin{cases}
-H^\bullet(\text{Sh}^{(m-1,0)}_{G,b,1})[\rho \otimes \delta_B^{-1}] \otimes \varphi_{x_1} \left( -\frac{1}{2} \right) & \text{if } m = 2l + 1 \\
-H^\bullet(\text{Sh}^{(m-1,0)}_{G,b,1})[\rho] \otimes \varphi_{x_1} \left( m - 2l - \frac{1}{2} \right) & \text{if } m \geq 2l + 2.
\end{cases}
\]

(2) Assume \( m = 2l \). If \( l = 0 \), then we have

\[
H^\bullet(\text{Sh}^{(0,0)}_{G,b,1})[\text{Ind}^{G(F)}_{B(F)}] = (\text{Ind}^{G(F)}_{B(F)} \rho) \boxtimes 1.
\]

If \( l \geq 1 \), then we have

\[
H^\bullet(\text{Sh}^{(m,0)}_{G,b,1})[\text{Ind}^{G(F)}_{B(F)}] \rho] = -H^\bullet(\text{Sh}^{(m-1,0)}_{G,b,1})[\rho] \otimes \varphi_{x_2} \left( -\frac{1}{2} \right)
\]

\[
-H^\bullet(\text{Sh}^{(m-1,0)}_{G,b,1})[\rho^{w} \otimes \delta_B^{-1}] \otimes \varphi_{x_1} \left( -\frac{3}{2} \right)
\]

\[
-H^\bullet(\text{Sh}^{(m-2,0)}_{G,b,2})[\text{Ind}^{G(F)}_{B(F)}] \rho].
\]

Proof. First we show the claim (1). If \( l = 0 \), we have

\[
R^\bullet \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}^{(m,0)}_{G,b,1}), \rho \right)
\]

\[
= (-1)^m R^\bullet \text{Hom}_{J_b(F)} \left( \text{Ind}^{G(F)}_{B(F)} R^\bullet \Gamma_c(\text{Sh}^{(m,0)}_{T,b,1}) \left( \frac{m}{2} \right), \rho \right)
\]

\[
= (-1)^m \text{Ind}^{G(F)}_{B(F)} R^\bullet \text{Hom}_{J_b(F)} \left( R^\bullet \Gamma_c(\text{Sh}^{(m,0)}_{T,b,1}) \left( \frac{m}{2} \right), \rho \right) \otimes \delta_B^{-1}
\]

\[
= (-1)^m \left( \text{Ind}^{G(F)}_{B(F)} \rho \otimes \delta_B^{-1} \right) \boxtimes \varphi_{x_1}^{m} \left( \frac{m}{2} \right),
\]

where we use [GI16, Theorem 4.25] at the first equality. We assume that \( l \geq 1 \). By Proposition 5.1 and Lemma 5.3, the sum

\[
R^\bullet \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}^{(m,0)}_{G,b,1}), \rho \right) + R^\bullet \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}^{(m-1,1)}_{G,b,1}), \rho \right)
\]

is equal to the sum

\[
R^\bullet \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}^{(m-1,0)}_{G,b,1}), R \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}^{(1,0)}_{G,b,1}), \rho \right) \right)
\]

\[
+ R^\bullet \text{Hom}_{J_{b_1}(F)} \left( R\Gamma_c(\text{Sh}^{(m-1,0)}_{G,b,2}), R \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}^{(1,0)}_{G,b,1}), \rho \right) \right).
\]

Since the fiber of the natural morphism \( \text{Sh}^{(1,0)}_{G,b,b_1} \rightarrow \text{Sh}^{(0,1)}_{T,b,b_1} \) is isomorphic to \( \mathbb{B}^{\varphi=x^{m+1-2l}} \), we have

\[
R^\bullet \Gamma_c(\text{Sh}^{(1,0)}_{G,b,b_1}) = -R^\bullet \Gamma_c(\text{Sh}^{(0,1)}_{T,b,b_1}) \left( 2l - m - \frac{1}{2} \right).
\]

Hence we have

\[
R^\bullet \text{Hom}_{J_b(F)} \left( R\Gamma_c(\text{Sh}^{(1,0)}_{G,b,b_1}), \rho \right) = -\rho \boxtimes \varphi_{x_2} \left( 2l - m - \frac{1}{2} \right).
\]
Further, we have

\[ R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(m-1,0)}), R \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(1,0)}), \rho \right) \right) \]

\[ = -R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(m-1,0)}), \rho \right) \varphi_{\chi_2} \left( m + \frac{1}{2} - 2l \right). \]

If \( m = 2l + 1 \), we have

\[ R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(1,0)}), \rho \right) = -R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{T, b_1}^{(1,0)}), \frac{1}{2} + 2l - m \right), \rho \]

by the claim in the case where \( l = 0 \).

If \( m \geq 2l + 2 \), since the fiber of the natural morphism \( Sht_{G, b_1}^{(1,0)} \to Sht_{T, b_1}^{(1,0)} \) is isomorphic to \( \mathbb{B}^{17} \), we have

\[ R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(1,0)}), \rho \right) = -R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{T, b_1}^{(1,0)}), \frac{1}{2} + 2l - m \right), \rho \]

\[ = -\rho \varphi_{\chi_1} \left( m - 2l - \frac{1}{2} \right). \]

Therefore

\[ R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma(Sht_{G, b_1}^{(m,0)}), \rho \right) \]

\[ = R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(m-1,0)}), R \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(1,0)}), \rho \right) \right) \]

\[ + R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(m-1,1)}), R \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(1,0)}), \rho \right) \right) \]

\[ - R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma(Sht_{G, b_1}^{(m-1,1)}), \rho \right) \]

\[ = \left\{ \begin{array}{ll}
-H^*\left( Sht_{G, b_1}^{(m-1,0)} \right)[\rho] \otimes \varphi_{\chi_2} \left( m + \frac{1}{2} - 2l \right) - H^*\left( Sht_{G, b_1}^{(m-1,1)} \right)[\rho] & \text{if } m = 2l + 1 \\
-H^*\left( Sht_{G, b_1}^{(m-1,0)} \right)[\rho] \otimes \varphi_{\chi_1} \left( m - 2l - \frac{1}{2} \right) & \text{if } m \geq 2l + 2.
\end{array} \right. \]

Next we show the claim (2). The claim is trivial if \( l = 0 \). Assume that \( l \geq 0 \). We put

\[ b'_1 = \begin{pmatrix} 0 & \varpi^{l-1} \\ \varpi^l & 0 \end{pmatrix}. \]

By Proposition 5.1 and Lemma 5.3, the sum

\[ R^* \text{Hom}_{G(F)} \left( R \Gamma(Sht_{G, b_1}^{(m,0)}), \pi \right) + R^* \text{Hom}_{G(F)} \left( R \Gamma(Sht_{G, b_1}^{(m-1,1)}), \pi \right) \]

is equal to the sum

\[ R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(m-1,0)}), R \text{Hom}_{G(F)} \left( R \Gamma_c(Sht_{G, b_1}^{(1,0)}), \pi \right) \right) \]

\[ + R^* \text{Hom}_{J_{b_1}}(F) \left( R \Gamma_c(Sht_{G, b_1}^{(m-1,1)}), R \text{Hom}_{G(F)} \left( R \Gamma_c(Sht_{G, b_1}^{(1,0)}), \pi \right) \right). \]
We have

\[ R^* \text{Hom}_{G(F)} \left( R\Gamma_c(\text{Sh}(1,0)_{G,b,b_1}), \pi \right) = 0 \]

by [Dat07, Théorème A].

By Lemma 8.1 and the geometric Lemma (cf. [Ren10, VI.5.1 Théorème]), we have

\[
R^* \text{Hom}_{J_{b_1}(F)} \left( R\Gamma_c(\text{Sh}(m-1,0)_{G,b_1,1}), R\text{Hom}_{G(F)} \left( R\Gamma_c(\text{Sh}(1,0)_{G,b,b_1}), \pi \right) \right)
\]

\[ = -R^* \text{Hom}_{J_{b_1}(F)} \left( R\Gamma_c(\text{Sh}(m-1,0)_{G,b_1,1}), R^* \text{Hom}_{T(F)} \left( R^* \Gamma_c(\text{Sh}(0,1)_{T,b_1}), \pi_{N_{op}} \otimes \delta_B^{-1} \right) \left( -\frac{1}{2} \right) \right) \]

\[ = -R^* \text{Hom}_{J_{b_1}(F)} \left( R\Gamma_c(\text{Sh}(m-1,0)_{G,b_1,1}), R^* \text{Hom}_{T(F)} \left( R^* \Gamma_c(\text{Sh}(0,1)_{T,b_1}), \rho + (\rho^w \otimes \delta_B^{-1}) \right) \left( -\frac{1}{2} \right) \right) \]

\[ = -R^* \text{Hom}_{J_{b_1}(F)} \left( R\Gamma_c(\text{Sh}(m-1,0)_{G,b_1,1}), \rho \right) \otimes \varphi_{\chi_2} \left( -\frac{1}{2} \right) - R^* \text{Hom}_{J_{b_1}(F)} \left( R\Gamma_c(\text{Sh}(m-1,0)_{G,b_1,1}), \rho \right) \otimes \varphi_{\chi_3} \left( -\frac{3}{2} \right). \]

Hence

\[ H^*(\text{Sh}(m,0)_{G,b_1,1})[\pi] = -H^*(\text{Sh}(m-1,1)_{G,b_1,1})[\pi] - H^*(\text{Sh}(m-1,0)_{G,b_1,1})[\rho] \otimes \varphi_{\chi_2} \left( -\frac{1}{2} \right) \]

\[ - H^*(\text{Sh}(m-1,0)_{G,b_1,1})[\rho \otimes \delta_B^{-1}] \otimes \varphi_{\chi_1} \left( -\frac{3}{2} \right) \]

\[ \square \]

By Proposition 8.2, we can calculate \( H^*(\text{Sh}(m,0)_{G,b_1,1})[\rho] \) and \( H^*(\text{Sh}(m,0)_{G,b_1,1})[\text{Ind}_{B(F)}^G(\rho)] \) in Proposition 8.2 inductively. We don’t pursue the explicit formula here, but record the following corollary.

**Corollary 8.3.** The \( GL_2(F) \)-representations \( H^*(\text{Sh}(m,0)_{G,b_1,1})[\rho] \) and \( H^*(\text{Sh}(m,0)_{G,b_1,1})[\text{Ind}_{B(F)}^G(\rho)] \) in Proposition 8.2 are linear combinations of proper parabolic inductions.

**Proof.** This follows from Proposition 8.2 by induction. \( \square \)

**Proposition 8.4.** We put

\[ b_1 = \begin{pmatrix} 0 & 1 \\ \varpi^{-1} & 0 \end{pmatrix} \]

and \( b_m = b_1^m \) for \( m \in \mathbb{Z} \). For an odd integer \( m \), we put

\[ b_m = \begin{pmatrix} \varpi^{m+1} & 0 \\ \varpi^{m+1} & \varpi^m \end{pmatrix} \]

Assume that \( m \geq 2 \). If \( m \) is odd or \( \varphi \) is cuspidal, we have

\[ H^*(\text{Sh}(m,0)_{b_m,1})[\pi_{b_m}] = H^*(\text{Sh}(m-1,0)_{b_{m-1},1})[\pi_{b_{m-1}}] \otimes \varphi - H^*(\text{Sh}(m-2,0)_{b_{m-2},1})[\pi_{b_{m-2}}] \otimes (r_{(1,1)} \circ \varphi). \]

If \( m \) is even and \( \varphi \) is not cuspidal, we have

\[ H^*(\text{Sh}(m,0)_{b_m,1})[\pi_{b_m}] = H^*(\text{Sh}(m-1,0)_{b_{m-1},1})[\pi_{b_{m-1}}] \otimes \varphi - H^*(\text{Sh}(m-2,0)_{b_{m-2},1})[\pi_{b_{m-2}}] \otimes (r_{(1,1)} \circ \varphi)
\]

\[ - H^*(\text{Sh}(m-1,0)_{b_{m-1},1})[\pi_{b_{m-1}}] \otimes \delta_B^{-1} \otimes \varphi \chi \left( -\frac{1}{2} \right) \]

where \( \chi \) is a character of \( F^\times \) such that \( \pi_{b_m} \simeq \text{St}_\chi \).
Proof. If $\kappa(b) \equiv 1 \mod 2$, this is proved in Theorem 7.5. Hence we may assume that $\kappa(b) \equiv 0 \mod 2$. By Proposition 4.1, we may assume that
$$b = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix}, \quad b' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
where $\varpi$ is a uniformizer of $F$. We put
$$b_1 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}, \quad b'_1 = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}.$$

By Proposition 5.1 and Lemma 5.3, the sum
$$R^* \text{Hom}_{\text{GL}_2(F)} \left( R\Gamma(\text{Sh}_{b,b'}^{(2,0)}), \pi_b \right) + R^* \text{Hom}_{\text{GL}_2(F)} \left( R\Gamma(\text{Sh}_{b,b'}^{(1,1)}), \pi_b \right)$$
is equal to the sum
$$R^* \text{Hom}_{\text{GL}_2(F)} \left( R\Gamma_c(\text{Sh}_{b,b'}^{(1,0)}), R \text{Hom}_{\text{GL}_2(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1}^{(1,0)}), \pi_b \right) \right)$$
$$+ R^* \text{Hom}_{\text{GL}_2(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1'}^{(1,0)}), R \text{Hom}_{\text{GL}_2(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1'}^{(1,0)}), \pi_b \right) \right).$$

Hence, by Corollary 7.6, it suffices to show that
$$R^* \text{Hom}_{\text{GL}_2(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1'}^{(1,0)}), R \text{Hom}_{\text{GL}_2(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1'}^{(1,0)}), \pi_b \right) \right)$$
$$= \left( \text{Ind}_{B(F)}^{GL_2(F)} \left( (\pi_b)_{\text{op}} \otimes \delta_B^{-1} \right) \right) \boxtimes (r_{1,1} \circ \varphi)$$
since $\pi_b = \pi_{b'}$. By [Cas82, A.11 Proposition, A.12 Theorem], [GI16, Theorem 4.25] (cf. [Han16]) and [Ren10, VI.9.6 Proposition], we have
$$R^* \text{Hom}_{\text{GL}_2(F)} \left( R\Gamma_c(\text{Sh}_{b,b_1'}^{(1,0)}), \pi_b \right)$$
$$= R^* \text{Hom}_{\text{GL}_2(F)} \left( (\pi_b)_N \otimes \delta_B, R^* \Gamma_c(\text{Sh}_{b,b_1'}^{(0,1)})^* \otimes \delta_B^{-1} \right) \left( \frac{-1}{2} \right)$$
$$= -R^* \text{Hom}_{T(F)} \left( (\pi_b)_N \otimes \delta_B, R^* \Gamma_c(\text{Sh}_{T,b,b_1'}^{(0,1)})^* \otimes \delta_B^{-1} \right) \left( \frac{-1}{2} \right)$$
$$= -R^* \text{Hom}_{T(F)} \left( R^* \Gamma_c(\text{Sh}_{T,b,b_1'}^{(0,1)}), (\pi_b)_{\text{op}} \otimes \delta_B^{-1} \right) \left( \frac{-1}{2} \right)$$
$$= -R^* \text{Hom}_{T(F)} \left( R^* \Gamma_c(\text{Sh}_{T,b,b_1'}^{(0,1)}), (\pi_b)_{\text{op}} \otimes \delta_B^{-1} \right) \left( \frac{-1}{2} \right).$$

If $\pi_b$ is supercuspidal, the last equation is zero and the claim follows. Assume that $\pi_b$ is not supercuspidal. Let $\varphi_T$ is the Langlands parameter of $(\pi_b)_{\text{op}} \otimes \delta_B^{-1}$. Then the last equation is equal to
$$- \left( (\pi_b)_{\text{op}} \otimes \delta_B^{-1} \right) \boxtimes (r_{T,(0,1)} \circ \varphi_T) \left( \frac{-1}{2} \right).$$
Further we have

\[
R^* \text{Hom}_{J'_{b_1}}(F) \left( R^* \Gamma_c(\text{Sh}_{b_1,b'},^{(1,0)}), - \left( (\pi_b)_{N^p} \otimes \delta^{-1}_B \right) \boxtimes (r_{T,(0,1)} \circ \varphi_T) \left( -\frac{1}{2} \right) \right)
\]

\[
= R^* \text{Hom}_{J'_{b_1}}(F) \left( \text{Ind}^{GL_2(F)}_{B(F)} R^* \Gamma_c(\text{Sh}_{b_1,b'},^{(1,0)}), \left( (\pi_b)_{N^p} \otimes \delta^{-1}_B \right) \boxtimes (r_{T,(0,1)} \circ \varphi_T)(-1) \right)
\]

\[
= \text{Ind}^{GL_2(F)}_{B(F)} R^* \text{Hom}_{J'_{b_1}}(F) \left( R^* \Gamma_c(\text{Sh}_{b_1,b'},^{(1,0)}), \left( (\pi_b)_{N^p} \otimes \delta^{-1}_B \right) \boxtimes (r_{T,(0,1)} \circ \varphi_T)(-1) \right) \otimes \delta^{-1}_B
\]

\[
= \left( \text{Ind}^{GL_2(F)}_{B(F)} \left( (\pi_b)_{N^p} \otimes \delta^{-2}_B \right) \right) \boxtimes (r_{T,(1,0)} \circ \varphi_T) \otimes (r_{T,(0,1)} \circ \varphi_T)(-1)
\]

\[
= \left( \text{Ind}^{GL_2(F)}_{B(F)} \left( (\pi_b)_{N^p} \otimes \delta^{-2}_B \right) \right) \boxtimes (r_{(1,1)} \circ \varphi)
\]

by [GI16, Theorem 4.25]. Therefore we obtain the claim. \(\square\)

**Proposition 8.5.** We put

\[
b_1 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}
\]

and \(b_m = b_1^m\) for \(m \in \mathbb{Z}\). For \(m \geq 1\), we have

\[
R^* \text{Hom}_{J_{b_m}}(F) \left( R^* \Gamma_c(\text{Sh}_{G, (m+1), b_{m-1}}, \pi_{b_m}) \right)
\]

\[
= R^* \text{Hom}_{GL_2(F)} \left( R^* \Gamma_c(\text{Sh}_{G, (m+1), b_{m-1}}), R \text{Hom}_{J_{b_m}}(F) \left( R^* \Gamma_c(\text{Sh}_{G, (m+1), b_{m-1}}, \pi_{b_m}) \right) \right)
\]

\[
- R^* \text{Hom}_{J_{b_{m-2}}}(F) \left( R^* \Gamma_c(\text{Sh}_{G, (m+1), b_{m-3}}, \pi_{b_{m-2}}) \right) \otimes \varphi.
\]

**Proof.** This follows from Proposition 5.1 and Lemma 5.3. \(\square\)

**Theorem 8.6.** Assume that \(n = 2\). Then Conjecture 7.1 is true if \(\kappa(b')\) is odd or \(\varphi\) is cuspidal.

**Proof.** We put

\[
b_1 = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}.
\]

To show the claim, we may assume that \(\mu = (m,0)\) for some \(m \geq 0\) and \(b' = 1\) or \(b_1^{-1}\) by twisting.

Assume that \(\varphi\) is cuspidal. If \(b' = 1\), we can show the claim by induction using Proposition 8.4. If \(b' = b_1^{-1}\), we can show the claim by induction using Proposition 8.5 and the case for \(b' = 1\).

It remains to treat the case where \(\varphi\) is not cuspidal and \(b' = b_1^{-1}\). First, we can show that

\[
H^\bullet(\text{Sh}_{b_1,0}^{(m,0)}, [\pi_b] - \pi_1 \boxtimes (r_{(m,0)} \circ \varphi)
\]

is a linear combination of proper parabolic inductions as representations of \(GL_2(F)\) using Corollary 8.3 and Proposition 8.4. Hence, the claim follows from Proposition 8.5 and [Dat07, Théorème A]. \(\square\)

On the other hand, the following example shows that Conjecture 7.2 is not true if \(\mu\) is not minuscule and \(\varphi\) is not cuspidal.
Example 8.7. Assume that $\mu = (2,0)$ and $b$ is basic element such that $\kappa(b) = 2$. Then we have

$$R^* \text{Hom}_{J_b(F)} (R\Gamma(\text{Sh}_b^\alpha), \pi_b)$$

$$= \pi_1 \boxtimes (r_\mu \circ \varphi) + \left(\text{Ind}_{B(F)}^{\text{GL}_2(F)} ((\pi_b)_{\text{Nop}} \otimes \delta_B^{-2})\right) \boxtimes (r_{(1,1)} \circ \varphi)$$

by Proposition 8.2 and Proposition 8.4.

Remark 8.8. Example 8.7 is compatible with the main theorem of [KW17], since the representation $\text{Ind}_{B(F)}^{\text{GL}_2(F)} ((\pi_b)_{\text{Nop}} \otimes \delta_B^{-2})$ has trace 0 on regular elliptic elements.

Remark 8.9. The error term in Example 8.7 supports that the expectation in [Far16, Remark 4.6] is true.

References

[BZ76] I. N. Bernštejn and A. V. Zelevinskiǐ, Representations of the group $\text{GL}(n, F)$, where $F$ is a local non-Archimedean field, Uspehi Mat. Nauk 31 (1976), no. 3, 5–70.


[Han16] D. Hansen, Moduli of local shtukas and Harris’s conjecture, 2016, preprint.


Naoki Imai
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
naoki@ms.u-tokyo.ac.jp