Cohomology of the supersingular locus of certain PEL Shimura varieties of Coxeter type

2023 NTU x UTokyo joint conference:
arithmetic geometry/number theory session

Joseph Muller

December 8th 2023

The University of Tokyo

Contents

Introduction: the notion of Coxeter type

Basic loci of Coxeter type (following [Görtz, He, Nie])

Consider a datum (G, μ, K) where

- G: connected reductive group over a p-adic field F,
- μ: conjugacy class of cocharacter $\mathbb{G}_{m,F^{\mathrm{ac}}} \to G_{F^{\mathrm{ac}}}$,
- K: parahoric subgroup of $G(F)$.

Let $[b_0] \in B(G, \mu)$ unique basic element and $X(\mu, b_0)_K$ the associated basic partial affine Deligne-Lusztig (DL) variety.

Introduction: the notion of Coxeter type

Definition

The pair (G, μ) is **fully Hodge-Newton (HN) decomposable** if $X(\mu, b_0)_K$ is "naturally" stratified by classical DL varieties for some (equivalently for all) K.
The triple (G, μ, K) is **of Coxeter type** if (G, μ) is fully HN decomposable and the DL varieties involved are of Coxeter type.

The stratification by classical DL varieties is the **Bruhat-Tits (BT) stratification**.
Remark: "of Coxeter type" depends on K.

Remark: [Görtz, He, Nie] classified all the fully HN decomposable pairs (G, μ), and all the triples (G, μ, K) of Coxeter type.

Motivation

Assume $F = \mathbb{Q}_p$. If (G, μ, K) comes from an integral Rapoport-Zink (RZ) datum, then $X(\mu, b_0)_K$ = perfection of the RZ space $\mathcal{M}(G, \mu, b_0)_K$.
If moreover (G, μ) is the p-local component of a Shimura datum, then $\mathcal{M}(G, \mu, b_0)_K$ provides a uniformization of the basic locus in the mod p reduction of a Shimura variety.

Introduction: the notion of Coxeter type

Cases where BT stratification is studied (non exhaustive):

- $\mathrm{GU}(1, n-1)$, p inert, K hyperspecial [Vollaard, Wedhorn], any max. K by [Cho],
- $\mathrm{GU}(1, n-1)$, ramified p, special K by [Rapoport, Terstiege, Wilson], level K of exotic good reduction by [Wu],
- $\mathrm{GU}(2, 2)$, K hyperspecial, p inert by [Howard, Pappas], split p by [Fox], see also [Wang],
- $\mathrm{GU}(2, 2)$, ramified p, special K [Oki],
- $\mathrm{GU}(2, n-2)$ for $n \geq 5$, p inert, hyperspecial K by [Fox, Imai],
- $\mathrm{GSpin}(n, 2)$, hyperspecial K by [Howard, Pappas], a non-hyperspecial K by [Oki],

All cases are of Coxeter type, except [Cho] (fully HN decomposable) and [Fox, Imai] (not fully HN decomposable).

Introduction: the notion of Coxeter type

In all the cases of Coxeter type, the basic loci is stratified by Coxeter varieties, and [Lusztig] computed the ℓ-adic cohomology of all such varieties.

Question

How can we use the cohomology of DL varieties of Coxeter type and the combinatorics of the BT stratification to study the cohomology of RZ spaces/of basic loci of Shimura varieties?

Motivated by results of [Mantovan] and [Shen], one expects that the $G(\mathbb{Q}_p)$-supercuspidal part of $\varinjlim_{\mathbb{K}} \mathrm{H}_c^\bullet(\mathrm{Sh}_{\mathbb{K}})$ is encoded in the cohomology of the basic loci (with higher levels at p).

Computing cohomology: a general strategy

Computing cohomology: a general strategy

$\mathcal{M} = \mathcal{M}(G, \mu, b_0)_K$ = formal scheme over $\mathrm{Spf}(\mathcal{O}_{\breve{E}})$ with action of $J(\mathbb{Q}_p)$, where

- E = local reflex field (a p-adic field),
- $\breve{E} = \widehat{E^{\mathrm{un}}}$,
- $J = J_{b_0}$ = a certain inner form of G.

$\mathcal{M}_{\mathrm{red}}$ = reduced special fiber over $\mathrm{Spec}(\kappa(E)^{\mathrm{ac}})$,
$\mathcal{M}^{\mathrm{an}}$ = generic fiber as an analytic space over $\breve{E}$.

Computing cohomology: a general strategy

From now on, assume (G, μ, K) is of Coxeter type.

BT stratification

$$\mathcal{M}_{\mathrm{red}} = \bigsqcup_{\Lambda \in \mathrm{BT}'} \mathcal{M}_\Lambda^\circ,$$

where

- $\mathcal{M}_\Lambda^\circ =$ a Coxeter variety,
- $\mathrm{BT}' =$ a polysimplicial complex closely related to the BT building of $J(\mathbb{Q}_p)$.

$\mathcal{M}_\Lambda^\circ$ is a smooth affine variety over $\mathrm{Spec}(\kappa(E))$ and $\mathcal{M}_\Lambda := \overline{\mathcal{M}_\Lambda^\circ}$ is a projective closure.

Computing cohomology: a general strategy

$\mathrm{red} : \mathcal{M}^{\mathrm{an}} \to \mathcal{M}_{\mathrm{red}}$, the reduction map, anticontinuous.
$U_\Lambda := \mathrm{red}^{-1}(\mathcal{M}_\Lambda)$ the **analytical tube over** $\mathcal{M}_\Lambda$, open in $\mathcal{M}^{\mathrm{an}}$.
$\{U_\Lambda\}_{\Lambda \in \mathrm{BT}'}$ is an open cover of $\mathcal{M}^{\mathrm{an}}$, inducing a
$J(\mathbb{Q}_p) \times W$-equivariant Čech spectral sequence ($\ell \neq p$)

$$E_1^{a,b} = \bigoplus_{\gamma \in I_{-a+1}} \mathrm{H}_c^b(U(\gamma) \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}),$$

where $W = W_E$ is the Weil group of E and for $a \leq 0$

$$I_{-a+1} := \left\{ \gamma \subset \mathrm{BT}' \,\middle|\, \#\gamma = -a+1 \text{ and } U(\gamma) := \bigcap_{\Lambda \in \gamma} U_\Lambda \neq \emptyset \right\}.$$

Note that $\exists \Lambda(\gamma) \in \mathrm{BT}'$ such that $U(\gamma) = U_{\Lambda(\gamma)}$.

Computing cohomology: a general strategy

$(\mathbb{G}, X) =$ a Shimura datum which recovers the local RZ datum at p,
$\mathbb{E} \subset \mathbb{C}$ the reflex field,
$v =$ a place of $\mathbb{E}$ above p such that $\mathbb{E}_v = E$,
$K^p \subset G(\mathbb{A}_f^p) =$ small enough open compact subgroup,
$\mathrm{Sh}_{\mathbb{K}} =$ the associated Shimura variety of level $\mathbb{K} = KK^p$, smooth quasi-projective over $\mathrm{Spec}(\mathbb{E})$.

We assume that there exists an integral model S_{K^p} over $\mathrm{Spec}(\mathcal{O}_E)$.
$\overline{\mathrm{S}_{K^p}} =$ the special fiber over $\mathrm{Spec}(\kappa(E))$,
$\overline{\mathrm{S}_{K^p}}(b_0) =$ the basic locus, closed subvariety of the special fiber.

Computing cohomology: a general strategy

p-adic uniformization theorem [Rapoport, Zink]

There is a natural isomorphism

$$I(\mathbb{Q})\backslash\left(\mathcal{M}^{\mathrm{an}} \times \mathbb{G}(\mathbb{A}_f^p)/K^p\right) \xrightarrow{\sim} \widehat{\mathrm{S}}_{K^p}(b_0)^{\mathrm{an}}.$$

I = a certain inner form of $\mathbb{G}$,
$\widehat{\mathrm{S}}_{K^p}(b_0)^{\mathrm{an}}$ = analytical tube over $\overline{\mathrm{S}_{K^p}}(b_0)$ inside the generic fiber of the formal completion of S_{K^p}.

Computing cohomology: a general strategy

Theorem [Fargues]

There is a $\mathbb{G}(\mathbb{A}_f^p) \times W$-equivariant spectral sequence

$$F_2^{a,b} = \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^a_J(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}})(n-1), \Pi_p) \otimes \Pi^{\infty,p} \Longrightarrow \mathrm{H}_c^{a+b}(\overline{\mathrm{S}}(b_0), \mathrm{R}\Psi_\eta \mathcal{L}_\xi).$$

$\overline{\mathrm{S}}(b_0) := \varprojlim_{K^p} \overline{\mathrm{S}}_{K^p}(b_0)$,

$\xi =$ finite dimensional irreducible algebraic representation over $\overline{\mathbb{Q}_\ell}$,

$\mathcal{L}_\xi =$ the associated local system on the Shimura variety,

$\mathrm{R}\Psi_\eta \mathcal{L}_\xi =$ the nearby cycles,

$\mathcal{A}(I) =$ multiset of automorphic representations of $I(\mathbb{A})$ counted with multiplicities,

$\mathcal{A}_\xi(I) := \{\Pi \in \mathcal{A}(I) \mid \Pi \text{ is } \xi\text{-cohomological}\}$.

Our strategy

Step 1: Compute the cohomology of $\mathcal{M}_\Lambda$, the projective closure of Coxeter varieties.

Step 2: Relate the cohomology of U_Λ with that of $\mathcal{M}_\Lambda$ via computation of the nearby cycles.

Step 3: Compute the cohomology of $\mathcal{M}^{\mathrm{an}}$ via the Čech spectral sequence E.

Step 4: Compute the cohomology of the basic locus via the spectral sequence F associated to the p-adic uniformization theorem.

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

We focus on the cases of $\mathrm{GU}(1, n-1)$ with

- p inert K hyperspecial as in [Vollaard, Wedhorn],
- p ramified K stabilizer of a self dual lattice as in [Rapoport, Terstiege, Wilson].

We summarize the main results obtained when trying to apply the strategy described in the previous slides.

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

Step 2: Nearby cycles

	Inert p	Ramified p
Integral model S_{K^p}	smooth	flat, isolated singularities
Nearby cycles $\mathrm{R}\Psi_\eta \overline{\mathbb{Q}_\ell}$	trivial	skyscraper at singular points

Computation for ramified p:

$\mathrm{Bl}(S_{K^p}) =$ blow-up at singular points, semi-stable reduction.

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

By [Pappas] and [Krämer], the local model of $\mathrm{Bl}(\overline{S_{K^p}})$ has 2 irreducible components Z_1, Z_2, and $Q := Z_1 \cap Z_2$, where

$Z_1 \simeq \mathbb{P}^{n-1}$, $Q \hookrightarrow Z_1$ given by $X_1^2 + \ldots + X_n^2 = 0$,

Z_2 is a $\mathbb{P}^1$-bundle over Q.

Proposition [M.]

We have

$$\mathrm{R}^i\Psi_\eta\overline{\mathbb{Q}_\ell} = \begin{cases} \overline{\mathbb{Q}_\ell} & \text{if } i = 0, \\ \bigoplus_{y \text{ singular}} i_{y,*}\overline{\mathbb{Q}_\ell}\left[\left(\frac{-1}{p}\right)^{n/2} p^{n/2}\right] & \text{if } i = n-1 \text{ and } n \text{ even}, \\ 0 & \text{else}, \end{cases}$$

where the scalar between brackets denotes the Frobenius eigenvalue.

Step 1: Cohomology of $\mathcal{M}_\Lambda$

The BT stratification is compatible with $J(\mathbb{Q}_p)$-action:

$$\forall g \in J(\mathbb{Q}_p), g : \mathcal{M}_\Lambda \xrightarrow{\sim} \mathcal{M}_{g\cdot\Lambda}.$$

$J_\Lambda =$ connected stabilizer of $\Lambda \in \mathrm{BT}'$. It is a parahoric subgroup of $J(\mathbb{Q}_p)$,
$J_\Lambda^+ =$ pro-p radical of J_Λ,
$\mathcal{J}_\Lambda := J_\Lambda / J_\Lambda^+ =$ connected reductive group over a finite field ($=$ **a finite group of Lie type**).
$\implies$ induced action $\mathcal{J}_\Lambda \curvearrowright \mathcal{M}_\Lambda$.

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

Any vertex $\Lambda \in \mathrm{BT}'$ has a **type** $0 \leq t(\Lambda) \leq n$.

	Inert p	Ramified p
Type of $\Lambda \in \mathrm{BT}'$	$t(\Lambda) = 2\theta + 1$	$t(\Lambda) = 2\theta$
Variety $\mathcal{M}_\Lambda$	irreducible, projective, smooth, $\dim = \theta$	irreducible, projective, isolated singularities, $\dim = \theta$
Group $\mathcal{J}_\Lambda$ acting on $\mathcal{M}_\Lambda$	$\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$	$\mathrm{GSp}_{2\theta}(\mathbb{F}_p)$

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

By [Lusztig], unipotent representations of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ and $\mathrm{GSp}_{2\theta}(\mathbb{F}_p)$ are classified by triples (d, α, β) where

	$\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$	$\mathrm{GSp}_{2\theta}(\mathbb{F}_p)$
Integer $d \geq 0$	$c := 2\theta + 1 - \frac{d(d+1)}{2} \geq 0$	$c := \theta - d(d+1) \geq 0$
Integer partitions α, β	$\lvert\alpha\rvert + \lvert\beta\rvert = c$	$\lvert\alpha\rvert + \lvert\beta\rvert = c$

Write $\rho_{d,\alpha\beta}$ for associated representation. We have

$$\rho_{d,\alpha,\beta} \text{ cuspidal } \iff (\alpha, \beta) = (\emptyset, \emptyset).$$

Moreover $\rho_{d,\alpha,\beta}$ and $\rho_{d',\alpha',\beta'}$ have same cuspidal support iff $d = d'$.

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

Theorem (M.)

Let $\Lambda \in \mathrm{BT}'$. Eigenvalues of Frobenius written in brackets.

1. **Inert p:** For $0 \leq i \leq \theta$

$$\mathrm{H}^{2i}(\mathcal{M}_\Lambda) \simeq \bigoplus_{s=0}^{\min(i,\theta-i)} \rho_{1,(\theta-s,s),\emptyset}[p^{2i}],$$
$$\mathrm{H}^{2i+1}(\mathcal{M}_\Lambda) \simeq \bigoplus_{s=0}^{\min(i,\theta-1-i)} \rho_{2,(\theta-1-s,s),\emptyset}[-p^{2i+1}].$$

2. **Ramified p:** For $0 \leq i \leq \theta$

$$\mathrm{H}^{2i}(\mathcal{M}_\Lambda) \simeq \bigoplus_{s=0}^{\min(i,\theta-i)} \rho_{0,(\theta-s,s),\emptyset}[p^{i}] \oplus \bigoplus_{s=0}^{\min(i-1,\theta-1-i)} \rho_{1,(\theta-2-s,s),\emptyset}[-p^{i}],$$
$$\mathrm{H}^{2i+1}(\mathcal{M}_\Lambda) = 0.$$

Step 3: On the cohomology of $\mathcal{M}^{\mathrm{an}}$

Terms $E_1^{a,b}$ of Čech spectral sequence can be written as finite sums of representations of the form

$$\mathrm{c}-\mathrm{Ind}_{J_\Lambda}^{J(\mathbb{Q}_p)}\,\mathrm{H}^\bullet(\mathcal{M}_\Lambda, \mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell}).$$

We can analyze admissibility of such representations, and inertial support of their irreducible subquotients via theory of types.

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

Notations: For $V \in \mathrm{Rep}(J(\mathbb{Q}_p))$ and χ character of $\mathrm{Z}(J(\mathbb{Q}_p))$,
$V_\chi =$ largest quotient of V on which $\mathrm{Z}(J(\mathbb{Q}_p))$ acts through χ.
$\theta_{\max} =$ maximal value of θ for $\Lambda \in \mathrm{BT}'$.

Proposition [M.]

For $n \geq 3$ and χ any unramified character of $\mathrm{Z}(J(\mathbb{Q}_p))$, the cohomology group $\mathrm{H}_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})_\chi$ is not $J(\mathbb{Q}_p)$-admissible.

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

Remark: By [Fargues, Scholze], $\forall \rho$ admissible rep'n of $J(\mathbb{Q}_p)$, the $G(\mathbb{Q}_p)$-module

$$\mathrm{Ext}^a_J(\mathrm{H}^\bullet_c(\mathcal{M}_\infty), \rho)$$

is admissible, where $\mathcal{M}_\infty = (\mathcal{M}_K)_{K \subset G(\mathbb{Q}_p)}$ is the RZ tower. Follows from $\mathrm{H}^\bullet_c(\mathcal{M}_K)$ being a fg $J(\mathbb{Q}_p)$-module.

Meanwhile, it is known that $\mathrm{H}^\bullet_c(\mathcal{M}^{\mathrm{an}})_\chi$ can have infinite length as $J(\mathbb{Q}_p)$-module for certain RZ spaces. "To be non admissible" is stronger.

Step 4: The cohomology of the basic locus for low n

Recalls: I = inner form of $\mathbb{G}$ with

$$I_{\mathbb{Q}_p} = J, \qquad I_{\mathbb{A}_f^p} = \mathbb{G}_{\mathbb{A}_f^p}, \qquad I_{\mathbb{R}} = \mathrm{GU}(0, n),$$

ξ: finite dimensional irreducible algebraic rep'n of $\mathbb{G}$,
$w(\xi) \geq 0$ the weight of ξ,
$\mathcal{A}_\xi(I)$ = multiset of automorphic rep'n of $I(\mathbb{A})$ cohomological for ξ.
$X^{\mathrm{un}}(J)$ = unramified characters of $J(\mathbb{Q}_p)$.
For $\Pi \in \mathcal{A}_\xi(I)$, can attach a scalar $\delta_\Pi \in \overline{\mathbb{Q}_\ell}$ such that $|\iota(\delta_\Pi)| = 1$ for all $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$.

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

We assume $\dim \overline{\mathrm{S}}(b_0) = 1$, ie.

- for p inert, $n = 3$ or 4,
- for p ramified, $n = 2$ "split", $n = 3$ or $n = 4$ "non-split".

$\mathrm{H}^0(\overline{\mathrm{S}}(b_0), \mathcal{L}_\xi)$ and $\mathrm{H}^2(\overline{\mathrm{S}}(b_0), \mathcal{L}_\xi)$ have same description in all cases. Define $\pi_\ell \in \overline{\mathbb{Q}_\ell}$ by

$$\pi_\ell = \begin{cases} p & \text{if } p \text{ inert,} \\ \sqrt{p} & \text{if } p \text{ ramified.} \end{cases}$$

$J_1 \subset J(\mathbb{Q}_p)$ stabilizer of self-dual lattice (hyperspecial when p inert).

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

Theorem (M.)

There are $G(\mathbb{A}_f^p) \times W$-equivariant isomorphisms

$$\mathrm{H}^0(\overline{\mathrm{S}}(b_0), \mathcal{L}_\xi) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^{\infty,p} \otimes \overline{\mathbb{Q}_\ell}[\delta_\Pi \pi_\ell^{w(\xi)}],$$

$$\mathrm{H}^2(\overline{\mathrm{S}}(b_0), \mathcal{L}_\xi) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^{\infty,p} \otimes \overline{\mathbb{Q}_\ell}[\delta_\Pi \pi_\ell^{w(\xi)+2}],$$

where the Frobenius eigenvalues are written between brackets.

Next we describe the H^1.

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

For inert p: $\sigma =$ depth 0 supercuspidal rep'n of $J(\mathbb{Q}_p)$ coming from the unipotent cuspidal rep'n $\rho_{2,\emptyset,\emptyset}$ of $\mathrm{GU}_3(\mathbb{F}_p)$.

Theorem (M.)

$$\mathrm{H}^1(\overline{S}(b_0), \mathcal{L}_\xi) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in {}^{\mathrm{un}}(J) \\ \Pi_p = \chi \mathrm{St}_J}} \Pi^{\infty,p} \otimes \overline{\mathbb{Q}_\ell}[\delta_\Pi \pi_\ell^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J) \\ \Pi_p = \chi \sigma}} \Pi^{\infty,p} \otimes \overline{\mathbb{Q}_\ell}[-\delta_\Pi \pi_\ell^{w(\xi)+1}],$$

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

For ramified p:

Theorem (M.)

$$\mathrm{H}^1(\overline{S}(b_0), \mathcal{L}_\xi) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J) \\ \Pi_p = \chi \mathrm{St}_J}} \Pi^{\infty,p} \otimes \overline{\mathbb{Q}_\ell}[\delta_\Pi \pi_\ell^{w(\xi)}].$$

Application to $\mathrm{GU}(1, n-1)$ over p inert or ramified

$J_0 \subset J(\mathbb{Q}_p)$ other parahoric not conjugate to J_1.

Theorem (M.)

Contribution of nearby cycles for n even:

$$\mathrm{H}^1(\overline{S}(b_0), \mathrm{R}\Psi_\eta \mathcal{L}_\xi) \simeq \mathrm{H}^1(\overline{S}(b_0), \mathcal{L}_\xi) \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_0} \neq 0}} \Pi^{\infty,p} \otimes \overline{\mathbb{Q}_\ell}\left[\left(\frac{-1}{p}\right) \delta_\Pi \pi_\ell^{w(\xi)+1}\right] \text{ if } n = 2,$$

$$\mathrm{H}^3(\overline{S}(b_0), \mathrm{R}\Psi_\eta \mathcal{L}_\xi) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_0} \neq 0}} \Pi^{\infty,p} \otimes \overline{\mathbb{Q}_\ell}[\delta_\Pi \pi_\ell^{w(\xi)+2}] \text{ if } n = 4.$$

Applications and further directions

Applications and further directions

In principle, can get similar results for any Coxeter type Shimura varieties. Only few cases non-maximal parahoric level. This could give non trivial action of inertia $I \subset W$.

Eg. $\mathrm{GU}(1, 2m-1)$ inert p with $K = K_0 \cap K_m$ where K_0, K_m are hyperspecial, stabilizers of lattices Λ_0 and Λ_m such that

$$\Lambda_0^\vee = \Lambda_0, \qquad \Lambda_m^\vee = p\Lambda_m.$$

In this case, semi-stable reduction and BT strata are products $\mathcal{M}_\Lambda \times \mathcal{M}_{\Lambda'}$ of BT strata for hyperspecial level, so Steps 1 and 2 are OK.

Applications and further directions

For inert p and $n = 3$, [De Shalit, Goren] described the geometry of $\mathrm{S}_0(p)_{K^p}$, the special fiber at Iwahori level of the Shimura variety. They describe the fibers of

$$\pi : \mathrm{S}_0(p)_{K^p} \to \overline{\mathrm{S}}_{K^p}$$

over the Ekedahl-Oort strata. Using this, [Fu] proved a Mazur principle for $\mathrm{GU}(1,2)$ at inert p.

With our work, we can derive the cohomology of the basic locus of $\mathrm{S}_0(p)$, and this could simplify some arguments in [Fu].

Thank you for your attention!