Nearby cycles on the local model for the $\mathrm{GU}(n-1,1)$ PEL Shimura variety over a ramified prime

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Abstract : *In this paper, we compute the cohomology sheaves of the ℓ-adic nearby cycles on the local model of the PEL $\mathrm{GU}(n-1,1)$ Shimura variety over a ramified prime. The local model is known to have isolated singularities. If $n=2$ it has semi-stable reduction, and if $n \geqslant 3$ the blow-up at the singular point has semi-stable reduction. Thus, in principle one may compute the nearby cycles at least on the blow-up, then use proper base change to describe them on the original local model. As a result, we prove that the nearby cycles are trivial when n is odd, and that only a single higher cohomology sheaf does not vanish when n is even. In this case, we also describe the Galois action by computing the associated eigenvalue of the Frobenius.*

Contents

1 Introduction

The $\mathrm{GU}(n-1,1)$ PEL Shimura variety is a variety over an imaginary quadratic number field $\mathbb{E}$ which can be classically defined as a moduli space of abelian varieties equipped with additional structures, ie. a polarization, an $\mathbb{E}$-action and a level structure. While smooth integral models of this Shimura variety can easily be defined over unramified primes of $\mathbb{E}$ by a natural integral formulation of the moduli problem as in [RZ96] for instance, the corresponding construction over a ramified prime results in an integral model which is not flat. To remedy this issue, Pappas suggested in [Pap00] to modify the moduli problem by adding a so-called "wedge condition", resulting in an integral model which is flat and has isolated singularities located in the special fiber. Moreover, he proved that the integral model is semi-stable when $n = 2$, and that its blow-up at the singular points is semi-stable. The proof actually consists in proving the corresponding properties on the associated local model, which is a moduli space defined purely in terms of linear algebra in so that it is represented by a closed subscheme of a grassmannian. The local model is related to the integral model of the Shimura variety via the local model diagramm, implying that both schemes share a common etale cover. When trying to prove a property on Shimura variety which is etale local, it is thus a common technique to reduce the problem to proving the corresponding statement on the local model, which is admittedly easier to work with. The local model has a single singular point located in its special fiber, and Krämer showed in [Krä03] that the special fiber consists of only two smooth divisors, one isomorphic to a projective space and the other having the structure of a $\mathbb{P}^1$-bundle over the scheme theoretic intersection, which itself is an explicit smooth projective quadric. In this paper, we compute the cohomology sheaves $\mathrm{R}^i\Psi_\eta\Lambda$ of the ℓ-adic complex of nearby cycles on the local model. To be more precise, let p be a fixed odd prime which ramifies in $\mathbb{E}$, and let $E := \mathbb{E}_p$ denote the p-adic completion. Thus E is a quadratic ramified extension of $\mathbb{Q}_p$. Let π denote a uniformizer of E such that $\overline{\pi} = -\pi$, where $\overline{\cdot}$ denotes the non-trivial element of $\mathrm{Gal}(E/\mathbb{Q}_p)$. Let M^{loc} denote the local model associated to Pappas' flat integral model of the Shimura variety. Then M^{loc} is a flat projective scheme over $\mathrm{Spec}(\mathcal{O}_E)$, whose special fiber we denote by $\overline{M^{\mathrm{loc}}}$. The special fiber has dimension $n-1$ and it has a unique singular point $y^{\mathrm{sg}} \in \overline{M^{\mathrm{loc}}}(\mathbb{F}_p)$. Let $b : M^{\mathrm{loc},K} \to M^{\mathrm{loc}}$ denote the blow-up of M^{loc} at y^{sg}. Eventually, fix a prime $\ell \neq p$ and let Λ denote a ring of coefficients (ie. $\Lambda = \mathbb{Z}/\ell^k\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$ or $\overline{\mathbb{Q}_\ell}$). Let $\mathrm{R}\Psi_\eta\Lambda$ (resp. $\mathrm{R}\Psi_\eta^K\Lambda$) denote the associated nearby cycle complex on M^{loc} (resp. on the blow-up $M^{\mathrm{loc},K}$), see [Ill94]. We will prove the following statement.

Theorem. *We have*

$$\mathrm{R}^k\Psi_\eta\Lambda = \begin{cases} \Lambda & \text{if } k=0, \\ i_{\overline{y^{\mathrm{sg}}}*}\Lambda[\epsilon p^{\frac{n}{2}}] & \text{if } n \text{ is even and } k=n-1, \\ 0 & \text{else,} \end{cases}$$

where, for n even, $i_{\overline{y^{\mathrm{sg}}}*}\Lambda[\epsilon p^{\frac{n}{2}}]$ is the skyscraper sheaf on $\overline{M^{\mathrm{loc}}}$ concentrated at the geometric singular point $\overline{y^{\mathrm{sg}}}$, with $\mathrm{Gal}(\overline{E}/E)$-action which is trivial on the inertia subgroup and such that the Frobenius element acts by multiplication by $\pm p^{\frac{n}{2}}$.

The sign of the Frobenius eigenvalue is explicitely determined depending on the nature of the algebraic datum underlying the definition of M^{loc}. Namely, it depends on whether a certain $E/\mathbb{Q}_p$-hermitian space is split or not. It is interesting to observe that, as a consequence of the computation above, we have $\mathrm{R}\Psi_\eta\Lambda \simeq \Lambda$ when n is odd. Thus the nearby cycles are trivial despite the presence of a singularity, which comes as rather unexpected.
The principle of the proof is simple, as it relies only on the proper base change formula applied to the blow-up morphism $b : M^{\mathrm{loc},K} \to M^{\mathrm{loc}}$. Since b induces an isomorphism on the generic fibers, we have

$$\mathrm{R}\Psi_\eta\Lambda \simeq \mathrm{R}b_{\overline{s}*}\mathrm{R}\Psi_\eta^K\Lambda,$$

where $b_{\overline{s}} : \overline{M^{\mathrm{loc},K}} \otimes \overline{\mathbb{F}_p} \to \overline{M^{\mathrm{loc}}} \otimes \overline{\mathbb{F}_p}$ denotes the morphism on geometric special fibers. Thus, if $\overline{x}$ is a geometric point of $\overline{M^{\mathrm{loc}}}$, we have

$$(\mathrm{R}^k\Psi_\eta\Lambda)_{\overline{x}} \simeq \mathrm{H}^k(b_{\overline{s}}^{-1}\{\overline{x}\}, \mathrm{R}\Psi_\eta^K\Lambda).$$

Clearly if $\overline{x}$ does not lie over y^{sg} then the stalk above vanishes for all $k \geqslant 1$. Denoting by $Z_1 := b_s^{-1}\{y^{\mathrm{sg}}\} \simeq \mathbb{P}^{n-1}$ the exceptional divisor, we are thus reduced to computing the cohomology of Z_1 with coefficient in the nearby cycles complex $\mathrm{R}\Psi_\eta^K\Lambda$. We carry out this computation by using functioriality of the spectral sequence associated to the monodromy filtration on the nearby cycles.
We remark that Krämer has computed the alternating semisimple trace of the Frobenius on the nearby cycles in [Krä03]. Namely, she computed the following sum

$$\mathrm{Tr}^{\mathrm{ss}}(\mathrm{Frob}, \mathrm{R}\Psi_t(\mathbb{Q}_\ell)^I_{y^{\mathrm{sg}}}) := \sum_i (-1)^i \mathrm{Tr}^{\mathrm{ss}}(\mathrm{Frob}, \mathrm{R}^i\Psi_t(\mathbb{Q}_\ell)^I_{y^{\mathrm{sg}}}).$$

Here $\mathrm{R}\Psi_t\mathbb{Q}_\ell := (\mathrm{R}\Psi_\eta\mathbb{Q}_\ell)^P$ denotes the tame nearby cycles, defined by taking the invariants under the largest p-subgroup $P \subset I$, and $I \subset \mathrm{Gal}(\overline{E}/E)$ denotes the inertia. Our computations recover Krämer's results and is more precise. Indeed, it turns out that the inertia acts trivially and that the Frobenius action is trivially semi-simple, so that taking I-invariants and semi-simple trace is actually superfluous. Moreover, the sum actually consists of only one or two non-zero terms depending on whether n is odd or even.

2 Geometry of the local model

As in the introduction, let p be an odd prime and let E be a ramified quadratic extension of $\mathbb{Q}_p$. Fix a uniformizer π such that $\overline{\pi} = -\pi$ where $\overline{\cdot} \in \mathrm{Gal}(E/\mathbb{Q}_p)$ is the non-trivial Galois involution on E. Let V be an E-vector space of dimension $n \geqslant 2$ equipped with a perfect hermitian form $(\cdot,\cdot) : V \times V \to E$. By convention, we assume that it is linear in the first variable and semilinear in the second. We also define a $\mathbb{Q}_p$-valued alternating bilinear form ψ via the formula

$$\psi(u,v) := \mathrm{Tr}_{E/\mathbb{Q}_p}\left(\frac{(u,v)}{\pi}\right),$$

for all $u, v \in V$. Note that one may also recover $(\cdot,\cdot)$ from ψ. Let L be a self-dual (for $(\cdot,\cdot)$ or equivalently for ψ) $\mathcal{O}_E$-lattice in V. The local model associated to this datum is given by the following moduli problem. Fix two non-negative integers r and s such that $r + s = n$. For an $\mathcal{O}_E$-scheme S, let $M^{\psi}_{r,s}(S)$ denote the set of subsheaves $\mathcal{F} \subset L \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules such that

- as a sheaf of $\mathcal{O}_S$-modules, $\mathcal{F}$ is locally a free direct summand of $L \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ of rank n,
- $\mathcal{F}$ is totally isotropic for the alternating bilinear form $\psi \otimes \mathcal{O}_S$ on $L \otimes_{\mathbb{Z}_p} \mathcal{O}_S$,
- the Kottwitz and Pappas conditions are satisfied:

$$\forall x \in \mathcal{O}_E,\ \det(T - x \,|\, \mathcal{F}) = (T - x)^s (T - \overline{x})^r \in \mathcal{O}_E[T],$$
$$\text{if } n \geqslant 3, \bigwedge^{r+1}(x \otimes 1 - 1 \otimes x) = 0 \text{ and } \bigwedge^{s+1}(x \otimes 1 - 1 \otimes \overline{x}) = 0 \text{ on } \mathcal{F}.$$

This functor is represented by a projective $\mathcal{O}_E$-scheme which we still denote by $M^{\psi}_{r,s}$.

Remark 2.1. In [Pap00], the moduli space above is denoted $M'^{\psi}_{r,s}$, and the definition is slightly different, in so that the Kottwitz and Pappas conditions are imposed on the quotient $\mathcal{Q} := L \otimes_{\mathbb{Z}_p} \mathcal{O}_S/\mathcal{F}$ instead of $\mathcal{F}$. However, both definitions coincide as it is not difficult to check that these conditions on $\mathcal{Q}$ are equivalent to the same conditions on $\mathcal{F}$ with the roles of r and s reversed.

From now on, we assume that $(r,s) = (n-1,1)$. We will write $M^{\psi} := M^{\psi}_{n-1,1}$ (corresponding to M^{loc} in the introduction). According to [Pap00], M^{ψ} is flat over $\mathrm{Spec}(\mathcal{O}_E)$, regular if $n = 2$, and regular outside of a single closed point y^{sg} if $n \geqslant 3$. Explicitely, y^{sg} lies in the special fiber of M^{ψ}. It is the $\mathbb{F}_p$-rational point given by $\mathcal{F}^{\mathrm{sg}} := (\pi \otimes 1)L \otimes_{\mathbb{Z}_p} \kappa(E)$. If $n \geqslant 3$, let $b : M^{\psi,K} \to M^{\psi}$ be the blow-up of M^{ψ} along y^{sg}. Let $\overline{M^{\psi}}$ and $\overline{M^{\psi,K}}$ denote the special fibers over $\kappa(E) = \mathbb{F}_p$ of M^{ψ} and of $M^{\psi,K}$ respectively. If $n = 2$, $\overline{M^{\psi}}$ is a geometrically irreducible divisor with simple normal crossings in M^ψ. If $n \geqslant 3$, then $M^{\psi,K}$ is regular and $\overline{M^{\psi,K}}$ is a geometrically irreducible divisor with simple normal crossings in $M^{\psi,K}$. In [Krä03], the author gives an explicit description of the geometry of the special fiber $\overline{M^{\psi,K}}$.

Theorem 2.2. *The special fiber $\overline{M^{\psi,K}}$ is the union of two smooth irreducible varieties Z_1 and Z_2. We have $Z_1 := b^{-1}\{y^{\mathrm{sg}}\} \simeq \mathbb{P}^{n-1}$ and Z_2 is a $\mathbb{P}^1$-bundle over the scheme theoretic intersection $Q := Z_1 \cap Z_2$. Moreover, the closed immersion $Q \hookrightarrow Z_1$ identifies Q with an explicit smooth quadric in $\mathbb{P}^{n-1}$.*

We may sum up the situation via the following diagram

$$\begin{tikzcd}
& \overline{M^{\psi,K}} & \\
Z_1 \simeq \mathbb{P}^{n-1} \arrow[ur, hook, "i_1"] & & Z_2 \simeq \mathbb{P}(\mathcal{E}) \arrow[ul, hook', "i_2"'] \arrow[dl, bend right, "\mathrm{pr}"'] \\
& Q \arrow[ul, hook, "\iota_1"] \arrow[ur, hook', "\iota_2"'] &
\end{tikzcd}$$

where $\mathcal{E}$ is a certain locally free sheaf of rank 2 on Q. Let us give more details. The blow-up $M^{\psi,K}$ can also be characterized as the moduli space classifying pairs $(\mathcal{F}_0, \mathcal{F})$ of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-submodules of $L \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ (for S an $\mathcal{O}_E$-scheme) such that

- $\mathcal{F} \in M^\psi(S)$,
- as a sheaf of $\mathcal{O}_S$-modules, $\mathcal{F}_0$ is a locally free direct summand of $L \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ of rank 1,
- $\mathcal{F}_0 \subset \mathcal{F}$,
- $(\pi \otimes 1 + 1 \otimes \pi)\mathcal{F} \subset \mathcal{F}_0$,
- $(\pi \otimes 1 - 1 \otimes \pi)\mathcal{F}_0 = 0$.

The blow-up map $b : M^{\psi,K} \to M^\psi$ corresponds to the forgetful functor $(\mathcal{F}, \mathcal{F}_0) \mapsto \mathcal{F})$.

Remark 2.3. Both papers [Pap00] and [Krä03] study $M^{\psi,K}$, but the former uses the blow-up construction and the latter used the moduli description. The equivalence between both definitions was well-known for a long time, and a proof may be found in [Shi22] Appendix A.

The map $\mathrm{pr} : \overline{M^{\psi,K}} \to \mathbb{P}((\pi \otimes 1)L \otimes_{\mathbb{Z}_p} \kappa(E)) \simeq \mathbb{P}^{n-1}$ induced by $(\mathcal{F}, \mathcal{F}_0) \mapsto \mathcal{F}_0$ restricts to an isomorphism on the fiber $Z_1 := b^{-1}\{y^{\mathrm{sg}}\}$. Krämer then defines a pairing on $(\pi \otimes 1)L \otimes_{\mathbb{Z}_p} \kappa(E)$ by the formula

$$\{(\pi \otimes 1)v, (\pi \otimes 1)w\} := \psi((\pi \otimes 1)v, w).$$

It is easy to check that it is well-defined, symmetric and non-degenerate. The closed subvariety $Q \subset Z_1$ consists of the isotropic lines in $(\pi \otimes 1)L \otimes_{\mathbb{Z}_p} \kappa(E)$.

Proposition 2.4. *The variety Q is smooth and geometrically irreducible of dimension $n-2$.*

Proof. Let $\delta \in \mathbb{Z}_p^\times$ which is not the norm of any element of $\mathcal{O}_E$. There is an E-basis $(e_1, \ldots, e_n)$ of V in which $(\cdot, \cdot)$ is given by the matrix $\mathrm{Diag}(1, \ldots, 1)$ or $\mathrm{Diag}(1, \ldots, 1, \delta)$. We may further assume that L is generated by the e_i's. A line of $(\pi \otimes 1)L \otimes_{\mathbb{Z}_p} \kappa(E)$ generated by a non-zero vector $(\pi \otimes 1)(a_1 e_1 + \ldots + a_n e_n)$ is then isotropic if and only if $\sum_{i=1}^n a_i^2 = 0$ or $\sum_{i=1}^{n-1} a_i^2 + \overline{\delta} a_n^2 = 0$, where $\overline{\delta} \in \mathbb{F}_p$ is the residue modulo p of δ. Since $\kappa(E)$ has odd characteristic, the proposition follows. $\square$

Let $Z_2 := \mathrm{pr}^{-1}(Q, \Lambda)$. Then Krämer builds a locally free sheaf $\mathcal{E}$ of rank 2 on Q and an isomorphism $Z_2 \simeq \mathbb{P}(\mathcal{E})$. Eventually, one checks that Q coincides with the scheme theoretical intersection $Z_1 \cap Z_2$.

3 The nearby cycles on the local model

Let ℓ be a prime different from p and let Λ be a coefficient ring as in the introduction. Let $\mathrm{R}\Psi_\eta\Lambda$ denote the nearby cycles on the local model M^ψ. This section is dedicated to proving the following statement.

Theorem 3.1. *We have*

$$\mathrm{R}^k\Psi_\eta\Lambda = \begin{cases} \Lambda & \text{if } k = 0, \\ i_{\overline{y^{\mathrm{sg}}}*}\Lambda[\epsilon p^{\frac{n}{2}}] & \text{if } n \text{ is even and } k = n-1, \\ 0 & \text{else,} \end{cases}$$

where, for n even, $i_{\overline{y^{\mathrm{sg}}}*}\Lambda[\epsilon p^{\frac{n}{2}}]$ is the skyscraper sheaf concentrated at the geometric singular point $\overline{y^{\mathrm{sg}}}$ with Frobenius action given by multiplication by $\epsilon p^{\frac{n}{2}}$, where $\epsilon = 1$ if $n = 2$ or if $n \geqslant 4$ and the hermitian space $(V, (\cdot, \cdot))$ is split, and $\epsilon = -1$ if $n \geqslant 4$ and the hermitian space is non-split.

Recall that $(V, (\cdot, \cdot))$ is said to be split if its discriminant $\mathrm{disc}(V) := (-1)^{\frac{n(n-1)}{2}}\det(V) \in \mathbb{Q}_p^\times/\mathrm{Norm}_{E/\mathbb{Q}_p^\times}(E^\times)$ is trivial, and non-split otherwise. The case $n = 2$ is easy since M^ψ already has semi-stable reduction. Thus in the remaining of this section, we assume that $n \geqslant 3$. Let $\mathrm{R}\Psi_\eta^K\Lambda$ denote the nearby cycles on the blow-up $M^{\psi,K}$ of the local model. Since $M^{\psi,K}$ has semi-stable reduction and the special fiber $\overline{M^{\psi,K}}$ has only two irreducible components, we have

$$\mathrm{R}^i\Psi_\eta^K\Lambda = \begin{cases} \Lambda & \text{if } i = 0, \\ i_{Q*}\Lambda(-1) & \text{if } i = 1, \\ 0 & \text{else,} \end{cases}$$

see [Ill94] Théorème 3.2. Here $i_Q : Q \hookrightarrow \overline{M^{\psi,K}}$ denotes the closed immersion. By proper base change and since b is an isomorphism on the generic fibers, we have

$$\mathrm{R}\Psi_\eta \Lambda \simeq \mathrm{R}b_{\overline{s}*}\mathrm{R}\Psi_\eta^K \Lambda,$$

where $b_{\overline{s}} : \overline{M^{\psi,K}} \otimes \overline{\mathbb{F}_p} \to \overline{M} \otimes \overline{\mathbb{F}_p}$ is the induced map on the geometric special fibers. Since $b_{\overline{s}}$ is an isomorphism away from y^{sg} and the fiber over y^{sg} is Z_1, we deduce that

$$(\mathrm{R}^i\Psi_\eta\Lambda)_y \simeq \begin{cases} \Lambda & \text{if } i = 0 \text{ and } y \neq y^{\mathrm{sg}}, \\ 0 & \text{if } i > 0 \text{ and } y \neq y^{\mathrm{sg}}, \\ \mathrm{H}^i(Z_1, i_1^*\mathrm{R}\Psi_\eta^K\Lambda) & \text{if } y = y^{\mathrm{sg}}. \end{cases}$$

Thus it remains to compute the cohomology of Z_1 with coefficients in $i_1^*\mathrm{R}\Psi_\eta^K\Lambda$. Recall that $\mathrm{R}\Psi_\eta^K\Lambda$ is equipped with a monodromy filtration

$$\ldots \subset F^i\mathrm{R}\Psi_\eta^K\Lambda \subset F^{i+1}\mathrm{R}\Psi_\eta^K\Lambda \subset \ldots,$$

see [Ill94] 3.8. According to [Sai03] Corollary 2.8, the graded pieces are given by

$$\mathrm{Gr}_r\mathrm{R}\Psi_\eta^K\Lambda \simeq \begin{cases} a_{1*}\Lambda[-1] & \text{if } r = -1, \\ a_{0*}\Lambda[0] & \text{if } r = 0, \\ a_{1*}\Lambda(-1)[-1] & \text{if } r = 1, \\ 0 & \text{else}, \end{cases}$$

where $a_0 : Z_1 \sqcup Z_2 \to \overline{M^{\psi,K}}$ and $a_1 = i_Q : Q \to \overline{M^{\psi,K}}$ are the natural maps. The restriction $i_1^*\mathrm{R}\Psi_\eta^K\Lambda$ also inherits a filtration, and since a_0 and a_1 are finite morphisms, by proper base change we have

$$\mathrm{Gr}_r i_1^*\mathrm{R}\Psi_\eta^K\Lambda \simeq \begin{cases} a'_{1*}\Lambda[-1] & \text{if } r = -1, \\ a'_{0*}\Lambda[0] & \text{if } r = 0, \\ a'_{1*}\Lambda(-1)[-1] & \text{if } r = 1, \\ 0 & \text{else}, \end{cases}$$

where $a'_0 : Z_1 \sqcup Q \to Z_1$ and $a'_1 = \iota_1 : Q \to Z_1$. These filtrations induce Galois equivariant spectral sequences

$$(E^K)_1^{a,b} = \mathrm{H}^{a+b}(\overline{M^{\psi,K}}, \mathrm{Gr}_{-a}\mathrm{R}\Psi_\eta^K\Lambda) \implies \mathrm{H}^{a+b}(\overline{M^{\psi,K}}, \mathrm{R}\Psi_\eta^K\Lambda),$$
$$(E^{Z_1})_1^{a,b} = \mathrm{H}^{a+b}(Z_1, \mathrm{Gr}_{-a}i_1^*\mathrm{R}\Psi_\eta^K\Lambda) \implies \mathrm{H}^{a+b}(Z_1, i_1^*\mathrm{R}\Psi_\eta^K\Lambda),$$

see Figure 1 and Figure 2 for a picture of their first pages given the previous computations of the graded pieces. The adjunction morphism $\mathrm{R}\Psi_\eta^K\Lambda \to i_{1*}i_1^*\mathrm{R}\Psi_\eta^K\Lambda$

is compatible with the filtrations, thus induces a morphism of spectral sequences $f_\bullet^{\bullet,\bullet} : (E^K) \to (E^{Z_1})$. On the first page it induces commutative diagrams for $0 \leqslant i \leqslant 2(n-2)$,

$$\begin{tikzcd} \mathrm{H}^{i-2}(Q,\Lambda)(-1) \arrow[r, "-\iota_{1*}+\iota_{2*}"] \arrow[d, "f_1^{-1,i}"] & \mathrm{H}^i(Z_1,\Lambda)\oplus \mathrm{H}^i(Z_2,\Lambda) \arrow[r, "-\iota_1^*+\iota_2^*"] \arrow[d, "f_1^{0,i}"] & \mathrm{H}^i(Q,\Lambda) \arrow[d, "f_1^{1,i}"] \\ \mathrm{H}^{i-2}(Q,\Lambda)(-1) \arrow[r, "\varphi"] & \mathrm{H}^i(Z_1,\Lambda)\oplus \mathrm{H}^i(Q,\Lambda) \arrow[r, "\psi"] & \mathrm{H}^i(Q,\Lambda) \end{tikzcd} \tag{1}$$

where the top maps are given by Gysin and restriction maps according to [Sai03] Proposition 2.10. Moreover, the morphisms $f_1^{\bullet,\bullet}$ are just the restriction maps. In particular, the vertical maps on the left and right are the identity. It follows that the bottom maps are given by

$$\begin{aligned} \mathrm{H}^{i-2}(Q,\Lambda)(-1) &\xrightarrow{\varphi} \mathrm{H}^i(Z_1,\Lambda)\oplus \mathrm{H}^i(Q,\Lambda) \xrightarrow{\psi} \mathrm{H}^i(Q,\Lambda) \\ x &\longmapsto (-\iota_{1*}(x), \iota_2^*\iota_{2*}(x)) \\ &\qquad\qquad (x,y) \longmapsto -\iota_1^*(x)+\alpha_i(y) \end{aligned}$$

where $\alpha_i : \mathrm{H}^i(Q,\Lambda) \to \mathrm{H}^i(Q,\Lambda)$ is some morphism which is identity on $\mathrm{Im}(i_2^* : \mathrm{H}^i(Z_2,\Lambda) \to \mathrm{H}^i(Q,\Lambda))$. Recall that $Z_1 \simeq \mathbb{P}^{n-1}$, and the closed immersion $\iota_1 : Q \hookrightarrow Z_1$ identifies Q with the smooth quadric $\psi((\pi\otimes 1)v, v) = 0$. The cohomology of Q will depend on whether $(V, (\cdot,\cdot))$ is split or not.

Proposition 3.2. *For $0 \leqslant i \leqslant 2(n-2)$, the restriction map $\iota_1^* : \mathrm{H}^i(Z_1,\Lambda) \to \mathrm{H}^i(Q,\Lambda)$ is an isomorphism for $i \neq n-2$, and is injective for $i = n-2$. The middle degree cohomology group decomposes naturally as*

$$\mathrm{H}^{n-2}(Q,\Lambda) \simeq \iota_1^*(\mathrm{H}^{n-2}(Z_1,\Lambda)) \oplus \mathrm{H}^{n-2}_{\mathrm{prim}}(Q,\Lambda),$$

and we have

$$\mathrm{H}^{n-2}_{\mathrm{prim}}(Q,\Lambda) = \mathrm{Ker}(\iota_{1*} : \mathrm{H}^{n-2}(Q,\Lambda) \to \mathrm{H}^n(Z_1,\Lambda)(1)) = \begin{cases} 0 & \textit{if } n \textit{ is odd,} \\ \Lambda[\epsilon p^{\frac{n-2}{2}}] & \textit{if } n \textit{ is even,} \end{cases}$$

where $\epsilon = 1$ if $(V,(\cdot,\cdot))$ is split and $\epsilon = -1$ when $(V,(\cdot,\cdot))$ is non-split.

Proof. The first statement on the restriction map ι_1^* and the decomposition $\mathrm{H}^{n-2}(Q,\Lambda) \simeq \iota_1^*(\mathrm{H}^{n-2}(Z_1,\Lambda)) \oplus \mathrm{H}^{n-2}_{\mathrm{prim}}(Q,\Lambda)$ where $\mathrm{H}^{n-2}_{\mathrm{prim}}(Q,\Lambda) = \mathrm{Ker}(\iota_{1*} : \mathrm{H}^{n-2}(Q,\Lambda) \to \mathrm{H}^n(Z_1,\Lambda)(1))$ follow from the weak Lefschetz theorem and Poincaré duality. Moreover, the primitive cohomology $\mathrm{H}^{n-2}_{\mathrm{prim}}(Q,\Lambda)$ is a free Λ-module whose rank depends only on the degree of the hypersurface Q, see [Mil13] Example 16.4. Since Q is a smooth quadric in $Z_1 \simeq \mathbb{P}^{n-1}$, we know that $\mathrm{H}^{n-2}_{\mathrm{prim}}(Q,\Lambda) = 0$ if n is odd and that $\mathrm{rank}(\mathrm{H}^{n-2}_{\mathrm{prim}}(Q,\Lambda)) = 1$ if n is even. Thus, the non-trivial part of the Proposition consists in the computation of the Frobenius eigenvalue on the primitive cohomology when $n = 2m \geqslant 4$ is even, which we know assume. By the Lefschetz trace formula, we have

$$\#Q(\mathbb{F}_p) = \sum_{i\geqslant 0}(-1)^i \mathrm{Trace}(\mathrm{Frob} \mid \mathrm{H}^i(Q,\Lambda)) = 1+p+\ldots+p^{n-2}+\mathrm{Trace}(\mathrm{Frob} \mid \mathrm{H}^{n-2}_{\mathrm{prim}}(Q,\Lambda)).$$

Remark 3.3. We point out that this is an equality in Λ. Thus, for instance if $\Lambda = \mathbb{Z}/\ell^k\mathbb{Z}$ then the left-hand side is the number of points of $Q(\mathbb{F}_p)$ modulo ℓ^k.

The left-hand side can be computed using [Wei49]. To do this, we need to fix an equation for Q. The hermitian space V is split if and only if $(\cdot,\cdot)$ is given by the matrix $\mathrm{Diag}(1,\ldots,1,-1,\ldots,-1)$ in some basis, where 1 and -1 occur m times each. It is non-split if and only if $(\cdot,\cdot)$ is given by $\mathrm{Diag}(1,\ldots,1,-1,\ldots,-1,-\delta)$ for some $\delta \in \mathbb{Z}_p^\times$ which is not the norm of any unit of $\mathcal{O}_E$. Assuming that L is generated by such a basis, the quadric $Q \subset \mathbb{P}^{n-1}$ is given by the equation $x_1^2+\ldots+x_m^2-x_{m+1}^2-\ldots-x_n^2 = 0$ in the split case, and by $x_1^2+\ldots+x_m^2-x_{m+1}^2-\ldots-x_{n-1}^2-\overline{\delta}x_n^2 = 0$ in the non-split case, where $\overline{\delta} \in \mathbb{F}_p$ is the residue modulo p of δ. Consider the following Jacobi sum

$$j_m := \frac{1}{p-1}\sum_{\substack{u_1+\ldots+u_{2m}=0\\ u_i\in\mathbb{F}_p^\times}} \left(\frac{u_1}{p}\right)\ldots\left(\frac{u_{2m}}{p}\right),$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol. According to [Wei49], we have

$$\#Q(\mathbb{F}_p) = \begin{cases} 1+\ldots+p^{n-2}+\left(\frac{-1}{p}\right)^m j_m & \text{if } (V,(\cdot,\cdot)) \text{ is split},\\ 1+\ldots+p^{n-2}+\left(\frac{-1}{p}\right)^m \left(\frac{\overline{\delta}}{p}\right)^{-1} j_m & \text{if } (V,(\cdot,\cdot)) \text{ is non-split}.\end{cases}$$

Since $\overline{\delta}$ is not a square in $\mathbb{F}_p$, we have $\left(\frac{\overline{\delta}}{p}\right) = -1$. Thus, the proof is over once we compute j_m, which is the object of the next Lemma. $\square$

Lemma 3.4. *Let q be a power of p and denote by $\left(\frac{\cdot}{q}\right)$ the Legendre symbol on $\mathbb{F}_q$. For $m \geqslant 1$, let $j_m := \frac{1}{q-1}\sum_{\substack{u_1+\ldots+u_{2m}=0\\ u_i\in\mathbb{F}_q^\times}} \left(\frac{u_1}{q}\right)\ldots\left(\frac{u_{2m}}{q}\right)$. Then*

$$j_m = \left(\frac{-1}{q}\right)^m q^{m-1}.$$

Proof. We establish a relation between j_m and j_{m-1}. Let us assume that $m \geqslant 2$. Observe that the equation $u_1 + \ldots + u_{2m} = 0$ is equivalent to $v_1 + \ldots + v_{2m-1} = 1$ where $v_i := -\frac{u_i}{u_{2m}}$. Thus, we rearrange the sum as

$$(q-1)j_m = \sum_{u_{2m} \in \mathbb{F}_q^\times} \left(\frac{-1}{q}\right)^{2m-1} \left(\frac{u_{2m}}{q}\right)^{2m} \sum_{\substack{v_1+\ldots+v_{2m-1}=1 \\ v_i \in \mathbb{F}_q^\times}} \left(\frac{v_1}{q}\right) \cdots \left(\frac{v_{2m-1}}{q}\right).$$

Since the Legendre symbol takes value in $\{\pm 1\}$, this simplifies to

$$j_m = \left(\frac{-1}{q}\right) \sum_{\substack{v_1+\ldots+v_{2m-1}=1 \\ v_i \in \mathbb{F}_q^\times}} \left(\frac{v_1}{q}\right) \cdots \left(\frac{v_{2m-1}}{q}\right). \tag{$*$}$$

By the change of variable $w := 1 - v_{2m-1} \in \mathbb{F}_q \backslash \{1\}$, we have

$$j_m = \left(\frac{-1}{q}\right) \sum_{\substack{v_1+\ldots+v_{2m-2}=w \\ v_i \in \mathbb{F}_q^\times \\ w \in \mathbb{F}_q \backslash \{1\}}} \left(\frac{v_1}{q}\right) \cdots \left(\frac{v_{2m-2}}{q}\right) \left(\frac{1-w}{q}\right).$$

Isolating the terms corresponding to $w = 0$, we obtain

$$j_m = \left(\frac{-1}{q}\right) (q-1) j_{m-1} + \left(\frac{-1}{q}\right) \sum_{\substack{v_1+\ldots+v_{2m-2}=w \\ v_i \in \mathbb{F}_q^\times \\ w \in \mathbb{F}_q \backslash \{0,1\}}} \left(\frac{v_1}{q}\right) \cdots \left(\frac{v_{2m-2}}{q}\right) \left(\frac{1-w}{q}\right). \tag{$**$}$$

Let us call S the sum on the right. For $1 \leqslant i \leqslant 2m-2$ we write $w_i := \frac{v_i}{w}$, so that

$$S = \sum_{w \in \mathbb{F}_q \backslash \{0,1\}} \left(\frac{1-w}{q}\right) \left(\frac{w}{q}\right)^{2m-2} \sum_{\substack{w_1+\ldots+w_{2m-2}=1 \\ w_i \in \mathbb{F}_q^\times}} \left(\frac{w_1}{q}\right) \cdots \left(\frac{w_{2m-2}}{q}\right).$$

Since $2m-2$ is even and since $\sum_{w \in \mathbb{F}_q \backslash \{0,1\}} \left(\frac{1-w}{q}\right) = -1$, this simplifies to

$$S = -\sum_{\substack{w_1+\ldots+w_{2m-2}=1 \\ w_i \in \mathbb{F}_q^\times}} \left(\frac{w_1}{q}\right) \cdots \left(\frac{w_{2m-2}}{q}\right).$$

If $m = 2$, then $S = -\sum_{x \in \mathbb{F}_q \backslash \{0,1\}} \left(\frac{x(1-x)}{q}\right)$. Notice that $\left(\frac{x(1-x)}{q}\right) = \left(\frac{x^{-1}-1}{q}\right)$. Since $x \mapsto x^{-1}$ defines a bijection on $\mathbb{F}_q \backslash \{0,1\}$, by a change of variable we obtain

$S = -\sum_{y \in \mathbb{F}_q \setminus \{0,1\}} \left(\frac{y-1}{q}\right) = \left(\frac{-1}{q}\right)$.
Now assume that $m \geqslant 3$. We introduce the change of variable $x := 1 - w_{2m-2}$ and write

$$S = - \sum_{\substack{w_1 + \ldots + w_{2m-3} = x \\ w_i \in \mathbb{F}_q^\times \\ x \in \mathbb{F}_q \setminus \{1\}}} \left(\frac{w_1}{q}\right) \cdots \left(\frac{w_{2m-3}}{q}\right) \left(\frac{1-x}{q}\right).$$

Isolating the terms corresponding to $x = 0$, we have

$$S = - \sum_{\substack{w_1 + \ldots + w_{2m-3} = 0 \\ w_i \in \mathbb{F}_q^\times}} \left(\frac{w_1}{q}\right) \cdots \left(\frac{w_{2m-3}}{q}\right) - \sum_{\substack{w_1 + \ldots + w_{2m-3} = x \\ w_i \in \mathbb{F}_q^\times \\ x \in \mathbb{F}_q \setminus \{0,1\}}} \left(\frac{w_1}{q}\right) \cdots \left(\frac{w_{2m-3}}{q}\right) \left(\frac{1-x}{q}\right).$$

First we compute the sum on the left. fix an element $\lambda \in \mathbb{F}_q^\times$ which is not a square, and make the change of variables $w_i \mapsto \lambda w_i$. Since $\left(\frac{\lambda}{q}\right)^{2m-3} = -1$, one finds out that this sum vanishes. Next, we proceed to the change of variables $x_i := \frac{w_i}{x}$ in the sum on the right. We obtain

$$S = - \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \left(\frac{1-x}{q}\right) \left(\frac{x}{q}\right)^{2m-3} \sum_{\substack{x_1 + \ldots + x_{2m-3} = 1 \\ x_i \in \mathbb{F}_q^\times}} \left(\frac{x_1}{q}\right) \cdots \left(\frac{x_{2m-3}}{q}\right).$$

Since $2m - 3$ is odd, the sum on x recovers the case $m = 2$ treated above, and simplifies to $\left(\frac{-1}{q}\right)$. On the other hand, according to $(*)$ with m replaced by $m - 1 \geqslant 2$, the inner sum is related to j_{m-1}. All in all, we obtain

$$S = j_{m-1}.$$

Replacing in $(**)$, we deduce that

$$j_m = \left(\frac{-1}{q}\right)(q-1) j_{m-1} + \left(\frac{-1}{q}\right) j_{m-1} = \left(\frac{-1}{q}\right) q j_{m-1}.$$

It remains to compute

$$j_1 = \frac{1}{q-1} \sum_{x \in \mathbb{F}_q^\times} \left(\frac{x}{q}\right) \left(\frac{-x}{q}\right) = \frac{1}{q-1} \sum_{x \in \mathbb{F}_q^\times} \left(\frac{-1}{q}\right) = \left(\frac{-1}{q}\right),$$

and the proof is over. □

Recall the commutative diagram (1). The composition $\iota_{1*} \circ \iota_1^* : \mathrm{H}^i(Z_1, \Lambda) \to \mathrm{H}^{i+2}(Z_1, \Lambda)(1)$ is equal to the cup product with $\mathrm{cl}_{Z_1}(Q, \Lambda) \in \mathrm{H}^2(Z_1, \Lambda)(1)$, the cycle class of Q in Z_1. Thus, the Gysin map $\iota_{1*} : \mathrm{H}^{i-2}(Q, \Lambda)(-1) \to \mathrm{H}^i(Z_1, \Lambda)$ is surjective and induces an isomorphism when restricted to $\iota_1^*(\mathrm{H}^{i-2}(Z_1, \Lambda))(-1)$. In particular, φ is injective for all even i different from n, since $\mathrm{H}^{i-2}(Q, \Lambda) \simeq \iota_1^*(\mathrm{H}^{i-2}(Z_1, \Lambda))$ in such case.
Next, recall that $Z_2 \simeq \mathbb{P}(\mathcal{E})$ is a projective bundle over Q. By the projective bundle formula, the pullback pr^* gives an isomorphism of graded $\mathrm{H}^*(Q, \Lambda)$-modules

$$\mathrm{H}^*(Q, \Lambda)[t]/(t^2) \xrightarrow{\sim} \mathrm{H}^*(Z_2, \Lambda),$$

sending t to the image $\zeta := [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)] \in \mathrm{H}^2(Z_2, \Lambda)(1)$ via the Kummer sequence. Since $\mathrm{pr} \circ \iota_2 = \mathrm{id}_Q$, we know that $\iota_2^* \circ \mathrm{pr}^*$ and $\mathrm{pr}_* \circ \iota_{2*}$ are the identity on $\mathrm{H}^i(Q, \Lambda)$. In particular, ι_{2*} is injective and ι_2^* is surjective, implying that $\alpha_i = \mathrm{id}$ for all $0 \leqslant i \leqslant 2(n-2)$. Therefore ψ is surjective for all $0 \leqslant i \leqslant 2(n-2)$.
Summing everything up, consider the bottom line for a given $0 \leqslant i \leqslant 2(n-1)$ with $i \neq n$

$$\mathrm{H}^{i-2}(Q, \Lambda)(-1) \overset{\varphi}{\hookrightarrow} \mathrm{H}^i(Z_1, \Lambda) \oplus \mathrm{H}^i(Q, \Lambda) \overset{\psi}{\twoheadrightarrow} \mathrm{H}^i(Q, \Lambda).$$

If i is odd, all these terms are 0. If $i = 0$, the LHS is 0 and ψ is $(x, y) \mapsto -\iota_1^*(x) + y$, where $\iota_1^* : \mathrm{H}^0(Z_1, \Lambda) \xrightarrow{\sim} \mathrm{H}^0(Q, \Lambda)$. The kernel is isomorphic to $\mathrm{H}^0(Z_1, \Lambda) = \Lambda$. If $i = 2(n-1)$, the RHS is 0 and φ is an isomorphism. Eventually, if $0 < i < 2(n-1)$ is even with $i \neq n$, the kernel of ψ consists of all the couples of the form $(x, \iota_1^*(x)) \in \mathrm{H}^i(Z_1, \Lambda) \oplus \mathrm{H}^i(Q, \Lambda)$. There exists a unique $x' \in H^{i-2}(Z_1, \Lambda)(-1)$ such that $x = -x' \cup \mathrm{cl}_{Z_1}(Q, \Lambda) = -\iota_{1*} \circ \iota_1^*(x')$. Let us check that $\varphi(\iota_1^*(x')) = (x, \iota_1^*(x))$. Since $\psi \circ \varphi = 0$ we have $\iota_1^* \iota_{1*} \iota_1^*(x') + \iota_2^* \iota_{2*} \iota_1^*(x') = 0$, ie. $\iota_2^* \iota_{2*} \iota_1^*(x') = -\iota_1^*(x)$ as required. Thus, we have proved that $\mathrm{Ker}(\psi) \subset \mathrm{Im}(\varphi)$, implying that the sequence is exact in the middle.
It remains to consider the case $i = n$ where $n = 2m$ is even. In this case the bottom line of (1) writes down as

$$H^{n-2}(Q, \Lambda)(-1) = \iota_1^*(\mathrm{H}^{n-2}(Z_1, \Lambda))(-1) \oplus \mathrm{H}^{n-2}_{\mathrm{prim}}(Q, \Lambda)(-1) \xrightarrow{\varphi} \mathrm{H}^n(Z_1, \Lambda) \oplus \mathrm{H}^n(Q, \Lambda) \overset{\psi}{\twoheadrightarrow} \mathrm{H}^n(Q, \Lambda).$$

The morphism φ is given by $x \mapsto (-\iota_{1*}(x), \iota_2^* \iota_{2*}(x))$. If $x \in \mathrm{H}^{n-2}_{\mathrm{prim}}(Q, \Lambda)(-1)$ belongs to the primitive part, then $\iota_{1*}(x) = 0$ by definition and $\iota_2^* \iota_{2*}(x) = 0$ since $\psi \circ \varphi = 0$. Thus, $\mathrm{H}^{n-2}_{\mathrm{prim}}(Q, \Lambda)(-1) \subset \mathrm{Ker}(\varphi)$. The reverse inclusion is obvious, and the equality $\mathrm{Ker}(\psi) = \mathrm{Im}(\varphi)$ is easily checked as above.
All in all, we have computed the second page $(E^{Z_1})_2$ entirely. Explicitely, we have

$$(E^{Z_1})_2^{a,b} = \begin{cases} \mathrm{H}^0(Z_1, \Lambda) \simeq \Lambda & \text{if } (a, b) = (0, 0), \\ \mathrm{H}^{n-2}_{\mathrm{prim}}(Q, \Lambda)(-1) \simeq \Lambda[\epsilon p^m] & \text{if } (a, b) = (-1, n) \text{ and } n = 2m \text{ is even}, \\ 0 & \text{else}, \end{cases}$$

where $\epsilon = 1$ if $(V, (\cdot, \cdot))$ is split and $\epsilon = -1$ if $(V, (\cdot, \cdot))$ is non-split. It is then clear that the spectral sequence degenerates on the second page, thus concluding the computation of the cohomology groups $\mathrm{H}^\bullet(Z_1, i_1^* \mathrm{R}\Psi_\eta^K \Lambda)$. This concludes the proof of Theorem 3.1.

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A Figures

$$
\begin{array}{ccccc}
\mathrm{H}^{2(n-2)}(Q,\Lambda)(-1) & \longrightarrow & \mathrm{H}^{2(n-1)}(Z_1,\Lambda)\oplus\mathrm{H}^{2(n-1)}(Z_2,\Lambda) & & \\
\mathrm{H}^{2(n-3)+1}(Q,\Lambda)(-1) & \longrightarrow & \mathrm{H}^{2(n-2)+1}(Z_1,\Lambda)\oplus\mathrm{H}^{2(n-2)+1}(Z_2,\Lambda) & & \\
\mathrm{H}^{2(n-3)}(Q,\Lambda)(-1) & \longrightarrow & \mathrm{H}^{2(n-2)}(Z_1,\Lambda)\oplus\mathrm{H}^{2(n-2)}(Z_2,\Lambda) & \longrightarrow & \mathrm{H}^{2(n-2)}(Q,\Lambda) \\
\vdots & & \vdots & & \vdots \\
\mathrm{H}^{0}(Q,\Lambda)(-1) & \longrightarrow & \mathrm{H}^{2}(Z_1,\Lambda)\oplus\mathrm{H}^{2}(Z_2,\Lambda) & \longrightarrow & \mathrm{H}^{2}(Q,\Lambda) \\
 & & \mathrm{H}^{1}(Z_1,\Lambda)\oplus\mathrm{H}^{1}(Z_2,\Lambda) & \longrightarrow & \mathrm{H}^{1}(Q,\Lambda) \\
 & & \mathrm{H}^{0}(Z_1,\Lambda)\oplus\mathrm{H}^{0}(Z_2,\Lambda) & \longrightarrow & \mathrm{H}^{0}(Q,\Lambda)
\end{array}
$$

Figure 1: The first page $(E^K)_1^{a,b}$ of the monodromy spectral sequence for $M^{\psi,K}$.

$$
\begin{array}{ccccc}
\mathrm{H}^{2(n-2)}(Q,\Lambda)(-1) & \longrightarrow & \mathrm{H}^{2(n-1)}(Z_1,\Lambda) & & \\
\mathrm{H}^{2(n-3)+1}(Q,\Lambda)(-1) & \longrightarrow & \mathrm{H}^{2(n-2)+1}(Z_1,\Lambda) & & \\
\mathrm{H}^{2(n-3)}(Q,\Lambda)(-1) & \longrightarrow & \mathrm{H}^{2(n-2)}(Z_1,\Lambda)\oplus\mathrm{H}^{2(n-2)}(Q,\Lambda) & \longrightarrow & \mathrm{H}^{2(n-2)}(Q,\Lambda) \\
\vdots & & \vdots & & \vdots \\
\mathrm{H}^{0}(Q,\Lambda)(-1) & \longrightarrow & \mathrm{H}^{2}(Z_1,\Lambda)\oplus\mathrm{H}^{2}(Q,\Lambda) & \longrightarrow & \mathrm{H}^{2}(Q,\Lambda) \\
 & & \mathrm{H}^{1}(Z_1,\Lambda)\oplus\mathrm{H}^{1}(Q,\Lambda) & \longrightarrow & \mathrm{H}^{1}(Q,\Lambda) \\
 & & \mathrm{H}^{0}(Z_1,\Lambda)\oplus\mathrm{H}^{0}(Q,\Lambda) & \longrightarrow & \mathrm{H}^{0}(Q,\Lambda)
\end{array}
$$

Figure 2: The first page $(E^{Z_1})_1^{a,b}$ of the restriction of the monodromy spectral sequence to Z_1.