The nearby cycles of the PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

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Joseph Muller

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The University of Tokyo

Contents

Introduction

Introduction

In general, **Shimura varieties** are smooth quasi-projective varieties defined over a certain number field. Such a variety is determined by a **Shimura datum**, ie. a pair $(\mathbb{G}, X)$ where

- $\mathbb{G}$ is a reductive group over $\mathbb{Q}$,
- X is a "conjugacy class of cocharacters of $\mathbb{G}_{\mathbb{R}}$".

The datum $(\mathbb{G}, X)$ determines

- a number field $\mathbb{E} \subset \mathbb{C}$ called the **reflex field**,
- for $K \subset \mathbb{G}(\mathbb{A}_f)$ open compact subgroup which is "small enough", a Shimura variety Sh_K over $\mathrm{Spec}(\mathbb{E})$.

Rk: Sh_K can be defined as a finite union of locally symmetric spaces for $\mathbb{G}$.

If $K' \subset K$, there is a natural finite étale transition morphism $\Pi_{K',K} : \mathrm{Sh}_{K'} \to \mathrm{Sh}_K$. The **Shimura tower** is the inverse system $\mathrm{Sh} := (\mathrm{Sh}_K)_K$. The group $\mathbb{G}(\mathbb{A}_f)$ acts on Sh by **Hecke correspondences**.

$\implies$ the cohomology of Sh is naturally a $\mathbb{G}(\mathbb{A}_f) \times \mathrm{Gal}(\overline{\mathbb{E}}/\mathbb{E})$-module. It is expected to give a geometric incarnation of the **Langlands correspondences**.

Introduction

For arithmetic applications, one wants to define **integral models** of Shimura varieties over some prime p, ie. a quasi-projective scheme S_K over $\mathrm{Spec}(\mathcal{O}_{\mathbb{E},(p)})$ such that

$$\mathrm{Sh}_K \simeq \mathrm{S}_K \times_{\mathrm{Spec}(\mathcal{O}_{\mathbb{E},(p)})} \mathrm{Spec}(\mathbb{E}),$$

where $\mathcal{O}_{\mathbb{E},(p)} := \mathcal{O}_{\mathbb{E}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

If such a model S_K exists, we say that the Shimura variety has **good reduction at** p if S_K is smooth, otherwise we say that it has **bad reduction at** p.

Introduction

Assume that an integral model S_K exists over p. Fix a place v of $\mathbb{E}$ above p and write $E := \mathbb{E}_v$ for the v-adic completion. We can think of Sh_K and S_K as schemes over $\mathrm{Spec}(E)$ and $\mathrm{Spec}(\mathcal{O}_E)$ respectively. If $\ell \neq p$ is a prime, we have an isomorphism of $\mathrm{Gal}(\overline{E}/E)$-modules

$$\mathrm{H}^\bullet(\mathrm{Sh}_K \otimes_E \overline{E}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^\bullet(\overline{\mathrm{S}}_K \otimes \overline{\kappa(E)}, \mathrm{R}\Psi_\eta \overline{\mathbb{Q}_\ell}),$$

where $\kappa(E)$ is the residue field of E, $\overline{\mathrm{S}}_K := \mathrm{S}_K \times \mathrm{Spec}(\kappa(E))$ is the special fiber and $\mathrm{R}\Psi_\eta \overline{\mathbb{Q}_\ell}$ is the **nearby cycle complex** on S_K.

Rk: It works even when S_K is not proper (Lan-Stroh).

Introduction

In many cases, the special fiber $\overline{\mathrm{S}}_K$ has nice geometric properties (stratifications, Igusa varieties, etc.). Thus one may expect that $\mathrm{H}^\bullet(\overline{\mathrm{S}}_K \otimes \overline{\kappa(E)}, \mathrm{R}\Psi_\eta \overline{\mathbb{Q}_\ell})$ is easier to understand.

In this talk, we consider one specific example: the **PEL $\mathrm{GU}(\mathbf{n}-\mathbf{1},\mathbf{1})$ Shimura variety over a ramified prime**. We will compute the cohomology sheaves $\mathrm{R}^i\Psi_\eta\overline{\mathbb{Q}_\ell}$ by using the theory of **local models**.

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

Shimura varieties of PEL type can be described as **moduli spaces of abelian varieties** with additional structures. Some notations:

- $\mathbb{E} := \mathbb{Q}[\sqrt{-\delta}]$ where $\delta > 1$ squarefree.
- p a prime dividing δ.
- $\overline{\cdot} \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q})$ the non-trivial Galois involution.
- $\pi := \sqrt{-\delta}$ so that $\overline{\pi} = -\pi$.
- $E := \mathbb{E} \otimes \mathbb{Q}_p = \mathbb{Q}_p[\sqrt{-\delta}]$ a quadratic ramified extension of $\mathbb{Q}_p$.
- $(\mathbb{V}, (\cdot,\cdot))$ an n-dimensional $\mathbb{E}/\mathbb{Q}$-hermitian space of signature $(n-1,1)$ at infinity.
- $V := \mathbb{V} \otimes \mathbb{Q}_p$ with induced $E/\mathbb{Q}_p$-hermitian pairing $(\cdot,\cdot)$.

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

- $\mathbb{G} := \mathrm{GU}(\mathbb{V}, (\cdot,\cdot))$ as a reductive group over $\mathbb{Q}$.
- $\Gamma \subset \mathbb{V}$ a self-dual $\mathcal{O}_{\mathbb{E}}$-lattice (the existence of Γ is a condition on $\mathbb{V}$).
- $L := \Gamma \otimes \mathbb{Z}_p \subset V$ a self-dual $\mathcal{O}_E$-lattice.
- $K \subset \mathbb{G}(\mathbb{A}_f)$ the stabilizer of $\Gamma \otimes \widehat{\mathbb{Z}}$.
- $K^p \subset K \cap \mathbb{G}(\mathbb{A}_f^p)$ an open compact subgroup.

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

For S an $\mathcal{O}_E$-scheme, $\mathrm{S}_{K^p}(S)$ is the set of isomorphism classes of tuples $(A, \iota, \overline{\lambda}, \overline{\eta})$ where

- A is **an abelian scheme** over S of relative dimension n "up to an isogeny of order prime to p",
- ι is an "$\mathcal{O}_{\mathbb{E}}$-**action** on A", ie. a ring morphism

$$\iota : \mathcal{O}_{\mathbb{E}} \otimes \mathbb{Z}_{(p)} \to \mathrm{End}_S(A) \otimes \mathbb{Z}_{(p)},$$

- $\overline{\lambda}$ is an $\mathcal{O}_{\mathbb{E}}$-linear **principal polarization** on A (seen up to a scalar in $\mathbb{Q}^\times$ locally on S),
- "$\overline{\eta} : \mathrm{H}_1(A, \mathbb{A}_f^p) \simeq \mathbb{V} \otimes \mathbb{A}_f^p \mod K^p$" a K^p-**level structure** compatible with the hermitian products,

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

Moreover, we add the following conditions on A, ι

- **Kottwitz' determinant condition**

$$\forall x \in \mathcal{O}_{\mathbb{E}}, \det(T - \iota(x) \mid \mathrm{Lie}_S(A)) = (T - x)^{n-1}(T - \overline{x})^1 \in \mathcal{O}_E[T],$$

- **Pappas' wedge condition** if $n \geq 3$

$$\forall x \in \mathcal{O}_{\mathbb{E}}, \bigwedge^2 (\iota(x) - x) = 0 \text{ and } \bigwedge^n (\iota(x) - \overline{x}) = 0 \text{ on } \mathrm{Lie}_S(A).$$

If K^p is small enough, S_{K^p} is a **flat** quasi-projective scheme over $\mathrm{Spec}(\mathcal{O}_E)$. It is an integral model over p of a Shimura variety Sh_K for $\mathbb{G} = \mathrm{GU}(\mathbb{V})$ where $K = K_p K^p$ and $K_p := \mathrm{Stab}(L) \subset \mathbb{G}(\mathbb{Q}_p)$.

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

For an $\mathcal{O}_E$-scheme S and $A, \iota \in (AV)_S$, write

$$\Pi := \iota(\pi) \in \mathrm{End}_S(A) \otimes \mathbb{Z}_{(p)}.$$

If p vanishes on S and $A \in \overline{\mathrm{S}}_{K^p}(S)$, by Pappas' condition we have $\bigwedge^2 \Pi = 0$ on $\mathrm{Lie}_S(A)$.

Let Z be the **zero dimensional** closed subscheme of $\overline{S}_{K^p}$ consisting of those A, ι such that $\Pi \cong 0$ on $\mathrm{Lie}(A)$.

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

Theorem (Pappas)

(1) The closed subscheme Z is the **singular locus** of the special fiber $\overline{\mathrm{S}}_{K^p}$.
(2) If $n = 2$, the scheme S_{K^p} is regular and has semi-stable reduction.
(3) If $n \geq 3$, the scheme S_{K^p} is regular outside of Z. The **blow up** $b_{K^p} : \mathrm{S}'_{K^p} \to \mathrm{S}_{K^p}$ **at** Z is regular and has semi-stable reduction.

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

From (1), it follows that the nearby cycle complex $\mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell}$ satisfy

- $\mathrm{R}^0\Psi_\eta\overline{\mathbb{Q}_\ell} = \overline{\mathbb{Q}_\ell}$ (constant sheaf over $\overline{\mathrm{S}}_{K^p}$),
- for $i \geq 1$, $\mathrm{R}^i\Psi_\eta\overline{\mathbb{Q}_\ell}$ is a **skyscraper sheaf concentrated on Z.**

$\implies$ it remains to compute $(\mathrm{R}^i\Psi_\eta\overline{\mathbb{Q}_\ell})_{\overline{z}}$ for all points $z \in Z$. To do this, we use the **local model**.

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

By general theory, there is a **local model diagram**

$$\begin{tikzcd} & \mathcal{N} \arrow[dl, "r"'] \arrow[dr, "q"] & \\ \mathrm{S}_{K^p} & & M^{\mathrm{loc}}, \end{tikzcd}$$

where, for an $\mathcal{O}_E$-scheme S, $\mathcal{N}(S)$ is the set of $(A, \iota, \overline{\lambda}, \overline{\eta}, \gamma)$ where

- $(A, \iota, \overline{\lambda}, \overline{\eta}) \in \mathrm{S}_{K^p}(S)$,
- $\gamma : \mathcal{H}^1_{\mathrm{dR}}(A^\vee/S) \xrightarrow{\sim} L \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ is an isomorphism of sheaves of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules compatible with the hermitian pairings.

The map r simply **forgets about** γ.

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

The scheme M^{loc} is the **local model** and can be defined purely in terms of linear algebra, ie. $M^{\mathrm{loc}}(S)$ is the set of subsheaves $\mathcal{F} \subset L \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules such that

- $\mathcal{F}$ is a **locally free** $\mathcal{O}_S$-direct summand of $L \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ of rank n,
- $\mathcal{F}$ is **totally isotropic** for the pairing $(\cdot,\cdot) \otimes \mathcal{O}_S$,
- it satisfies **Kottwitz and Pappas conditions**, $\forall x \in \mathcal{O}_E$,

$$\det(T - x \otimes 1 \,|\, \mathcal{F})) = (T - 1 \otimes x)^1 (T - 1 \otimes \overline{x})^{n-1} \in \mathcal{O}_E[T],$$

$$\bigwedge^n (x \otimes 1 - 1 \otimes x) = 0 \text{ and } \bigwedge^2 (x \otimes 1 - 1 \otimes \overline{x}) = 0 \text{ on } \mathcal{F}.$$

Then q sends $(A, \iota, \overline{\lambda}, \overline{\eta}, \gamma)$ to the image of the submodule $\mathcal{H}^0(A^\vee, \Omega^1_{A^\vee}) \subset \mathcal{H}^1_{\mathrm{dR}}(A^\vee/S)$ via γ.

The PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime

To say that M^{loc} is **the local model** of S_{K^p} means that there exists an etale cover $\mathcal{V} \to \mathrm{S}_{K^p}$ and a section $s : \mathcal{V} \to \mathcal{N}$ such that the composition $qs : \mathcal{V} \to M^{\mathrm{loc}}$ is an etale cover.

$$\begin{tikzcd} \mathcal{V} \arrow[r, "s"] \arrow[d] & \mathcal{N} \arrow[dl, "r"'] \arrow[dr, "q"] & \\ \mathrm{S}_{K^p} & & M^{\mathrm{loc}}, \end{tikzcd}$$

$\implies$ By restricting $\mathrm{R}\Psi_\eta \overline{\mathbb{Q}_\ell}$ to the etale cover, **it is enough to compute the nearby cycles on M^{loc} instead.**

Nearby cycles and semi-stable reduction

Nearby cycles and semi-stable reduction

In this section, let E be any p-adic field with ring of integers $\mathcal{O}_E$ and residue field κ.

$$\begin{tikzcd}
 & & \overline{\eta} \arrow[dl, "\bar{j}"'] \arrow[d] \\
\widetilde{s} \arrow[r, "\widetilde{i}"] \arrow[d] & \widetilde{S} \arrow[d] & \widetilde{\eta} \arrow[l, "\widetilde{j}"'] \arrow[d] \\
s \arrow[r] & S & \eta \arrow[l]
\end{tikzcd}$$

where

$$s = \mathrm{Spec}(\kappa), \qquad S = \mathrm{Spec}(\mathcal{O}_E), \qquad \eta = \mathrm{Spec}(E),$$
$$\widetilde{s} = \mathrm{Spec}(\overline{\kappa}), \qquad \widetilde{S} = \mathrm{Spec}(\mathcal{O}_{E^{\mathrm{un}}}), \qquad \widetilde{\eta} = \mathrm{Spec}(E^{\mathrm{un}}),$$
$$\overline{\eta} = \mathrm{Spec}(\overline{E^{\mathrm{un}}}).$$

Nearby cycles and semi-stable reduction

$$\begin{tikzcd}
 & & \overline{\eta} \arrow[dl, "\bar{j}"'] \arrow[d] \\
\widetilde{s} \arrow[r, "\widetilde{i}"] \arrow[d] & \widetilde{S} \arrow[d] & \widetilde{\eta} \arrow[l, "\widetilde{j}"'] \arrow[d] \\
s \arrow[r] & S & \eta \arrow[l]
\end{tikzcd}$$

Fix $\ell \neq p$ and let Λ be a **coefficient ring** eg. $\mathbb{Z}/\ell^k\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, or $\overline{\mathbb{Q}_\ell}$. Let $f : X \to S$ be a morphism of finite type. By base change, we have morphisms

$$X_{\widetilde{s}} \xrightarrow{\widetilde{i}} X_{\widetilde{S}} \xleftarrow{\bar{j}} X_{\overline{\eta}}$$

Nearby cycles and semi-stable reduction

$$X_{\widetilde{s}} \xrightarrow{\widetilde{i}} X_{\widetilde{S}} \xleftarrow{\bar{j}} X_{\overline{\eta}}$$

For a scheme Y, let $D^+(Y, \Lambda)$ denote the **derived category** of bounded-below complexes of (etale) sheaves of Λ-modules on Y.

Definition: For $K \in D^+(X_\eta, \Lambda)$, the **nearby cycles complex of** K is

$$\mathrm{R}\Psi_\eta K := \widetilde{i}^* \mathrm{R}\bar{j}_*(K_{|X_{\overline{\eta}}}).$$

Then $\mathrm{R}\Psi_\eta K \in D^+(X_{\widetilde{s}}, \Lambda, \mathrm{Gal}(\overline{\eta}/\eta))$, the bounded-below derived category of sheaves of Λ-modules on $X_{\widetilde{s}}$ equipped with a **continuous action** of $\mathrm{Gal}(\overline{\eta}/\eta)$.

Nearby cycles and semi-stable reduction

If $h : X \to Y$ is smooth, we have the **smooth base change theorem**, ie. the natural map

$$h_{\tilde{s}}^* \mathrm{R}\Psi_\eta^Y \xrightarrow{\sim} \mathrm{R}\Psi_\eta^X h_{\overline{\eta}}^*$$

is an isomorphism. In particular if $X \to S$ is smooth then

$$\mathrm{R}\Psi_\eta \Lambda \simeq \Lambda.$$

Slogan: if X is smooth then the nearby cycles are trivial.

Nearby cycles and semi-stable reduction

If $h : X \to Y$ is proper, we have the **proper base change theorem**, ie. the natural map

$$\mathrm{R}h_{\widetilde{s},*}\mathrm{R}\Psi_\eta^X \xrightarrow{\sim} \mathrm{R}\Psi_\eta^Y \mathrm{R}h_{\overline{\eta},*}$$

is an isomorphism. In particular if $X \to S$ is proper then

$$\mathrm{R}\Gamma(X_{\widetilde{s}}, \mathrm{R}\Psi_\eta K) \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\overline{\eta}}, K),$$

for all $K \in D^+(X_\eta, \Lambda)$, compatible with the Galois action.

Slogan: cohomology of the generic fiber = cohomology of the special fiber with coefficients in the nearby cycles.

Definition: We say that $X \to S$ is (strictly) semi-stable of relative dimension n if

1. $X \to S$ is regular and flat,
2. $X_\eta \to \eta$ is smooth of relative dimension n,
3. $X_s \hookrightarrow X$ is a **divisor with simple normal crossings**.

Write $X_s = \sum X_k$ as a sum of smooth divisors X_k's for $1 \leq k \leq m$. Write Λ_{X_k} for the **constant sheaf Λ with support on X_k**.

Nearby cycles and semi-stable reduction

Theorem (Grothendieck)

We have $\mathrm{R}^0\Psi_\eta\Lambda = \Lambda$ and

$$\mathrm{R}^1\Psi_\eta\Lambda \simeq \Big(\bigoplus_{1\leq k\leq m} \Lambda_{X_{k,\widetilde{s}}}/\Lambda\Big)(-1).$$

Moreover for all $i \geq 2$ we have

$$\mathrm{R}^i\Psi_\eta\Lambda \simeq \Lambda^i\mathrm{R}^1\Psi_\eta\Lambda.$$

In particular $\mathrm{R}^i\Psi_\eta\Lambda = 0$ if $i \geq m$.

Example: if $m = 2$ then $\mathrm{R}^0\Psi_\eta\Lambda \simeq \Lambda$ and $\mathrm{R}^1\Psi_\eta\Lambda \simeq \Lambda_{Q_{\widetilde{s}}}(-1)$ where $Q := X_1 \cap X_2$ is the **scheme-theoretic intersection**.

Nearby cycles on the local model

Nearby cycles on the local model

From now on, recover our notations for the local model (E is quadratic ramified, etc.)

Let $\Pi := \pi \otimes 1 \in \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{F}_p$.
Let $\overline{M^{\mathrm{loc}}} := M^{\mathrm{loc}} \times \mathrm{Spec}(\mathbb{F}_p)$ denote the **special fiber**.
Let $y := [\Pi(L \otimes \mathbb{F}_p)] \in \overline{M^{\mathrm{loc}}}(\mathbb{F}_p)$.

Theorem (Pappas)

The special fiber $\overline{M^{\mathrm{loc}}}$ is smooth outside of y, the unique singular point.

$\implies$ We have $\mathrm{R}^0\Psi_\eta\overline{\mathbb{Q}_\ell} \simeq \overline{\mathbb{Q}_\ell}$ and for $i \geq 1$, the sheaves $\mathrm{R}^i\Psi_\eta\overline{\mathbb{Q}_\ell}$ on $\overline{M^{\mathrm{loc}}}$ are **skyscraper**, concentrated at $\overline{y}$ (a geometric point over y).

Nearby cycles on the local model

Let us state our main result.

Theorem (M.)

For $i \geq 0$ we have

$$(\mathrm{R}^i\Psi_\eta\overline{\mathbb{Q}_\ell})_{\overline{y}} = \begin{cases} \overline{\mathbb{Q}_\ell} & \text{if } i = 0, \\ \overline{\mathbb{Q}_\ell} & \text{if } n \text{ is even and } i = n-1, \\ 0 & \text{else.} \end{cases}$$

If n is even, the action of $\mathrm{Gal}(\overline{E}/E)$ on $(\mathrm{R}^{n-1}\Psi_\eta\overline{\mathbb{Q}_\ell})_{\overline{y}}$ is trivial on the inertia subgroup, and the Frobenius acts by multiplication by the scalar $\epsilon p^{\frac{n}{2}}$, where

$$\epsilon = \begin{cases} 1 & \text{if } n = 2 \text{ or if } n \geq 4 \text{ and } (V, (\cdot,\cdot)) \text{ is split,} \\ -1 & \text{if } n \geq 4 \text{ and } (V, (\cdot,\cdot)) \text{ is non-split.} \end{cases}$$

Nearby cycles on the local model

Remarks: (1) The **discriminant** of the hermitian space $(V, (\cdot,\cdot))$ is

$$\mathrm{disc}(V) := (-1)^{\frac{n(n+1)}{2}}\det(V) \in \mathbb{Q}_p^\times/\mathrm{Norm}_{E/\mathbb{Q}_p}(E^\times) \simeq \mathbb{Z}/2\mathbb{Z}.$$

We say that $(V, (\cdot,\cdot))$ is **split** if $\mathrm{disc}(V) = 1$, and that it is **non-split** otherwise.

(2) In particular, if n is odd then $\mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell} \simeq \overline{\mathbb{Q}_\ell}$, as in the case of good reduction. The singularities do not disrupt the cohomology.

(2) In 2003, Krämer computed the **alternating trace of the Frobenius**

$$\mathrm{Tr}^{\mathrm{ss}}(\mathrm{Frob}, (\mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell})_{\overline{y}}) := \sum_{i\geq 0}(-1)^i\mathrm{Trace}(\mathrm{Frob}, (\mathrm{R}^i\Psi_\eta\overline{\mathbb{Q}_\ell})_{\overline{y}}).$$

Our computations agree with her's.

Outline of the proof

Outline of the proof

If $n = 2$, the proof is easy since M^{loc} has semi-stable reduction. From now on, **we assume** $n \geq 3$.

Let $b : (M^{\mathrm{loc}})' \to M^{\mathrm{loc}}$ be the **blow-up at the singular point** y. Let $\mathrm{R}\Psi'_\eta\overline{\mathbb{Q}_\ell}$ denote the **nearby cycles on** $(M^{\mathrm{loc}})'$. By **proper base change** and since b is an isomorphism on the generic fibers, we have

$$\mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell} \simeq \mathrm{R}b_{\overline{s}*}\mathrm{R}\Psi'_\eta\overline{\mathbb{Q}_\ell},$$

where $b_{\overline{s}}$ is the induced map on geometric special fibers. In particular,

$$(\mathrm{R}^i\Psi_\eta\overline{\mathbb{Q}_\ell})_{\overline{y}} = \mathrm{H}^i(b_{\overline{s}}^{-1}\{\overline{y}\}, \mathrm{R}\Psi'_\eta\overline{\mathbb{Q}_\ell}).$$

Outline of the proof

The blow-up $(M^{\mathrm{loc}})'$ has **semi-stable reduction**. Moreover, Krämer gives an explicit description of the special fiber.

Theorem (Krämer)

The special fiber $(\overline{M^{\mathrm{loc}}})'$ is the union of **two smooth irreducible varieties** Z_1 and Z_2. We have $Z_1 := b_s^{-1}\{y\} \simeq \mathbb{P}^{n-1}$ and Z_2 is a $\mathbb{P}^1$-bundle over the scheme theoretic intersection $Q := Z_1 \cap Z_2$. Moreover, the closed immersion $Q \hookrightarrow Z_1 \simeq \mathbb{P}^{n-1}$ identifies Q with an explicit smooth quadric in $\mathbb{P}^{n-1}$.

The situation is summed up in the following diagram.

$$\begin{tikzcd}
 & (\overline{M^{\mathrm{loc}}})' & \\
Z_1 \simeq \mathbb{P}^{n-1} \arrow[ur, hook, "i_1"] & & Z_2 \simeq \mathbb{P}(\mathcal{E}) \arrow[ul, hook', "i_2"'] \arrow[dl, bend right, "\varphi"'] \\
 & Q \arrow[ul, hook', "\iota_1"] \arrow[ur, hook, "\iota_2"'] &
\end{tikzcd}$$

where $\mathcal{E}$ is a locally free sheaf of rank 2 on Q and φ is the associated projective bundle morphism.

Recall that **we want to compute** $\mathrm{H}^i(Z_1, i_1^*\mathrm{R}\Psi'_\eta\overline{\mathbb{Q}_\ell})$.

Since $(M^{\mathrm{loc}})'$ has semi-stable reduction, we have

$$\mathrm{R}^i\Psi'_\eta\overline{\mathbb{Q}_\ell} = \begin{cases} \overline{\mathbb{Q}_\ell} & \text{if } i = 0, \\ i_{Q*}\overline{\mathbb{Q}_\ell}(-1) & \text{if } i = 1, \\ 0 & \text{else,} \end{cases}$$

where $i_Q : Q \hookrightarrow (\overline{M^{\mathrm{loc}}})'$ is the closed immersion.

From general theory, the nearby cycles $\mathrm{R}\Psi'_\eta\overline{\mathbb{Q}_\ell}$ are equipped with **the monodromy filtration**. Saito T. computed the graded components of this filtration in the case of semi-stable reduction.

We have

$$\mathrm{Gr}_r \mathrm{R}\Psi'_\eta \Lambda \simeq \begin{cases} i_{Q*}\overline{\mathbb{Q}_\ell}[-1] & \text{if } r = -1, \\ (i_{1*}\overline{\mathbb{Q}_\ell} \oplus i_{2*}\overline{\mathbb{Q}_\ell})[0] & \text{if } r = 0, \\ i_{Q*}\overline{\mathbb{Q}_\ell}(-1)[-1] & \text{if } r = 1, \\ 0 & \text{else,} \end{cases}$$

where for $k = 1, 2$, $i_{k*}\overline{\mathbb{Q}_\ell}$ is the constant sheaf $\overline{\mathbb{Q}_\ell}$ concentrated on Z_k.

Outline of the proof

By restriction, we also have an induced filtration on $i_{1*}i_1^* \mathrm{R}\Psi'_\eta \overline{\mathbb{Q}_\ell}$ and likewise, the graded pieces are

$$\mathrm{Gr}_r i_{1*}i_1^* \mathrm{R}\Psi'_\eta \Lambda \simeq \begin{cases} i_{Q*}\overline{\mathbb{Q}_\ell}[-1] & \text{if } r = -1, \\ (i_{1*}\overline{\mathbb{Q}_\ell} \oplus i_{Q*}\overline{\mathbb{Q}_\ell})[0] & \text{if } r = 0, \\ i_{Q*}\overline{\mathbb{Q}_\ell}(-1)[-1] & \text{if } r = 1, \\ 0 & \text{else.} \end{cases}$$

The natural **adjunction morphism** $\mathrm{R}\Psi'_\eta \overline{\mathbb{Q}_\ell} \to i_{1*}i_1^* \mathrm{R}\Psi'_\eta \overline{\mathbb{Q}_\ell}$ is compatible with the filtrations.

Outline of the proof

The filtrations induce **spectral sequences** computing the cohomology with coefficients in the nearby cycles:

$$\begin{aligned}(E)_1^{a,b} &= \mathrm{H}^{a+b}((\overline{M^{\mathrm{loc}}})', \mathrm{Gr}_{-a}\mathrm{R}\Psi'_\eta\overline{\mathbb{Q}_\ell}) &\implies \mathrm{H}^{a+b}((\overline{M^{\mathrm{loc}}})', \mathrm{R}\Psi'_\eta\overline{\mathbb{Q}_\ell}),\\ (i_1^*E)_1^{a,b} &= \mathrm{H}^{a+b}(Z_1, \mathrm{Gr}_{-a}i_1^*\mathrm{R}\Psi'_\eta\overline{\mathbb{Q}_\ell}) &\implies \mathrm{H}^{a+b}(Z_1, i_1^*\mathrm{R}\Psi'_\eta\overline{\mathbb{Q}_\ell}).\end{aligned}$$

The adjunction morphism now induces a **morphism of spectral sequences** $(E)_\bullet^{a,b} \to (i_1^*E)_\bullet^{a,b}$. In the first page, it leads to commutative diagrams as follows.

For $0 \leq i \leq 2(n-1)$,

$$\begin{tikzcd}
\mathrm{H}^{i-2}(Q,\overline{\mathbb{Q}_\ell})(-1) \arrow[r, "-\iota_{1*}+\iota_{2*}"] \arrow[d, "\mathrm{id}"] & \mathrm{H}^{i}(Z_1,\overline{\mathbb{Q}_\ell}) \oplus \mathrm{H}^{i}(Z_2,\overline{\mathbb{Q}_\ell}) \arrow[r, "-\iota_1^*+\iota_2^*"] \arrow[d, "\mathrm{id}\oplus\iota_2^*"] & \mathrm{H}^{i}(Q,\overline{\mathbb{Q}_\ell}) \arrow[d, "\mathrm{id}"] \\
\mathrm{H}^{i-2}(Q,\overline{\mathbb{Q}_\ell})(-1) \arrow[r, "f"] & \mathrm{H}^{i}(Z_1,\overline{\mathbb{Q}_\ell}) \oplus \mathrm{H}^{i}(Q,\overline{\mathbb{Q}_\ell}) \arrow[r, "g"] & \mathrm{H}^{i}(Q,\overline{\mathbb{Q}_\ell})
\end{tikzcd}$$

Top maps are restriction and Gysin maps for etale cohomology. Vertical maps are restrictions. We want to understand f, g and **compute the cohomology of the bottom chain complex.**

Outline of the proof

By commutativity, there exists $\alpha_i : \mathrm{H}^i(Q) \to \mathrm{H}^i(Q)$ which is identity on $\mathrm{Im}(\iota_2^*)$ such that

$$\begin{aligned}
\mathrm{H}^{i-2}(Q)(-1) &\xrightarrow{\quad f \quad} \mathrm{H}^i(Z_1) \oplus \mathrm{H}^i(Q) \xrightarrow{\qquad g \qquad} \mathrm{H}^i(Q) \\
x &\longmapsto (-\iota_{1*}(x), \iota_2^*\iota_{2*}(x)) \\
& \qquad\qquad (x, y) \longmapsto -\iota_1^*(x) + \alpha_i(y)
\end{aligned}$$

Outline of the proof

By the **projective bundle formula** for $\varphi : Z_2 \to Q$, we know that φ^* induces an isomorphism

$$\mathrm{H}^\bullet(Q)[t]/(t^2) \xrightarrow{\sim} \mathrm{H}^\bullet(Z_2).$$

Since $\varphi \circ \iota_2 = \mathrm{id}_Q$, we have $\iota_2^* \varphi^* = \mathrm{id}$ on $\mathrm{H}^i(Q)$. In particular, $\alpha_i \equiv \mathrm{id}$ and g **is surjective.**

Since Q is a smooth quadric in $Z_1 \simeq \mathbb{P}^{n-1}$, it is well known that

$$\mathrm{H}^\bullet(Q) = \iota_1^* \mathrm{H}^\bullet(Z_1) \oplus \mathrm{H}^{n-2}_{\mathrm{prim}}(Q),$$

where the **primitive cohomology** is defined by $H^{n-2}_{\mathrm{prim}}(Q) := \mathrm{Ker}(\iota_{1*} : \mathrm{H}^{n-2}(Q) \to \mathrm{H}^n(Z_1)(1))$. By general theory, $H^{n-2}_{\mathrm{prim}}(Q)$ is zero if n is odd, and is 1-dimensional if n is even.

Thus, the bottom row of the commutative diagram has the form

$$\mathrm{H}^{i-2}(Q)(-1) \xrightarrow{f} \mathrm{H}^{i}(Z_1) \oplus \mathrm{H}^{i}(Q) \twoheadrightarrow \mathrm{H}^{i}(Q),$$

and **the sequence is always exact in the middle**.

Moreover the map f is **injective except when n is even and $i = n$**. In this case, the kernel is given by $H^{n-2}_{\mathrm{prim}}(Q) \simeq \overline{\mathbb{Q}_\ell}$ (up to Tate twist). It only remains to **compute the Frobenius action**. To do this, we use Lefschetz' trace formula.

Outline of the proof

Assume now that $n = 2m$. Then Q is cut out by the equation

- $X_1^2 + \ldots + X_m^2 - X_{m+1}^2 - \ldots - X_n^2$ if $(V, (\cdot,\cdot))$ is split,
- $X_1^2 + \ldots + X_m^2 - X_{m+1}^2 - \ldots - X_{n-1}^2 - \delta X_n^2$ for some $\delta \in \mathbb{F}_p^\times$ which is not a square if $(V, (\cdot,\cdot))$ is not split.

Consider the **Jacobi sum**

$$j_m := \frac{1}{p-1} \sum_{\substack{u_1+\ldots+u_{2m}=0 \\ u_i \in \mathbb{F}_p^\times}} \left(\frac{u_1}{p}\right) \cdots \left(\frac{u_{2m}}{p}\right).$$

Proposition (Weil)

$$\#Q(\mathbb{F}_p) = \frac{p^{n-1}-1}{p-1} + \epsilon \left(\frac{-1}{p}\right)^m j_m,$$

where $\epsilon = 1$ if $(V, (\cdot,\cdot))$ is split, -1 otherwise.

Outline of the proof

Proposition

We have $j_m = \left(\frac{-1}{p}\right)^m p^{m-1}$.

$\implies$ By **Lefschetz's trace formula**, we deduce that the Frobenius acts like multiplication by ϵp^{m-1} on $\mathrm{H}^{n-2}_{\mathrm{prim}}(Q)$.

This concludes the proof!

Thank you for listening!

ご清聴ありがとうございました。