Cohomology of the Bruhat-Tits strata in the supersingular locus of the $\mathrm{GU}(1, n-1)$ Shimura variety at a ramified prime

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Abstract : *The supersingular locus of the* $\mathrm{GU}(1, n-1)$ *Shimura variety at a ramified prime* p *is stratified by Coxeter varieties attached to finite symplectic groups. In this paper, we compute the* ℓ*-adic cohomology of the Zariski closure of any such stratum. These are known as closed Bruhat-Tits strata. We prove that the cohomology groups of odd degree vanish, and those of even degree are explicitely determined as representations of the symplectic group with a Frobenius action. Each closed Bruhat-Tits stratum is linearly stratified by Coxeter varieties attached to smaller symplectic groups. Thanks to results of Lusztig who computed the cohomology of Coxeter varieties for classical groups, we make use of the spectral sequence associated to this stratification and describe explicitely all the terms at infinity. We point out that the closed Bruhat-Tits strata have isolated singularities when the dimension is greater than 1. Our analysis requires discussing the smoothness of the blow-up at the singular points, as well as comparing the ordinary* ℓ*-adic cohomology with intersection cohomology. A by-product of our computations is that these two cohomologies actually coincide, so that surprisingly the presence of singularities does not interfere with the cohomology.*

Contents

Introduction: Shimura varieties are objects of central interest in number theory and arithmetic geometry. When an integral model is given, the geometry of the special fiber is particularly interesting. In particular, the supersingular locus of the special fiber has been extensively studied in recent years. In many situations which are precisely listed in [7] and [8], the supersingular locus admits a so-called Bruhat-Tits stratification, whose strata are isomorphic to classical Deligne-Lusztig varieties for certain finite groups of Lie type. Cohomology plays an important role both in the world of Shimura varieties and in Deligne-Lusztig theory. Thus, exploiting the geometry of the Bruhat-Tits stratification in order to connect both cohomology theories sounds like a promising idea. In [14] and [13], we investigated the case of the $\mathrm{GU}(1, n-1)$ PEL Shimura variety over an inert prime $p > 2$, whose supersingular locus was described in [17] and [18]. More precisely, we explicitely determined the cohomology of the closed Bruhat-Tits strata as representations of finite unitary groups. We used this result to prove the non-admissibility of the cohomology of the associated Rapoport-Zink space, and to determine the cohomology of the supersingular locus for low n in terms of automorphic representations. In this paper, we focus on the case of a ramified prime $p > 2$ as studied in [15]. Our goal is to replicate the same approach as in the inert case, and find out how to deal with the new technical difficulties caused by the non-smoothness of the integral model. The exposition is divided into two papers, and the present paper is the first of the series. It is devoted to the computation of the cohomology of a given closed Bruhat-Tits stratum using Deligne-Lusztig theory. Let us explain the results in more details.
Let q be a power of an odd prime number p. Let V be a symplectic space over $\mathbb{F}_q$ of dimension 2θ. For any field extension $k/\mathbb{F}_q$, let $\tau = \mathrm{id}\otimes\sigma$ denote the semi-linear automorphism of $V_k := V \otimes k$, where $\sigma : x \mapsto x^q$. Let $L(V)$ denote the Lagrangian Grassmanian variety of V. We consider the closed subvariety $S_\theta \subset L(V)$ whose k-points are given by

$$S_\theta(k) = \{U \subset V_k \,|\, U = U^\perp \text{ and } \dim(U \cap \tau(U)) \geqslant \theta - 1\}.$$

It turns out that the closed Bruhat-Tits strata mentioned in the previous paragraph are actually isomorphic to S_θ for some $\theta \geqslant 0$. The variety S_θ is projective, irreducible, normal and of dimension θ. It has isolated singularities when $\theta \geqslant 2$.

When $\theta = 1$, we have $S_1 \simeq \mathbb{P}^1$. Moreover it is equipped with a natural action of the finite symplectic group $\mathrm{Sp}(V)$. Up to fixing a basis of V, we may identify $\mathrm{Sp}(V)$ with the usual group $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ of symplectic matrices. Unipotent representations of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ are naturally classified by Lusztig's notion of symbols, whose definition is recalled in Section 2. Given a symbol S, the associated unipotent representation is denoted ρ_S. The main theorem is the following.

Theorem. (1) *All the cohomology groups of S_θ of odd degree vanish.*

(2) *For $0 \leqslant i \leqslant \theta$, we have an $\mathrm{Sp}(2\theta, \mathbb{F}_q)$-equivariant isomorphism*

$$\mathrm{H}^{2i}(S_\theta, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{s=0}^{\min(i,\theta-i)} \rho_{\begin{pmatrix} s & \theta+1-s \\ 0 & \end{pmatrix}} \oplus \bigoplus_{s=0}^{\min(i-1,\theta-1-i)} \rho_{\begin{pmatrix} 0 & s+1 & \theta-s \\ & & \end{pmatrix}}.$$

The Frobenius acts like multiplication by q^i on the first summand, and multiplication by $-q^i$ on the second summand.

Let us explain the main steps of the proof. The variety S_θ admits a stratification

$$S_\theta = \bigsqcup_{\theta'=0}^{\theta} X_{I_{\theta'}}(w_{\theta'}),$$

where the $X_{I_{\theta'}}(w_{\theta'})$ are certain Deligne-Lusztig varieties which are "parabolically induced" from the Coxeter variety $X^{\theta'}$ of the finite group $\mathrm{Sp}(2\theta', \mathbb{F}_q)$, see Section 1 for the precise definitions. There is an induced spectral sequence

$$E_1^{\theta',i} = \mathrm{H}_c^{\theta'+i}(X_{I_{\theta'}}(w_{\theta'})) \implies \mathrm{H}^{\theta'+i}(S_\theta).$$

See Figure 1 for a drawing of E_1. The term $E_1^{\theta',i}$ is the parabolic induction of the degree i cohomology group of the Coxeter variety $X^{\theta'}$. The cohomology of such Coxeter varieties has been computed in [11], and the parabolic inductions can be computed explicitely via the comparison theorem of [9]. In particular, we can determine $E_1^{\theta',i}$ explicitely, see Lemma 27. The Frobenius acts semi-simply on $E_1^{\theta',i}$ with at most 2 eigenvalues. These eigenvalues are equal to q^i and $-q^{i+1}$, the latter only occuring if $0 \leqslant i \leqslant \theta' - 2$. Since terms on different rows do not carry any common eigenvalue, the spectral sequence degenerates in E_2 and the resulting filtration on the abutment splits. Thus, we are reduced to computing the terms $E_2^{\theta',i}$ explicitely.

The most effective way to determine most of the terms $E_2^{\theta',i}$ is to find restrictions on the eigenvalues of the Frobenius on the abutment of the spectral sequence. To do so, we seek a good resolution of the singularities of S_θ when $\theta > 1$. Such a resolution is afforded by the blow-up at singular points, as we prove in Section 5. In fact, we exhibit a certain affine open neighborhood of any given singular point, and observe that it is a finite étale cover of the symmetric determinantal variety of rank $\leqslant 1$. Incidentally, desingularizations of such determinantal varieties have been studied in [4] by means of successive blow-ups. In particular, in our case a single blow-up is required to resolve the singularities. As a consequence, we prove that the Frobenius action on $\mathrm{H}^k(S_\theta, \overline{\mathbb{Q}_\ell})$ is pure of weight $2\lfloor \frac{k}{2} \rfloor$. In particular, all terms $E_2^{\theta',i}$ which do not carry any eigenvalue of compatible weight must vanish. In order to determine the remaining terms $E_2^{\theta',i}$, we introduce the intersection cohomology of S_θ. Since S_θ has only isolated singularities (when $\theta > 1$), it is well-known that intersection cohomology and ℓ-adic cohomology agree above the middle degree. In particular, the cohomology groups $\mathrm{H}^k(S_\theta)$ for $k > \theta$ with k odd vanish, since all weights of the Frobenius are both odd (by intersection cohomology) and even (by the spectral sequence) at the same time. The hypercohomology spectral sequence associated to the intersection complex, as represented in Figure 2, allows us to remove the restriction $k > \theta$, thus proving the first part of the main theorem. The second part follows easily given the shape of the spectral sequence represented in Figure 1. As a by-product, we find out that the intersection complex of S_θ has vanishing cohomology in higher degrees. In particular, the intersection cohomology and the ℓ-adic cohomology of S_θ actually agree in all degrees.

Notations: In this paper, p will always denote an odd prime number and q will be a power of p. If M is a matrix with coefficients in a field of characteristic p, then $M^{(q)}$ denotes the matrix M with entries raised to the power q. The trivial representation of a given group will be denoted $\mathbf{1}$. Given a reductive group G with Levi complement L, the associated Harish-Chandra induction and restriction functors are denoted R_L^G and ${}^*\mathrm{R}_L^G$ respectively.

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1 The closed Deligne-Lusztig variety S_θ for $\mathrm{Sp}(2\theta, \mathbb{F}_q)$

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$, together with a split $\mathbb{F}_q$-structure given by a geometric Frobenius morphism F. For $\mathbf{H}$ any F-stable subgroup of $\mathbf{G}$, we write $H := \mathbf{H}^F$ for its group of $\mathbb{F}_q$-rational points. Let $(\mathbf{T}, \mathbf{B})$ be a pair consisting of a maximal F-stable torus $\mathbf{T}$ contained in an F-stable Borel subgroup $\mathbf{B}$. Let $(\mathbf{W}, \mathbf{S})$ be the associated Coxeter system, where $\mathbf{W} = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Since the $\mathbb{F}_q$-structure on $\mathbf{G}$ is split, the Frobenius F acts trivially on $\mathbf{W}$. For $I \subset$

$\mathbf{S}$, let $\mathbf{P}_I, \mathbf{U}_I, \mathbf{L}_I$ be respectively the standard parabolic subgroup of type I, its unipotent radical and its unique Levi complement containing $\mathbf{T}$. Let $\mathbf{W}_I$ be the subgroup of $\mathbf{W}$ generated by I. For $\mathbf{P}$ any parabolic subgroup of $\mathbf{G}$, the associated **generalized parabolic Deligne-Lusztig variety** is

$$X_{\mathbf{P}} := \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}F(\mathbf{P})\}.$$

We say that the variety is **classical** (as opposed to generalized) when in addition the parabolic subgroup $\mathbf{P}$ contains an F-stable Levi complement. Note that $\mathbf{P}$ itself need not be F-stable. We may give an equivalent definition using the Coxeter system $(\mathbf{W}, \mathbf{S})$. For $I \subset \mathbf{S}$, let ${}^I\mathbf{W}^I$ be the set of elements $w \in \mathbf{W}$ which are I-reduced-I. For $w \in {}^I\mathbf{W}^I$, the associated generalized parabolic Deligne-Lusztig variety is

$$X_I(w) := \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \mid g^{-1}F(g) \in \mathbf{P}_I w \mathbf{P}_I\}.$$

The variety $X_I(w)$ is classical when $w^{-1}Iw = I$, and it is defined over $\mathbb{F}_q$. The dimension is given by $\dim X_I(w) = l(w) + \dim \mathbf{G}/\mathbf{P}_{I\cap wIw^{-1}} - \dim \mathbf{G}/\mathbf{P}_I$ where $l(w)$ denotes the length of w with respect to $\mathbf{S}$.
Let $\theta \geqslant 0$ and let V be a 2θ-dimensional $\mathbb{F}_q$-vector space equipped with a non-degenerate symplectic form $(\cdot,\cdot) : V \times V \to \mathbb{F}_q$. Fix a basis $(e_1, \ldots, e_{2\theta})$ in which $(\cdot,\cdot)$ is described by the matrix

$$\Omega := \begin{pmatrix} 0 & A_\theta \\ -A_\theta & 0 \end{pmatrix},$$

where A_θ denotes the matrix having 1 on the anti-diagonal and 0 everywhere else. If k is a field extension of $\mathbb{F}_q$, let $V_k := V \otimes_{\mathbb{F}_q} k$ denote the scalar extension to k equipped with its induced k-symplectic form $(\cdot,\cdot)$. Let $\tau : V_k \xrightarrow{\sim} V_k$ denote the map $\mathrm{id} \otimes \sigma$, where $\sigma(x) := x^q$. If $U \subset V_k$, let $U^\perp$ denote its orthogonal. We consider the finite symplectic group $\mathrm{Sp}(V, (\cdot,\cdot)) \simeq \mathrm{Sp}(2\theta, \mathbb{F}_q)$, where the RHS is defined with respect to Ω. It can be identified with $G = \mathbf{G}^F$ where $\mathbf{G}$ is the symplectic group $\mathrm{Sp}(V_{\mathbb{F}}, (\cdot,\cdot)) \simeq \mathrm{Sp}(2\theta, \mathbb{F})$ and $F(M) := M^{(q)}$. Let $\mathbf{T} \subset \mathbf{G}$ be the maximal torus of diagonal symplectic matrices and let $\mathbf{B} \subset \mathbf{G}$ be the Borel subgroup of upper-triangular symplectic matrices. The Weyl system of $(\mathbf{T}, \mathbf{B})$ is identified with $(W_\theta, \mathbf{S})$ where W_θ is the finite Coxeter group of type B_θ and $\mathbf{S} = \{s_1, \ldots, s_\theta\}$ is the set of simple reflexions. They satisfy the following relations

$$(s_\theta s_{\theta-1})^4 = 1, \qquad (s_i s_{i-1})^3 = 1, \qquad \forall\, 2 \leqslant i \leqslant \theta - 1,$$
$$(s_i s_j)^2 = 1, \qquad \forall\, |i-j| \geqslant 2.$$

Concretely, the simple reflexion s_i acts on V by exchanging e_i and e_{i+1} as well as $e_{2\theta-i}$ and $e_{2\theta-i+1}$ for $1 \leqslant i \leqslant \theta - 1$, whereas s_θ exchanges e_θ and $e_{\theta+1}$. We define

$$I := \{s_1, \ldots, s_{\theta-1}\} = \mathbf{S}\backslash\{s_\theta\}.$$

We consider the generalized Deligne-Lusztig variety $X_I(s_\theta)$. Since $s_\theta s_{\theta-1}s_\theta \notin I$, it is not a classical Deligne-Lusztig variety. Let $S_\theta := \overline{X_I(s_\theta)}$ be its closure in $\mathbf{G}/\mathbf{P}_I$. This variety has been introduced in [15], as it is isomorphic to the closed Bruhat-Tits strata of type θ in the supersingular locus of the $\mathrm{GU}(1,n-1)$ Shimura variety over a ramified prime. We recall some geometric facts on S_θ which are proved in [15] Section 5.

Proposition 1. *The variety S_θ is normal and projective. The variety S_1 is isomorphic to the projective line $\mathbb{P}^1$, and for $\theta \geqslant 2$ the variety S_θ has isolated singularities. If k is a field extension of $\mathbb{F}_q$, we have*

$$S_\theta(k) \simeq \{U \subset V_k \,|\, U^\perp = U \text{ and } U \cap \tau(U) \overset{\leqslant 1}{\subset} U\},$$

where $\overset{\leqslant 1}{\subset}$ denotes an inclusion of subspaces with index at most 1. There is a decomposition

$$S_\theta = X_I(\mathrm{id}) \sqcup X_I(s_\theta),$$

where $X_I(\mathrm{id})$ is closed and of dimension 0, and $X_I(s_\theta)$ is open, dense of dimension θ. They correspond respectively to k-points U having $U = \tau(U)$ or $U \cap \tau(U) \subsetneq U$. If $\theta \geqslant 2$ then S_θ is singular at the points of $X_I(\mathrm{id})$.

For $0 \leqslant \theta' \leqslant \theta$, define

$$I_{\theta'} := \{s_1, \ldots, s_{\theta-\theta'-1}\}$$

and $w_{\theta'} := s_{\theta+1-\theta'} \ldots s_\theta$. In particular $I_0 = I$, $I_{\theta-1} = I_\theta = \varnothing$, $w_0 = \mathrm{id}$ and $w_1 = s_\theta$.

Proposition 2. *There is a stratification into locally closed subvarieties*

$$S_\theta = \bigsqcup_{\theta'=0}^{\theta} X_{I_{\theta'}}(w_{\theta'}).$$

The stratum $X_{I_{\theta'}}(w_{\theta'})$ corresponds to points U such that $\dim(U \cap \tau(U) \cap \ldots \cap \tau^{\theta'+1}(U)) = \theta - \theta'$. The closure in S_θ of a stratum $X_{I_{\theta'}}(w_{\theta'})$ is the union of all the strata $X_{I_t}(w_t)$ for $t \leqslant \theta'$. The stratum $X_{I_{\theta'}}(w_{\theta'})$ is of dimension θ', and $X_{I_\theta}(w_\theta)$ is open, dense and irreducible.

It follows in particular that S_θ is irreducible as well. It turns out that the strata $X_{I_{\theta'}}(w_{\theta'})$ are related to Coxeter varieties for symplectic groups of smaller sizes. For $0 \leqslant \theta' \leqslant \theta$, define

$$K_{\theta'} := \{s_1, \ldots s_{\theta-\theta'-1}, s_{\theta-\theta'+1}, \ldots, s_\theta\} = \mathbf{S}\backslash\{s_{\theta-\theta'}\}.$$

Note that $K_0 = I_0 = I$ and $K_\theta = \mathbf{S}$. We have $I_{\theta'} \subset K_{\theta'}$ with equality if and only if $\theta' = 0$.

Proposition 3. *There is an* $\mathrm{Sp}(2\theta, \mathbb{F}_p)$*-equivariant isomorphism*

$$X_{I_{\theta'}}(w_{\theta'}) \simeq \mathrm{Sp}(2\theta, \mathbb{F}_q)/U_{K_{\theta'}} \times_{L_{K_{\theta'}}} X^{\mathbf{L}_{K_{\theta'}}}_{I_{\theta'}}(w_{\theta'}),$$

where $X^{\mathbf{L}_{K_{\theta'}}}_{I_{\theta'}}(w_{\theta'})$ *is a Deligne-Lusztig variety for* $\mathbf{L}_{K_{\theta'}}$*. The zero-dimensional variety* $\mathrm{Sp}(2\theta, \mathbb{F}_q)/U_{K_{\theta'}}$ *has a left action of* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ *and a right action of* $L_{K_{\theta'}}$*.*

Proof. It is a special case of the geometric identity used to prove transitivity of the Deligne-Lusztig induction functor. We refer to [2] Proposition 7.19 or [3] Proposition 9.1.8. □

The Levi complement $\mathbf{L}_{K_{\theta'}}$ is isomorphic to $\mathrm{GL}(\theta - \theta') \times \mathrm{Sp}(2\theta')$, and its Weyl group is isomorphic to $\mathfrak{S}_{\theta-\theta'} \times W_{\theta'}$. Via this decomposition, the permutation $w_{\theta'}$ corresponds to $\mathrm{id} \times w_{\theta'}$. The Deligne-Lusztig variety $X^{\mathbf{L}_{K_{\theta'}}}_{I_{\theta'}}(w_{\theta'})$ decomposes as a product

$$X^{\mathbf{L}_{K_{\theta'}}}_{I_{\theta'}}(w_{\theta'}) = X^{\mathrm{GL}(\theta-\theta')}_{\mathbf{I}_{\theta'}}(\mathrm{id}) \times X^{\mathrm{Sp}(2\theta')}_{\varnothing}(w_{\theta'}).$$

The variety $X^{\mathrm{GL}(\theta-\theta')}_{\mathbf{I}_{\theta'}}(\mathrm{id})$ is just a single point, but $X^{\mathrm{Sp}(2\theta')}_{\varnothing}(w_{\theta'})$ is the Coxeter variety for the symplectic group of size $2\theta'$. Indeed, $w_{\theta'}$ is a Coxeter element, ie. the product of all the simple reflexions of the Weyl group of $\mathrm{Sp}(2\theta')$.

2 Unipotent representations of the finite symplectic group

Recall that a (complex) irreducible representation of a finite group of Lie type $G = \mathbf{G}^F$ is said to be **unipotent**, if it occurs in the Deligne-Lusztig induction of the trivial representation of some maximal rational torus. Equivalently, it is unipotent if it occurs in the cohomology (with coefficient in $\overline{\mathbb{Q}_\ell}$ and $\ell \neq p$) of some Deligne-Lusztig variety of the form $X_{\mathbf{B}}$, with $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ containing a maximal rational torus. In this section, we recall the classification of the unipotent representations of the finite symplectic groups. The underlying combinatorics is described by Lusztig's notion of symbols. Our main reference is [5] Section 4.4.

Definition 4. Let $\theta \geqslant 1$ and let d be an odd positive integer. The set of **symbols of rank** θ **and defect** d is

$$\mathcal{Y}^1_{d,\theta} := \left\{ S = (X,Y) \;\middle|\; \begin{matrix} X = (x_1, \ldots, x_{r+d}) \\ Y = (y_1, \ldots, y_r) \end{matrix}, x_i, y_j \in \mathbb{Z}_{\geqslant 0}, \begin{matrix} x_{i+1} - x_i \geqslant 1, \\ y_{j+1} - y_j \geqslant 1, \end{matrix} \mathrm{rk}(S) = \theta \right\} \Big/ (\mathrm{shift}),$$

where the shift operation is defined by $\mathrm{shift}(X,Y) := (\{0\} \sqcup (X+1), \{0\} \sqcup (Y+1))$, and where the rank of S is given by

$$\mathrm{rk}(S) := \sum_{s \in S} s - \left\lfloor \frac{(\#S - 1)^2}{4} \right\rfloor.$$

Note that the formula defining the rank is invariant under the shift operation, therefore it is well defined. By [12], we have $\mathrm{rk}(S) \geqslant \left\lfloor \frac{d^2}{4} \right\rfloor$ so in particular $\mathcal{Y}^1_{d,\theta}$ is empty for d big enough. We write $\mathcal{Y}^1_\theta$ for the union of the $\mathcal{Y}^1_{d,\theta}$ with d odd, this is a finite set.

Example 5. In general, a symbol $S = (X, Y)$ will be written

$$S = \begin{pmatrix} x_1 & \dots & x_r & \dots & x_{r+d} \\ y_1 & \dots & y_r & & \end{pmatrix}.$$

We refer to X and Y as the first and second rows of S. The 6 elements of $\mathcal{Y}^1_2$ are given by

$$\begin{pmatrix} 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ & & \end{pmatrix}.$$

The last symbol has defect 3 whereas all the other symbols have defect 1.

The symbols can be used to classify the unipotent representations of the finite symplectic group, cf [12] Theorem 8.2.

Theorem 6. *There is a natural bijection between $\mathcal{Y}^1_\theta$ and the set of equivalence classes of unipotent representations of* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

If $S \in \mathcal{Y}^1_\theta$ we write ρ_S for the associated unipotent representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$. The classification is done so that the symbols

$$\begin{pmatrix} \theta \end{pmatrix}, \qquad \begin{pmatrix} 0 & \dots & \theta-1 & \theta \\ 1 & \dots & \theta & \end{pmatrix},$$

correspond respectively to the trivial and to the Steinberg representations. Let $S = (X, Y)$ be a symbol and let $k \geqslant 1$. A k**-hook** h in S is an integer $z \geqslant k$ such that $z \in X, z - k \notin X$ or $z \in Y, z - k \notin Y$. A k**-cohook** c in S is an integer $z \geqslant k$ such that $z \in X, z - k \notin Y$ or $z \in Y, z - k \notin X$. The integer k is referred to as the **length** of the hook h or the cohook c, and it is denoted $\ell(h)$ or $\ell(c)$. The **hook formula** gives an expression of $\dim(\rho_S)$ in terms of hooks and cohooks.

Proposition 7. *We have*

$$\dim(\rho_S) = q^{a(S)} \frac{\prod_{i=1}^{\theta} (q^{2i} - 1)}{2^{b'(S)} \prod_h (q^{\ell(h)} - 1) \prod_c (q^{\ell(c)} + 1)},$$

where the products in the denominator run over all the hooks h and all the cohooks c in S, and the numbers $a(S)$ and $b'(S)$ are given by

$$a(S) = \sum_{\{s,t\} \subset S} \min(s, t) - \sum_{i \geqslant 1} \binom{\#S - 2i}{2}, \quad b'(S) = \left\lfloor \frac{\#S - 1}{2} \right\rfloor - \#(X \cap Y).$$

For $\delta \geqslant 0$, we define the symbol

$$S_\delta := \begin{pmatrix} 0 & \dots & 2\delta \\ & & \end{pmatrix} \in \mathcal{Y}^1_{2\delta+1,\delta(\delta+1)}.$$

Definition 8. The **core** of a symbol $S \in \mathcal{Y}^1_{d,\theta}$ is defined by $\mathrm{core}(S) := S_\delta$ where $d = 2\delta + 1$. We say that S is **cuspidal** if $S = \mathrm{core}(S)$.

Remark 9. In general, we have $\mathrm{rk}(\mathrm{core}(S)) \leqslant \mathrm{rk}(S)$ with equality if and only if S is cuspidal.

The next theorem states that cuspidal unipotent representations correspond to cuspidal symbols.

Theorem 10. *The group* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ *admits a cuspidal unipotent representation if and only if* $\theta = \delta(\delta+1)$ *for some* $\delta \geqslant 0$. *When this is the case, the cuspidal unipotent representation is unique and given by* ρ_{S_δ}.

The determination of the cuspidal unipotent representations leads to a description of the unipotent Harish-Chandra series.

Definition 11. Let $\delta \geqslant 0$ such that $\theta = \delta(\delta+1) + a$ for some $a \geqslant 0$. We write

$$L_\delta \simeq \mathrm{GL}(1, \mathbb{F}_q)^a \times \mathrm{Sp}(2\delta(\delta+1), \mathbb{F}_q)$$

for the block-diagonal Levi complement in $\mathrm{Sp}(2\theta, \mathbb{F}_q)$, with one middle block of size $2\delta(\delta+1)$ and other blocks of size 1. We write $\rho_\delta := (\mathbf{1})^a \boxtimes \rho_{S_\delta}$, which is a cuspidal representation of L_δ.

Proposition 12. *Let* $S \in \mathcal{Y}^1_{d,\theta}$. *The cuspidal support of* ρ_S *is* (L_δ, ρ_δ) *where* $d = 2\delta + 1$.

In particular, the defect of the symbol S of rank θ classifies the unipotent Harish-Chandra series of $\mathrm{Sp}(2\theta, \mathbb{F}_p)$. If $\delta \geqslant 0$ is such that $\delta(\delta+1) \leqslant \theta$, we write $\mathcal{E}_\delta$ for the Harish-Chandra series determined by (L_δ, ρ_δ). The previous proposition says that $\mathcal{E}_\delta$ is in bijection with $\mathcal{Y}^1_{2\delta+1,\theta}$. Representations in a given series $\mathcal{E}_\delta$ can also be labelled in an alternative way.

Definition 13. A **partition** of an integer $n \geqslant 0$ is a sequence of positive integers $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ with $r \geqslant 0$ such that $n = \lambda_1 + \ldots + \lambda_r$. The integer $|\lambda| := n$ is called the length of λ. A **bipartition** of n is a pair (λ, μ) of partitions such that $|\lambda| + |\mu| = n$.

Let $S \in \mathcal{Y}^1_{2\delta+1,\theta}$ where $\delta(\delta+1) \leqslant \theta$. Up to taking suitable shifts, we may consider representatives of S and of S_δ whose rows have the same length. We obtain a bipartition (α, β) of $\theta - \delta(\delta+1)$ by subtracting component-wise the rows of S_δ from the rows of S, and reordering them decreasingly while ignoring the potential 0 components.

Example 14. Recall the 6 symbols of $\mathcal{Y}_2^1$ described in Exemple 5. The associated bipartitions are respectively

$$((2), \varnothing), \qquad (\varnothing, (2)), \qquad ((1), (1)), \qquad ((1^2), \varnothing), \qquad (\varnothing, (1^2)) \qquad (\varnothing, \varnothing).$$

The five symbols of defect 1 correspond to bipartitions of 2, and the last symbol of defect 3 corresponds to the empty bipartition of 0.

Proposition 15. *The process described above defines a natural bijection between* $\mathcal{Y}_{2\delta+1,\theta}^1$ *and the set of bipartitions of* $\theta - \delta(\delta+1)$.

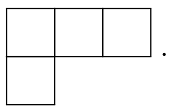
It is well-known that the set $\mathrm{Irr}(W_n)$ of equivalence classes of irreducible representations of the Coxeter group W_n are in bijection with the set of bipartitions of n, cf. [6] Section 5.5. Through this bijection, the bipartitions $((n), \varnothing)$ and $(\varnothing, (1^n))$ correspond respectively to the trivial and to the signature characters of W_n.

Corollary 16. *Let* $\theta, \delta \geqslant 0$ *such that* $\delta(\delta+1) \leqslant \theta$. *There is a natural bijection between* $\mathcal{E}_\delta$ *and* $\mathrm{Irr}(W_{\theta-\delta(\delta+1)})$.

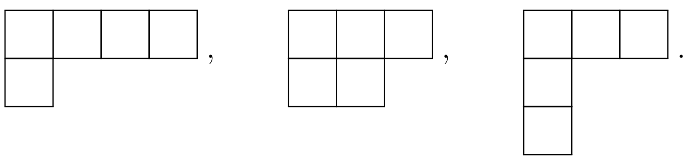
One may check that the trivial (resp. the sign) character of W_θ corresponds to the trivial (resp. the Steinberg) representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ in $\mathcal{E}_0$. Given a symbol $S \in \mathcal{Y}_{2\delta+1,\theta}^1$ and the corresponding bipartition (α, β) of $\theta - \delta(\delta+1)$, we will sometimes also write $\rho_{\delta,\alpha,\beta}$ instead of ρ_S. It turns out that the labelling of the unipotent representations of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ in terms of bipartitions is particularly well suited in order to compute Harish-Chandra inductions and restrictions. To this end, it is also convenient to identify partitions with their Young diagrams.

Definition 17. A **Young diagram** T is a finite collection of boxes which are organized in rows of non-increasing lengths, justified on the left side. The size $|T|$ of a Young diagram is the number of boxes it contains. If T and T' are two Young diagrams such that $|T| = |T'| + 1$, we say that T is obtained from T' by adding one box, or that T' is obtained from T by removing one box, if we can overlay T' on the top of T so that the difference consists of just one box.

Example 18. Let us consider the following Young diagram of size 4



The Young diagrams which can be obtained by adding a box are given by



Clearly the set of partitions of n is in bijection with the set of Young diagrams of size n. Likewise, bipartitions of n correspond to pairs of Young diagrams of combined sizes n. In the following, we will have to compute inductions of the following form

$$\mathrm{R}_{\mathrm{GL}(a,\mathbb{F}_q)\times\mathrm{Sp}(2\theta',\mathbb{F}_q)}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)}\,\mathbf{1}\boxtimes\rho_{S'},$$

where $\theta = a + \theta'$ and $S' \in \mathcal{Y}^1_{d,\theta'}$ is a symbol. In particular we assume that $\theta' \geqslant \delta(\delta+1)$ where $d = 2\delta+1$.

Theorem 19. *Let $S' \in \mathcal{Y}^1_{d,\theta'}$ and (α',β') the associated bipartition of $\theta' - \delta(\delta+1)$. We have*

$$\mathrm{R}_{\mathrm{GL}(a,\mathbb{F}_q)\times\mathrm{Sp}(2\theta',\mathbb{F}_q)}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)}\,\mathbf{1}\boxtimes\rho_{S'} = \sum_{\alpha,\beta}\rho_{\delta,\alpha,\beta},$$

where (α,β) runs over all the bipartitions of $\theta-\delta(\delta+1)$ such that for some $0 \leqslant d \leqslant a$, the Young diagram of α (resp. of β) can be obtained from the Young diagram of α' (resp. of β') by adding a succession of d boxes (resp. $a-d$ boxes), no two of them lying in the same column.

This computation is a consequence of the Howlett-Lehrer comparison theorem [9], stating that induction in a finite group of Lie type can be computed inside a corresponding Weyl group. We then apply Pieri's rule for Coxeter groups of type B_n, see [6] 6.1.9. We will see concrete exemples of such computations in the following sections. There is a similar rule in order to compute certain Harish-Chandra restrictions. We write ${}^*\mathrm{R}_{\mathrm{Sp}(2\theta',\mathbb{F}_q)}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)}$ for the restriction to the symplectic part of the Harish-Chandra restriction functor from $\mathrm{Sp}(2\theta,\mathbb{F}_q)$ to the Levi complement $\mathrm{GL}(a,\mathbb{F}_q)\times\mathrm{Sp}(2\theta',\mathbb{F}_q)$.

Theorem 20. *Let $S \in \mathcal{Y}^1_{d,\theta}$ and (α,β) the associated bipartition of $\theta-\delta(\delta+1)$. We have*

$${}^*\mathrm{R}_{\mathrm{Sp}(2\theta',\mathbb{F}_q)}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)}\,\rho_S = \sum_{\alpha',\beta'}\rho_{\delta,\alpha',\beta'},$$

where (α',β') runs over all the bipartitions of $\theta'-\delta(\delta+1)$ such that for some $0 \leqslant d \leqslant a$, the Young diagram of α' (resp. of β') can be obtained from the Young diagram of α (resp. of β) by removing a succession of d boxes (resp. $a-d$ boxes), no two of them lying in the same column.

3 The cohomology of the Coxeter variety for the symplectic group

In this section we compute the cohomology of Coxeter varieties of finite symplectic groups, in terms of the classification of the unipotent characters that we recalled in Theorem 6.

Notation. We write $X^\theta := X_\varnothing(\mathrm{cox})$ for the Coxeter variety attached to the symplectic group $\mathrm{Sp}(2\theta, \mathbb{F}_q)$, and $\mathrm{H}_c^\bullet(X^\theta)$ instead of $\mathrm{H}_c^\bullet(X^\theta \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ where $\ell \neq p$.

We first recall known facts on the cohomology of X^θ from Lusztig's work in [11].

Theorem 21. *The following statements hold.*

(1) *The variety X^θ has dimension θ and is affine. The cohomology group $\mathrm{H}_c^i(X^\theta)$ is zero unless $\theta \leqslant i \leqslant 2\theta$.*

(2) *The Frobenius F acts in a semisimple manner on the cohomology of X^θ.*

(3) *The groups $\mathrm{H}_c^{2\theta-1}(X^\theta)$ and $\mathrm{H}_c^{2\theta}(X^\theta)$ are irreducible as $\mathrm{Sp}(2\theta, \mathbb{F}_q)$-representations, and the latter is the trivial representation. The Frobenius F acts with eigenvalues respectively $q^{\theta-1}$ and q^θ.*

(4) *The group $\mathrm{H}_c^{\theta+i}(X^\theta)$ for $0 \leqslant i \leqslant \theta - 2$ is the direct sum of two eigenspaces of F, for the eigenvalues q^i and $-q^{i+1}$. Each eigenspace is an irreducible unipotent representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.*

(5) *The sum $\bigoplus_{i\geqslant 0} \mathrm{H}_c^i(X^\theta)$ is multiplicity-free as a representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.*

In other words, there exists a uniquely determined family of pairwise distinct symbols $S_0^\theta, \ldots, S_\theta^\theta$ and $T_0^\theta, \ldots, T_{\theta-2}^\theta$ in $\mathcal{Y}_\theta^1$ such that

$$\forall 0 \leqslant i \leqslant \theta - 2, \qquad \mathrm{H}_c^{\theta+i}(X^\theta) \simeq \rho_{S_i^\theta} \oplus \rho_{T_i^\theta},$$

$$\text{for } i = \theta - 1, \theta, \qquad \mathrm{H}_c^{\theta+i}(X^\theta) \simeq \rho_{S_i^\theta}.$$

The representation $\rho_{S_i^\theta}$ (resp. $\rho_{T_i^\theta}$) corresponds to the eigenspace of the Frobenius F on $\bigoplus_{i\geqslant 0} \mathrm{H}_c^{\theta+i}(X^\theta)$ attached to q^i (resp. to $-q^{i+1}$). Moreover, we know that $\rho_{S_\theta^\theta}$ is the trivial representation, therefore

$$S_\theta^\theta = \binom{\theta}{\ }.$$

Lusztig also gives a formula computing the dimension of the eigenspaces. Specializing to the case of the symplectic group, it reduces to the following statement.

Proposition 22 ([11])**.** *For $0 \leqslant i \leqslant \theta$ we have*

$$\deg(\rho_{S_i^\theta}) = q^{(\theta-i)^2} \prod_{s=1}^{\theta-i} \frac{q^{s+i}-1}{q^s-1} \prod_{s=0}^{\theta-i-1} \frac{q^{s+i}+1}{q^s+1}.$$

For $0 \leqslant j \leqslant \theta - 2$ we have

$$\deg(\rho_{T_j^\theta}) = q^{(\theta-j-1)^2} \frac{(q^{\theta-1}-1)(q^\theta-1)}{2(q+1)} \prod_{s=1}^{\theta-j-2} \frac{q^{s+j}-1}{q^s-1} \prod_{s=2}^{\theta-j-1} \frac{q^{s+j}+1}{q^s+1}.$$

Our goal in this section is to determine the symbols S_i^θ and T_j^θ explicitly. This is done in the following proposition.

Proposition 23. *For* $0 \leqslant i \leqslant \theta$ *and* $0 \leqslant j \leqslant \theta - 2$*, we have*

$$S_i^\theta = \begin{pmatrix} 0 \ldots \theta - i - 1 & \theta \\ 1 \ldots \quad \theta - i & \end{pmatrix}, \quad T_j^\theta = \begin{pmatrix} 0 & \ldots & \theta - j - 3 & \theta - j - 2 & \theta - j - 1 & \theta \\ 1 & \ldots & \theta - j - 2 & & & \end{pmatrix}.$$

Remark 24. In terms of bipartitions, S_i^θ corresponds to $((i), (1^{\theta - i}))$ and T_j^θ corresponds to $((j), (1^{\theta - 2 - j}))$.

We note that the statement is coherent with the two dimension formulae that we provided earlier. That is, the degree of $\rho_{S_i^\theta}$ (resp. of $\rho_{T_j^\theta}$) computed with the hook formula of Proposition 7, agrees with the dimension of the eigenspace of q^i (resp. of $-q^{j+1}$) in the cohomology of X^θ as given in the previous paragraph.

Proof. We use induction on $\theta \geqslant 0$. Since we already know that S_θ^θ is the symbol corresponding to the trivial representation, the proposition is proved for $\theta = 0$. Thus we may assume $\theta \geqslant 1$. We consider the block diagonal Levi complement $L \simeq \mathrm{GL}(1, \mathbb{F}_q) \times \mathrm{Sp}(2(\theta - 1), \mathbb{F}_q)$, and we write ${}^*\mathrm{R}_{\theta-1}^\theta$ for the restriction to $\mathrm{Sp}(2(\theta-1), \mathbb{F}_q)$ of the Harish-Chandra restriction from $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ to L. According to [11] Corollary 2.10, for all $0 \leqslant i \leqslant \theta$ we have an $\mathrm{Sp}(2(\theta - 1), \mathbb{F}_q) \times \langle F \rangle$-equivariant isomorphism

$$^*\mathrm{R}_{\theta-1}^\theta \left(\mathrm{H}_c^{\theta+i}(X^\theta) \right) \simeq \mathrm{H}_c^{\theta-1+i}(X^{\theta-1}) \oplus \mathrm{H}_c^{\theta-1+(i-1)}(X^{\theta-1})(-1). \tag{$*$}$$

The right-hand side can be computed by induction hypothesis whereas the left-hand side can be computed using Theorem 20. We fix $0 \leqslant i \leqslant \theta - 1$ and $0 \leqslant j \leqslant \theta - 2$. We denote by (δ, α, β) and (ν, γ, δ) the alternate labelling by bipartitions of the representations $\rho_{S_i^\theta}$ and $\rho_{T_j^\theta}$ respectively. Recall that the restriction ${}^*\mathrm{R}_{\theta-1}^\theta \, \rho_{\delta,\alpha,\beta}$ is the sum of all the representations $\rho_{\delta,\alpha',\beta'}$ where (α', β') is obtained from (α, β) by removing a single box in one of their Young diagrams. The similar description also holds for ${}^*\mathrm{R}_{\theta-1}^\theta \, \rho_{\nu,\gamma,\delta}$.
First we determine S_i^θ by identifying the q^i-eigenspace of the Frobenius in ($*$). We distinguish different cases depending on the values of θ and i.

- **Case** $\theta = 1$. In this case $i = 0$. The right-hand side of ($*$) is $\rho_{S_0^0} \simeq \rho_{0,\varnothing,\varnothing}$ with eigenvalue 1. Thus, $\delta = 0$ and the bipartition (α, β) consists of a single box. Therefore $(\alpha, \beta) = ((1), \varnothing)$ or $(\varnothing, (1))$. By Theorem 3, we know that $\rho_{0,\alpha,\beta}$ has degree q. This forces $(\alpha, \beta) = (\varnothing, (1))$.
- **Case** $\theta = 2$ **and i** $= 0$. The eigenspace attached to 1 on the right-hand side of ($*$) is $\rho_{S_0^1} \simeq \rho_{0,\varnothing,(1)}$. Thus, $\delta = 0$ and there is a single removable box in the bipartition (α, β). When we remove it, we obtain $(\varnothing, (1))$. Therefore, $(\alpha, \beta) = (\varnothing, (2))$ or $(\varnothing, (1^2))$. By Theorem 3, we know that $\rho_{0,\alpha,\beta}$ has degree q^4, thus $(\alpha, \beta) = (\varnothing, (1^2))$.

- **Case $\theta > 2$ and $\mathbf{i} = \mathbf{0}$**. The eigenspace attached to 1 on the right-hand side of ($*$) is $\rho_{S_0^{\theta-1}} \simeq \rho_{0,\varnothing,(1^{\theta-1})}$. Thus, $\delta = 0$ and there is a single removable box in the bipartition (α,β). When we remove it, we obtain $(\varnothing,(1^{\theta-1}))$. The only such bipartition is $(\alpha,\beta) = (\varnothing,(1^{\theta}))$.
- **Case $\theta > 2$ and $\mathbf{1} \leqslant \mathbf{i} \leqslant \mathbf{k} - \mathbf{1}$**. The eigenspace attached to p^i on the right-hand side of ($*$) is $\rho_{S_i^{\theta-1}} \oplus \rho_{S_{i-1}^{\theta-1}} \simeq \rho_{0,(i),(1^{\theta-1-i})} \oplus \rho_{0,(i-1),(1^{\theta-i})}$. Thus, $\delta = 0$ and there are exactly two removable boxes in the bipartition (α,β). When we remove one of them, we obtain either $((i),(1^{\theta-1-i}))$ or $((i-1),(1^{\theta-i}))$. The only such bipartition is $(\alpha,\beta) = ((i),(1^{\theta-i}))$.

It remains to determine T_j^{θ} for $0 \leqslant j \leqslant \theta-2$.

- **Case $\theta = 2$**. The eigenspace attached to $-p$ on the right-hand side of ($*$) is 0. Thus, the symbol $T_0^2 \in \mathcal{Y}_2^1$ has no hook at all, implying that it is cuspidal. Since $\mathrm{Sp}(4,\mathbb{F}_q)$ admits only one unipotent cuspidal representation, we deduce that $\nu = 1$ and $(\gamma,\delta) = (\varnothing,\varnothing)$.
- **Case $\mathbf{k} = \mathbf{3}$**. First when $j = 0$, the eigenspace attached to $-p$ on the right-hand side of ($*$) is $\rho_{T_0^2} \simeq \rho_{1,\varnothing,\varnothing}$. Thus, $\nu = 1$ and there is a single box in the bipartition (γ,δ). Therefore $(\gamma,\delta) = ((1),\varnothing)$ or $(\varnothing,(1))$. By Theorem 3, we know that $\rho_{1,\gamma,\delta}$ has degree $q^4\frac{(q^2-1)(q^3-1)}{2(q+1)}$, thus $(\gamma,\delta) = (\varnothing,(1))$.
  Then when $j = 1$, the eigenspace attached to $-p^2$ on the right-hand side of ($*$) is $\rho_{T_0^2} \simeq \rho_{1,\varnothing,\varnothing}$. Thus, $\nu = 1$ and as in the case $j = 0$ we have $(\gamma,\delta) = ((1),\varnothing)$ or $(\varnothing,(1))$. We can deduce that it is equal to the former by comparing the dimensions or by using the fact that the symbols T_j^{θ} are pairwise distinct.
- **Case $\theta = 4$ and $\mathbf{j} = \mathbf{0}$**. The eigenspace attached to $-p$ on the right-hand side of ($*$) is $\rho_{T_0^3} \simeq \rho_{1,\varnothing,(1)}$. Thus, $\nu = 1$ and there is a single removable box in the bipartition (γ,δ). When we remove it, we obtain $(\varnothing,(1))$. Therefore, $(\gamma,\delta) = (\varnothing,(2))$ or $(\varnothing,(1^2))$. By Theorem 3, we know that $\rho_{1,\gamma,\delta}$ has degree $q^9\frac{(q^3-1)(q^4-1)}{2(q+1)}$, thus $(\gamma,\delta) = (\varnothing,(1^2))$.
- **Case $\theta > 4$ and $\mathbf{j} = \mathbf{0}$**. The eigenspace attached to $-p$ on the right-hand side of ($*$) is $\rho_{T_0^{\theta-1}} \simeq \rho_{1,\varnothing,(1^{\theta-3})}$. Thus, $\nu = 1$ and there is a single removable box in the bipartition (γ,δ). When we remove it, we obtain $(\varnothing,(1^{\theta-3}))$. The only such bipartition is $(\gamma,\delta) = (\varnothing,(1^{\theta-2}))$.
- **Case $\theta = 4$ and $\mathbf{j} = \theta - \mathbf{2}$**. The eigenspace attached to $-p^3$ on the right-hand side of ($*$) is $\rho_{T_1^3} \simeq \rho_{1,(1),\varnothing}$. Thus, $\nu = 1$ and there is a single removable box in the bipartition (γ,δ). When we remove it, we obtain $((1),\varnothing)$. Therefore, $(\gamma,\delta) = ((2),\varnothing)$ or $((1^2),\varnothing)$. By Theorem 3, we know that $\rho_{1,\gamma,\delta}$ has degree $q\frac{(q^3-1)(q^4-1)}{2(q+1)}$, thus $(\gamma,\delta) = ((2),\varnothing)$.
- **Case $\theta > 4$ and $\mathbf{j} = \theta - \mathbf{2}$**. The eigenspace attached to $-p^{\theta-1}$ on the right-hand side of ($*$) is $\rho_{T_{\theta-3}^{\theta-1}} \simeq \rho_{1,(\theta-3),\varnothing}$. Thus, $\nu = 1$ and there is a single

removable box in the bipartition (γ,δ). When we remove it, we obtain $((\theta-3),\varnothing)$. The only such bipartition is $(\gamma,\delta)=((\theta-2),\varnothing)$.

- **Case $\mathbf{1\leqslant j\leqslant \theta-3}$.** The eigenspace attached to $-p^{j+1}$ on the right-hand side of $(*)$ is $\rho_{T_j^{\theta-1}}\oplus\rho_{T_{j-1}^{\theta-1}}\simeq\rho_{1,(j),(1^{\theta-3-j})}\oplus\rho_{1,(j-1),(1^{\theta-2-j})}$. Thus, $\nu=1$ and there are exactly two removable boxes in the bipartition (γ,δ). When we remove one of them, we obtain either $((j),(1^{\theta-3-j}))$ or $((j-1),(1^{\theta-2-j}))$. The only such bipartition is $(\gamma,\delta)=((j),(1^{\theta-2-j}))$.

□

4 The cohomology of S_θ

The last three sections of the paper are devoted to proving the main theorem below, which describes the cohomology of the variety S_θ. Since S_0 is a point and $S_1\simeq\mathbb{P}^1$, the cases $\theta=0$ or 1 are trivial. **From now and up to the end of the paper, we assume that $\theta\geqslant 2$.**

Theorem 25. *The following statements hold.*

(1) *All the cohomology groups of S_θ of odd degree vanish.*

(2) *For $0\leqslant i\leqslant\theta$, we have an $\mathrm{Sp}(2\theta,\mathbb{F}_q)$-equivariant isomorphism*

$$\mathrm{H}^{2i}(S_\theta)\simeq\bigoplus_{s=0}^{\min(i,\theta-i)}\rho_{\begin{pmatrix}s & \theta+1-s\\ 0 & \end{pmatrix}}\oplus\bigoplus_{s=0}^{\min(i-1,\theta-1-i)}\rho_{\begin{pmatrix}0 & s+1 & \theta-s\\ & & \end{pmatrix}}.$$

The Frobenius acts like multiplication by q^i on the first summand, and multiplication by $-q^i$ on the second summand.

Remarks 26. Let us make a few comments.

- We may rewrite the formula in terms of the alternate labelling of the unipotent representations of $\mathrm{Sp}(2\theta,\mathbb{F}_p)$. We obtain

$$\mathrm{H}^{2i}(S_\theta)\simeq\bigoplus_{s=0}^{\min(i,\theta-i)}\rho_{0,(\theta-s,s),\varnothing}\oplus\bigoplus_{s=0}^{\min(i-1,\theta-1-i)}\rho_{1,(\theta-2-s,s),\varnothing}.$$

- A unipotent cuspidal representation occurs in the cohomology of S_θ only in the cases $\theta=0$ and $\theta=2$. When $\theta=0$ it corresponds to $\mathrm{H}^0(S_0)$ which is trivial. When $\theta=2$ it occurs in $\mathrm{H}^2(S_2)$ with the eigenvalue $-p$. All the representations occuring in the cohomology of S_θ have cuspidal support given by one of these two cuspidal unipotent representations.

– Even though S_θ has isolated singularities for $\theta \geqslant 2$, its cohomology looks like the cohomology of a smooth projective variety, in so that it satisfies Poincaré duality, hard Lefschetz and purity of the Frobenius action.

In order to compute the cohomology of S_θ, we use the stratification by classical Deligne-Lusztig varieties which we recalled in Proposition 2, and we analyze the associated spectral sequence. It is given in its first page by

$$E_1^{\theta',i} = \mathrm{H}_c^{\theta'+i}(X_{I_{\theta'}}(w_{\theta'})) \implies \mathrm{H}^{\theta'+i}(S_\theta). \qquad (E)$$

Let us first determine each term explicitely. By Proposition 3 and using the notations introduced there, we have an isomorphism

$$X_{I_{\theta'}}(w_{\theta'}) \simeq \mathrm{Sp}(2\theta, \mathbb{F}_q)/U_{K_{\theta'}} \times_{L_{K_{\theta'}}} X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'}).$$

Taking cohomology, this identity translates into some Harish-Chandra induction

$$\mathrm{H}_c^\bullet(X_{I_{\theta'}}(w_{\theta'})) \simeq \mathrm{R}_{L_{K_{\theta'}}}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)} \mathrm{H}_c^\bullet(X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'})).$$

The Deligne-Lusztig variety $X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'})$ for the Levi complement $L_{K_{\theta'}} \simeq \mathrm{GL}(\theta - \theta', \mathbb{F}_q) \times \mathrm{Sp}(2\theta', \mathbb{F}_q)$ is isomorphic to the Coxeter variety $X^{\theta'}$ with the GL-part acting trivially. Thus, the terms $E_1^{\theta',i}$ are the Harish-Chandra inductions of the cohomology groups of the Coxeter varieties $X^{\theta'}$, which we have determined in Proposition 23. Let us compute these inductions explicitely.

Lemma 27. *Let $0 \leqslant i \leqslant \theta' \leqslant \theta$. We have $E_1^{\theta',i} = A^{\theta',i} \oplus B^{\theta',i}$ with*

$$A^{\theta',i} = \bigoplus_{\alpha,\beta} \rho_{0,\alpha,\beta}, \qquad B^{\theta',i} = \bigoplus_{\gamma,\delta} \rho_{1,\gamma,\delta},$$

where (α,β) runs over all the bipartitions of θ such that, for some $0 \leqslant d \leqslant \theta - \theta'$, we have

$$\begin{cases} \alpha = (i+d-s, s) \text{ for some } 0 \leqslant s \leqslant \min(d,i), \\ \beta = (\theta - \theta' - d, 1^{\theta'-i}) \text{ or } (\theta - \theta' - d + 1, 1^{\theta'-1-i}), \end{cases}$$

and if $i \leqslant \theta' - 2$, (γ,δ) runs over all the bipartitions of $\theta - 2$ such that, for some $0 \leqslant d \leqslant \theta - \theta'$, we have

$$\begin{cases} \gamma = (i+d-s, s) \text{ for some } 0 \leqslant s \leqslant \min(d,i), \\ \delta = (\theta - \theta' - d, 1^{\theta'-2-i}) \text{ or } (\theta - \theta' - d + 1, 1^{\theta'-3-i}). \end{cases}$$

The summand $A^{\theta',i}$ (resp. $B^{\theta',i}$) is the eigenspace of the Frobenius for the eigenvalue q^i (resp. the eigenvalue $-q^{i+1}$).

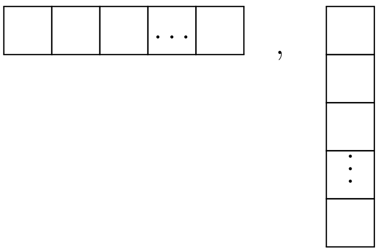
Remark 28. In particular, for each d there are at most two possibilities for β and δ. To remove ambiguity, let us point out that there are just two situations, possibly overlapping, where the given possibilities for β (resp. δ) actually coincide, that is

- if $i = \theta'$ (resp. $i = \theta'-2$), in which case $\beta = (\theta-\theta'-d)$ (resp. $\delta = (\theta-\theta'-d)$),
- if $d = \theta - \theta'$, in which case $\beta = (1^{\theta'-i})$ and $\delta = (1^{\theta'-2-i})$.

Proof. Using Pieri's rule for Coxeter groups of type B_n as we recalled in Proposition 19, we must decompose the Harish-Chandra inductions

$$R^{\mathrm{Sp}(2\theta,\mathbb{F}_q)}_{\mathrm{GL}(\theta-\theta',\mathbb{F}_q)\times\mathrm{Sp}(2\theta',\mathbb{F}_q)} \mathbf{1} \boxtimes \rho_{S_i^{\theta'}}, \qquad R^{\mathrm{Sp}(2\theta,\mathbb{F}_q)}_{\mathrm{GL}(\theta-\theta',\mathbb{F}_q)\times\mathrm{Sp}(2\theta',\mathbb{F}_q)} \mathbf{1} \boxtimes \rho_{T_j^{\theta'}},$$

where $S_i^{\theta'}$ and $T_j^{\theta'}$ are the symbols determined in Proposition 23. The Young diagrams of the bipartitions associated to these symbols have the form



The problem is to determine all the pairs of Young diagrams one may obtain after adding a succession of $\theta - \theta'$ boxes to the pair of diagrams above, with no two boxes in the same column. This computation has already been done in [14] Section 5, and leads to the claimed formula. $\square$

Corollary 29. *The spectral sequence* (E) *degenerates in the second page and the resulting filtration on the abutment splits. The weights of the eigenvalues of the Frobenius action on* $\mathrm{H}^k(S_\theta)$ *are even and at most equal to* $2\lfloor \frac{k}{2} \rfloor$.

Recall that an eigenvalue $\alpha \in \overline{\mathbb{Q}_\ell}$ of the Frobenius on the cohomology of a variety defined over $\mathbb{F}_q$ is said to be of weight $w \in \mathbb{Z}$ if, for any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$, we have $|\iota(\alpha)| = q^{\frac{w}{2}}$.

Proof. By the previous lemma, two terms of the sequence lying on different rows have no common eigenvalues for the Frobenius morphism. The Frobenius equivariance of the differentials force them to vanish in pages after E_1. Therefore the spectral sequence degenerates in the second page. For any $0 \leqslant k \leqslant 2\theta$, we deduce the existence of a filtration $\mathrm{Fil}^\bullet$ on $\mathrm{H}^k(S_\theta)$ such that the graded pieces $\mathrm{Gr}^p := \mathrm{Fil}^p/\mathrm{Fil}^{p+1}$ are isomorphic to $E_2^{p,k-p}$. In particular the non-zero graded pieces are concentrated in degree $k \leqslant 2p \leqslant 2\min(k,\theta)$. Each term $E_2^{p,k-p}$ is a subquotient of $E_1^{p,k-p}$. The Frobenius acts semisimply on the latter space with at most 2 distinct eigenvalues, which are q^{k-p}, and $-q^{k-p+1}$ if $k+2 \leqslant 2p$. Since there is no common eigenvalue of the Frobenius in two different graded pieces, the filtration splits and the Frobenius acts semi-simply on $\mathrm{H}^k(S_\theta)$. Moreover, the eigenvalues form a subset of $\{q^i, -q^j \mid 0 \leqslant i, j \leqslant \lfloor \frac{k}{2} \rfloor\}$. □

Analyzing the $\mathrm{Sp}(2\theta, \mathbb{F}_q)$-action, we may decompose each term $E_1^{\theta',i}$ in the following way,

$$A^{\theta',i} = A_0^{\theta',i} \oplus A_1^{\theta',i}, \qquad B^{\theta',i} = B_0^{\theta',i} \oplus B_1^{\theta',i},$$

where for $\epsilon = 0, 1$ the term $A_\epsilon^{\theta',i}$ is the sum of all the irreducible components $\rho_{0,\alpha,\beta}$ with the partition β, written as $\beta = (\beta_1 \geqslant \ldots \geqslant \beta_r > 0)$, satisfies $r = \theta' + \epsilon - i$, and if $i \leqslant \theta' - 2$ the term $B_\epsilon^{\theta',i}$ is the sum of all the irreducible components $\rho_{1,\gamma,\delta}$ with the partition δ, written as $\delta = (\delta_1 \geqslant \ldots \geqslant \delta_s > 0)$, satisfies $s = \theta' - 2 + \epsilon - i$. We observe that $A_1^{\theta,i} = B_1^{\theta,i} = 0$, and for $0 \leqslant i \leqslant \theta' < \theta$ (resp. $0 \leqslant j + 2 \leqslant \theta' < \theta$) we have isomorphisms $A_1^{\theta',i} \simeq A_0^{\theta'+1,i}$ and $B_1^{\theta',j} \simeq B_0^{\theta'+1,j}$. Consider a differential

$$d^{\theta',i} : E_1^{\theta',i} \to E_1^{\theta'+1,i}.$$

Since the differentials are Frobenius equivariant, they decompose as a sum $d^{\theta',i} = d_A^{\theta',i} \oplus d_B^{\theta',i}$ where

$$d_A^{\theta',i} : A^{\theta',i} \to A^{\theta'+1,i}, \qquad d_B^{\theta',i} : B^{\theta',i} \to B^{\theta'+1,i}.$$

The $\mathrm{Sp}(2\theta, \mathbb{F}_q)$-equivariance then forces

$$\mathrm{Im}(d_A^{\theta'-1,i}) \subset A_0^{\theta',i} \subset \mathrm{Ker}(d_A^{\theta',i}), \qquad \mathrm{Im}(d_B^{\theta'-1,i}) \subset B_0^{\theta',i} \subset \mathrm{Ker}(d_B^{\theta',i}).$$

In order to help visualize the situation, the page E_1 is drawn in Figure 1. Since the sequence degenerates in E_2 and the resulting filtration splits, it is clear that for all $0 \leqslant i \leqslant \theta$, the cohomology group $\mathrm{H}^{2i}(S_\theta)$ contains $A_0^{i,i} \oplus B_0^{i+1,i-1}$ (the term $B_0^{i+1,i-1}$ is non-zero if and only if $0 < i < \theta$). We point out that in Theorem 25, (2) can be rephrased as $\mathrm{H}^{2i}(S_\theta) \simeq A_0^{i,i} \oplus B_0^{i+1,i-1}$ for all $0 \leqslant i \leqslant \theta$.

Lemma 30. *In the statement of Theorem 25, (1) is equivalent to (2).*

Proof. Let us fix $0 \leqslant i \leqslant \theta' \leqslant \theta$. The equality $\mathrm{Im}(d_A^{\theta'-1,i}) = A_0^{\theta',i}$ (resp. $\mathrm{Im}(d_B^{\theta'-1,i}) = B_0^{\theta',i}$) is equivalent to $\mathrm{Ker}(d_A^{\theta'-1,i}) = A_0^{\theta'-1,1}$ (resp. $\mathrm{Ker}(d_B^{\theta'-1,i}) = B_0^{\theta'-1,i}$). Thus, the vanishing of a term $E_2^{\theta',i} = \mathrm{Ker}(d_A^{\theta',i})/\mathrm{Im}(d_A^{\theta'-1,i}) \oplus \mathrm{Ker}(d_B^{\theta',i})/\mathrm{Im}(d_B^{\theta'-1,i})$ is equivalent to the equalities

$$\mathrm{Ker}(d_A^{\theta'-1,i}) = A_0^{\theta'-1,i}, \qquad \mathrm{Im}(d_A^{\theta',i}) = A_0^{\theta'+1,i},$$
$$\mathrm{Ker}(d_B^{\theta'-1,i}) = B_0^{\theta'-1,i}, \qquad \mathrm{Im}(d_B^{\theta',i}) = B_0^{\theta'+1,i}.$$

The lemma follows easily. □

5 Desingularization of S_θ and purity of the Frobenius action on cohomology

In order to make out how the spectral sequence simplifies in the second page, it is necessary to get more information on the expected weights of the Frobenius on the abutment. To this end, we introduce the blow-up $\pi : S'_\theta \to S_\theta$ at its singular points. We denote by $E := \pi^{-1}(Z)$ the exceptional divisor, where in the notations of Proposition 1, $Z = X_I(\mathrm{id})$ is the singular locus of S_θ. Recall that $\dim Z = 0$. In this section, we prove the following Proposition.

Proposition 31. *The varieties S'_θ and E are smooth.*

Since $S'_\theta \backslash E$ is isomorphic to the smooth locus of S_θ, it is enough to prove that the blow-up is smooth at points of the exceptional divisor. To this end, we exhibit a certain affine neighborhood of any singular point of S_θ. Recall the symplectic space V introduced in the first section. Let $L(\theta)$ denote the Lagrangian Grassmannian of V, ie. the smooth projective variety defined over $\mathbb{F}_q$ whose k-rational points, for any field extension $k/\mathbb{F}_q$, correspond to subspaces of $U \subset V_k$ such that $U^\perp = U$. Recall from Proposition 1 that S_θ is the closed subvariety of $L(\theta)$ consisting of those U such that $U \cap \tau(U) \overset{\leqslant 1}{\subset} U$.
Let $\mathrm{Sym}_\theta \simeq \mathbb{A}^{\frac{\theta(\theta+1)}{2}}$ denote the variety of $\theta \times \theta$ symmetric matrices over $\mathbb{F}_q$. Let V_θ denote the closed subvariety consisting of all $M \in \mathrm{Sym}_\theta$ such that $M^{(q)} - M$ has rank at most one.

Lemma 32. *Any singular point $U \in S_\theta$ has an open affine neighborhood isomorphic to V_θ.*

Proof. If $U \in L(\theta)$ is a closed point defined over a finite extension $k/\mathbb{F}_q$, one may choose an isotropic supplement U' so that we have a decomposition $V_k = U \oplus U'$. The symplectic pairing induces an identification between U' and the k-linear dual of U. We may consider the affine variety $\mathrm{Hom}(U, U')$ defined over $\mathrm{Spec}(k)$, and the subvariety $\mathrm{Hom}(U, U')^{\mathrm{sym}}$ consisting of morphisms $\varphi \in \mathrm{Hom}(U, U')$ such that $\varphi^* = \varphi$, where $\varphi^* : U'^* \simeq U^{**} \simeq U \to U^*$ is the dual of φ. According to [10] Lemma 2.8, we have $\varphi \in \mathrm{Hom}(U, U')^{\mathrm{sym}}$ if and only $\Gamma_\varphi \in L(\theta)$ where Γ_φ denotes the graph of φ. The assignment $\varphi \mapsto \Gamma_\varphi$ defines an open immersion $\mathrm{Hom}(U, U')^{\mathrm{sym}} \hookrightarrow L(\theta)$, identifying the former with an open affine neighborhood of U. We have an identification $\mathrm{Hom}(U, U')^{\mathrm{sym}} \simeq \mathrm{Sym}_\theta \otimes k$ upon fixing a basis of U and equipping U' with the dual basis.
Now assume that $U \in S_\theta$ is a singular point, equivalently an $\mathbb{F}_q$-rational point of S_θ. Let us fix a basis $(f_i)_{1 \leqslant i \leqslant \theta}$ of U and equip U' with the dual basis $(f'_i)_{1 \leqslant i \leqslant \theta}$.

Let $M \in \mathrm{Sym}_\theta$. The graphs Γ_M and $\Gamma_{M^{(q)}}$ are generated by the vectors

$$g_j := f_j + \sum_{i=1}^{\theta} M_{ij} f'_i, \qquad\qquad g_j^{(q)} = f_j + \sum_{i=1}^{\theta} M_{ij}^q f'_i,$$

respectively. A direct computation shows that the intersection $\Gamma_M \cap \Gamma_{M^{(q)}}$ is isomorphic to $\mathrm{Ker}(M^{(q)} - M)$. Since U is defined over $\mathbb{F}_q$, the vectors f_j and f'_j are fixed by τ, thus we have $\tau(\Gamma_M) = \Gamma_{M^{(q)}}$. It follows that $M \in S_\theta \cap \mathrm{Sym}_\theta$ if and only if $\dim \mathrm{Ker}(M^{(q)} - M) \geqslant \theta - 1$, ie. if and only if $M \in V_\theta$. □

Let $V'_\theta \subset \mathrm{Sym}_\theta$ denote the subvariety of symmetric matrices M of rank at most 1. The variety V'_θ is known as a symmetric determinantal variety. It admits a single singular point corresponding to $M = 0$.

Proposition 33. *The blow-up of V'_θ at the point $M = 0$ is smooth with smooth exceptional divisor.*

Proof. This is Theorem B of [4] where, in their notations, we have $R_0 = \mathbb{F}_q$, $m = \theta$ and $r = 2$. □

Proof of Proposition 31. The variety V_θ contains the $\mathbb{F}_q$-points of Sym_θ and is singular precisely at them. By flat base change, the blow-up of V_θ at these singular points is an open neighborhood in S'_θ of the exceptional divisors above them. Thus it is enough to check smoothness of the blow-up of V_θ. Moreover, the Lang map $M \mapsto M^{(q)} - M$ defines a finite étale cover $V_\theta \to V'_\theta$. By flat base change again, we are reduced to Proposition 33. □

Proposition 31 has the following consequence regarding the cohomology of S_θ.

Corollary 34. *For $0 \leqslant k \leqslant 2\theta$, the Frobenius action on $\mathrm{H}^k(S_\theta)$ is pure of weight $2\lfloor \frac{k}{2} \rfloor$.*

Proof. By [16, Lemma 0EW3], there is a long exact sequence

$$\ldots \to \mathrm{H}^k(S_\theta) \to \mathrm{H}^k(S'_\theta) \oplus \mathrm{H}^k(Z) \to \mathrm{H}^k(E) \to \mathrm{H}^{k+1}(S_\theta) \to \ldots$$

Since S'_θ and E are projective and smooth, the Frobenius action on their k-th cohomology group is pure of weight k. For any $0 \leqslant j \leqslant \theta$, the maps $\mathrm{H}^{2j-1}(E) \to \mathrm{H}^{2j}(S_\theta)$ and $\mathrm{H}^{2j+1}(S_\theta) \to \mathrm{H}^{2j+1}(S'_\theta)$ vanish since the cohomology of S_θ only contains even weights by Corollary 29. Thus, we have in fact exact sequences

$$0 \to \mathrm{H}^{2j}(S_\theta) \to \mathrm{H}^{2j}(S'_\theta) \oplus \mathrm{H}^{2j}(Z) \to \mathrm{H}^{2j}(E) \to \mathrm{H}^{2j+1}(S_\theta) \to 0.$$

It follows that $\mathrm{H}^{2j}(S_\theta)$ and $\mathrm{H}^{2j+1}(S_\theta)$ are pure of weight $2j$. □

6 Intersection cohomology of S_θ

Let $U := S_\theta \backslash Z$ denote the smooth locus of S_θ. Let $j : U \hookrightarrow S_\theta$ be the open immersion. The **intersection complex** of S_θ is the intermediate image $j_{!*}\overline{\mathbb{Q}_\ell}$. We write $\mathrm{IH}^\bullet(S_\theta) := \mathrm{H}^\bullet(S_\theta, j_{!*}\overline{\mathbb{Q}_\ell})$ for the intersection cohomology of S_θ. It is a standard fact that $j_{!*}\overline{\mathbb{Q}_\ell}$ is pure of weight 0, and therefore $\mathrm{IH}^k(S_\theta)$ is pure of weight k for all k. Since Z is 0-dimensional, it follows from [1] Proposition 2.1.11 that the intersection complex is given by

$$j_{!*}\overline{\mathbb{Q}_\ell} \simeq \tau_{\theta-1}\mathrm{R}j_*\overline{\mathbb{Q}_\ell}.$$

The hypercohomology spectral sequence for intersection cohomology reads

$$F_2^{a,b} = \mathrm{H}^a(S_\theta, \mathrm{H}^b(j_{!*}\overline{\mathbb{Q}_\ell})) \implies \mathrm{IH}^{a+b}(S_\theta). \tag{F}$$

For $k \geqslant 1$, the sheaf $\mathrm{R}^k j_*\overline{\mathbb{Q}_\ell}$ is skyscraper at the points of Z. Furthermore we have $j_*\overline{\mathbb{Q}_\ell} = \overline{\mathbb{Q}_\ell}$ since S_θ is irreducible and normal. It follows that

$$F_2^{a,b} = \begin{cases} \mathrm{H}^a(S_\theta) & \text{if } b = 0, \\ \bigoplus_{\overline{z}\in Z}(\mathrm{R}^b j_*\overline{\mathbb{Q}_\ell})_{\overline{z}} & \text{if } a = 0 \text{ and } 1 \leqslant b \leqslant \theta - 1, \\ 0 & \text{else.} \end{cases}$$

In particular, the spectral sequence degenerate in $F_{\theta+1}$, and we have $\mathrm{H}^k(S_\theta) = \mathrm{IH}^k(S_\theta)$ for all $k > \theta$. The second page F_2 is drawn in Figure 2.

Proposition 35. *For $k > \theta$, we have*

$$\mathrm{H}^k(S_\theta) \simeq \begin{cases} A_0^{i,i} \oplus B_0^{i+1,i-1} & \textit{if } k = 2i \textit{ is even,} \\ 0 & \textit{if } k \textit{ is odd.} \end{cases}$$

The statement also holds when $k = \theta$ is even.

Proof. If $k > \theta$ is odd, the cohomology group $\mathrm{H}^k(S_\theta) = \mathrm{IH}^k(S_\theta)$ is pure of weight k. Since the cohomology of S_θ consists of only even weights by Proposition 29, we have $\mathrm{H}^k(S_\theta) = 0$. Assume now that $k = 2i \geqslant \theta$ is even. Given the repartition of the Frobenius weights in the first page of the spectral sequence (E), we have $H^{2i}(S_\theta) \simeq \mathrm{Ker}(d_A^{i,i}) \oplus \mathrm{Ker}(d_B^{i+1,i-1})$, the first (resp. second) summand corresponding to the eigenvalue q^i (resp. $-q^i$) of the Frobenius. Furthermore, since $H^{2i+1}(S_\theta) = 0$, the restriction of $d_A^{i,i}$ to $A_1^{i,i}$ (resp. of $d_B^{i+1,i-1}$ to $B_1^{i+1,i-1}$) defines an isomorphism onto $A_0^{i+1,i}$ (resp. onto $B_0^{i+2,i-1}$). Thus the kernel of $d_A^{i,i}$ (resp. of $d_B^{i+1,i-1}$) is reduced to $A_0^{i,i}$ (resp. to $B_0^{i+1,i-1}$). ☐

It remains to compute the cohomology of S_θ up to the middle degree. To do so, for $1 \leqslant k \leqslant \theta - 1$ we consider the differential $\delta_k : F_{k+1}^{0,k} \to F_{k+1}^{k+1,0}$ in the $(k+1)$-th page of the spectral sequence (F). We note that $F_{k+1}^{0,k} = F_2^{0,k}$ and $F_{k+1}^{k+1,0} = F_2^{k+1,0}$ since, up to the $(k+1)$-th page, both terms have not been touched by any non zero differential.

Proposition 36. *The differential δ_k vanishes for odd k and is surjective for even k.*

Proof. Assume that k is odd. We have $F_2^{0,k} = \bigoplus_{\overline{z} \in Z} \mathrm{H}^k(j_{!*}\overline{\mathbb{Q}_\ell})_{\overline{z}}$. Since $j_{!*}\overline{\mathbb{Q}_\ell}$ is pure of weight 0, the cohomology sheaf $\mathrm{H}^k(j_{!*}\overline{\mathbb{Q}_\ell})$ is mixed of weights $\leqslant k$. On the other hand, by Corollary 34, we know that $F_2^{0,k+1} = \mathrm{H}^{k+1}(S_\theta)$ is pure of weight $k+1$. Therefore δ_k must vanish.
Assume now that k is even. We know that $H^{k+1}(S_\theta)$ is pure of even weight, whereas $\mathrm{IH}^{k+1}(S_\theta)$ is pure of odd weight. Thus δ_k must be surjective. □

Proof of Theorem 25. Let $k = 2i < \theta$ be even. Since the differential δ_{2i-1} vanishes, the term of coordinate $(2i, 0)$ in the spectral sequence (F) is unchanged through the deeper pages. In particular, $\mathrm{IH}^{2i}(S_\theta)$ contains a subspace isomorphic to $\mathrm{H}^{2i}(S_\theta)$. Thus, we have

$$\mathrm{H}^{2i}(S_\theta) \hookrightarrow \mathrm{IH}^{2i}(S_\theta) \simeq \mathrm{IH}^{2(\theta-i)}(S_\theta)(\theta - 2i),$$

where the isomorphism follows from the hard Lefschetz theorem for intersection cohomology. Since $2(\theta - i) > \theta$, the RHS is isomorphic to $A_0^{\theta-i,\theta-i} \oplus B_0^{\theta-i+1,\theta-i-1}$ by Proposition 35. But $A_0^{\theta-i,\theta-i} \simeq A_0^{i,i}$ and $B_0^{\theta-i+1,\theta-i-1} \simeq B_0^{i+1,i-1}$ as $\mathrm{Sp}(2\theta, \mathbb{F}_q)$-modules. As $\mathrm{H}^{2i}(S_\theta)$ already contains $A_0^{i,i} \oplus B_0^{i+1,i-1}$, we actually have isomorphisms $\mathrm{IH}^{2i}(S_\theta) = \mathrm{H}^{2i}(S_\theta) \simeq A_0^{i,i} \oplus B_0^{i+1,i-1}$.
At this stage, we have proved statement (2) of Theorem 25. According to Lemma 30, the proof is over. □

As a by-product, we have proved the following statement.

Proposition 37. *We have $j_{!*}\overline{\mathbb{Q}_\ell} \simeq \overline{\mathbb{Q}_\ell}$ and $\mathrm{IH}^k(S_\theta) = \mathrm{H}^k(S_\theta)$ for all k.*

Proof. The natural map $\overline{\mathbb{Q}_\ell} \to j_{!*}\overline{\mathbb{Q}_\ell} \simeq \tau_{\theta-1}\mathrm{R}j_*\overline{\mathbb{Q}_\ell}$ is a quasi-isomorphism, as we have incidentally proved that $\mathrm{H}^k(j_{!*}\overline{\mathbb{Q}_\ell}) = 0$ for all $k \geqslant 1$. □

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A Figures

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & A_0^{\theta,\theta} \\
 & & & & & & & & & & A_0^{\theta-1,\theta-1} \oplus A_1^{\theta-1,\theta-1} & \longrightarrow & A_0^{\theta,\theta-1} \\
 & & & & & & & & A_0^{\theta-2,\theta-2} \oplus A_1^{\theta-2,\theta-2} & \longrightarrow & A_0^{\theta-1,\theta-2} \oplus A_1^{\theta-1,\theta-2} & \longrightarrow & A_0^{\theta,\theta-2} \oplus B_0^{\theta,\theta-2} \\
 & & & & & & \iddots & & & & & & \vdots \\
 & & & & A_0^{2,2} \oplus A_1^{2,2} & \longrightarrow & \dots & \longrightarrow & \begin{matrix} A_0^{\theta-2,2} \oplus A_1^{\theta-2,2} \\ B_0^{\theta-2,2} \oplus B_1^{\theta-2,2} \end{matrix} & \longrightarrow & \begin{matrix} A_0^{\theta-1,2} \oplus A_1^{\theta-1,2} \\ B_0^{\theta-1,2} \oplus B_1^{\theta-1,2} \end{matrix} & \longrightarrow & A_0^{\theta,2} \oplus B_0^{\theta,2} \\
 & & A_0^{1,1} \oplus A_1^{1,1} & \longrightarrow & A_0^{2,1} \oplus A_1^{2,1} & \longrightarrow & \dots & \longrightarrow & \begin{matrix} A_0^{\theta-2,1} \oplus A_1^{\theta-2,1} \\ B_0^{\theta-2,1} \oplus B_1^{\theta-2,1} \end{matrix} & \longrightarrow & \begin{matrix} A_0^{\theta-1,1} \oplus A_1^{\theta-1,1} \\ B_0^{\theta-1,1} \oplus B_1^{\theta-1,1} \end{matrix} & \longrightarrow & A_0^{\theta,1} \oplus B_0^{\theta,1} \\
A_0^{0,0} \oplus A_1^{0,0} & \longrightarrow & A_0^{1,0} \oplus A_1^{1,0} & \longrightarrow & \begin{matrix} A_0^{2,0} \oplus A_1^{2,0} \\ B_0^{2,0} \oplus B_1^{2,0} \end{matrix} & \longrightarrow & \dots & \longrightarrow & \begin{matrix} A_0^{\theta-2,0} \oplus A_1^{\theta-2,0} \\ B_0^{\theta-2,0} \oplus B_1^{\theta-2,0} \end{matrix} & \longrightarrow & \begin{matrix} A_0^{\theta-1,0} \oplus A_1^{\theta-1,0} \\ B_0^{\theta-1,0} \oplus B_1^{\theta-1,0} \end{matrix} & \longrightarrow & A_0^{\theta,0} \oplus B_0^{\theta,0}
\end{array}
$$

Figure 1: The first page of the spectral sequence (E).

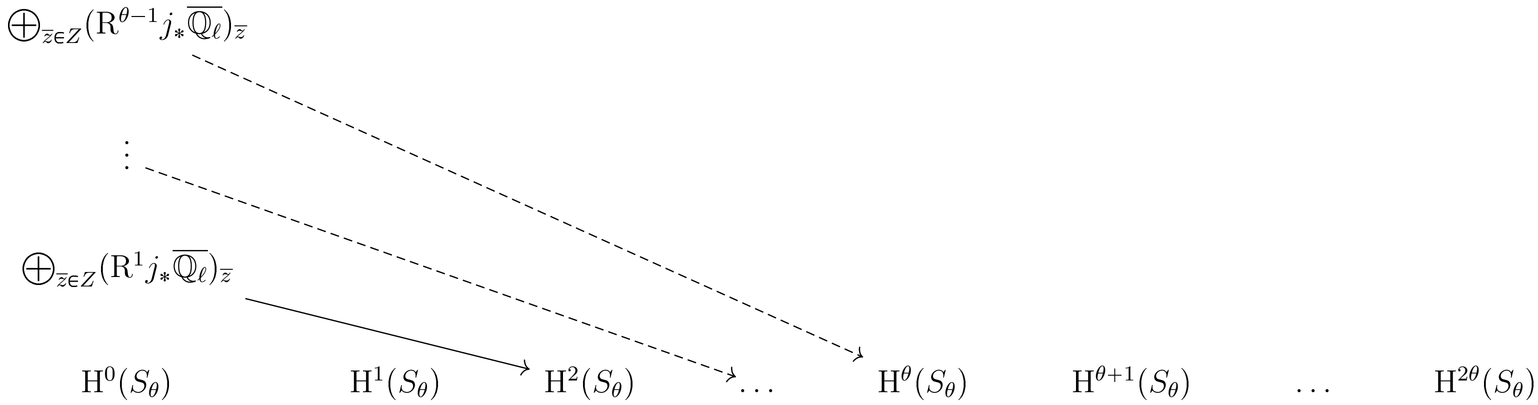

Figure 2: The second page of the spectral sequence (F) (the differentials in dashed lines correspond to deeper pages).