On the cohomology of the basic unramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$

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Abstract : *In this paper, we study the cohomology of the unitary unramified PEL Rapoport-Zink space of signature $(1, n-1)$ at maximal level. Our method revolves around the spectral sequence associated to the open cover by the analytical tubes of the closed Bruhat-Tits strata in the special fiber, which were constructed by Vollaard and Wedhorn. The cohomology of these strata, which are isomorphic to generalized Deligne-Lusztig varieties, has been computed in [Mul23]. This spectral sequence allows us to prove the semisimplicity of the Frobenius action and the non-admissibility of the cohomology in general. Via p-adic uniformization, we relate the cohomology of the Rapoport-Zink space to the cohomology of the supersingular locus of a Shimura variety with no level at p. In the case $n = 3$ or 4, we give a complete description of the cohomology of the supersingular locus in terms of automorphic representations.*

Contents

Introduction: By defining moduli problems classifying deformations of p-divisible groups with additional structures, Rapoport and Zink have constructed their eponymous spaces which consist in a projective system $(\mathcal{M}_{K_p})$ of non-archimedean analytic spaces. The set of data defining the moduli problem determines two p-adic groups $G(\mathbb{Q}_p)$ and $J(\mathbb{Q}_p)$ which both act on the tower. Its cohomology is therefore equipped with an action of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W$ where W is the absolute Weil group of a finite extension of $\mathbb{Q}_p$, called the local reflex field. This is expected to give a geometric incarnation of the local Langlands correspondence. So far, relatively little is known about the cohomology of Rapoport-Zink spaces in general. The Kottwitz conjecture describes the $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p)$-supercuspidal part of the cohomology but it is only known in a handful of cases. It was first proved for the Lubin-Tate tower in [Boy99] and in [HT01], from which the Drinfeld case follows by duality. The case of basic unramified EL Rapoport-Zink spaces has been treated in [Far04] and [Shi12]. As for the PEL case, it was proved for basic unramified unitary Rapoport-Zink spaces with signature $(1, n-1)$ with n odd in [Ngu19], and in [BMN21] for an arbitrary signature with an odd number of variables. Beyond the Kottwitz conjecture, one would like to understand the individual cohomology groups of the Rapoport-Zink spaces entirely. This has been done in [Boy09] for the Lubin-Tate case (and, dually, for the Drinfeld case as well) using a vanishing cycle approach. Boyer's results were later used in [Dat07] to recover the action of the monodrony and give an elegant form of geometric Jacquet-Langlands correspondence. However, this method relied heavily on the particuliar geometry of the Lubin-Tate tower, and we are faced with technical issues in other situations where we do not have a satisfactory understanding of the geometry of the Rapoport-Zink spaces.

In this paper, we aim at pursuing the goal of describing the individual cohomology groups of the Rapoport-Zink spaces in the basic PEL unramified unitary case with signature $(1, n-1)$. Here, $G(\mathbb{Q}_p)$ is an unramified group of unitary similitudes in n variables and $J(\mathbb{Q}_p)$ is an inner form of $G(\mathbb{Q}_p)$. In fact, $J(\mathbb{Q}_p)$ is isomorphic to $G(\mathbb{Q}_p)$ when n is odd and $J(\mathbb{Q}_p)$ is the non quasi-split inner form when n is even. Our approach is based on the geometric description of the reduced special fiber $\mathcal{M}_{\mathrm{red}}$ given in [Vol10] and [VW11]. In these papers, Vollaard and Wedhorn built the Bruhat-Tits stratification $\{\mathcal{M}_\Lambda\}_\Lambda$ on $\mathcal{M}_{\mathrm{red}}$ which is interesting for two reasons:

- the closed strata $\mathcal{M}_\Lambda$ are indexed by the vertices of the Bruhat-Tits building $\mathrm{BT}(J,\mathbb{Q}_p)$ of $J(\mathbb{Q}_p)$. The combinatorics of the stratification can be read on the building.
- each individual stratum $\mathcal{M}_\Lambda$ is isomorphic to a generalized Deligne-Lusztig variety for a finite group of Lie type of the form $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$, arising in the maximal reductive quotient of the maximal parahoric subgroup $J_\Lambda := \mathrm{Fix}_J(\Lambda)$. Here $1 \leqslant 2\theta+1 =: t(\Lambda) \leqslant n$ is an odd integer called the orbit type of $\Lambda \in \mathrm{BT}(J,\mathbb{Q}_p)$.

Let $\theta_{\max} := \left\lfloor \frac{n-1}{2} \right\rfloor$ so that we have $0 \leqslant \theta \leqslant \theta_{\max}$ for all vertices $\Lambda \in \mathrm{BT}(J,\mathbb{Q}_p)$. In [Mul23], by exploiting the Ekedahl-Oort stratification on a given stratum $\mathcal{M}_\Lambda$, we computed the cohomology groups $\mathrm{H}^\bullet(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$ in terms of representations of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ with a Frobenius action. We consider the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}} := \mathcal{M}_{K_0}$ at maximal level, where $K_0 \subset G(\mathbb{Q}_p)$ is a hyperspecial maximal open compact subgroup. Then $\mathcal{M}^{\mathrm{an}}$ is an analytic space of dimension $n-1$. It admits an open cover by the analytical tubes U_Λ of the closed Bruhat-Tits strata $\mathcal{M}_\Lambda$. This induces a $J(\mathbb{Q}_p) \times W$-equivariant Čech spectral sequence computing the cohomology of $\mathcal{M}^{\mathrm{an}}$

$$E_1^{a,b} : \bigoplus_{\gamma \in I_{-a+1}} \mathrm{H}_c^b(U_{\Lambda(\gamma)} \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}),$$

where for $s \geqslant 1$ the index set is given by

$$I_s := \left\{ \gamma = (\Lambda^1, \dots, \Lambda^s) \in \mathrm{BT}(J,\mathbb{Q}_p)^s \,|\, \forall i, t(\Lambda^i) = 2\theta_{\max}+1 \text{ and } U(\gamma) := \bigcap_{i=1}^s U_{\Lambda^i} \neq \varnothing \right\}.$$

In the remaining of the introduction, we omit the Using Berkovich's comparison theorem, the cohomology of the tubes U_Λ can be identified, up to a shift in indices and a suitable Tate twist, with the cohomology of the closed Bruhat-Tits strata $\mathcal{M}_\Lambda$. Let $\mathrm{Frob} \in W$ be a lift of the geometric Frobenius and let τ denote the action of the element $(p \cdot \mathrm{id}, \mathrm{Frob}) \in J(\mathbb{Q}_p) \times W$ on the cohomology. We refer to τ as the "rational Frobenius". Then the action of τ on the cohomology of U_Λ is identified with the Frobenius action on the cohomology of $\mathcal{M}_\Lambda$.

Proposition. *The spectral sequence degenerates on the second page E_2. For $0 \leqslant b \leqslant 2(n-1)$, the induced filtration on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}} \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ splits, ie. we have an* *isomorphism*

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}\widehat{\otimes}\,\mathbb{C}_p,\overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{b\leqslant b'\leqslant 2(n-1)} E_2^{b-b',b'}.$$

The action of W on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}\widehat{\otimes}\,\mathbb{C}_p,\overline{\mathbb{Q}_\ell})$ is trivial on the inertia subgroup and the action of the rational Frobenius τ is semisimple. The subspace $E_2^{b-b',b'}$ is identified with the eigenspace of τ associated to the eigenvalue $(-p)^{b'}$.

Let us fix a maximal simplex $\{\Lambda_0,\dots,\Lambda_{\theta_{\max}}\}$ in $\mathrm{BT}(J,\mathbb{Q}_p)$ such that $t(\Lambda_\theta) = 2\theta+1$ for all $0 \leqslant \theta \leqslant \theta_{\max}$, and let us write J_θ instead of J_{Λ_θ}. In order to study the $J(\mathbb{Q}_p)$-action, we rewrite the terms $E_1^{a,b}$ using compactly induced representations

$$E_1^{a,b} \simeq \bigoplus_{\theta=0}^{\theta_{\max}} \mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\left(\mathrm{H}_c^b(U_{\Lambda_\theta}\widehat{\otimes}\,\mathbb{C}_p,\overline{\mathbb{Q}_\ell})\otimes\overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}]\right).$$

Here for $s\geqslant 1$ and $0\leqslant\theta\leqslant\theta_{\max}$ the finite set $K_s^{(\theta)}\subset I_s$ is given by

$$K_s^{(\theta)} := \{\gamma\in I_s \,|\, U(\gamma) = U_{\Lambda_\theta}\}.$$

It is equipped with an action of J_θ and $\overline{\mathbb{Q}_\ell}[K_s^{(\theta)}]$ is the associated permutation module. The various J_θ's are maximal parahoric subgroups of $J(\mathbb{Q}_p)$, and the representations $\mathrm{H}_c^b(U_{\Lambda_\theta}\widehat{\otimes}\,\mathbb{C}_p,\overline{\mathbb{Q}_\ell})\otimes\overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}]$ are trivial on the unipotent radical J_θ^+. In particular, they are representations of the finite group of Lie type $\mathcal{J}_\theta := J_\theta/J_\theta^+ \simeq \mathrm{G}(\mathrm{U}_{2\theta+1}(\mathbb{F}_p)\times\mathrm{U}_{n-2\theta-1}(\mathbb{F}_p))$. By exploiting this spectral sequence and the underlying combinatorics of the Bruhat-Tits building of $J(\mathbb{Q}_p)$, we are able to compute the cohomology groups of $\mathcal{M}^{\mathrm{an}}$ of highest degree $2(n-1)$, and when $n=3$ or 4 the group of degree $2(n-1)-1$ as well. We denote by J° the subgroup of $J(\mathbb{Q}_p)$ consisting of all the unitary similitudes in $J(\mathbb{Q}_p)$ whose multipliers are a unit. We note that J° is normal in $J(\mathbb{Q}_p)$ with quotient $J/J^\circ\simeq\mathbb{Z}$.

Proposition. *There is an isomorphism*

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}\widehat{\otimes}\,\mathbb{C}_p,\overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J^\circ}^{J}\mathbf{1},$$

and the rational Frobenius τ acts via multiplication by $p^{2(n-1)}$.

For λ a partition of $2\theta_{\max}+1$, we denote by ρ_λ the associated irreducible unipotent representation of $\mathrm{GU}_{2\theta_{\max}+1}(\mathbb{F}_p)$ via the classification of [LS77] which we recall in Section 2. We also write ρ_λ for its inflation to the maximal parahoric subgroup $J_{\theta_{\max}}$. In particular, if $2\theta_{\max}+1$ is equal to $\frac{t(t+1)}{2}$ for some integer $t\geqslant 1$, we write $\Delta_t := (t,t-1,\dots,1)$ for the partition of $2\theta_{\max}+1$ whose Young diagram is a staircase. The unipotent representation ρ_{Δ_t} of $\mathrm{GU}_{2\theta_{\max}+1}(\mathbb{F}_p)$ is cuspidal.

Theorem. *Assume that $\theta_{\max} = 1$, ie. $n = 3$ or 4. We have*

$$\mathrm{H}_c^{2(n-1)-1}(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J_1}^{J} \rho_{\Delta_2},$$

with the rational Frobenius τ acting via multiplication by $-p^{2(n-1)-1}$.

In general, the terms $E_2^{a,b}$ in the second page may be difficult to compute. However, the terms corresponding to $a = 0$ and $b \in \{2(n-1-\theta_{\max}), 2(n-1-\theta_{\max})+1\}$ are not touched by any non-zero differential in the alternating version of the Čech spectral sequence, making their computations accessible. We note that $2(n-1-\theta_{\max})$ is equal to the middle degree when n is odd, and to one plus the middle degree when n is even.

Proposition. *We have an isomorphism of $J(\mathbb{Q}_p)$-representations*

$$E_2^{0,2(n-1-\theta_{\max})} \simeq \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_{(2\theta_{\max}+1)}.$$

If $n \geqslant 3$ then we also have an isomorphism

$$E_2^{0,2(n-1-\theta_{\max})+1} \simeq \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_{(2\theta_{\max},1)}.$$

We note that the representation $\rho_{(2\theta_{\max}+1)}$ is the trivial representation. Using type theory, we may describe the inertial supports of the irreducible subquotients of such compactly induced representations. An inertial class is a pair $[L,\tau]$ where L is a Levi complement of $J(\mathbb{Q}_p)$ and τ is a supercuspidal representation of L, up to conjugation and twist by an unramified character. Any smooth irreducible representation π of $J(\mathbb{Q}_p)$ determines a unique inertial class $\ell(\pi)$. If $\mathfrak{s}$ is an inertial class, let $\mathrm{Rep}^{\mathfrak{s}}(J(\mathbb{Q}_p))$ be the category of smooth representations of $J(\mathbb{Q}_p)$ all of whose irreducible subquotients π satisfy $\ell(\pi) = \mathfrak{s}$. For $\mathfrak{S}$ a set of inertial classes, let $\mathrm{Rep}^{\mathfrak{S}}(J(\mathbb{Q}_p))$ be the direct product of the categories $\mathrm{Rep}^{\mathfrak{s}}(J(\mathbb{Q}_p))$ for $\mathfrak{s} \in \mathfrak{S}$. Let $(\mathbf{V}, \{\cdot,\cdot\})$ be the n-dimensional $\mathbb{Q}_{p^2}$-hermitian space whose group of unitary similitudes is $J(\mathbb{Q}_p)$. The Witt index of $\mathbf{V}$ is $\theta_{\max}$. Let

$$\mathbf{V} = H_1 \oplus \ldots \oplus H_{\theta_{\max}} \oplus \mathbf{V}^{\mathrm{an}}$$

be a Witt decomposition, where for all $1 \leqslant i \leqslant \theta_{\max}$, H_i is a hyperbolic plane and where $\mathbf{V}^{\mathrm{an}}$ is anisotropic. Note that $\mathbf{V}^{\mathrm{an}}$ has dimension 1 or 2 depending on whether n is odd or even respectively. For $0 \leqslant f \leqslant \theta_{\max}$, we consider

$$L_f := \mathrm{G}\left(\mathrm{U}(H_1 \oplus \ldots \oplus H_f \oplus \mathbf{V}^{\mathrm{an}}) \times T_{f+1} \times \ldots \times T_{\theta_{\max}}\right),$$

where for $1 \leqslant i \leqslant \theta_{\max}$, $T_i \subset \mathrm{GU}(H_i)$ is a maximal torus. Then L_f can be seen as a Levi complement in $J(\mathbb{Q}_p)$, and $L_{\theta_{\max}} = J(\mathbb{Q}_p)$. In particular L_0 is a minimal Levi complement. Let τ_0 denote the trivial representation of L_0, and let τ_1 denote the representation of L_1 obtained by letting the T_i's for $i \geqslant 2$ act trivially, and $\mathrm{GU}(H_1 \oplus \mathbf{V}^{\mathrm{an}})$ act through the compact induction of the inflation to a special maximal parahoric subgroup of the unique cuspidal unipotent representation ρ_{Δ_2} of $\mathrm{GU}_3(\mathbb{F}_p)$. For $f = 0, 1$, the irreducible representation τ_f of L_f is supercuspidal. For V a smooth representation of $J(\mathbb{Q}_p)$ and χ a continuous character of the center $\mathrm{Z}(J(\mathbb{Q}_p))$, we denote by V_χ the maximal quotient of V on which the center acts like χ. Combining our previous proposition with an analysis of the inertial supports via type theory, we obtain the following proposition.

Proposition. *Let χ be an unramified character of $\mathrm{Z}(J)$.*

- *Assume that $n \geqslant 3$. The representation $(E_2^{0,2(n-1-\theta_{\max})})_\chi$ contains no non-zero admissible subrepresentation, and is not $J(\mathbb{Q}_p)$-semisimple. Moreover, any irreducible subquotient has inertial support $[L_0, \tau_0]$. If $n \geqslant 5$, then the same statement holds for $(E_2^{0,2(n-1-\theta_{\max})+1})_\chi$ with the inertial support being $[L_1, \tau_1]$.*
- *For $n = 1, 2, 3, 4$, let $b = 0, 2, 3, 5$ respectively. We have $\theta_{\max} = 0$ if $n = 1, 2$ and $\theta_{\max} = 1$ if $n = 3, 4$. Let χ be an unramified character of $\mathrm{Z}(J(\mathbb{Q}_p))$. The twist $\tau_{\theta_{\max},\chi}$ of $\tau_{\theta_{\max}}$ by χ is an irreducible supercuspidal representation of $J(\mathbb{Q}_p)$, and we have*

$$(E_2^{0,b})_\chi \simeq \begin{cases} \tau_{\theta_{\max},\chi} & \text{if } n = 1, 3, 4, \\ \tau_{\theta_{\max},\chi} \oplus \chi_0 \tau_{m,\chi} & \text{if } n = 2. \end{cases}$$

Here, when $n = 2$ the subgroup $\mathrm{Z}(J(\mathbb{Q}_p))J_0$ has index 2 in $J(\mathbb{Q}_p)$. In this situation, χ_0 denotes the unique non-trivial character of $J(\mathbb{Q}_p)$ which is trivial on $\mathrm{Z}(J)J_0$. This proposition yields the following important corollary.

Corollary. *Let χ be an unramified character of $\mathrm{Z}(J(\mathbb{Q}_p))$. If $n \geqslant 3$ then $\mathrm{H}_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell})_\chi$ is not $J(\mathbb{Q}_p)$-admissible. If $n \geqslant 5$ then the same holds for $\mathrm{H}_c^{2(n-1-\theta_{\max})+1}(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell})_\chi$.*

Thus the cohomology of Rapoport-Zink spaces need not be admissible nor $J(\mathbb{Q}_p)$-semisimple in general. Lastly, we introduce the unramified unitary PEL Shimura variety of signature $(1, n-1)$ with no level structure at p. It is defined over a quadratic extension F of $\mathbb{Q}$ in which the prime p is inert. The corresponding Shimura datum gives rise to a reductive group $\mathbb{G}$ over $\mathbb{Q}$ such that $\mathbb{G}_{\mathbb{Q}_p} = G$ and $\mathbb{G}(\mathbb{R}) \simeq \mathrm{GU}(1, n-1)$. The Shimura varieties are indexed by the open compact subgroups $K^p \subset \mathbb{G}(\mathbb{A}_f^p)$ which are small enough. Kottwitz constructed integral models S_{K^p} at p of these Shimura varieties. Their special fibers are stratified by the Newton strata, and the unique closed stratum is called the supersingular locus, which we denote by $\overline{\mathrm{S}}_{K^p}^{\mathrm{ss}}$ since it coincides with the supersingular locus. It has dimension $\theta_{\max}$. The p-adic uniformization theorem of [RZ96] gives a geometric identity between the special fiber $\mathcal{M}_{\mathrm{red}}$ of the Rapoport-Zink space $\mathcal{M}$ and the supersingular locus $\overline{\mathrm{S}}^{\mathrm{ss}}_{K^p}$. In [Far04], Fargues constructed a Hochschild-Serre spectral sequence associated to this geometric identity, computing the cohomology of the supersingular locus.

Let ξ be an irreducible algebraic finite dimensional representation of $\mathbb{G}$, and let $\overline{\mathcal{L}_\xi}$ be the associated local system on the Shimura variety, restricted to the special fiber. It is a pure sheaf of some weight $w(\xi) \in \mathbb{Z}_{\geqslant 0}$. Let I be the inner form of $\mathbb{G}$ such that $I_{\mathbb{Q}_p} = J$, $I_{\mathbb{A}_f^p} = \mathbb{G}_{\mathbb{A}_f^p}$ and $I(\mathbb{R}) \simeq \mathrm{GU}(0,n)$. We denote by $\mathcal{A}_\xi(I)$ the set of automorphic representations of I of type $\check{\xi}$ at infinity, and counted with multiplicities. Fargues' spectral sequence is given in the second page by

$$F_2^{a,b} = \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^a_J \left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}} \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p \implies \mathrm{H}^{a+b}(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \overline{\mathcal{L}_\xi}),$$

where $\mathrm{H}^\bullet(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \overline{\mathcal{L}_\xi}) := \varinjlim_{K^p} \mathrm{H}^\bullet(\overline{\mathrm{S}}^{\mathrm{ss}}_{K^p} \otimes \overline{\mathbb{F}_p}, \overline{\mathcal{L}_\xi})$. We point out that the abutment is just the cohomology of the supersingular locus with coefficients in $\overline{\mathcal{L}_\xi}$ because the nearby cycles are trivial thanks to the smoothness of the integral model S_{K^p}. It is $\mathbb{G}(\mathbb{A}_f^p) \times W$-equivariant. When $n = 3$ or 4 this sequence degenerates on the second page, and our knowledge on the cohomology of the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}}$ allows us to compute every term. We obtain a description of the cohomology of the supersingular locus in terms of automorphic representations.

A smooth character of $J(\mathbb{Q}_p)$ is said to be unramified if it is trivial on all compact subgroups of $J(\mathbb{Q}_p)$. Let $X^{\mathrm{un}}(J(\mathbb{Q}_p))$ denote the set of unramified characters of $J(\mathbb{Q}_p)$. Let St_J denote the Steinberg representation of $J(\mathbb{Q}_p)$. If $\Pi \in \mathcal{A}_\xi(I)$, we define $\delta_{\Pi_p} := \omega_{\Pi_p}(p^{-1} \cdot \mathrm{id})p^{-w(\xi)} \in \overline{\mathbb{Q}_\ell}^\times$ where ω_{Π_p} is the central character of Π_p, and $p^{-1} \cdot \mathrm{id}$ lies in the center of $J(\mathbb{Q}_p)$. For any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$ we have $|\iota(\delta_{\Pi_p})| = 1$. Eventually, if $x \in \overline{\mathbb{Q}_\ell}^\times$, we denote by $\overline{\mathbb{Q}_\ell}[x]$ the 1-dimensional representation of the Weil group W where the inertia acts trivially and Frob acts like multiplication by the scalar x.

Theorem. *Assume that $n = 3$ or 4, so that $\overline{\mathrm{S}}^{\mathrm{ss}}$ is one dimensional. There are $\mathbb{G}(\mathbb{A}_f^p) \times W$-equivariant isomorphisms*

$$\mathrm{H}^0(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

$$\mathrm{H}^1(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \mathrm{St}_J}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}],$$

$$\mathrm{H}^2(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

Notations: Throughout the paper, we fix an integer $n \geqslant 1$ and we write $\theta_{\max} := \lfloor \frac{n-1}{2} \rfloor$ so that $n = 2\theta_{\max} + 1$ or $2(\theta_{\max} + 1)$ according to whether n is odd or even. We also fix an odd prime number p. If k is a perfect field of characteristic p, we denote by $W(k)$ the ring of Witt vectors and by $W(k)_{\mathbb{Q}}$ its fraction field, which is an unramified extension of $\mathbb{Q}_p$. We denote by $\sigma : x \mapsto x^p$ the Frobenius on k or its lift to $W(k)$. If $q = p^e$ is a power of p, we write $\mathbb{F}_q$ for the field with q elements. In the special case where $q = p^2$, we also use the alternative notation $\mathbb{Z}_{p^2} = W(\mathbb{F}_{p^2})$ and $\mathbb{Q}_{p^2} = W(\mathbb{F}_{p^2})_{\mathbb{Q}}$. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. For $k \geqslant 1$, the $k \times k$ identity matrix is denoted by I_k, and the matrix with 1 in the antidiagonal and 0 everywhere else is denoted by A_k. In various situations, the symbol $\mathbf{1}$ will always represent the trivial representation of the group we are considering. The symmetric group of $\{1, \ldots, k\}$ is denoted $\mathfrak{S}_k$.

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1 The Bruhat-Tits stratification on the PEL unitary Rapoport-Zink space of signature $(1, n-1)$

1.1 The PEL unitary Rapoport-Zink space $\mathcal{M}$ of signature $(1, n-1)$

In [VW11], the authors introduce the PEL unitary Rapoport-Zink space $\mathcal{M}$ of signature $(1, n-1)$ as a moduli space, classifying the deformations of a given p-divisible group equipped with additional structures. We briefly recall the construction. Let E be a quadratic unramified extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$ and with nontrivial Galois involution $a \mapsto a^*$. Let $\varphi_0 : K \xrightarrow{\sim} \mathbb{Q}_{p^2}$ be a $\mathbb{Q}_p$-linear isomorphism and let $\varphi_1 := \sigma \circ \varphi_0$. Let Nilp denote the category of schemes over $\mathbb{Z}_{p^2}$ where p is locally nilpotent. For $S \in \mathrm{Nilp}$, a **unitary p-divisible group of signature** $(1, n-1)$ over S is a triple (X, ι_X, λ_X) where

- X is a p-divisible group over S.
- $\iota_X : \mathcal{O}_E \to \mathrm{End}(X)$ is a $\mathcal{O}_E$-action on X such that the induced action on its Lie algebra satisfies the **signature** $(1, n-1)$ **condition**: for every $a \in \mathcal{O}_E$, the characteristic polynomial of $\iota_X(a)$ acting on $\mathrm{Lie}(X)$ is given by

$$(T - \varphi_0(a))^1 (T - \varphi_1(a))^{n-1} \in \mathbb{Z}_{p^2}[T] \subset \mathcal{O}_S[T].$$

– $\lambda_X : X \xrightarrow{\sim} {}^tX$ is an $\mathcal{O}_E$-linear polarization where ${}^tX$ denotes the Serre dual of X.

The $\mathcal{O}_E$-linearity of λ_X is with respect to the $\mathcal{O}_E$-actions ι_X and the induced action $\iota_{{}^tX}$ on the dual. A specific example of unitary p-divisible group over $\mathbb{F}_{p^2}$ is given in [VW11] 2.4 by means of covariant Dieudonné theory. We denote it by $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and call it the **standard unitary p-divisible group**. The p-divisible group $\mathbb{X}$ is superspecial. The following set-valued functor $\mathcal{M}$ defines a moduli problem classifying deformations of $\mathbb{X}$ by quasi-isogenies. More precisely, for $S \in \mathrm{Nilp}$ the set $\mathcal{M}(S)$ consists of all isomorphism classes of tuples $(X, \iota_X, \lambda_X, \rho_X)$ such that

– (X, λ_X, ρ_X) is a unitary p-divisible group of signature $(1, n-1)$ over S.

– $\rho_X : X \times_S \overline{S} \to \mathbb{X} \times_{\mathbb{F}_{p^2}} \overline{S}$ is an $\mathcal{O}_E$-linear quasi-isogeny compatible with the polarizations, in the sense that ${}^t\rho_X \circ \lambda_{\mathbb{X}} \circ \rho_X$ is a $\mathbb{Q}_p^\times$-multiple of λ_X.

In the second condition, $\overline{S}$ denotes the special fiber of S. By [RZ96] Corollary 3.40, this moduli problem is represented by a separated formal scheme $\mathcal{M}$ over $\mathrm{Spf}(\mathbb{Z}_{p^2})$, called a **Rapoport-Zink space**. It is formally locally of finite type, and since the associated PEL datum is unramified it is also formally smooth over $\mathbb{Z}_{p^2}$. The **reduced special fiber** of $\mathcal{M}$ is the reduced $\mathbb{F}_{p^2}$-scheme $\mathcal{M}_{\mathrm{red}}$ defined by the maximal ideal of definition. Rational points of $\mathcal{M}$ over a perfect field extension k of $\mathbb{F}_{p^2}$ can be understood in terms of semi-linear algebra by means of Dieudonné theory. We denote by $M(\mathbb{X})$ the (covariant) Dieudonné module of $\mathbb{X}$, this is a free $\mathbb{Z}_{p^2}$-module of rank $2n$. We denote by $N(\mathbb{X}) := M(\mathbb{X}) \otimes \mathbb{Q}_{p^2}$ its isocrystal. By construction, the Frobenius $\mathbf{F}$ and the Verschiebung $\mathbf{V}$ agree on $N(\mathbb{X})$. In particular, we have $\mathbf{F}^2 = p \cdot \mathrm{id}$ on the isocrystal. The $\mathcal{O}_E$-action $\iota_{\mathbb{X}}$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading $M(\mathbb{X}) = M(\mathbb{X})_0 \oplus M(\mathbb{X})_1$ as a sum of two free $\mathbb{Z}_{p^2}$-modules of rank n, such that $a \in \mathcal{O}_E$ acts via $\varphi_i(a)$ on $M(\mathbb{X})_i$ for $i = 0, 1$. The same goes for the isocrystal $N(\mathbb{X}) = N(\mathbb{X})_0 \oplus N(\mathbb{X})_1$ where $N(\mathbb{X})_i = M(\mathbb{X})_i \otimes \mathbb{Q}_{p^2}$ for $i = 0, 1$. The polarization $\lambda_{\mathbb{X}}$ induces a perfect alternating $\mathbb{Q}_{p^2}$-bilinear pairing $\langle \cdot, \cdot \rangle$ on $N(\mathbb{X})$ such that

$$\forall x, y \in N(\mathbb{X}), \forall a \in E, \qquad \langle \mathbf{F}x, y \rangle = \langle x, \mathbf{V}y \rangle^\sigma \text{ and } \langle ax, y \rangle = \langle x, a^* y \rangle.$$

Moreover $\langle \cdot, \cdot \rangle$ restricts to a perfect $\mathbb{Z}_{p^2}$-pairing on the lattice $M(\mathbb{X})$. The pieces $N(\mathbb{X})_i$ are totally isotropic for $i = 0, 1$ and dual of each other. Moreover, the Frobenius $\mathbf{F}$ is 1-homogeneous with respect to this grading. As in [VW11] 2.6, we define

$$\forall x, y \in N(\mathbb{X})_0, \qquad \{x, y\} := \delta \langle x, \mathbf{F}y \rangle,$$

where $\delta \in \mathbb{Z}_{p^2}^\times$ is a scalar satisfying $\delta^\sigma = -\delta$. The pairing $\{\cdot, \cdot\}$ is a perfect σ-hermitian form on $N(\mathbb{X})_0$.

Notation. From now on, we will write $\mathbf{V} := N(\mathbb{X})_0$ and $\mathbf{M} := M(\mathbb{X})_0$.

Then $\mathbf{V}$ is a $\mathbb{Q}_{p^2}$-hermitian space of dimension n, and $\mathbf{M}$ is a given $\mathbb{Z}_{p^2}$-lattice, ie. a finitely generated $\mathbb{Z}_{p^2}$-submodule containing a basis of $\mathbf{V}$. Given two lattices M_1 and M_2, the notation $M_1 \overset{d}{\subset} M_2$ means that $M_1 \subset M_2$ and the quotient module M_2/M_1 has length d. The integer d is called the **index** of M_1 in M_2, and is denoted $d = [M_2 : M_1]$. Given a lattice $M \subset \mathbf{V}$, we define the dual lattice $M^\vee := \{v \in \mathbf{V} \mid \{v, M\} \subset \mathbb{Z}_{p^2}\}$.By construction the lattice $\mathbf{M}$ satisfies

$$p\mathbf{M}^\vee \overset{1}{\subset} \mathbf{M} \overset{n-1}{\subset} \mathbf{M}^\vee.$$

Consider the matrices

$$T_{\text{odd}} := A_{2\theta_{\max}+1}, \qquad T_{\text{even}} := \begin{pmatrix} & & & A_{\theta_{\max}} \\ & 1 & 0 & \\ & 0 & p & \\ A_{\theta_{\max}} & & & \end{pmatrix}.$$

By [Vol10] Proposition 1.15, there exists a basis of $\mathbf{V}$ such that $\{\cdot,\cdot\}$ is represented by the matrix T_{odd} is n is odd and by T_{even} if n is even. A **Witt decomposition** on $\mathbf{V}$ is a set $\{L_i\}_{i\in I}$ of isotropic lines in $\mathbf{V}$ such that the following conditions are satisfied:

- for every $i \in I$, there is a unique $i' \in I$ such that $\{L_i, L_{i'}\} \neq 0$,
- the sum of the L_i's is direct,
- the orthogonal of the direct sum of the L_i's is an anisotropic subspace of $\mathbf{V}$.

Since each line L_i is isotropic, in the first condition one necessarily has $(i')' = i$ and $i \neq i'$. As a consequence, we have $\#I = 2w(\mathbf{V})$ for some integer $w = w(\mathbf{V}$ called the **Witt index** of $\mathbf{V}$. It does not depend on the choice of a Witt decomposition. We write L^{an} for the orthogonal of the direct sum of the L_i's. The dimension of L^{an} is $n^{\text{an}} := n - 2w$. Given a Witt decomposition of $\mathbf{V}$, one may find vectors $e_i \in L_i$ such that $\{e_i, e_j\} = \delta_{j,i'}$. Together with a choice of an orthogonal basis for L^{an}, these vectors define a basis of $\mathbf{V}$ which is said to be adapted to the Witt decomposition. For any $i \in I$, the direct sum $L_i \oplus L_{i'}$ is isometric to the hyperbolic plane $\mathbf{H}$. Therefore, we obtain a decomposition

$$\mathbf{V} = w\mathbf{H} \oplus L^{\text{an}}.$$

We may always rearrange the index set so that $I = \{-w, \dots, -1, 1, \dots, w\}$ and $i' = -i$ for all $i \in I$. In this context, we write L_0 instead of L^{an}.
We fix once and for all a basis e of $\mathbf{V}$ in which the hermitian form is represented by the matrix T_{odd} or T_{even}. In the case $n = 2\theta_{\max} + 1$ is odd, we will denote it

$$e = (e_{-\theta_{\max}}, \dots, e_{-1}, e_0^{\text{an}}, e_1, \dots, e_{\theta_{\max}}),$$

and in the case $n = 2(\theta_{\max} + 1)$ is even we will denote it

$$e = (e_{-\theta_{\max}}, \dots, e_{-1}, e_0^{\mathrm{an}}, e_1^{\mathrm{an}}, e_1, \dots, e_{\theta_{\max}}).$$

The choice of such a basis gives a Witt decomposition with $L_i := \mathbb{Q}_{p^2} e_i$ and L_0 the subspace generated by e_0^{an}, and when n is even by e_1^{an} as well. In particular, $w(\mathbf{V}) = \theta_{\max}$ and $n^{\mathrm{an}} = 1$ or 2 depending on whether n is odd or even respectively.

Given a perfect field extension k of $\mathbb{F}_{p^2}$, we denote by $\mathbf{V}_k$ the base change $\mathbf{V} \otimes_{\mathbb{Q}_{p^2}} W(k)_{\mathbb{Q}}$. The form may be extended to $\mathbf{V}_k$ by the formula

$$\{v \otimes x, w \otimes y\} := xy^{\sigma}\{v, w\} \in W(k)_{\mathbb{Q}}$$

for all $v, w \in \mathbf{V}$ and $x, y \in W(k)_{\mathbb{Q}}$. The notions of index and duality for $W(k)$-lattices can be extended as well. By [Vol10] Proposition 1.10, the rational points of the Rapoport-Zink space are described by the following statement.

Proposition 1.1. *Let k be a perfect field extension of $\mathbb{F}_{p^2}$. There is a natural bijection between $\mathcal{M}(k)$ and the set of $W(k)$-lattices M in $\mathbf{V}_k$ such that for some integer $i \in \mathbb{Z}$, we have*

$$p^{i+1} M^{\vee} \overset{1}{\subset} M \overset{n-1}{\subset} p^i M^{\vee}.$$

There is a decomposition $\mathcal{M} = \bigsqcup_{i \in \mathbb{Z}} \mathcal{M}_i$ into formal connected subschemes which are open and closed. The rational points of $\mathcal{M}_i$ are those lattices M satisfying the relation above with the given integer i. In particular, the lattice $\mathbf{M}$ defined in the previous paragraph is an element of $\mathcal{M}_0(\mathbb{F}_{p^2})$. By [Vol10] Proposition 1.7, the formal scheme $\mathcal{M}_i$ is empty if ni is odd.
Let $J = \mathrm{GU}(\mathbf{V})$ be the group of unitary similitudes of $\mathbf{V}$, seen as a reductive group over $\mathbb{Q}_p$. Then $J(\mathbb{Q}_p)$ consists of all $g \in \mathrm{GL}_{\mathbb{Q}_{p^2}}(\mathbf{V})$ which preserve the hermitian form up to a unit $c(g) \in \mathbb{Q}_p^{\times}$, called the **multiplier**. By Dieudonné theory, the group $J(\mathbb{Q}_p)$ is also identified with the group of quasi-isogenies $\mathbb{X} \to \mathbb{X}$ of unitary p-divisible groups. The space $\mathcal{M}$ is endowed with a natural action of $J(\mathbb{Q}_p)$. At the level of points, the element g acts by sending a lattice M to $g(M)$. For $g \in J(\mathbb{Q}_p)$, let $\alpha(g)$ be the p-adic valuation of $c(g)$. This defines a continous homomorphism

$$\alpha : J \to \mathbb{Z}$$

where $\mathbb{Z}$ is given the discrete topology. Then g induces an isomorphism $\mathcal{M}_i \xrightarrow{\sim} \mathcal{M}_{i+\alpha(g)}$. According to [Vol10] 1.17 the image of α is $\mathbb{Z}$ if n is even, and $2\mathbb{Z}$ if n is odd. The center $\mathrm{Z}(J(\mathbb{Q}_p))$ consists of all the scalar matrices, so that it is identified with $\mathbb{Q}_{p^2}^{\times}$. If $\lambda \in \mathbb{Q}_{p^2}^{\times}$, then $c(\lambda \cdot \mathrm{id}) = \mathrm{Norm}(\lambda) \in \mathbb{Q}_p^{\times}$, where Norm is the norm map relative to the quadratic extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$. In particular, $\alpha(\mathrm{Z}(J)) = 2\mathbb{Z}$. Thus, the restriction of α to the center is surjective onto $\mathrm{Im}(\alpha)$ only when n is odd. When n is even, we define the following element

$$g_0 := \begin{pmatrix} & & & I_{\theta_{\max}} \\ & 0 & p & \\ & 1 & 0 & \\ pI_{\theta_{\max}} & & & \end{pmatrix}.$$

Then $g_0 \in J(\mathbb{Q}_p)$ and $c(g_0) = p$ so that $\alpha(g_0) = 1$. Moreover $g_0^2 = p \cdot \mathrm{id}$ belongs to $\mathrm{Z}(J(\mathbb{Q}_p))$. Let i and i' be two integers such that ni and ni' are even. We consider the multiplication $p^{\frac{i'-i}{2}} : \mathbb{X} \to \mathbb{X}$ when $i \equiv i' \mod 2$, and the quasi-isogeny $p^{\frac{i'-i-1}{2}} g_0 : \mathbb{X} \to \mathbb{X}$ when $i \not\equiv i' \mod 2$. This is well defined as the second case may only happen when n is even. It induces a morphism $\psi_{i,i'} : \mathcal{M}_i \to \mathcal{M}_{i'}$. By [Vol10] Proposition 1.18, the map $\psi_{i,i'}$ is an isomorphism between $\mathcal{M}_i$ and $\mathcal{M}_{i'}$, and if i, i' and i'' are three integers such that ni, ni' and ni'' are even, then we have $\psi_{i',i''} \circ \psi_{i,i'} = \psi_{i,i''}$.

1.2 The Bruhat-Tits stratification of the special fiber $\mathcal{M}_{\mathrm{red}}$

We now recall the construction of the Bruhat-Tits stratification on $\mathcal{M}_{\mathrm{red}}$ as in [VW11]. Let i be an integer such that ni is even. We define

$$\mathcal{L}_i := \{\Lambda \subset \mathbf{V} \text{ a } \mathbb{Z}_{p^2}-\text{lattice} \,|\, p^{i+1}\Lambda^\vee \subsetneq \Lambda \subset p^i\Lambda^\vee\}.$$

If $\Lambda \in \mathcal{L}_i$, we define its **orbit type** $t(\Lambda) := [\Lambda : p^{i+1}\Lambda^\vee]$. We also call it the type of Λ. In particular, the lattices in $\mathcal{L}_i$ of type 1 are precisely the $\mathbb{F}_{p^2}$-rational points of $\mathcal{M}_i$. By sending Λ to $g(\Lambda)$, an element $g \in J(\mathbb{Q}_p)$ defines a map $\mathcal{L}_i \to \mathcal{L}_{i+\alpha(g)}$. The following Proposition follows from [Vol10] Remark 2.3 and [VW11] Remark 4.1.

Proposition 1.2. *Let i be an integer such that ni is even and let $\Lambda \in \mathcal{L}_i$.*
- *The map $\mathcal{L}_i \to \mathcal{L}_{i+\alpha(g)}$ induced by an element $g \in J(\mathbb{Q}_p)$ is an inclusion preserving, type preserving bijection.*
- *We have $1 \leqslant t(\Lambda) \leqslant n$. Furthermore $t(\Lambda)$ is odd.*
- *The sets $\mathcal{L}_i$'s for various i's are pairwise disjoint.*

Moreover, two lattices $\Lambda, \Lambda' \in \bigsqcup_{ni\in 2\mathbb{Z}} \mathcal{L}_i$ are in the same orbit under the action of $J(\mathbb{Q}_p)$ if and only if $t(\Lambda) = t(\Lambda')$.

We write $\mathcal{L} := \bigsqcup_{ni\in 2\mathbb{Z}} \mathcal{L}_i$. For any odd number t between 1 and n, there exists a lattice $\Lambda \in \mathcal{L}_0$ of orbit type t. Write $t_{\max} := 2\theta_{\max} + 1$, so that the orbit type t of any lattice in $\mathcal{L}$ satisfies $1 \leqslant t \leqslant t_{\max}$. The following lemma will be useful later.

Lemma 1.3. *Let $i \in \mathbb{Z}$ such that ni is even, and let $\Lambda \in \mathcal{L}_i$. We have $\Lambda^\vee \in \mathcal{L}$ if and only if either n is even, either n is odd and $t(\Lambda) = t_{\max}$. If $\Lambda^\vee \in \mathcal{L}$ and n is even, then $\Lambda^\vee \in \mathcal{L}_{-i-1}$ and $t(\Lambda^\vee) = n - t(\Lambda)$. If on the contrary n is odd, then $\Lambda^\vee \in \mathcal{L}_{-i}$ and $t(\Lambda^\vee) = t(\Lambda)$.*

Proof. First we prove the converse. We have the following chain of inclusions

$$p^{-i}\Lambda \overset{n-t(\Lambda)}{\subset} \Lambda^\vee \overset{t(\Lambda)}{\subset} p^{-i-1}\Lambda.$$

If n is even, then $-n(i+1)$ is also even and $n - t(\Lambda) \neq 0$. Since $(\Lambda^\vee)^\vee = \Lambda$, we deduce that $\Lambda^\vee \in \mathcal{L}_{-i-1}$ with orbit type $n - t(\Lambda)$. Assume now that n is odd and that $t(\Lambda) = t_{\max} = n$. Then $\Lambda^\vee = p^{-i}\Lambda \in \mathcal{L}_{-i}$.
Let us now assume that $\Lambda^\vee \in \mathcal{L}$ and that n is odd. Let $i' \in 2\mathbb{Z}$ such that $\Lambda^\vee \in \mathcal{L}_{i'}$. We have

$$\Lambda^\vee \overset{n-t(\Lambda^\vee)}{\subset} p^{i'}\Lambda \overset{n-t(\Lambda)}{\subset} p^{i'+i}\Lambda^\vee, \qquad \Lambda^\vee \overset{t(\Lambda)}{\subset} p^{-i-1}\Lambda \overset{t(\Lambda^\vee)}{\subset} p^{-i-i'-2}\Lambda^\vee,$$

therefore $-2 \leqslant i + i' \leqslant 0$. Since $i + i'$ is even it is either -2 or 0. If it were -2, then we would have $t(\Lambda) = t(\Lambda^\vee) = 0$ which is absurd. Therefore $i + i' = 0$, and we have $n - t(\Lambda) = n - t(\Lambda^\vee) = 0$. □

With the help of $\mathcal{L}_i$, one may construct an abstract simplicial complex $\mathcal{B}_i$. For $s \geqslant 0$, an s-simplex of $\mathcal{B}_i$ is a subset $S \subset \mathcal{L}_i$ of cardinality $s+1$ such that for some ordering $\Lambda_0, \ldots, \Lambda_s$ of its elements, we have a chain of inclusions $p^{i+1}\Lambda_s^\vee \subsetneq \Lambda_0 \subsetneq \Lambda_1 \subsetneq \ldots \subsetneq \Lambda_s$. We must have $0 \leqslant s \leqslant m$ for such a simplex to exist. Let $\tilde{J} = \mathrm{SU}(\mathbf{V})$ be the derived group of J. We consider the abstract simplicial complex $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ of the Bruhat-Tits building of $\tilde{J}$ over $\mathbb{Q}_p$. A concrete description of this complex is given in [Vol10] Theorem 3.5.

Theorem 1.4. *The Bruhat-Tits building $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ is naturally identified with $\mathcal{B}_i$ for any fixed integer i such that ni is even. The set $\mathcal{L}_i$ is identified with the set of vertices of $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$. The identification is $\tilde{J}(\mathbb{Q}_p)$-equivariant.*

Apartments in the Bruhat-Tits building $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ are in 1 to 1 correspondence with Witt decompositions of $\mathbf{V}$. Let $L = \{L_i\}_{i \in I}$ be a Witt decomposition of $\mathbf{V}$ and let $f = (f_i)_{i \in I} \sqcup B^{\mathrm{an}}$ be a basis of $\mathbf{V}$ adapted to the decomposition, where $f_i \in L_i$ and B^{an} is an orthogonal basis of L^{an}. Under the identification of $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ with $\mathcal{B}_i$, the vertices inside the apartment associated to L correspond to the lattices $\Lambda \in \mathcal{L}_i$ which are equal to the direct sum of $\Lambda \cap L^{\mathrm{an}}$ and of the modules $p^{r_i}\mathbb{Z}_{p^2} f_i$ for some integers $(r_i)_{i \in I}$. The subset of $\mathcal{L}_i$ consisting of all such lattices will be denoted $\mathcal{A}_i^L$ or, with an abuse of notations, $\mathcal{A}_i^f$. We call such a set $\mathcal{A}_i^L$ the **apartment associated to L in $\mathcal{L}_i$.** We also define $\mathcal{A}^L := \bigsqcup_{ni \in 2\mathbb{Z}} \mathcal{A}_i^L$. We recall a general result regarding Bruhat-Tits buildings.

Proposition 1.5. *Let i be an integer such that ni is even. Any two lattices Λ and Λ' in $\mathcal{L}_i$ lie inside a common apartment $\mathcal{A}_i^L$ for some Witt decomposition L. Moreover, the action of $\tilde{J}(\mathbb{Q}_p)$ acts transitively on the set of apartments $\{\mathcal{A}_i^L\}_L$.*

Recall the basis e of $\mathbf{V}$ that we have fixed earlier. We will denote by

$$\Lambda(r_{-\theta_{\max}}, \dots, r_{-1}, s, r_1, \dots, r_{\theta_{\max}})$$

the $\mathbb{Z}_{p^2}$-lattice generated by the vectors $p^{r_j} e_j$ for all $j = \pm 1, \dots, \pm\theta_{\max}$, by $p^{s_0} e_0^{\mathrm{an}}$ and if n is even, by $p^{s_1} e_1^{\mathrm{an}}$ too. Here, the r_j's are integers and s denotes either the integer s_0 if n is odd or the pair of integers (s_0, s_1) if n is even.

Proposition 1.6. *Let i be an integer such that ni is even. Let (r_j, s) be a family of integers as above. The corresponding lattice $\Lambda = \Lambda(r_{-\theta_{\max}}, \dots, r_{-1}, s, r_1, \dots, r_{\theta_{\max}})$ belongs to $\mathcal{L}_i$ if and only if the following conditions are satisfied*
- *for all $1 \leqslant j \leqslant \theta_{\max}$, we have $r_{-j} + r_j \in \{i, i+1\}$,*
- *$s_0 = \lfloor \frac{i+1}{2} \rfloor$,*
- *if n is even, then $s_1 = \lfloor \frac{i}{2} \rfloor$.*

Moreover, when $\Lambda \in \mathcal{L}_i$ then

$$t(\Lambda) = 1 + 2\#\{1 \leqslant j \leqslant \theta_{\max} \,|\, r_{-j} + r_j = i\}.$$

Proof. The lattice Λ belongs to $\mathcal{L}_i$ if and only if the following chain of inclusions holds

$$p^{i+1}\Lambda^\vee \subsetneq \Lambda \subset p^i \Lambda^\vee.$$

The dual lattice $\Lambda^\vee$ is equal to the lattice $\Lambda(-r_{\theta_{\max}}, \dots, -r_1, s', -r_{-1}, \dots, -r_{-\theta_{\max}})$, where $s' = -s_0$ when n is odd, and $s' = (-s_0, -s_1 - 1)$ when n is even. Thus, the inclusions above are equivalent to the following inequalities:

$$i - r_{-j} \leqslant r_j \leqslant i + 1 - r_{-j}, \qquad i - s_0 \leqslant s_0 \leqslant i + 1 - s_0,$$
$$i - 1 - s_1 \leqslant s_1 \leqslant i - s_1 \text{ (if } n \text{ is even)}.$$

This proves the desired condition on the integers r_j's and on s. Let us now assume that $\Lambda \in \mathcal{L}_i$. Its orbit type is equal to the index $[\Lambda, p^{i+1}\Lambda^\vee]$. This corresponds to the number of times equality occurs with the left-hand side in all the inequalities above. Of course, if the equality $i - r_{-j} = r_j$ occurs for some j, then it occurs also for $-j$. Moreover, if i is even then the equality $i - s_0 = s_0$ occurs whereas $i - 1 - s_1 \neq s_1$. On the contrary if i is odd, then the equality $i - 1 - s_1 = s_1$ occurs whereas $i - s_0 \neq s_0$. Thus in all cases, only one of s_0 and s_1 contributes to the index. Putting things together, we deduce the desired formula. □

We deduce the following corollary.

Corollary 1.7. *The apartment A_i^e (resp. A^e) consists of all the lattices of the form*

$$\Lambda = \Lambda(r_{-\theta_{\max}}, \ldots, r_{-1}, s, r_1, \ldots, r_{\theta_{\max}})$$

which belong to $\mathcal{L}_i$ (resp. to $\mathcal{L}$).

Proof. According to the previous proposition, it is clear that all lattices which belong to $\mathcal{L}_i$ and are of the form $\Lambda(r_{-\theta_{\max}}, \ldots, r_{-1}, s, r_1, \ldots, r_{\theta_{\max}})$ are elements of $\mathcal{A}_i^e$. We shall prove the converse. Let $\Lambda \in \mathcal{A}_i^e$. By definition, there exists integers $(r_j)_j$ such that

$$\Lambda = \Lambda \cap \mathbf{V}^{\mathrm{an}} \oplus \bigoplus_{1 \leqslant j \leqslant \theta_{\max}} (p^{r_{-j}} \mathbb{Z}_{p^2} e_{-j} \oplus p^{r_j} \mathbb{Z}_{p^2} e_j) \,.$$

Write $\Lambda' = \Lambda \cap \mathbf{V}^{\mathrm{an}}$. This is a lattice in $\mathbf{V}^{\mathrm{an}}$ which satisfies the chain of inclusions

$$p^{i+1} \Lambda'^{\vee} \subset \Lambda' \subset p^i \Lambda'^{\vee},$$

where the duals are taken with respect to the restriction of $\{\cdot,\cdot\}$ to $\mathbf{V}^{\mathrm{an}}$. Since $\mathbf{V}^{\mathrm{an}}$ is anisotropic, there is only a single lattice satisfying the chain of inclusions above. If we write $a := \lfloor \frac{i+1}{2} \rfloor$ and $b := \lfloor \frac{i}{2} \rfloor$, it is given by $p^a \mathbb{Z}_{p^2} e_0^{\mathrm{an}}$ if n is odd, and by $p^a \mathbb{Z}_{p^2} e_0^{\mathrm{an}} \oplus p^b \mathbb{Z}_{p^2} e_1^{\mathrm{an}}$ if n is even. Thus, it must be equal to Λ' and it concludes the proof. ☐

We fix a maximal simplex in $\mathcal{L}_0$ lying inside the apartment $\mathcal{A}_0^e$. For $0 \leqslant \theta \leqslant \theta_{\max}$ we define

$$\Lambda_\theta := \Lambda(\underbrace{0, \ldots, 0}_{\theta_{\max}}, 0, \underbrace{0, \ldots, 0}_{\theta}, \underbrace{1, \ldots, 1}_{\theta_{\max} - \theta}).$$

Here, the 0 in the middle stands for $(0,0)$ in case n is even. We have $t(\Lambda_\theta) = 2\theta + 1$ and

$$p\Lambda_0^{\vee} \subsetneq \Lambda_0 \subset \ldots \subset \Lambda_{\theta_{\max}}.$$

Thus, they form an $\theta_{\max}$-simplex in $\mathcal{L}_0$. Given a lattice $\Lambda \in \mathcal{L}_i$, a subfunctor $\mathcal{M}_\Lambda$ of $\mathcal{M}_{i,\mathrm{red}}$ is defined in [VW11], classifying those p-divisible groups for which a certain quasi-isogeny, depending on Λ, is in fact an actual isogeny. In Lemma 4.2, the authors prove that it is representable by a projective scheme over $\mathbb{F}_{p^2}$, and that the natural morphism $\mathcal{M}_\Lambda \hookrightarrow \mathcal{M}_{i,\mathrm{red}}$ is a closed immersion. The schemes $\mathcal{M}_\Lambda$ are called the **closed Bruhat-Tits strata of $\mathcal{M}$**. Their rational points are described as follows, see Lemma 4.3 of loc. cit.

Proposition 1.8. *Let k be a perfect field extension of $\mathbb{F}_{p^2}$, and let $M \in \mathcal{M}_{i,\mathrm{red}}(k)$. Then*

$$M \in \mathcal{M}_\Lambda(k) \iff M \subset \Lambda_k := \Lambda \otimes_{\mathbb{Z}_{p^2}} W(k).$$

The set of lattices satisfying the condition above was conjectured in [Vol10] to be the set of points of a subscheme of $\mathcal{M}_{i,\mathrm{red}}$, and it was proved in the special cases $n = 2, 3$. In [VW11], the general argument is given by the construction of $\mathcal{M}_\Lambda$. The action of an element $g \in J(\mathbb{Q}_p)$ on $\mathcal{M}_{\mathrm{red}}$ induces an isomorphism $\mathcal{M}_\Lambda \xrightarrow{\sim} \mathcal{M}_{g\cdot\Lambda}$.
Let $\Lambda \in \mathcal{L}$. We denote by J_Λ the fixator of Λ under the action of $J(\mathbb{Q}_p)$. If $\Lambda = \Lambda_\theta$ for some $0 \leqslant \theta \leqslant \theta_{\max}$, we will write J_θ instead. These are **maximal parahoric subgroups** of $J(\mathbb{Q}_p)$. In unramified unitary similitude groups, maximal parahoric subgroups and maximal compact subgroups are the same. A general **parahoric subgroup** is an intersection $J_{\Lambda_1} \cap \ldots \cap J_{\Lambda_s}$ where $\{\Lambda_1, \ldots, \Lambda_s\}$ is an s-simplex in $\mathcal{L}_i$ for some i. Any parahoric subgroup is compact and open in $J(\mathbb{Q}_p)$. Let i be the integer such that $\Lambda \in \mathcal{L}_i$. We define $V_\Lambda^0 := \Lambda/p^{i+1}\Lambda^\vee$ and $V_\Lambda^1 := p^i\Lambda^\vee/\Lambda$. Since $p\Lambda \subset p\cdot p^i\Lambda^\vee$ and $p\cdot p^i\Lambda^\vee \subset \Lambda$, these are both $\mathbb{F}_{p^2}$-vector space of dimensions respectively $t(\Lambda)$ and $n-t(\Lambda)$. Both spaces come together with a non-degenerate σ-hermitian form $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ with values in $\mathbb{F}_{p^2}$, respectively induced by $p^{-i}\{\cdot,\cdot\}$ and by $p^{-i+1}\{\cdot,\cdot\}$. If k is a perfect field extension of $\mathbb{F}_{p^2}$ and if $\epsilon \in \{0,1\}$, we may extend the pairings to $(V_\Lambda^\epsilon)_k = V_\Lambda^\epsilon \otimes_{\mathbb{F}_{p^2}} k$ by setting

$$(v \otimes x, w \otimes y)_\epsilon := xy^\sigma(v,w)_\epsilon \in k$$

for all $v, w \in V_\Lambda^\epsilon$ and $x, y \in k$. If U is a subspace of $(V_\Lambda^\epsilon)_k$ we denote by $U^\perp$ its orthogonal.
Denote by J_Λ^+ the pro-unipotent radical of J_Λ and write $\mathcal{J}_\Lambda := J_\Lambda/J_\Lambda^+$. This is a finite group of Lie type, called the **maximal reductive quotient** of J_Λ. We have an identification $\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{U}(V_\Lambda^0) \times \mathrm{U}(V_\Lambda^1))$, that is the group of pairs (g_0, g_1) where for $\epsilon \in \{0,1\}$ we have $g_\epsilon \in \mathrm{GU}(V_\Lambda^\epsilon)$ and $c(g_0) = c(g_1)$. Here, $c(g_\epsilon) \in \mathbb{F}_p^\times$ denotes the multiplier of g_ϵ. For $0 \leqslant \theta \leqslant \theta_{\max}$ and $\epsilon \in \{0,1\}$, we will write V_θ^ϵ and $\mathcal{J}_\theta$ instead of $V_{\Lambda_\theta}^\epsilon$ and $\mathcal{J}_{\Lambda_\theta}$.

Let $\Lambda \in \mathcal{L}_i$ where ni is even. We write $t(\Lambda) = 2\theta + 1$. Let k be a perfect field extension of $\mathbb{F}_{p^2}$. Let T be any $W(k)$-lattice in $\mathbf{V}_k$ such that

$$p^{i+1}T^\vee \overset{2\theta'+1}{\subset} T \subset \Lambda_k$$

where $0 \leqslant \theta' \leqslant \theta$. Then T must contain $p^{i+1}\Lambda_k^\vee$ and $[\Lambda_k : T] = \theta - \theta'$. We may consider $\overline{T} := T/p^{i+1}\Lambda_k^\vee$ the image of T in $V_\Lambda^{(0)}$. Then $\overline{T}$ is an $\mathbb{F}_{p^2}$-subspace of dimension $\theta + \theta' + 1$. Moreover, one may check that $\overline{p^{i+1}T^\vee} = \overline{T}^\perp$, therefore the subspace $\overline{T}$ contains its orthogonal. These observations lead to the following proposition, see [Vol10] Proposition 2.7.

Proposition 1.9. *The mapping $T \mapsto \overline{T}$ defines a bijection between the set of $W(k)$-lattices T in $\mathbf{V}_k$ such that $p^{i+1}T^\vee \overset{2\theta'+1}{\subset} T \subset \Lambda_k$ and the set*

$$\{U \subset (V_\Lambda^0)_k \mid \dim U = \theta + \theta' + 1 \text{ and } U^\perp \subset U\}.$$

In particular taking $\theta' = 0$, this set is in bijection with $\mathcal{M}_\Lambda(k)$.

Remark 1.10. Similarly, the set of $W(k)$-lattices T such that $\Lambda_k \subset T \overset{n-2\theta'-1}{\subset} p^i T^\vee$ for some $\theta \leqslant \theta' \leqslant \theta_{\max}$ is in bijection with

$$\{U \subset (V_\Lambda^1)_k \mid \dim U = n - \theta' - \theta - 1 \text{ and } U^\perp \subset U\}.$$

The bijection is given by $T \mapsto \overline{T}^\perp$ where $\overline{T} := T/\Lambda_k \subset V_k^{(1)}$. These sets can be seen as the k-rational points of some flag variety for $\mathrm{GU}(V_\Lambda^{(0)})$ and $\mathrm{GU}(V_\Lambda^{(1)})$, which are special instances of Deligne-Lusztig varieties. This is accounted for in the next paragraph.

Let $\Lambda \in \mathcal{L}$. The action of $J(\mathbb{Q}_p)$ on the Rapoport-Zink space $\mathcal{M}$ restricts to an action of the parahoric subgroup J_Λ on the closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$. This action factors through the maximal reductive quotient $\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{U}(V_\Lambda^0) \times \mathrm{U}(V_\Lambda^1))$. This action is trivial on the normal subgroup $\{\mathrm{id}\} \times \mathrm{U}(V_\Lambda^1) \subset \mathcal{J}_\Lambda$, thus it factors again through the quotient which is isomorphic to $\mathrm{GU}(V_\Lambda^0)$. With respect to this action, the variety $\mathcal{M}_\Lambda$ is isomophic to a generalized Deligne-Lusztig variety, see [VW11] Theorem 4.8.

Theorem 1.11. *There is an isomorphism between $\mathcal{M}_\Lambda$ and a certain generalized parabolic Deligne-Lusztig variety for the finite group of Lie type $\mathrm{GU}(V_\Lambda^0)$, compatible with the actions. In particular, if $t(\Lambda) = 2\theta + 1$ then the variety $\mathcal{M}_\Lambda$ is projective, smooth, geometrically irreducible of dimension θ.*

We refer to [Mul23] Section 1 for the definition of Deligne-Lusztig varieties. In particular, the adjective "generalized" is understood according to loc. cit. The Deligne-Lusztig variety isomorphic to $\mathcal{M}_\Lambda$ is introduced in [VW11] Section 4.5, and it is denoted by Y_Λ there. Theorem 5.1 of loc. cit. describes the incidence relations between the different strata.

Theorem 1.12. *Let $i \in \mathbb{Z}$ such that ni is even. Let $\Lambda, \Lambda' \in \mathcal{L}_i$. The following statements hold.*

(1) *The inclusion $\Lambda \subset \Lambda'$ is equivalent to the scheme-theoretic inclusion $\mathcal{M}_\Lambda \subset \mathcal{M}_{\Lambda'}$. It also implies $t(\Lambda) \leqslant t(\Lambda')$ and there is equality if and only if $\Lambda = \Lambda'$.*

(2) *The three following assertions are equivalent.*

(i) $\Lambda \cap \Lambda' \in \mathcal{L}_i$. (ii) $\Lambda \cap \Lambda'$ *contains a lattice of* $\mathcal{L}_i$. (iii) $\mathcal{M}_\Lambda \cap \mathcal{M}_{\Lambda'} \neq \emptyset$.

If these conditions are satisfied, then $\mathcal{M}_\Lambda \cap \mathcal{M}_{\Lambda'} = \mathcal{M}_{\Lambda \cap \Lambda'}$, where we understand the left hand side as the scheme theoretic intersection inside $\mathcal{M}_{i,\mathrm{red}}$.

(3) *The three following assertions are equivalent*

(i) $\Lambda + \Lambda' \in \mathcal{L}_i$. (ii) $\Lambda + \Lambda'$ *is contained in a lattice of* $\mathcal{L}_i$.

(iii) $\mathcal{M}_\Lambda, \mathcal{M}_{\Lambda'} \subset \mathcal{M}_{\widetilde{\Lambda}}$ *for some* $\widetilde{\Lambda}$ *in* $\mathcal{L}_i$.

If these conditions are satisfied, then $\mathcal{M}_{\Lambda+\Lambda'}$ *is the smallest subscheme of the form* $\mathcal{M}_{\widetilde{\Lambda}}$ *containing both* $\mathcal{M}_\Lambda$ *and* $\mathcal{M}_{\Lambda'}$.

(4) *If* k *is a perfect field extension of* $\mathbb{F}_{p^2}$ *then* $\mathcal{M}_i(k) = \bigcup_{\Lambda \in \mathcal{L}_i} \mathcal{M}_\Lambda(k)$.

In essence, the previous statements explain how the stratification given by the $\mathcal{M}_\Lambda$ mimics the combinatorics of the Bruhat-Tits building of $\tilde{J}$, hence the name.

1.3 Normalizers of maximal parahoric subgroups of $J(\mathbb{Q}_p)$

In this section we compute the normalizer of the maximal parahoric subgroups J_Λ.

Lemma 1.13. *Let* $\Lambda, \Lambda' \in \mathcal{L}$.

(i) The parahoric subgroup J_Λ *acts transitively on the set of apartments containing* Λ.

(ii) We have $J_\Lambda = J_{\Lambda'}$ *if and only if there exists* $k \in \mathbb{Z}$ *such that* $\Lambda = p^k\Lambda'$ *or* $\Lambda = p^k\Lambda'^{\vee}$.

Proof. The first point is a general fact from the theory of Bruhat-Tits buildings. For the second point, the converse is clear. Indeed, if $x \in \mathbb{Q}_{p^2}^\times$ then $J_{x\Lambda} = J_\Lambda$, and an element $g \in J(\mathbb{Q}_p)$ fixes a lattice Λ if and only if it fixes its dual $\Lambda^\vee$. Now, let $\Lambda, \Lambda' \in \mathcal{L}$ such that $J_\Lambda = J_{\Lambda'}$. Up to replacing Λ' with an appropriate lattice $g \cdot \Lambda'$, it is enough to treat the case $\Lambda' = \Lambda_\theta$ for some $0 \leqslant \theta \leqslant \theta_{\max}$. By Proposition 1.5, we can find an apartment $\mathcal{A}^L$ containing both Λ_θ and Λ. By the first point, we can find $g \in J_\theta = J_\Lambda$ which sends $\mathcal{A}^L$ to $\mathcal{A}^e$. Therefore $g \cdot \Lambda = \Lambda$ belongs to $\mathcal{A}^e$. According to Proposition 1.7, we may write

$$\Lambda = \Lambda(r_{-\theta_{\max}}, \ldots, r_{-1}, s, r_1, \ldots, r_{\theta_{\max}})$$

for some integers (r_j, s). Let i be the integer such that $\Lambda \in \mathcal{L}_i$. Then according to Proposition 1.6 we have

- $\forall 1 \leqslant j \leqslant \theta_{\max}, r_{-j} + r_j \in \{i, i+1\}$.
- $s_0 = \lfloor \frac{i+1}{2} \rfloor$.
- if n is even then $s_1 = \lfloor \frac{i}{2} \rfloor$.

For $1 \leqslant j \leqslant \theta$, let g_j be the automorphism of $\mathbf{V}$ which exchanges e_{-j} and e_j while fixing all the other vectors in the basis e. Then, from the definition of Λ_θ we have $g_j \in J_\theta$. Therefore g_j must fix Λ too, which implies that $r_{-j} = r_j$. And for $\theta + 1 \leqslant j \leqslant \theta_{\max}$, let g_j be the automorphism sending e_j to $p^{-1}e_{-j}$ and e_{-j} to pe_j

while fixing all the other vectors in the basis e. Then again we have $g_j \in J_\theta = J_\Lambda$ which implies that $r_{-j} = r_j - 1$.
Assume first that $i = 2i'$ is even. Combining the previous observations, we have $r_j = i'$ for all $1 \leqslant j \leqslant \theta$ and $r_j = i' + 1$ for all $\theta + 1 \leqslant j \leqslant \theta_{\max}$. Moreover we have $s_0 = i'$ and if n is even, we have $s_1 = i'$. In other words, we have $\Lambda = p^{i'}\Lambda_\theta$.
Assume now that $i = 2i' + 1$ is odd. This implies that n is even. Combining the previous observations, we have $r_j = i' + 1$ for all $1 \leqslant j \leqslant \theta_{\max}$. Moreover we have $s_0 = i' + 1$ and if n is even, we have $s_1 = i'$. In other words, we have $\Lambda = p^{i'+1}\Lambda_\theta^\vee$. □

Proposition 1.14. *Let $\Lambda \in \mathcal{L}$. If $t(\Lambda) \neq n - t(\Lambda)$ then the normalizer of J_Λ in $J(\mathbb{Q}_p)$ is $\mathrm{N}_J(J_\Lambda) = \mathrm{Z}(J(\mathbb{Q}_p))J_\Lambda$. Otherwise, n is even and there exists an element $h_0 \in J(\mathbb{Q}_p)$ such that $h_0^2 = p \cdot \mathrm{id}$ and $N_J(J_\lambda)$ is the subgroup generated by J_Λ and h_0. In particular, $\mathrm{Z}(J(\mathbb{Q}_p))J_\Lambda$ is a subgroup of index 2 in $N_J(J_\Lambda)$.*

Remark 1.15. The condition $t(\Lambda) \neq n - t(\Lambda)$ is automatically satisfied if n is odd. If n is even, it is satisfied when $t(\Lambda) \neq \theta_{\max} + 1$, this is the case in particular when $\theta_{\max}$ is odd.

Proof. It is clear that $\mathrm{Z}(J(\mathbb{Q}_p))J_\Lambda \subset \mathrm{N}_J(J_\Lambda)$. Conversely, let $g \in N_J(J_\Lambda)$, so that we have $J_\Lambda = {}^gJ_\Lambda = J_{g\cdot\Lambda}$. We apply Lemma 1.13 to deduce the existence of $k \in \mathbb{Z}$ such that $g \cdot \Lambda = p^k\Lambda$ (case 1) or $g \cdot \Lambda = p^k\Lambda^\vee$ (case 2). If we are in case 1, then $g \in p^kJ_\Lambda \subset \mathrm{Z}(J(\mathbb{Q}_p))J_\Lambda$ and we are done. If n is even, the assumption that $t(\Lambda) \neq n - t(\Lambda)$ makes the case 2 impossible. If n is odd and we are in case 2, then in particular $\Lambda^\vee \in \mathcal{L}$. By Lemma 1.3, we must have $\Lambda = p^i\Lambda^\vee$ for some even $i \in \mathbb{Z}$. In particular, we are also in case 1. Therefore, no matter the parity of n, we are always in case 1.
Assume now that $t(\Lambda) = n - t(\Lambda)$, in particular n and $\theta_{\max}$ are both even. We write $\theta_{\max} = 2\theta'_{\max}$ so that $t(\Lambda) = 2\theta'_{\max} + 1$ and we solve the case $\Lambda = \Lambda_{\theta'_{\max}}$ first. Recall the element g_0 that was defined earlier. By direct computation, we see that $g_0 \cdot \Lambda_{\theta'_{\max}} = p\Lambda_{\theta'_{\max}}^\vee$. Therefore ${}^{g_0}J_{\theta'_{\max}} = J_{p\Lambda^\vee_{\theta'_{\max}}} = J_{\theta'_{\max}}$ so that $g_0 \in \mathrm{N}_J(J_{\theta'_{\max}})$. Now let g be any element normalizing $J_{\theta_{\max}}$, so that $J_{\theta'_{\max}} = {}^gJ_{\theta'_{\max}} = J_{g\cdot\Lambda_{\theta'_{\max}}}$. According to 1.13 there exists $k \in \mathbb{Z}$ such that $g \cdot \Lambda_{\theta'_{\max}} = p^k\Lambda_{\theta'_{\max}}$ or $g \cdot \Lambda_{\theta'_{\max}} = p^k\Lambda^\vee_{\theta'_{\max}} = p^{k-1}g_0 \cdot \Lambda_{\theta'_{\max}}$. In the first case we have $g \in p^kJ_{\theta'_{\max}}$ and in the second case we have $g \in p^{k-1}g_0J_{\theta'_{\max}}$. Since $g_0^2 = p \cdot \mathrm{id}$, the claim is proved with $h_0 = g_0$.
In the general case, we have $t(\Lambda) = 2\theta'_{\max} + 1 = t(\Lambda_{\theta'_{\max}})$. There exists some $g \in J(\mathbb{Q}_p)$ such that $\Lambda = g \cdot \Lambda_{\theta'_{\max}}$. Then $\mathrm{N}_J(\Lambda) = {}^g\mathrm{N}_J(\Lambda_{\theta'_{\max}})$ so that the claim follows with $h_0 := gg_0g^{-1}$. □

1.4 Counting the closed Bruhat-Tits strata

In this section we count the number of closed Bruhat-Tits strata which contain or which are contained in another given one. Let $d \geqslant 0$ and consider V a d-dimensional $\mathbb{F}_{p^2}$-vector space equipped with a non degenerate hermitian form. This structure is uniquely determined up to isomorphism as we are working over a finite field. For $\left\lceil \frac{d}{2} \right\rceil \leqslant r \leqslant d$, we define

$$\begin{aligned} N(r,V) &:= \{U \,|\, U \text{ is an } r\text{-dimensional subspace of } V \text{ such that } U^\perp \subset U\}, \\ \nu(r,d) &:= \#N(r,V), \end{aligned}$$

where $U^\perp$ denotes the orthogonal of U with respect to the hermitian form on V. By Proposition 1.9 and the following Remark, we have the following statement, see also [VW11] Corollary 5.7.

Proposition 1.16. *Let $\Lambda \in \mathcal{L}$. Write $t(\Lambda) = 2\theta + 1$ for some $0 \leqslant \theta \leqslant \theta_{\max}$.*

- *Let θ' be an integer such that $0 \leqslant \theta' \leqslant \theta$. The number of closed Bruhat-Tits strata of dimension θ' contained in $\mathcal{M}_\Lambda$ is $\nu(\theta + \theta' + 1, 2\theta + 1)$.*
- *Let θ' be an integer such that $\theta \leqslant \theta' \leqslant \theta_{\max}$. The number of closed Bruhat-Tits strata of dimension θ' containing $\mathcal{M}_\Lambda$ is $\nu(n - \theta - \theta' - 1, n - 2\theta - 1)$.*

Another way to formulate the proposition is to say that $\nu(\theta + \theta' + 1, 2\theta + 1)$ (resp. $\nu(n - \theta - \theta' - 1, n - 2\theta - 1)$) is the number of vertices of type $2\theta' + 1$ in the Bruhat-Tits building of $\tilde{J}$ which are neighbors of a given vertex of type $2\theta + 1$ for $\theta' \leqslant \theta$ (resp. $\theta' \geqslant \theta$). In [VW11], an explicit formula is given for $\nu(d-1,d)$. The next proposition gives a formula to compute $\nu(r,d)$ for general r and d.

Proposition 1.17. *Let $d \geqslant 0$ and let $\left\lceil \frac{d}{2} \right\rceil \leqslant r \leqslant d$. We have*

$$\nu(r,d) = \frac{\prod_{j=1}^{2(d-r)} \left(p^{2r-d+j} - (-1)^{2r-d+j}\right)}{\prod_{j=1}^{d-r} \left(p^{2j} - 1\right)}$$

Proof. We fix a basis $(e_1, \ldots, e_d)$ of V in which the hermitian form is represented by the matrix A_d. We denote by U_0 the subspace generated by the vectors $e_1, \ldots, e_r$. The orthogonal of U_0 is generated by $e_1, \ldots, e_{d-r}$. Since $\left\lceil \frac{d}{2} \right\rceil \leqslant r \leqslant d$, U_0 contains its orthogonal, thus $U_0 \in N(r,V)$. The unitary group $\mathrm{U}(V) \simeq \mathrm{U}_d(\mathbb{F}_p)$ acts transitively on the set $N(r,V)$ (since $p \neq 2$). The stabilizer of U_0 in $\mathrm{U}_d(\mathbb{F}_p)$ is the standard parabolic subgroup

$$P_0 := \left\{ \begin{pmatrix} B & * & * \\ 0 & M & * \\ 0 & 0 & F(B) \end{pmatrix} \in \mathrm{U}_d(\mathbb{F}_p) \,\middle|\, B \in \mathrm{GL}_{d-r}(\mathbb{F}_{p^2}), M \in \mathrm{U}_{2r-d}(\mathbb{F}_p) \right\}.$$

Here, $F(B) = A_{d-r}(B^{(p)})^{-T}A_{d-r}$ where $B^{(p)}$ is the matrix B with all coefficients raised to the power p. The order of $\mathrm{U}_d(\mathbb{F}_p)$ is well known and given by the formula

$$\#\mathrm{U}_d(\mathbb{F}_p) = p^{\frac{d(d-1)}{2}} \prod_{j=1}^{d} \left(p^j - (-1)^j\right).$$

It remains to compute the order of P_0. We have a Levi decomposition $P_0 = L_0N_0$ with $L_0 \cap N_0 = \{1\}$ where

$$L_0 := \left\{ \begin{pmatrix} B & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & F(B) \end{pmatrix} \in \mathrm{U}_d(\mathbb{F}_p) \;\middle|\; B \in \mathrm{GL}_{d-r}(\mathbb{F}_{p^2}), M \in \mathrm{U}_{2r-d}(\mathbb{F}_p) \right\},$$

$$N_0 := \left\{ \begin{pmatrix} 1 & X & Z \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{U}_d(\mathbb{F}_p) \;\middle|\; X \in \mathrm{M}_{d-r,2r-d}(\mathbb{F}_{p^2}), Y \in \mathrm{M}_{2r-d,d-r}(\mathbb{F}_{p^2}), Z \in \mathrm{M}_{d-r}(\mathbb{F}_{p^2}) \right\}.$$

The order of L_0 is given by

$$\#L_0 = \#\mathrm{GL}_{d-r}(\mathbb{F}_{p^2})\#\mathrm{U}_{2r-d}(\mathbb{F}_p) = p^{(d-r)(d-r-1)+\frac{(2r-d)(2r-d-1)}{2}} \prod_{j=1}^{d-r} \left(p^{2j} - 1\right) \prod_{j=1}^{2r-d} \left(p^j - (-1)^j\right).$$

As for N_0, we need some more conditions on the matrices X, Y and Z. By direct computations, one may check that

$$\begin{pmatrix} 1 & X & Z \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{pmatrix} \in N_0 \iff Y = -A_{2r-d}(X^{(p)})^T A_{d-r} \text{ and } Z + A_{d-r}(Z^{(p)})^T A_{d-r} = XY \in \mathrm{M}_{d-r}(\mathbb{F}_{p^2}).$$

Thus, X is any matrix of size $(d-r) \times (2r-d)$ and Y is determined by X. In the second equation, the matrix $A_{d-r}(Z^{(p)})^T A_{d-r}$ is the reflexion of $Z^{(p)}$ with respect to the antidiagonal. The equation implies that the coefficients below the antidiagonal of Z determine those above the antidiagonal. Furthermore, if z is a coefficient in the antidiagonal then the equation determines the value of $\mathrm{Tr}(z) = z + z^p$, where $\mathrm{Tr} : \mathbb{F}_{p^2} \to \mathbb{F}_p$ is the trace relative to the extension $\mathbb{F}_{p^2}/\mathbb{F}_p$. The trace is surjective and its kernel has order p. Thus, there are only p possibilities for each antidiagonal coefficient. Putting things together, the order of N_0 is given by

$$\#N_0 = p^{2(d-r)(2r-d)} \cdot p^{2\frac{(d-r)(d-r-1)}{2}} \cdot p^{d-r} = p^{(d-r)(3r-d)}$$

where the three terms take account respectively of the choice of X, the choice of the coefficients below the antidiagonal of Z and the choice of the coefficients in the antidiagonal of Z. Hence the order of P_0 is given by

$$\#P_0 = \#L_0\#N_0 = p^{\frac{d(d-1)}{2}} \prod_{j=1}^{d-r} \left(p^{2j} - 1\right) \prod_{j=1}^{2r-d} \left(p^j - (-1)^j\right).$$

Upon taking the quotient $\nu(r,d) = \#\mathrm{U}_d(\mathbb{F}_p)/\#P_0$, the result follows. $\square$

In particular with $r = d-1$, we obtain

$$\nu(d-1,d) = \frac{(p^{d-1} - (-1)^{d-1})(p^d - (-1)^d)}{p^2-1}.$$

If $d = 2\delta$ is even, it is equal to $(p^{d-1}+1)\sum_{j=0}^{\delta-1} p^{2j}$, and if $d = 2\delta+1$ is odd, it is equal to $(p^d+1)\sum_{j=0}^{\delta-1} p^{2j}$. This coincides with the formula given in [VW11] Example 5.6.

2 The cohomology of a closed Bruhat-Tits stratum

In [Mul23], we computed the cohomology groups $\mathrm{H}^\bullet_c(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ of the closed Bruhat-Tits strata. The computation relies on the Ekedahl-Oort stratification on $\mathcal{M}_\Lambda$ which, in the language of Deligne-Lusztig varieties, translates into a stratification by Coxeter varieties for unitary groups of smaller sizes. The cohomology of Coxeter varieties is well known thanks to the work of Lusztig in [Lus76]. In order to state our results, we recall the classification of unipotent representations of the finite unitary group.

Let q be a power of prime number p, and let $\mathbf{G}$ be a reductive connected group over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Assume that $\mathbf{G}$ is equipped with an $\mathbb{F}_q$-structure induced by a Frobenius morphism F. Let $G = \mathbf{G}^F$ be the associated finite group of Lie type. Let $(\mathbf{T},\mathbf{B})$ be a pair consisting of an F-stable maximal torus $\mathbf{T}$ and an F-stable Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. Let $\mathbf{W} = \mathbf{W}(\mathbf{T})$ denote the Weyl group of $\mathbf{G}$. The Frobenius F induces an action on $\mathbf{W}$. For $w \in \mathbf{W}$, let $\dot{w}$ be a representative of w in the normalizer $\mathrm{N}_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$. By the Lang-Steinberg theorem, one can find $g \in \mathbf{G}$ such that $\dot{w} = g^{-1}F(g)$. Then ${}^g\mathbf{T} := g\mathbf{T}g^{-1}$ is another F-stable maximal torus, and $w \in \mathbf{W}$ is said to be the **type** of ${}^g\mathbf{T}$ with respect to $\mathbf{T}$. Every F-stable maximal torus arises in this manner. According to [DL76] Corollary 1.14, the G-conjugacy class of ${}^g\mathbf{T}$ only depends on the F-conjugacy class of w in the Weyl group $\mathbf{W}$. Here, two elements w and w' in $\mathbf{W}$ are said to be F-conjugate if there exists some element $\tau \in \mathbf{W}$ such that $w = \tau w' F(\tau)^{-1}$. For every $w \in \mathbf{W}$, we fix $\mathbf{T}_w$ an F-stable maximal torus of type w with respect to $\mathbf{T}$. The Deligne-Lusztig induction of the trivial representation of $\mathbf{T}_w$ is the virtual representation of G defined by the formula

$$R_w := \sum_{i \geqslant 0} (-1)^i \mathrm{H}^i_c(X(w) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}),$$

where $X(w)$ is the Deligne-Lusztig variety for $\mathbf{G}$ given by

$$X(w) := \{g\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid g^{-1}F(g) \in \mathbf{B}w\mathbf{B}\}.$$

According to [DL76] Theorem 1.6, the virtual representation R_w only depends on the F-conjugacy class of w in $\mathbf{W}$. An irreducible representation of G is said to be **unipotent** if it occurs in R_w for some $w \in \mathbf{W}$. The set of isomorphism classes of unipotent representations of G is denoted by $\mathcal{E}(G, 1)$.

Remark 2.1. Since the center $\mathrm{Z}(G)$ acts trivially on the variety $X(w)$, any irreducible unipotent representation of G has trivial central character.

Let $\mathbf{G}$ and $\mathbf{G}'$ be two reductive connected group over $\mathbb{F}$ both equipped with an $\mathbb{F}_q$-structure. We denote by F and F' the respective Frobenius morphisms. Let $f : \mathbf{G} \to \mathbf{G}'$ be an $\mathbb{F}_q$-isotypy, that is a homomorphism defined over $\mathbb{F}_q$ whose kernel is contained in the center of $\mathbf{G}$ and whose image contains the derived subgroup of $\mathbf{G}'$. Then, according to [DM20] Proposition 11.3.8, we have an equality

$$\mathcal{E}(G, 1) = \{\rho \circ f \mid \rho \in \mathcal{E}(G', 1)\}.$$

Thus, the irreducible unipotent representations of G and of G' can be identified. We will use this observation in the case $G = \mathrm{U}_k(\mathbb{F}_q)$ and $G' = \mathrm{GU}_k(\mathbb{F}_q)$. The corresponding reductive groups are $\mathbf{G} = \mathrm{GL}_k$ and $\mathbf{G}' = \mathrm{GL}_k \times \mathrm{GL}_1$. The Frobenius morphisms can be defined as

$$F(M) = \dot{w}_0(M^{(q)})^{-T}\dot{w}_0, \qquad F'(M, c) = (c^q\dot{w}_0(M^{(q)})^{-T}\dot{w}_0, c^q).$$

Here, $\dot{w}_0$ is the $k \times k$ matrix with only 1's in the antidiagonal and $M^{(q)}$ is the matrix M whose entries are all raised to the power q. The isotypy $f : \mathbf{G} \to \mathbf{G}'$ is defined by $f(M) = (M, 1)$. It satisfies $F' \circ f = f \circ F$, it is injective and its image contains the derived subgroup $\mathrm{SL}_n \times \{1\} \subset \mathbf{G}'$. Hence, we obtain the following result.

Proposition 2.2. *The irreducible unipotent representations of the finite groups of Lie type $\mathrm{U}_k(\mathbb{F}_q)$ and $\mathrm{GU}_k(\mathbb{F}_q)$ can be naturally identified.*

Assume that the Coxeter graph of the reductive group $\mathbf{G}$ is a union of subgraphs of type A_m (for various m). Let $\breve{\mathbf{W}}$ be the set of isomorphism classes of irreducible representations of its Weyl group $\mathbf{W}$. The action of the Frobenius F on $\mathbf{W}$ induces an action on $\breve{\mathbf{W}}$, and we consider the fixed point set $\breve{\mathbf{W}}^F$. The following theorem of [LS77] classifies the irreducible unipotent representations of G.

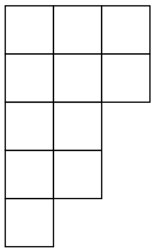
Theorem 2.3. *There is a bijection between $\breve{\mathbf{W}}^F$ and the set of isomorphism classes of irreducible unipotent representations of G.*

We recall how the bijection is constructed. According to loc. cit. if $V \in \widecheck{\mathbf{W}}^F$ there is a unique automorphism $\widetilde{F}$ of V of finite order such that

$$R(V) := \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \mathrm{Trace}(w \circ \widetilde{F} \,|\, V) R_w$$

is an irreducible representation of G. Then the map $V \mapsto R(V)$ is the desired bijection. In the case of $\mathrm{U}_k(\mathbb{F}_q)$ or $\mathrm{GU}_k(\mathbb{F}_q)$, the Weyl group $\mathbf{W}$ is identified with the symmetric group $\mathfrak{S}_k$ and we have an equality $\widecheck{\mathbf{W}}^F = \widecheck{\mathbf{W}}$. Moreover, the automorphism $\widetilde{F}$ is the multiplication by w_0, where w_0 is the element of maximal length in $\mathbf{W}$. Thus, in both cases the irreducible unipotent representations of G are classified by the irreducible representations of the Weyl group $\mathbf{W} \simeq \mathfrak{S}_k$, which in turn are classified by partitions of k or equivalently by Young diagrams, as we briefly recall in the next paragraph.

A partition of k is a tuple of integers $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r > 0)$ with $r \geqslant 1$ such that $\lambda_1 + \ldots + \lambda_r = k$. The integer k is called the length of the partition, and it is denoted by $|\lambda|$. A Young diagram of size k is a top left justified collection of k boxes, arranged in rows and columns. There is a correspondence between Young diagrams of size k and partitions of k, by associating to a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ the Young diagram having r rows consisting successively of $\lambda_1, \ldots, \lambda_r$ boxes. We will often identify a partition with its Young diagram, and conversely. For example, the Young diagram associated to $\lambda = (3^2, 2^2, 1)$ is the following one.



To any partition λ of k, one can naturally associate an irreducible character χ_λ of the symmetric group $\mathfrak{S}_k$. An explicit construction is given, for instance, by the notion of Specht modules as explained in [Jam84] 7.1.

The irreducible unipotent representation of $\mathrm{U}_k(\mathbb{F}_q)$ (resp. $\mathrm{GU}_k(\mathbb{F}_q)$) associated to χ_λ by the bijection of Theorem 2.3 is denoted by $\rho_\lambda^{\mathrm{U}}$ (resp. $\rho_\lambda^{\mathrm{GU}}$). In virtue of Proposition 2.2, for every λ we have $\rho_\lambda^{\mathrm{U}} = \rho_\lambda^{\mathrm{GU}} \circ f$, where $f : \mathrm{U}_k(\mathbb{F}_q) \to \mathrm{GU}_k(\mathbb{F}_q)$ is the inclusion. Thus, it is harmless to identify $\rho_\lambda^{\mathrm{U}}$ and $\rho_\lambda^{\mathrm{GU}}$ so that from now on, we will omit the superscript. The partition (k) corresponds to the trivial representation and (1^k) to the Steinberg representation. Given a box $\square$ in the Young diagram of λ, its **hook length** $h(\square)$ is 1 plus the number of boxes lying below it or on its right. For instance, in the following figure the hook length of every box of the Young diagram of $\lambda = (3^2, 2^2, 1)$ has been written inside it.

7	5	2
6	4	1
4	2	
3	1	
1		

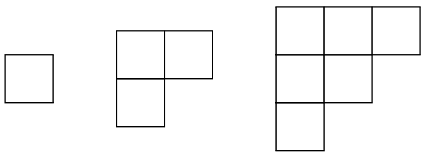
The degree of the representations ρ_λ is given by expressions known as **hook formula**, see for instance [GP00] Proposition 4.3.5.

Proposition 2.4. *Let $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r > 0)$ be a partition of k. The degree of the irreducible unipotent representation ρ_λ is given by the following formula*

$$\deg(\rho_\lambda) = q^{a(\lambda)} \frac{\prod_{i=1}^{k} q^i - (-1)^i}{\prod_{\square \in \lambda} q^{h(\square)} - (-1)^{h(\square)}}$$

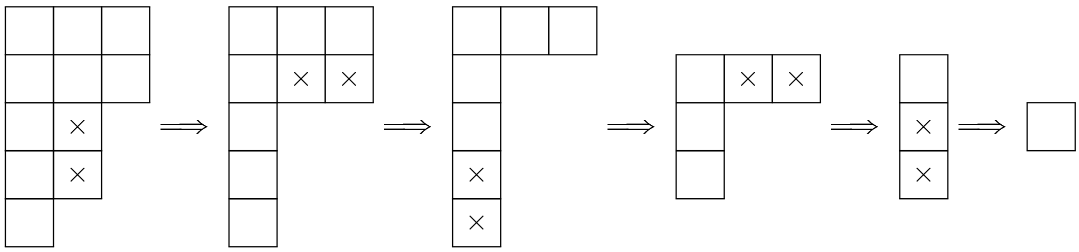
where $a(\lambda) = \sum_{i=1}^{r} (i-1)\lambda_i$.

We may describe the cuspidal support of the unipotent representations ρ_λ. According to [Lus77] Propositions 9.2 and 9.4 there exists an irreducible unipotent cuspidal representation of $\mathrm{U}_k(\mathbb{F}_q)$ (or $\mathrm{GU}_k(\mathbb{F}_q)$) if and only if k is an integer of the form $k = \frac{t(t+1)}{2}$ for some $t \geqslant 0$. When k is an integer of this form, the unique unipotent cuspidal representation is associated to the partition $\Delta_t := (t, t-1, \ldots, 1)$, whose Young diagram has the distinctive shape of a staircase. Here, as a convention $U_0(\mathbb{F}_q) = \mathrm{GU}_0(\mathbb{F}_q)$ denotes the trivial group. For example, here are the Young diagrams of Δ_1, Δ_2 and Δ_3. Of course, the one of Δ_0 the empty diagram.



We consider an integer $t \geqslant 0$ such that k decomposes as $k = 2e + \frac{t(t+1)}{2}$ for some $e \geqslant 0$. Let G denote $\mathrm{U}_k(\mathbb{F}_q)$ or $\mathrm{GU}_k(\mathbb{F}_q)$, and consider L_t the subgroup consisting of block-diagonal matrices having one middle block of size $\frac{t(t+1)}{2}$ and all other blocks of size 1. This is a standard Levi subgroup of G. For $\mathrm{U}_k(\mathbb{F}_q)$, it is isomorphic to $\mathrm{GL}_1(\mathbb{F}_{q^2})^e \times \mathrm{U}_{\frac{t(t+1)}{2}}(\mathbb{F}_q)$ whereas in the case of $\mathrm{GU}_k(\mathbb{F}_q)$ it is isomorphic to $\mathrm{G}\left(\mathrm{U}_1(\mathbb{F}_q)^e \times \mathrm{U}_{\frac{t(t+1)}{2}}(\mathbb{F}_q)\right)$. In both cases, L_t admits a quotient which is isomorphic to a group of the same type as G but of size $\frac{t(t+1)}{2}$. We write ρ_t for the inflation to L_t of the unipotent cuspidal representation ρ_{Δ_t} of this quotient. If λ is a partition of k, the cuspidal support of the representation ρ_λ is given by exactly one of the pair (L_t, ρ_t) up to conjugation, where $t \geqslant 0$ is an integer such that for some $e \geqslant 0$ we have $k = 2e + \frac{t(t+1)}{2}$. Note that in particular k and $\frac{t(t+1)}{2}$ must have the same parity. With these notations, the irreducible unipotent representations belonging to the principal series (ie. those whose cuspidal support is supported on a minimal parabolic subgroup) are those with cuspidal support (L_0, ρ_0) if k is even and (L_1, ρ_1) if k is odd.

Given an irreducible unipotent representation ρ_λ, there is a combinatorical way to determine the Harish-Chandra series to which it belongs, as we recalled in [Mul23] Section 2. We consider the Young diagram of λ. We call **domino** any pair of adjacent boxes in the diagram. It may be either vertical or horizontal. We remove dominoes from the diagram of λ so that the resulting shape is again a Young diagram, until one can not proceed further. This process results in the Young diagram of the partition Δ_t for some $t \geqslant 0$, and it is called the **2-core** of λ. It does not depend on the successive choices for the dominoes. Then, the representation ρ_λ has cuspidal support (L_t, ρ_t) if and only if λ has 2-core Δ_t. For instance, the diagram $\lambda = (3^2, 2^2, 1)$ has 2-core Δ_1, as it can be determined by the following steps. We put crosses inside the successive dominoes that we remove from the diagram.



Thus, the unipotent representation ρ_λ of $\mathrm{U}_{11}(\mathbb{F}_q)$ or $\mathrm{GU}_{11}(\mathbb{F}_q)$ has cuspidal support (L_1, ρ_1), so in particular it is a principal series representation.

From now on, we take $q = p$. Let $\Lambda \in \mathcal{L}$ with orbit type $t(\Lambda) = 2\theta + 1$. Recall that the stratum $\mathcal{M}_\Lambda$ is equipped with an action of the finite group of Lie type $\mathrm{GU}(V_\Lambda^0)$. Upon choosing a basis, we identify this group with $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$. Let $\mathrm{Frob} = \sigma^{-2} \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^2})$ be the geometric Frobenius. Then Frob is a topological generator of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^2})$. In [Mul23], we computed the cohomology groups $\mathrm{H}^\bullet(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ in terms of a $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p) \times \langle \mathrm{Frob} \rangle$-representations. The result is summed up in the following Theorem.

Theorem 2.5. *Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$ for some $0 \leqslant \theta \leqslant \theta_{\max}$.*

(1) *The cohomology group $\mathrm{H}^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ is zero unless $0 \leqslant j \leqslant 2\theta$.*

(2) *The Frobenius* Frob *acts like multiplication by $(-p)^j$ on $\mathrm{H}^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$.*

(3) *For $0 \leqslant j \leqslant \theta$ we have*

$$\mathrm{H}^{2j}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-j)} \rho_{(2\theta+1-2s,2s)}.$$

For $0 \leqslant j \leqslant \theta - 1$ *we have*

$$\mathrm{H}^{2j+1}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-1-j)} \rho_{(2\theta-2s,2s+1)}.$$

Thus, the cohomology of $\mathcal{M}_\Lambda$ consists only of unipotent representations whose associated Young diagram has at most two rows.

Remarks 2.6. Let us make a few comments.

- The cohomology groups of index 0 and 2θ are the trivial representation of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$.
- All irreducible representations in the cohomology groups of even index belong to the unipotent principal series, whereas all the ones in the groups of odd index have cuspidal support (L_2, ρ_2).
- The cohomology group $\mathrm{H}^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ contains no cuspidal representation unless $\theta = j = 0$ or $\theta = j = 1$. If $\theta = 0$ then H^0 is the trivial representation of $\mathrm{GU}_1(\mathbb{F}_p) = \mathbb{F}_{p^2}^\times$, and if $\theta = 1$ then H^1 is the representation ρ_{Δ_2} of $\mathrm{GU}_3(\mathbb{F}_p)$. Both of them are cuspidal.

3 Shimura variety and p-adic uniformization of the supersingular locus

In this section, we introduce the PEL unitary Shimura variety with signature $(1, n-1)$ as in [VW11] Sections 6.1 and 6.2, and we recall the p-adic uniformization theorem of its basic (or supersingular) locus. The Shimura variety can be defined as a moduli problem classifying abelian varieties with additional structures, as follows. Let $\mathbb{E}$ be a quadratic imaginary extension of $\mathbb{Q}$ such that $\mathbb{E}_p \simeq E$. In particular p **is inert** in $\mathbb{E}$. Let $B/\mathbb{E}$ be a simple central algebra of degree $d \geqslant 1$ which splits over p and at infinity. Let $*$ be a positive involution of the second kind on B, and let $\mathbb{V}$ be a non-zero finitely generated left B-module equipped with a non-degenerate $*$-alternating form $\langle \cdot, \cdot \rangle$ taking values in $\mathbb{Q}$. Assume also that $\dim_{\mathbb{E}}(\mathbb{V}) = nd$. Let $\mathbb{G}$ be the connected reductive group over $\mathbb{Q}$ whose points over a $\mathbb{Q}$-algebra R are given by

$$\mathbb{G}(R) := \{g \in \mathrm{GL}_{\mathbb{E}\otimes R}(\mathbb{V}\otimes R) \mid \exists c \in R^\times \text{ such that for all } v, w \in \mathbb{V}\otimes R, \langle gv, gw \rangle = c\langle v, w \rangle\}.$$

We denote by $c : \mathbb{G} \to \mathbb{G}_m$ the **multiplier** character. The base change $\mathbb{G}_{\mathbb{R}}$ is isomorphic to a group of unitary similitudes $\mathrm{GU}(r, s)$ of a hermitian space with signature (r, s) where $r + s = n$. We assume that $r = 1$ and $s = n - 1$. We consider a Shimura datum of the form $(\mathbb{G}, X)$, where X denotes the unique $\mathbb{G}(\mathbb{R})$-conjugacy class of homorphisms $h : \mathbb{C}^\times \to \mathbb{G}_{\mathbb{R}}$ such that for all $z \in \mathbb{C}^\times$ we have

$\langle h(z)\cdot,\cdot\rangle = \langle\cdot,h(\overline{z})\cdot\rangle$, and such that the $\mathbb{R}$-pairing $\langle\cdot,h(i)\cdot\rangle$ is positive definite. Such a homomorphism h induces a decomposition $\mathbb{V}\otimes\mathbb{C} = \mathbb{V}_1\oplus\mathbb{V}_2$. Concretely, $\mathbb{V}_1$ (resp. $\mathbb{V}_2$) is the subspace where $h(z)$ acts like z (resp. like $\overline{z}$). Let F be the unique subfield of $\mathbb{C}$ isomorphic to $\mathbb{E}$. The reflex field associated to the PEL data, that is the field of definition of $\mathbb{V}_1$ as a complex representation of B, is equal to F unless $n = 2$, in which case it is $\mathbb{Q}$. Nonetheless, for simplicity we will consider the associated Shimura varieties over F even in the case $n = 2$.

Remark 3.1. As remarked in [Vol10] Section 6, the group G satisfies the Hasse principle, ie. $\ker^1(\mathbb{Q},\mathbb{G})$ is a singleton. Therefore, the Shimura variety associated to the Shimura datum (G,X) coincides with the moduli space of abelian varieties that we are going to define.

Let $\mathbb{A}_f$ denote the ring of finite adèles over $\mathbb{Q}$ and let $K\subset G(\mathbb{A}_f)$ be an open compact subgroup. We define a functor Sh_K by associating to an F-scheme S the set of isomorphism classes of tuples $(A,\lambda_A,\iota_A,\overline{\eta}_A)$ where

– A is an abelian scheme over S.
– $\lambda_A : A\to\widehat{A}$ is a polarization.
– $\iota_A : B\to \mathrm{End}(A)\otimes\mathbb{Q}$ is a morphism of algebras such that $\iota_A(b^*) = \iota_A(b)^\dagger$ where $\cdot^\dagger$ denotes the Rosati involution associated to λ_A, and such that the Kottwitz determinant condition is satisfied:

$$\forall b\in B,\ \det(\iota_A(b)) = \det(b\,|\,\mathbb{V}_1).$$

– $\overline{\eta}_A$ is a K-level structure, that is a K-orbit of isomorphisms of $B\otimes\mathbb{A}_f$-modules $\mathrm{H}_1(A,\mathbb{A}_f)\xrightarrow{\sim}\mathbb{V}\otimes\mathbb{A}_f$ that is compatible with the other data.

The Kottwitz condition in the third point is independent on the choice of $h\in X$. If K is sufficiently small, this moduli problem is represented by a smooth quasi-projective scheme Sh_K over F. When the level K varies, the Shimura varieties form a projective system $(\mathrm{Sh}_K)_K$ equipped with an action of $\mathbb{G}(\mathbb{A}_f)$ by Hecke correspondences.

We assume the existence of a $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ in B, stable under the involution $*$, such that its p-adic completion is a maximal order in $B_{\mathbb{Q}_p}$. We also assume that there is a $\mathbb{Z}_p$-lattice Γ in $\mathbb{V}\otimes\mathbb{Q}_p$, invariant under $\mathcal{O}_B$ and self-dual for $\langle\cdot,\cdot\rangle$. We may fix isomorphisms $\mathbb{E}_p\simeq E$ and $B_{\mathbb{Q}_p}\simeq \mathrm{M}_d(E)$ such that $\mathcal{O}_B\otimes\mathbb{Z}_p$ is identified with $\mathrm{M}_d(\mathcal{O}_E)$.

As a consequence of the existence of Γ, the group $G := \mathbb{G}_{\mathbb{Q}_p}$ is unramified. Let $K_0 := \mathrm{Fix}(\Gamma)$ be the subgroup of $G(\mathbb{Q}_p)$ consisting of all g such that $g\cdot\Gamma = \Gamma$. It is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. We will consider levels of the form $K = K_0K^p$ where K^p is an open compact subgroup of $\mathbb{G}(\mathbb{A}_f^p)$. Note that K is sufficiently small as soon as K^p is sufficiently small. By the work of Kottwitz in [Kot92], the Shimura varieties $\mathrm{Sh}_{K_0K^p}$ admit integral models over $\mathcal{O}_{F,(p)}$ which have the following moduli interpretation. We define a functor S_{K^p} by associating to an $\mathcal{O}_{F,(p)}$-scheme S the set of isomorphism classes of tuples $(A, \lambda_A, \iota_A, \overline{\eta}_A^p)$ where

– A is an abelian scheme over S.
– $\lambda_A : A \to \widehat{A}$ is a polarization whose order is prime to p.
– $\iota_A : \mathcal{O}_B \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is a morphism of algebras such that $\iota_A(b^*) = \iota_A(b)^\dagger$ where $\cdot^\dagger$ denotes the Rosati involution associated to λ_A, and such that the Kottwitz determinant condition is satisfied:

$$\forall b \in \mathcal{O}_B,\ \det(\iota_A(b)) = \det(b \,|\, \mathbb{V}_1).$$

– $\overline{\eta}_A^p$ is a K^p-level structure, that is a K^p-orbit of isomorphisms of $B \otimes \mathbb{A}_f^p$-modules $\mathrm{H}_1(A, \mathbb{A}_f^p) \xrightarrow{\sim} \mathbb{V} \otimes \mathbb{A}_f^p$ that is compatible with the other data.

If K^p is sufficiently small, this moduli problem is also representable by a smooth quasi-projective scheme over $\mathcal{O}_{F,(p)}$. When the level K^p varies, these integral Shimura varieties form a projective system $(\mathrm{S}_{K^p})_{K^p}$ equipped with an action of $\mathbb{G}(\mathbb{A}_f^p)$ by Hecke correspondences. We have a family of isomorphisms

$$\mathrm{Sh}_{K_0 K^p} \simeq \mathrm{S}_{K^p} \otimes_{\mathcal{O}_{F,(p)}} F$$

which are compatible as the level K^p varies.

Notation. From now on, we identify F_p with $\mathbb{Q}_{p^2}$ and $\mathcal{O}_{F_p}$ with $\mathbb{Z}_{p^2}$. Moreover, the notation S_{K^p} will refer to the base change $\mathrm{S}_{K^p} \otimes_{\mathcal{O}_{F,(p)}} \mathbb{Z}_{p^2}$.

Therefore, under this convention we have isomorphisms $\mathrm{Sh}_{K_0 K^p} \otimes_F \mathbb{Q}_{p^2} \simeq \mathrm{S}_{\mathrm{K^p}} \otimes_{\mathbb{Z}_{p^2}} \mathbb{Q}_{p^2}$ compatible as the level K^p varies. Let $\overline{\mathrm{S}}_{K^p} := \mathrm{S}_{K^p} \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ denote the special fiber of the Shimura variety. Let $\overline{\mathrm{S}}_{K^p}^{\mathrm{ss}}$ denote the **supersingular locus** of the Shimura variety, ie. the locus of points $x \in \overline{\mathrm{S}}_{K^p}$ such that the universal abelian scheme is supersingular at x. Then $\overline{\mathrm{S}}_{K^p}^{\mathrm{ss}}$ is a closed subvariety of $\overline{\mathrm{S}}_{K^p}$, and its geometry can be described using the Rapoport-Zink space $\mathcal{M}$ in a process called p-adic uniformization, see [RZ96] and [Far04].
Let $x = [\mathcal{A}_x, \lambda_x, \iota_x, \overline{\eta}_x^p]$ be a geometric point of $\overline{\mathrm{S}}_{K^p}^{\mathrm{ss}}$. Since $\mathbb{G}$ satisfies the Hasse principle, according to [Far04] Proposition 3.1.8 the isogeny class of $(\mathcal{A}_x, \lambda_x, \iota_x)$ does not depend on the choice of x. The p-divisible group $\mathcal{A}_x[p^\infty]$ inherits an $\mathcal{O}_B \otimes \mathbb{Z}_p \simeq \mathrm{M}_d(\mathcal{O}_E)$-action from ι_A. Let $\mathbb{X}_x := \mathcal{O}_E^d \otimes_{\mathrm{M}_d(\mathcal{O}_E)} \mathcal{A}_x[p^\infty]$ with $\mathcal{O}_E$-action induced by the diagonal inclusion $\mathcal{O}_E \hookrightarrow \mathcal{O}_E^d$. According to [VW11] Section 6.3, $\mathbb{X}_x$ is a unitary p-divisible group of signature $(1, n-1)$ over $\mathbb{F}$ in the sense of Section 1. Let also $\mathcal{M}_x$ be the Rapoport-Zink space defined as in Section 1, but using $\mathbb{X}_x$ as a framing object. In particular $\mathcal{M}_x$ is a formal scheme over $\mathrm{Spf}(W(\mathbb{F}))$. There exists an isogeny $\mathbb{X} \otimes \mathbb{F} \to \mathbb{X}_x$ of unitary p-divisible group, inducing an isomorphism $\mathcal{M}_{W(\mathbb{F})} := \mathcal{M} \otimes_{\mathbb{Z}_{p^2}} W(\mathbb{F}) \xrightarrow{\sim} \mathcal{M}_x$, see [VW11] Section 6.4. The Rapoport-Zink space $\mathcal{M}_x$ is equipped with an action of the group $J_x(\mathbb{Q}_p)$ where J_x is the group of

quasi-isogenies of the unitary p-divisible group $\mathbb{X}_x$. The quasi-isogeny $\mathbb{X}\otimes\mathbb{F} \to \mathbb{X}_x$ identifies J_x with J and makes the isomorphism between the Rapoport-Zink spaces $J(\mathbb{Q}_p)$-equivariant. We define $I := \mathrm{Aut}(\mathcal{A}_x, \lambda_x, \iota_x)$ as a reductive group over $\mathbb{Q}$. Since x is in the supersingular locus, the group I is the inner form of $\mathbb{G}$ such that $I_{\mathbb{Q}_p} = J$ (in fact J_x, which is identified with J), $I_{\mathbb{A}_f} = \mathbb{G}_{\mathbb{A}_f^p}$ and $I(\mathbb{R}) \simeq \mathrm{GU}(0,n)$, which is the unique inner form of $G(\mathbb{R})$ that is compact modulo center. In particular, one can think of $I(\mathbb{Q})$ as a subgroup both of $J(\mathbb{Q}_p)$ and of $G(\mathbb{A}_f^p)$. Let $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss}}$ denote the formal completion of S_{K^p} along the supersingular locus. The p-adic uniformization theorem relates $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss}}$ with a certain quotient of $\mathcal{M}_x$, see [RZ96] Theorem 6.23. Using the isomorphism above, we may replace $\mathcal{M}_x$ with $\mathcal{M}_{W(\mathbb{F})}$ and obtain the following statement.

Theorem 3.2. *There is an isomorphism of formal schemes over* $\mathrm{Spf}(W(\mathbb{F}))$

$$\Theta_{K^p} : I(\mathbb{Q})\backslash \left(\mathcal{M}_{W(\mathbb{F})} \times \mathbb{G}(\mathbb{A}_f^p)/K^p\right) \xrightarrow{\sim} (\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss}} \otimes_{\mathbb{Z}_{p^2}} W(\mathbb{F})$$

which is compatible with the $\mathbb{G}(\mathbb{A}_f^p)$-action by Hecke correspondences as the level K^p varies.

This isomorphism is known as the **p-adic uniformization** of the supersingular locus. The induced map on the special fiber is an isomorphism

$$(\Theta_{K^p})_s : I(\mathbb{Q})\backslash \left(\mathcal{M}_{\mathrm{red}} \otimes_{\mathbb{F}_{p^2}} \mathbb{F} \times G(\mathbb{A}_f^p)/K^p\right) \xrightarrow{\sim} \overline{\mathrm{S}}_{K^p}^{\mathrm{ss}} \otimes_{\mathbb{F}_{p^2}} \mathbb{F}$$

of schemes over $\mathbb{F}$. The double coset space $I(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_f^p)/K^p$ is finite, so that we may fix a system of representatives $g_1, \ldots, g_s \in \mathbb{G}(\mathbb{A}_f^p)$. For every $1 \leqslant k \leqslant s$, we define $\Gamma_k := I(\mathbb{Q}) \cap g_k K^p g_k^{-1}$, which we see as a discrete subgroup of $J(\mathbb{Q}_p)$ that is cocompact modulo the center. The left hand side of the p-adic uniformization theorem is isomorphic to the disjoint union of the quotients $\Gamma_k\backslash\mathcal{M}_{W(\mathbb{F})}$. In particular for the special fiber, it is an isomorphism

$$(\Theta_{K^p})_s : \bigsqcup_{k=1}^{s} \Gamma_k\backslash(\mathcal{M}_{\mathrm{red}} \otimes \mathbb{F}) \xrightarrow{\sim} \overline{\mathrm{S}}_{K^p}^{\mathrm{ss}} \otimes \mathbb{F}.$$

Let $\Phi_{K^p}^k$ be the composition $\mathcal{M}_{\mathrm{red}} \otimes \mathbb{F} \to \Gamma_k\backslash(\mathcal{M}_{\mathrm{red}} \otimes \mathbb{F}) \to \overline{\mathrm{Sh}}_{C^p}^{\mathrm{ss}} \otimes \mathbb{F}$ and let Φ_{K^p} be the disjoint union of the $\Phi_{K^p}^k$. The map Φ_{K^p} is surjective. According to [VW11] Section 6.4, it is a local isomorphism which can be used to transport the Bruhat-Tits stratification from $\mathcal{M}_{\mathrm{red}}$ to $\overline{\mathrm{S}}_{K^p}^{\mathrm{ss}}$.

Proposition 3.3. *Let $\Lambda \in \mathcal{L}$. For any $1 \leqslant k \leqslant s$, the restriction of $\Phi_{K^p}^k$ to $\mathcal{M}_\Lambda \otimes \mathbb{F}$ is an isomorphism onto its image.*

We will denote by $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ the scheme theoretic image of $\mathcal{M}_\Lambda \otimes \mathbb{F}$ through Φ^k. A subscheme of the form $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ is called a **closed Bruhat-Tits stratum** of the Shimura variety. Together, they form the Bruhat-Tits stratification of the supersingular locus, whose combinatorics is described by the union of the complexes $\Gamma_k\backslash\mathcal{L}$.

4 The cohomology of the Rapoport-Zink space at maximal level

4.1 The spectral sequence associated to an open cover of $\mathcal{M}^{\mathrm{an}}$

The formal scheme $\mathcal{M}$ is special in the sense of [Ber96] since it is formally locally of finite type. Thus, we may consider the associated analytic space $\mathcal{M}^{\mathrm{an}}$ over $\mathbb{Q}_{p^2}$ in the sense of loc. cit. We note that $\mathcal{M}^{\mathrm{an}}$ is smooth, as follows from [RZ96] Proposition 5.17 (to be precise, this statement is about the rigid space $\mathcal{M}^{\mathrm{rig}}$ in the sense of Berthelot, but it is equivalent to the corresponding statement for $\mathcal{M}^{\mathrm{an}}$, see for instance [Far04] Lemme 2.3.24, or Appendice D for a brief summary of various comparisons between analytic, rigid and adic spaces). We refer to $\mathcal{M}^{\mathrm{an}}$ as the generic fiber of $\mathcal{M}$. It is equipped with a reduction (or specialization) map $\mathrm{red} : \mathcal{M}^{\mathrm{an}} \to \mathcal{M}_{\mathrm{red}}$ which is anticontinuous, ie. the preimage of a closed (resp. open) subset is open (resp. closed). If Z is a locally closed subset of $\mathcal{M}_{\mathrm{red}}$, then the preimage $\mathrm{red}^{-1}(Z)$ is called the **analytical tube over** Z. It is an analytic domain in $\mathcal{M}^{\mathrm{an}}$ and it coincides with the generic fiber of the formal completion of $\mathcal{M}_{\mathrm{red}}$ along Z. If $i \in \mathbb{Z}$ such that ni is even, then the tube $\mathrm{red}^{-1}(\mathcal{M}_i) = \mathcal{M}_i^{\mathrm{an}}$ is open and closed in $\mathcal{M}^{\mathrm{an}}$ and we have $\mathcal{M}^{\mathrm{an}} = \bigsqcup_{ni\in 2\mathbb{Z}} \mathcal{M}_i^{\mathrm{an}}$. If $\Lambda \in \mathcal{L}$, we define

$$U_\Lambda := \mathrm{red}^{-1}(\mathcal{M}_\Lambda),$$

the tube over $\mathcal{M}_\Lambda$. The action of $J(\mathbb{Q}_p)$ on $\mathcal{M}$ induces an action on the generic fiber $\mathcal{M}^{\mathrm{an}}$ such that red is $J(\mathbb{Q}_p)$-equivariant. By restriction it induces an action of J_Λ on U_Λ. The analytic space $\mathcal{M}^{\mathrm{an}}$ and each of the open subspaces U_Λ have dimension $n-1$.

We fix a prime number $\ell \neq p$. In [Ber93], Berkovich developped a theory of étale cohomology for his analytic spaces. Using it we may define the cohomology of the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}}$ by the formula

$$\begin{aligned}\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}}\widehat{\otimes}\,\mathbb{C}_p, \overline{\mathbb{Q}_\ell}) &:= \varinjlim_U \mathrm{H}_c^\bullet(U\widehat{\otimes}\,\mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \\ &= \varinjlim_U \varprojlim_n \mathrm{H}_c^\bullet(U\widehat{\otimes}\,\mathbb{C}_p, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes \overline{\mathbb{Q}_\ell}\end{aligned}$$

where U goes over all relatively compact open of $\mathcal{M}^{\mathrm{an}}$. These cohomology groups are equipped with commuting actions of $J(\mathbb{Q}_p)$ and of W, the absolute Weil group of $\mathbb{Q}_{p^2}$. The $J(\mathbb{Q}_p)$-action causes no problem of interpretation, but the W-action requires some explanations, see [Far04] Section 4.4.1. Let $\mathrm{Frob} = \sigma^{-2}$ be the geometric Frobenius in W. The inertia subgroup $I \subset W$ acts on $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}} \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ via the coefficients $\mathbb{C}_p$, whereas Frob acts via the **Weil descent datum** defined by Rapoport and Zink in [RZ96] 3.48. Let

$$F_{\mathbb{X}} : \mathbb{X} \otimes \mathbb{F} \to (\mathbb{X} \otimes \mathbb{F})^{(p^2)}$$

denote the Frobenius morphism relative to $\mathbb{F}_{p^2}$. Let $(\mathcal{M} \otimes W(\mathbb{F}))^{(p^2)}$ be the functor defined by

$$(\mathcal{M} \otimes W(\mathbb{F}))^{(p^2)}(S) := \mathcal{M}(S^{(p^2)}),$$

for all $W(\mathbb{F})$-scheme S where p is locally nilpotent. The Weil descent datum is the isomorphism $\alpha_{\mathrm{RZ}} : \mathcal{M} \otimes W(\mathbb{F}) \xrightarrow{\sim} (\mathcal{M} \otimes W(\mathbb{F}))^{(p^2)}$ given by $(X, \iota, \lambda, \rho) \in \mathcal{M}(S) \mapsto (X, \iota, \lambda, F_{\mathbb{X}} \circ \rho)$. We may describe this in terms of rational points and Dieudonné modules. If $k/\mathbb{F}$ is a perfect field extension, let $\tau := \mathrm{id} \otimes \sigma^2$ on $\mathbf{V}_k = \mathbf{V} \otimes_{\mathbb{Q}_{p^2}} W(k)_{\mathbb{Q}}$. Since we use covariant Dieudonné theory, the relative Frobenius $F_{\mathbb{X}}$ corresponds to the Verschiebung $\mathbf{V}^2$. By construction of $\mathbb{X}$, we have $\mathbf{V}^2 = p\tau^{-1}$ in $\mathbf{V}_k$. Therefore, α_{RZ} sends a Dieudonné module $M \in \mathcal{M}(k)$ to $p\tau^{-1}(M)$.

Remark 4.1. We stress that the Weil descent datum α_{RZ} is not effective, however the Rapoport-Zink space is defined over $\mathbb{Z}_{p^2}$, and this rational structure is induced by the effective descent datum $p^{-1}\alpha_{\mathrm{RZ}}$, with $p = p \cdot \mathrm{id} \in \mathrm{Z}(J(\mathbb{Q}_p))$.

We define

$$\varphi = (p^{-1} \cdot \mathrm{id}, \mathrm{Frob}) \in J(\mathbb{Q}_p) \times W.$$

The action of φ on the cohomology of $\mathcal{M}^{\mathrm{an}}$ coincides with the action of a geometric Frobenius induced by the effective descent datum $p^{-1}\alpha_{\mathrm{RZ}}$. Thus, we refer to φ as the **rational Frobenius element**.

Notation. To alleviate the notations, we will omit the coefficients $\mathbb{C}_p$. Thefore we write $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ and similarly for subspaces of $\mathcal{M}^{\mathrm{an}}$.

The cohomology groups $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ are concentrated in degrees 0 to $2(n-1)$. According to [Far04] Corollaire 4.4.7, these groups are smooth for the $J(\mathbb{Q}_p)$-action and continous for the I-action. For $g \in J(\mathbb{Q}_p)$, we have an isomorphism

$$g : \mathrm{H}_c^\bullet(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^\bullet(\mathcal{M}_{i+\alpha(g)}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}),$$

which is induced by g^{-1} and contravariance of cohomology. In particular, the action of Frob gives an isomorphism $\mathrm{H}^\bullet(\mathcal{M}_i, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^\bullet(\mathcal{M}_{i+2}, \overline{\mathbb{Q}_\ell})$. Let $(J(\mathbb{Q}_p) \times W)^\circ$ be the subgroup of $J(\mathbb{Q}_p) \times W$ consisting of all elements of the form $(g, u\mathrm{Frob}^j)$

with $u \in I$ and $\alpha(g) = -2j$. In fact, we have $(J(\mathbb{Q}_p) \times W)^\circ = (J^\circ \times I)\varphi^{\mathbb{Z}}$ where $J^\circ := \mathrm{Ker}(\alpha) \subset J(\mathbb{Q}_p)$, and $\alpha = v_p \circ c$ was introduced in Section 1.1. Each group $\mathrm{H}_c^\bullet(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is a $(J(\mathbb{Q}_p) \times W)^\circ$-representation, and we have an isomorphism

$$\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c} - \mathrm{Ind}_{(J(\mathbb{Q}_p)\times W)^\circ}^{J(\mathbb{Q}_p)\times W} \mathrm{H}_c^\bullet(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

In particular, when $\mathrm{H}_c^k(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is non-zero it is infinite dimensional. However, by [Far04] Proposition 4.4.13, these cohomology groups are always of finite type as $J(\mathbb{Q}_p)$-modules.

We introduce the Čech spectral sequence associated to the locally finite covering of $\mathcal{M}^{\mathrm{an}}$ by the U_Λ's. For $i \in \mathbb{Z}$ such that ni is even and for $0 \leqslant \theta \leqslant \theta_{\max}$, we denote by $\mathcal{L}_i^{(\theta)}$ the subset of $\mathcal{L}_i$ whose elements are those lattices of orbit type $2\theta + 1$. We also write $\mathcal{L}^{(\theta)}$ for the union of the $\mathcal{L}_i^{(\theta)}$. Then $\{U_\Lambda\}_{\Lambda \in \mathcal{L}^{(\theta_{\max})}}$ is an open cover of $\mathcal{M}^{\mathrm{an}}$. We may apply [Far04] Proposition 4.2.2 to deduce the existence of the following Čech spectral sequence computing the cohomology of the Rapoport-Zink space, concentrated in degrees $a \leqslant 0$ and $0 \leqslant b \leqslant 2(n-1)$,

$$E_1^{a,b} : \bigoplus_{\gamma \in I_{-a+1}} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}). \tag{E}$$

Here, for $s \geqslant 1$ the set I_s is defined by

$$I_s := \left\{ \gamma = (\Lambda^1, \ldots, \Lambda^s) \,\middle|\, \forall 1 \leqslant j \leqslant s, \Lambda^j \in \mathcal{L}^{(\theta_{\max})} \text{ and } U(\gamma) := \bigcap_{j=1}^{s} U_{\Lambda^j} \neq \varnothing \right\}.$$

Necessarily, if $\gamma = (\Lambda^1, \ldots, \Lambda^s) \in I_s$ then there exists a unique i such that ni is even and $\Lambda^j \in \mathcal{L}_i^{(\theta_{\max})}$ for all $J(\mathbb{Q}_p)$. We then define $\Lambda(\gamma) := \bigcap_{j=1}^s \Lambda^j \in \mathcal{L}_i$ so that $U(\gamma) = U_{\Lambda(\gamma)}$. In particular, the open subspace $U(\gamma)$ depends only on the intersection $\Lambda(\gamma)$ of the elements in the s-tuple γ.

For $s \geqslant 2$ and $\gamma = (\Lambda^1, \ldots, \Lambda^s) \in I_s$, define $\gamma_j := (\Lambda^1, \ldots, \widehat{\Lambda^j}, \ldots, \Lambda^s) \in I_{s-1}$ for the $(s-1)$-tuple obtained from γ by removing the j-th term. Besides, for $\Lambda, \Lambda' \in \mathcal{L}_i$ with $\Lambda' \subset \Lambda$, we write $f_{\Lambda',\Lambda}^b$ for the natural map $\mathrm{H}_c^b(U_{\Lambda'}, \overline{\mathbb{Q}_\ell}) \to \mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$ induced by the open immersion $U_{\Lambda'} \subset U_\Lambda$. For $a \leqslant -1$, the differential $E_1^{a,b} \to E_1^{a+1,b}$ is denoted by $\varphi^{a,b}$. It is the direct sum over all $\gamma \in I_{-a+1}$ of the maps

$$\begin{aligned} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) &\to \bigoplus_{\delta \in \{\gamma_1, \ldots \gamma_{-a+1}\}} \mathrm{H}_c^b(U(\delta), \overline{\mathbb{Q}_\ell}) \\ v &\mapsto \left(\sum_{\substack{j=1 \\ \gamma_j = \delta}}^{-a+1} (-1)^{j+1} f_{\Lambda(\gamma),\Lambda(\gamma_j)}^b(v) \right)_{\delta \in \{\gamma_1, \ldots, \gamma_{-a+1}\}} \end{aligned}.$$

An element $g \in J(\mathbb{Q}_p)$ acts on the set I_s by sending γ to $g\cdot\gamma := (g\Lambda^1, \dots, g\Lambda^s)$. The action of g^{-1} induces an isomorphism

$$\mathrm{H}^\bullet_c(U(\gamma), \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^\bullet_c(U(g\cdot\gamma), \overline{\mathbb{Q}_\ell}).$$

Likewise, Frob $\in W$ induces an isomorphism $\mathrm{H}^\bullet_c(U(\gamma), \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^\bullet_c(U(p\cdot\gamma), \overline{\mathbb{Q}_\ell})$. This defines a natural $J(\mathbb{Q}_p) \times W$-action on the terms $E_1^{a,b}$, with respect to which the spectral sequence is equivariant.

In order to analyze the spectral sequence (E), we begin by relating the cohomology of a tube U_Λ to the cohomology of the corresponding closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$. Note that by restriction, $\mathrm{H}^\bullet_c(U_\Lambda, \overline{\mathbb{Q}_\ell})$ is naturally a representation of the subgroup $(J_\Lambda \times I)\varphi^{\mathbb{Z}} \subset J(\mathbb{Q}_p) \times W$.

Proposition 4.2. *Let $\Lambda \in \mathcal{L}$ and let $0 \leqslant b \leqslant 2(n-1)$. There is a $(J_\Lambda \times I)\varphi^{\mathbb{Z}}$-equivariant isomorphism*

$$\mathrm{H}^b(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$$

where, on the left-hand side, the inertia I acts trivially and φ acts like the geometric Frobenius Frob.

In particular, the inertia acts trivially on the cohomology of U_Λ.

Proof. The closed subvariety $\mathcal{M}_\Lambda \subset \mathcal{M}_{\mathrm{red}}$ is bounded in the sense of [RZ96] Paragraph 2.30. Indeed, it is irreducible and all irreducible components of $\mathcal{M}_{\mathrm{red}}$ are bounded by the proof of loc. cit. Proposition 2.32. Thus, there exists a quasi-compact open formal subscheme $\mathcal{U}$ of $\mathcal{M}$ containing $\mathcal{M}_\Lambda$ (these are denoted by U^f and are introduced in the proof of Theorem 2.16 in loc. cit.). The formal scheme $\mathcal{U}$ is of finite type, in particular the structure morphism $\mathcal{U} \to \mathrm{Spf}(\mathbb{Z}_{p^2})$ is adic. Since $\mathcal{M}$ is formally smooth, $\mathcal{U}$ is actually a smooth formal scheme. Replacing $\mathcal{U}$ by $J_\Lambda \cdot \mathcal{U}$, we may assume that $\mathcal{U}$ is stable under the action of J_Λ.
Let $\mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell}$ denote Berkovich's nearby cycles on $\mathcal{U}_{\mathrm{red}}$ as defined in [Ber94]. Since $\mathcal{U}$ is smooth, by Corollary 5.4 of loc. cit. we actually have $\mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell} \simeq \overline{\mathbb{Q}_\ell}$. Besides, let $\mathrm{R}\widetilde{\lambda}_*\overline{\mathbb{Q}_\ell}$ denote Huber's nearby cycles as defined in [Hub98] Paragraph 3.12, where $\widetilde{\lambda} : \widetilde{d}(\mathcal{U}) \to \mathcal{U}$ is the natural reduction map attached to the adic space $\widetilde{d}(\mathcal{U})$ associated to the formal scheme $\mathcal{U}$. Since the etale sites of $\mathcal{U}$ and of $\mathcal{U}_{\mathrm{red}}$ are naturally identified, we can think of $\mathrm{R}\widetilde{\lambda}_*\overline{\mathbb{Q}_\ell}$ as an object of the derived category of ℓ-adic sheaves on $\mathcal{U}_{\mathrm{red}}$. According to [Far04] Section 5.4.2, both notions of nearby cycles coincide, ie.

$$\mathrm{R}\widetilde{\lambda}_*\overline{\mathbb{Q}_\ell} \simeq \mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell} \simeq \overline{\mathbb{Q}_\ell}.$$

In particular, the inertia acts trivially on the nearby cycles. Let $\mathcal{U}^\wedge_{|\mathcal{M}_\Lambda}$ denote the formal completion of $\mathcal{U}$ along $\mathcal{M}_\Lambda$. Since $\mathcal{U}$ is open in $\mathcal{M}$, it coincides with the formal completion of $\mathcal{M}$ along $\mathcal{M}_\Lambda$. Thus, we have $(\mathcal{U}^{\wedge}_{|\mathcal{M}_\Lambda})^{\mathrm{an}} = U_\Lambda$. Moreover, $\widetilde{d}(\mathcal{U}^{\wedge}_{|\mathcal{M}_\Lambda}) = U_\Lambda^{\mathrm{rig}}$ according to [Far04] Appendice D, where $(\,\cdot\,)^{\mathrm{rig}}$ is the natural functor from the category of Hausdorff analytic spaces to the category of quasiseparated adic spaces. Therefore, by [Hub96] Theorem 8.3.5.iii) we have an isomorphism $\mathrm{H}^b(U_\Lambda, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^b(\widetilde{d}(\mathcal{U}^{\wedge}_{|\mathcal{M}_\Lambda}) \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$. Moreover, by [Hub98] Proposition 3.15 applied to the smooth formal scheme $\mathcal{U}$, we have

$$\mathrm{H}^b(\widetilde{d}(\mathcal{U}^{\wedge}_{|\mathcal{M}_\Lambda}) \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^b(\mathcal{M}_\Lambda \otimes \mathbb{F}, (\mathrm{R}\widetilde{\lambda}_* \overline{\mathbb{Q}_\ell})_{|\mathcal{M}_\Lambda}) = \mathrm{H}^b(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}).$$

The isomorphisms are compatible with the actions of J_Λ and of the Frobenius. □

Corollary 4.3. *Let $\Lambda \in \mathcal{L}$ and let $0 \leqslant b \leqslant 2(n-1)$. There is a $(J_\Lambda \times I)\varphi^{\mathbb{Z}}$-equivariant isomorphism*

$$\mathrm{H}^b_c(U_\Lambda, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^{b-2(n-1-\theta)}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})(n-1-\theta)$$

where $t(\Lambda) = 2\theta + 1$.

Proof. This is a consequence of algebraic and analytic Poincaré duality, respectively for U_Λ and for $\mathcal{M}_\Lambda$. Indeed, we have

$$\begin{aligned}
\mathrm{H}^b_c(U_\Lambda, \overline{\mathbb{Q}_\ell}) &\simeq \mathrm{H}^{2(n-1)-b}(U_\Lambda, \overline{\mathbb{Q}_\ell})^\vee(n-1)\\
&\simeq \mathrm{H}^{2(n-1)-b}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})^\vee(n-1)\\
&\simeq \mathrm{H}^{b-2(n-1-\theta)}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})(n-1-\theta).
\end{aligned}$$

□

Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$. If λ is a partition of $2\theta + 1$, recall the unipotent irreducible representation ρ_λ of $\mathrm{GU}(V_\Lambda^0) \simeq \mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ that we introduced in Section 2. It can be inflated to the maximal reductive quotient $\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{U}(V_\Lambda^0) \times \mathrm{U}(V_\Lambda^1))$, and then to the maximal parahoric subgroup J_Λ. With an abuse of notation, we still denote this inflated representation by ρ_λ. In virtue of Theorem 2.5, the isomorphism in the last paragraph translates into the following result.

Proposition 4.4. *Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$. The following statements hold.*

1. *The cohomology group $\mathrm{H}^b_c(U_\Lambda, \overline{\mathbb{Q}_\ell})$ is zero unless $2(n-1-\theta) \leqslant b \leqslant 2(n-1)$.*
2. *The action of J_Λ on the cohomology factors through an action of the finite group of Lie type $\mathrm{GU}(V_\Lambda^0)$. The rational Frobenius φ acts like multiplication by $(-p)^b$ on $\mathrm{H}^b_c(U_\Lambda, \overline{\mathbb{Q}_\ell})$.*

(3) *For $0 \leqslant b \leqslant \theta$ we have*

$$\mathrm{H}_c^{2b+2(n-1-\theta)}(U_\Lambda, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(b,\theta-b)} \rho_{(2\theta+1-2s,2s)}.$$

For $0 \leqslant b \leqslant \theta - 1$ we have

$$\mathrm{H}_c^{2b+1+2(n-1-\theta)}(U_\Lambda, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(b,\theta-1-b)} \rho_{(2\delta-2s,2s+1)}.$$

The description of the rational Frobenius action yields the following corollary.

Corollary 4.5. *The spectral sequence degenerates on the second page E_2. For $0 \leqslant b \leqslant 2(n-1)$, the induced filtration on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ splits, ie. we have an isomorphism*

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{b \leqslant b' \leqslant 2(n-1)} E_2^{b-b',b'}.$$

The action of W on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is trivial on the inertia subgroup and the action of the rational Frobenius element φ is semisimple. The subspace $E_2^{b-b',b'}$ is identified with the eigenspace of φ associated to the eigenvalue $(-p)^{b'}$.

We stress that in the previous statement, the terms $E_2^{b-b',b'}$ may be zero.

Proof. The (a,b)-term in the first page of the spectral sequence is the direct sum of the cohomology groups $H_c^b(U(\gamma), \overline{\mathbb{Q}_\ell})$ for all $\gamma \in I_{-a+1}$. On each of these cohomology groups, the rational Frobenius φ acts via multiplication by $(-p)^b$. This action is in particular independant of γ and of a. Thus, on the b-th row of the first page of the sequence, the Frobenius acts everywhere as multiplication by $(-p)^b$. Starting from the second page, the differentials in the sequence connect two terms lying in different rows. Since the differentials are equivariant for the φ-action, they must all be zero. Thus, the sequence degenerates on the second page. By the machinery of spectral sequences, there is a filtration on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ whose graded factors are given by the terms $E_2^{b-b',b'}$ of the second page. Only a finite number of these terms are non-zero, and since they all lie on different rows, the Frobenius φ acts via multiplication by a different scalar on each graded factor of the filtration. It follows that the filtration splits, ie. the abutment is the direct sum of the graded pieces of the filtration, as they correspond to the eigenspaces of φ. Consequently, its action is semisimple. $\square$

The spectral sequence $E_1^{a,b}$ has non-zero terms extending indefinitely in the range $a \leqslant 0$. For instance, if $\Lambda \in \mathcal{L}^{(\theta_{\max})}$ then $(\Lambda, \dots, \Lambda) \in I_{-a+1}$ so that $E_1^{a,b} \neq 0$ for all $a \leqslant 0$ and $2(n-1-\theta_{\max}) \leqslant b \leqslant 2(n-1)$. To rectify this, we introduce the alternating Čech spectral sequence. If $v \in E_1^{a,b}$ and $\gamma \in I_{-a+1}$, we denote by $v_\gamma \in \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell})$ the component of v in the summand of $E_1^{a,b}$ indexed by γ. Besides, if $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1}) \in I_{-a+1}$ and if $\sigma \in \mathfrak{S}_{-a+1}$ then we write $\sigma(\gamma) := (\Lambda^{\sigma(1)}, \ldots, \Lambda^{\sigma(-a+1)}) \in I_{-a+1}$. For all a, b we define

$$E_{1,\mathrm{alt}}^{a,b} := \{v \in E_1^{a,b} \,|\, \forall \gamma \in I_{-a+1}, \forall \sigma \in \mathfrak{S}_{-a+1}, v_{\sigma(\gamma)} = \mathrm{sgn}(\sigma) v_\gamma\}.$$

In particular, if $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1})$ with $\Lambda^j = \Lambda^{j'}$ for some $j \neq j'$ then $v \in E_{1,\mathrm{alt}}^{a,b} \implies v_\gamma = 0$. The subspace $E_{1,\mathrm{alt}}^{a,b} \subset E_1^{a,b}$ is stable under the action of $J(\mathbb{Q}_p) \times W$, and the differential $\varphi^{a,b} : E_1^{a,b} \to E_1^{a+1,b}$ sends $E_{1,\mathrm{alt}}^{a,b}$ to $E_{1,\mathrm{alt}}^{a+1,b}$. Thus, for all b we have a chain complex $E_{1,\mathrm{alt}}^{\bullet,b}$ and the following proposition is well-known, see eg. [Sta23] Lemma 01FM.

Proposition 4.6. *The inclusion map $E_{1,\mathrm{alt}}^{\bullet,b} \hookrightarrow E_1^{\bullet,b}$ is a homotopy equivalence. In particular we have canonical isomorphisms $E_{2,\mathrm{alt}}^{a,b} \simeq E_2^{a,b}$ for all a, b.*

The advantage of the alternating Čech spectral sequence is that it is concentrated in a finite strip. Indeed, if $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1}) \in I_{-a+1}$, let $i \in \mathbb{Z}$ such that $\Lambda(\gamma) \in \mathcal{L}_i$. Then all the Λ^j's belong to the set of lattices in $\mathcal{L}_i^{(\theta_{\max})}$ containing $\Lambda(\gamma)$. This set is finite of cardinality $\nu(n-\theta-\theta_{\max}-1, n-2\theta-1)$ where $t(\Lambda(\gamma)) = 2\theta+1$ according to Proposition 1.16. Thus, if $-a+1$ is big enough then all the γ's in I_{-a+1} will have some repetition, so that $E_{1,\mathrm{alt}}^{a,b} = 0$.

Remark 4.7. The Lemma 01FM of [Sta23] is stated in the context of Čech cohomology of an abelian presheaf $\mathcal{F}$ on a topological space X. However, the proof may be adapted to Čech homology of precosheaves such as $U \mapsto \mathrm{H}_c^b(U, \overline{\mathbb{Q}_\ell})$.

For $a = 0$, we have $E_{1,\mathrm{alt}}^{0,b} = E_1^{0,b}$ by definition. Let us consider the cases $b = 2(n-1-\theta_{\max})$ and $b = 2(n-1-\theta_{\max})+1$. For such b, it follows from 4.4 that $\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell}) = 0$ if $t(\Lambda) < t_{\max}$. If $a \leqslant -1$, we have $-a+1 \geqslant 2$ so that for all $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1}) \in I_{-a+1}$, if there exists $j \neq j'$ such that $\Lambda^j \neq \Lambda^{j'}$, then $t(\Lambda(\gamma)) < t_{\max}$ so that $\mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) = 0$. It follows that $E_{1,\mathrm{alt}}^{a,b} = 0$ for all $a \leqslant -1$ and b as above. This observation, along with the previous paragraph, yields the following proposition.

Proposition 4.8. *We have $E_2^{0,2(n-1-\theta_{\max})} \simeq E_1^{0,2(n-1-\theta_{\max})}$. If moreover $\theta_{\max} \geqslant 1$ (ie. $n \geqslant 3$), then we have $E_2^{0,2(n-1-\theta_{\max})+1} \simeq E_1^{0,2(n-1-\theta_{\max})+1}$ as well.*

In order to study the action of $J(\mathbb{Q}_p)$, we may rewrite $E_1^{a,b}$ conveniently in terms of compactly induced representations. To do this, let us introduce a few more notations. For $0 \leqslant \theta \leqslant \theta_{\max}$ and $s \geqslant 1$, we define

$$I_s^{(\theta)} := \{\gamma \in I_s \,|\, t(\Lambda(\gamma)) = 2\theta + 1\}.$$

The subset $I_s^{(\theta)} \subset I_s$ is stable under the action of $J(\mathbb{Q}_p)$. We denote by $N(\Lambda_\theta)$ the set of lattices $\Lambda \in \mathcal{L}_0$ of maximal orbit type containing Λ_θ. For $s \geqslant 1$ we define

$$K_s^{(\theta)} := \{\delta = (\Lambda^1, \ldots, \Lambda^s) \,|\, \forall 1 \leqslant j \leqslant s, \Lambda^j \in N(\Lambda_\theta) \text{ and } \Lambda(\delta) = \Lambda_\theta\}.$$

Then $K_s^{(\theta)}$ is a finite subset of $I_s^{(\theta)}$ and it is stable under the action of J_θ. If $\gamma \in I_s^{(\theta)}$, there exists some $g \in J(\mathbb{Q}_p)$ such that $g \cdot \Lambda(\gamma) = \Lambda_\theta$ since both lattices share the same orbit type. Moreover, the coset $J_\theta \cdot g$ is uniquely determined, and $g \cdot \gamma$ is an element of $K_s^{(\theta)}$. This mapping results in a natural bijection between the orbit sets

$$J \backslash I_s^{(\theta)} \xrightarrow{\sim} J_\theta \backslash K_s^{(\theta)}.$$

The bijection sends the orbit $J \cdot \alpha$ to the orbit $J_\theta \cdot (g \cdot \alpha)$ where g is chosen as above. The inverse sends an orbit $J_\theta \cdot \beta$ to $J \cdot \beta$. We note that both orbit sets are finite. We may now rearrange the terms in the spectral sequence.

Proposition 4.9. *We have an isomorphism*

$$\begin{aligned} E_1^{a,b} &\simeq \bigoplus_{\theta=0}^{\theta_{\max}} \bigoplus_{[\delta] \in J_\theta \backslash K_{-a+1}^{(\theta)}} \mathrm{c}-\mathrm{Ind}_{\mathrm{Fix}(\delta)}^{J} \mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)} \\ &\simeq \bigoplus_{\theta=0}^{\theta_{\max}} \mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \left(\mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell}) \otimes \overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}] \right), \end{aligned}$$

where $\overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}]$ is the permutation representation associated to the action of J_θ on the finite set $K_{-a+1}^{(\theta)}$.

Remark 4.10. For $\delta \in K_s^{(\theta)}$, the group $\mathrm{Fix}(\delta)$ consists of the elements $g \in J(\mathbb{Q}_p)$ such that $g \cdot \delta = \delta$. Any such g satisfies $g\Lambda(\delta) = \Lambda(\delta)$, and since $\Lambda(\delta) = \Lambda_\theta$ we have $\mathrm{Fix}(\delta) \subset J_\theta$. If $\delta = (\Lambda^1, \ldots, \Lambda^s)$ then $\mathrm{Fix}(\delta)$ is the intersection of the maximal parahoric subgroups $J_{\Lambda^1}, \ldots, J_{\Lambda^s}$. We note that in general, $\mathrm{Fix}(\delta)$ is itself not a parahoric subgroup of $J(\mathbb{Q}_p)$ since the lattices $\Lambda^1, \ldots, \Lambda^s$ need not form a simplex in $\mathcal{L}$, as they all share the same orbit type. If however $\Lambda^1 = \ldots = \Lambda^s$ then $\mathrm{Fix}(\delta) = J_{\Lambda^1}$ is a conjugate of the maximal parahoric subgroup $J_{\theta_{\max}}$.

Proof. First, by decomposing I_{-a+1} as the disjoint union of the $I_{-a+1}^{(\theta)}$ for $0 \leqslant \theta \leqslant \theta_{\max}$, we may write

$$E_1^{a,b} = \bigoplus_{\theta=0}^{\theta_{\max}} \bigoplus_{\gamma \in I_{-a+1}^{(\theta)}} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}).$$

For each orbit $X \in J\backslash I^{(\theta)}_{-a+1}$, we fix a representative δ_X which lies in $K^{(\theta)}_{-a+1}$. We may write

$$E_1^{a,b} = \bigoplus_{\theta=0}^{\theta_{\max}} \bigoplus_{X \in J\backslash I^{(\theta)}_{-a+1}} \bigoplus_{\gamma \in X} \mathrm{H}^b_c(U(\gamma), \overline{\mathbb{Q}_\ell}) = \bigoplus_{\theta=0}^{\theta_{\max}} \bigoplus_{X \in J\backslash I^{(\theta)}_{-a+1}} \bigoplus_{g \in J/\mathrm{Fix}(\delta_X)} g \cdot \mathrm{H}^b_c(U(\delta_X), \overline{\mathbb{Q}_\ell}).$$

The rightmost sum can be identified with a compact induction from $\mathrm{Fix}(\delta_X)$ to $J(\mathbb{Q}_p)$. Identifying the orbit sets $J\backslash I^{(\theta)}_{-a+1} \xrightarrow{\sim} J_\theta\backslash K^{(\theta)}_{-a+1}$, we have

$$E_1^{a,b} \simeq \bigoplus_{\theta=0}^{\theta_{\max}} \bigoplus_{[\delta] \in J_\theta\backslash K^{(\theta)}_{-a+1}} \mathrm{c}-\mathrm{Ind}^J_{\mathrm{Fix}(\delta)} \mathrm{H}^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)}.$$

By transitivity of compact induction, we have

$$\mathrm{c}-\mathrm{Ind}^J_{\mathrm{Fix}(\delta)} \mathrm{H}^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)} = \mathrm{c}-\mathrm{Ind}^J_{J_\theta} \mathrm{c}-\mathrm{Ind}^{J_\theta}_{\mathrm{Fix}(\delta)} \mathrm{H}^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)}.$$

Since $H^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)}$ is the restriction of a representation of J_θ to $\mathrm{Fix}(\delta)$, applying compact induction from $\mathrm{Fix}(\delta)$ to J_θ results in tensoring with the permutation representation of $J_\theta/\mathrm{Fix}(\delta)$. Thus

$$\begin{aligned} E_1^{a,b} &\simeq \bigoplus_{\theta=0}^{\theta_{\max}} \bigoplus_{[\delta] \in J_\theta\backslash K^{(\theta)}_{-a+1}} \mathrm{c}-\mathrm{Ind}^J_{J_\theta} \left(\mathrm{H}^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell}) \otimes \overline{\mathbb{Q}_\ell}[J_\theta/\mathrm{Fix}(\delta)]\right) \\ &\simeq \bigoplus_{\theta=0}^{\theta_{\max}} \mathrm{c}-\mathrm{Ind}^J_{J_\theta} \left(\mathrm{H}^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell}) \otimes \bigoplus_{[\delta] \in J_\theta\backslash K^{(\theta)}_{-a+1}} \overline{\mathbb{Q}_\ell}[J_\theta/\mathrm{Fix}(\delta)]\right), \end{aligned}$$

where on the second line we used additivity of compact induction. Now, $J_\theta/\mathrm{Fix}(\delta)$ is identified with the J_θ-orbit $J_\theta \cdot \delta$ of δ in $K^{(\theta)}_{-a+1}$, so that

$$\bigoplus_{[\delta] \in J_\theta\backslash K^{(\theta)}_{-a+1}} \overline{\mathbb{Q}_\ell}[J_\theta/\mathrm{Fix}(\delta)] \simeq \overline{\mathbb{Q}_\ell}[\bigsqcup_{[\delta] \in J_\theta\backslash K^{(\theta)}_{-a+1}} J_\theta \cdot \delta] \simeq \overline{\mathbb{Q}_\ell}[K^{(\theta)}_{-a+1}],$$

which concludes the proof. □

By Proposition 1.9, we may identify $N(\Lambda_\theta)$ with the set $N(n-\theta-\theta_{\max}-1, V^1_\theta)$ as defined in Section 1.4. Thus, for $s \geqslant 1$, $K^{(\theta)}_s$ is naturally identified with

$$\overline{K}^{(\theta)}_s \simeq \left\{ \overline{\delta} = (U^1, \ldots, U^s) \,\middle|\, \forall 1 \leqslant j \leqslant s, U^j \in N(n-\theta-\theta_{\max}-1, V^1_\theta) \text{ and } \sum_{j=1}^s U^j = V^1_\theta \right\}.$$

The action of J_θ on $K_s^{(\theta)}$ corresponds to the natural action of $\mathrm{GU}(V_\theta^1)$ on $\overline{K}_s^{(\theta)}$, which factors through an action of the finite projective unitary group $\mathrm{PU}(V_\theta^1) := \mathrm{U}(V_\theta^1)/\mathrm{Z}(\mathrm{U}(V_\theta^1)) \simeq \mathrm{GU}(V_\theta^1)/\mathrm{Z}(\mathrm{GU}(V_\theta^1))$. Thus, the representation $\overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}]$ is the inflation to J_θ of the representation $\overline{\mathbb{Q}_\ell}[\overline{K}_{-a+1}^{(\theta)}]$ of the finite projective unitary group $\mathrm{PU}(V_\theta^1)$. When $\theta = \theta_{\max}$ or when $s = 1$, we trivially have the following proposition.

Proposition 4.11. *For $s \geqslant 1$, we have $\overline{\mathbb{Q}_\ell}[K_s^{(\theta_{\max})}] = \mathbf{1}$. For $0 \leqslant \theta \leqslant \theta_{\max} - 1$, we have $\overline{\mathbb{Q}_\ell}[K_1^{(\theta)}] = 0$.*

Proof. If $\delta = (\Lambda^1, \ldots, \Lambda^s) \in K_s^{(\theta_{\max})}$ then $\Lambda(\delta) = \Lambda_{\theta_{\max}}$ has maximal orbit type $t_{\max} = 2\theta_{\max} + 1$. For any $1 \leqslant j \leqslant s$ we have $\Lambda_{\theta_{\max}} \subset \Lambda^j$, therefore $\Lambda^1 = \ldots = \Lambda^s = \Lambda_{\theta_{\max}}$. Thus $K_s^{(\theta_{\max})}$ is a singleton and so $\overline{\mathbb{Q}_\ell}[K_s^{(\theta_{\max})}]$ is trivial. Besides, if $\theta < \theta_{\max}$ then $K_1^{(\theta)}$ is clearly empty. □

Recall Proposition 4.8. We obtain the following corollary.

Corollary 4.12. *We have*

$$E_1^{0,b} \simeq \mathrm{c} - \mathrm{Ind}_{J_{\theta_{\max}}}^{J} \mathrm{H}_c^b(U_{\Lambda_{\theta_{\max}}}, \overline{\mathbb{Q}_\ell}).$$

In particular, we have

$$E_2^{0,b} \simeq \begin{cases} \mathrm{c} - \mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_{(2\theta_{\max}+1)} & \text{if } b = 2(n-1-\theta_{\max}), \\ \mathrm{c} - \mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_{(2\theta_{\max},1)} & \text{if } m \geqslant 1 \text{ and } b = 2(n-1-\theta_{\max}) + 1. \end{cases}$$

Remark 4.13. The representation $\rho_{(2\theta_{\max}+1)} = \mathbf{1}$ is the trivial representation of $J_{\theta_{\max}}$.

Let us now consider the top row of the spectral sequence, corresponding to $b = 2(n-1)$. For $\Lambda' \subset \Lambda$, recall the map $f_{\Lambda',\Lambda}^{2(n-1)} : \mathrm{H}_c^{2(n-1)}(U_{\Lambda'}, \overline{\mathbb{Q}_\ell}) \to \mathrm{H}_c^{2(n-1)}(U_\Lambda, \overline{\mathbb{Q}_\ell})$. By Poincaré duality, it is the dual map of the restriction morphism $\mathrm{H}^0(U_\Lambda, \overline{\mathbb{Q}_\ell}) \to \mathrm{H}^0(U_{\Lambda'}, \overline{\mathbb{Q}_\ell})$. Both spaces are one-dimensional by Proposition 4.2, and the restriction morphism is the identity. Thus, $E_1^{a,2(n-1)}$ is the $\overline{\mathbb{Q}_\ell}$-vector space generated by I_{-a+1}, and the differential $\varphi^{a,2(n-1)}$ is given by

$$\gamma \in I_{-a+1} \mapsto \sum_{j=1}^{-a+1} (-1)^{j+1} \gamma_j.$$

Using this description, we may compute the highest cohomology group $\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ explicitely.

Proposition 4.14. *There is an isomorphism*

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c-Ind}_{J^\circ}^{J} \mathbf{1},$$

and the rational Frobenius φ acts via multiplication by $p^{2(n-1)}$.

Proof. The statement on the Frobenius action is already known by Corollary 4.5. Besides, we have $\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq E_2^{0,2(n-1)} = \mathrm{Coker}(\varphi^{-1,2(n-1)})$. The differential $\varphi^{-1,2(n-1)}$ is described by

$$\begin{aligned} (\Lambda, \Lambda) &\mapsto 0, && \forall \Lambda \in \mathcal{L}^{(\theta_{\max})}, \\ (\Lambda, \Lambda') &\mapsto (\Lambda') - (\Lambda), && \forall \Lambda, \Lambda' \in \mathcal{L}^{(\theta_{\max})} \text{ such that } U_\Lambda \cap U_{\Lambda'} \neq \varnothing. \end{aligned}$$

Let $i \in \mathbb{Z}$ such that ni is even, and let $\Lambda, \Lambda' \in \mathcal{L}_i^{(\theta_{\max})}$. Since the Bruhat-Tits building $\mathrm{BT}(\widetilde{J}, \mathbb{Q}_p) \simeq \mathcal{L}_i$ is connected, there exists a sequence $\Lambda = \Lambda^0, \ldots, \Lambda^d = \Lambda'$ of lattices in $\mathcal{L}_i$ such that for all $0 \leqslant j \leqslant d-1$, $\{\Lambda^j, \Lambda^{j+1}\}$ is an edge in $\mathcal{L}_i$. Assume that $d \geqslant 0$ is minimal satisfying this property. Since $t(\Lambda) = t(\Lambda') = t_{\max}$, the integer d is even and we may assume that $t(\Lambda^j)$ is equal to $t_{\max}$ when j is even, and equal to 1 when j is odd. In particular, for all $0 \leqslant j \leqslant \frac{d}{2} - 1$ we have $\Lambda^{2j}, \Lambda^{2j+2} \in \mathcal{L}_i^{(\theta_{\max})}$ and $U_{\Lambda^{2j}} \cap U_{\Lambda^{2j+2}} \neq \varnothing$. Consider the vector

$$w := \sum_{j=0}^{\frac{d}{2}-1} (\Lambda^{2j}, \Lambda^{2j+2}) \in E_1^{-1,2(n-1)}.$$

Then we compute $\varphi^{-1,2(n-1)}(w) = (\Lambda') - (\Lambda)$. It follows that for all $\Lambda, \Lambda' \in \mathcal{L}_i$, we have $(\Lambda) \cong (\Lambda')$ in $\mathrm{Coker}(\varphi^{-1,2(n-1)})$. Thus, $\mathrm{Coker}(\varphi_1^{2(n-1)})$ consists of one copy of $\overline{\mathbb{Q}_\ell}$ for each $i \in \mathbb{Z}$ such that ni is even. Considering the action of $J(\mathbb{Q}_p)$ as well, it readily follows that $\mathrm{Coker}(\varphi^{-1,2(n-1)}) \simeq \mathrm{c-Ind}_{J^\circ}^{J} \mathbf{1}$. □

Remark 4.15. The cohomology group $\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ can also be computed in another way which does not require the spectral sequence. Indeed, we have an isomorphism

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c-Ind}_{J^\circ}^{J} \mathrm{H}_c^{2(n-1)}(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

By definition, we have

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) = \varinjlim_{U} \mathrm{H}_c^{2(n-1)}(U \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell}),$$

where U runs over the relatively compact open subspaces of $\mathcal{M}_0^{\mathrm{an}}$. Since U is smooth, Poincaré duality gives

$$\mathrm{H}_c^{2(n-1)}(U \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^0(U \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})^\vee.$$

Using the connectedness of the Bruhat-Tits building $\mathrm{BT}(\widetilde{J}, \mathbb{Q}_p) \simeq \mathcal{L}_0$, one may prove that $\mathcal{M}_0^{\mathrm{an}}$ is connected. Thus we can insure that all the U's involved in the limit are connected as well. Therefore $\mathrm{H}^0(U \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \simeq \overline{\mathbb{Q}_\ell}$, and all the transition maps in the direct limit are identity. It follows that $\mathrm{H}_c^{2(n-1)}(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is trivial.

4.2 Compactly induced representations and type theory

Let $\mathrm{Rep}(J(\mathbb{Q}_p))$ denote the category of smooth $\overline{\mathbb{Q}_\ell}$-representations of $J(\mathbb{Q}_p)$. Let χ be a continuous character of the center $\mathrm{Z}(J(\mathbb{Q}_p)) \simeq \mathbb{Q}_{p^2}^\times$ and let $V \in \mathrm{Rep}(J(\mathbb{Q}_p))$. We define **the maximal quotient of V on which the center acts like** χ as follows. Let us consider the set

$$\Omega := \{W \mid W \text{ is a subrepresentation of } V \text{ and } \mathrm{Z}(J(\mathbb{Q}_p)) \text{ acts like } \chi \text{ on } V/W\}.$$

The set Ω is stable under arbitrary intersection, so that $W_\circ := \bigcap_{W \in \Omega} W \in \Omega$. The maximal quotient is defined by

$$V_\chi := V/W_\circ.$$

It satisfies the following universal property.

Proposition 4.16. *Let χ be a continuous character of $\mathrm{Z}(J(\mathbb{Q}_p))$ and let $V, V' \in \mathrm{Rep}(J(\mathbb{Q}_p))$. Assume that $\mathrm{Z}(J(\mathbb{Q}_p))$ acts like χ on V'. Then any morphism $V \to V'$ factors through V_χ.*

Proof. Let $f : V \to V'$ be a morphism of $J(\mathbb{Q}_p)$-representations. Since $V/\mathrm{Ker}(f) \simeq \mathrm{Im}(f) \subset V'$, the center $\mathrm{Z}(J(\mathbb{Q}_p))$ acts like χ on the quotient $V/\mathrm{Ker}(f)$. Therefore $\mathrm{Ker}(f) \in \Omega$. It follows that $\mathrm{Ker}(f)$ contains $W_\circ$ and as a consequence, f factors through V_χ. □

The terms $E_1^{a,b}$ of the spectral sequence (E) consist of representations of the form

$$\mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \rho,$$

where ρ is the inflation to J_θ of a representation of the finite group of Lie type $\mathcal{J}_\theta$. We note that such a compactly induced representation does not contain any smooth irreducible subrepresentation of $J(\mathbb{Q}_p)$. Indeed, the center $\mathrm{Z}(J(\mathbb{Q}_p)) \simeq \mathbb{Q}_{p^2}^\times$ does not fix any finite dimensional subspace. In order to rectify this, it is customary to fix a continuous character χ of $\mathrm{Z}(J(\mathbb{Q}_p))$ which agrees with the central character of ρ on $\mathrm{Z}(J(\mathbb{Q}_p)) \cap J_\theta \simeq \mathbb{Z}_{p^2}^\times$, and to describe the space $(\mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \rho)_\chi$ instead.

Lemma 4.17. *We have $(\mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \rho)_\chi \simeq \mathrm{c} - \mathrm{Ind}_{\mathrm{Z}(J(\mathbb{Q}_p))J_\theta}^{J} \chi \otimes \rho$.*

Proof. By Frobenius reciprocity, the identity map on $\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J(\mathbb{Q}_p))J_\theta}^{J}\chi\otimes\rho$ gives a morphism $\chi\otimes\rho \to \left(\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J(\mathbb{Q}_p))J_\theta}^{J}\chi\otimes\rho\right)_{|\mathrm{Z}(J(\mathbb{Q}_p))J_\theta}$ of $\mathrm{Z}(J(\mathbb{Q}_p))J_\theta$-representations. Restricting further to J_θ, we obtain a morphism $\rho \to \left(\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J(\mathbb{Q}_p))J_\theta}^{J}\chi\otimes\rho\right)_{|J_\theta}$. This corresponds to a morphism $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho \to \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J(\mathbb{Q}_p))J_\theta}^{J}\chi\otimes\rho$ of $J(\mathbb{Q}_p)$-representations by Frobenius reciprocity. Since $\mathrm{Z}(J(\mathbb{Q}_p))$ acts via the character χ on the target space, this morphism factors through a map $(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi \to \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J(\mathbb{Q}_p))J_\theta}^{J}\chi\otimes\rho$. In order to prove that this is an isomorphism, we build its inverse. The quotient morphism $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi$ corresponds, via Frobenius reciprocity, to a morphism $\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_{\chi\,|J_\theta}$ of J_θ-representations. Because $\mathrm{Z}(J(\mathbb{Q}_p))$ acts via the character χ on the target space, this arrow may be extended to a morphism $\chi\otimes\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_{\chi\,|\mathrm{Z}(J(\mathbb{Q}_p))J_\theta}$ of $\mathrm{Z}(J(\mathbb{Q}_p))J_\theta$-representations. By Frobenius reciprocity, this corresponds to a morphism $\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J(\mathbb{Q}_p))J_\theta}^{J}\chi\otimes\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi$, and this is our desired inverse. □

We recall Theorem 2 (supp) from [Bus90] describing certain compactly induced representations. In this paragraph only, let G be any p-adic group, and let L be an open subgroup of G which contains the center $\mathrm{Z}(G)$ and which is compact modulo $\mathrm{Z}(G)$.

Theorem 4.18. *Let (σ, V) be an irreducible smooth representation of L. There is a canonical decomposition*

$$\mathrm{c}-\mathrm{Ind}_L^G\sigma \simeq V_0 \oplus V_\infty,$$

where V_0 is the sum of all supercuspidal subrepresentations of $\mathrm{c}-\mathrm{Ind}_L^G\sigma$, and where V_∞ contains no non-zero admissible subrepresentation. Moreover, V_0 is a finite sum of irreducible supercuspidal subrepresentations of G.

The spaces V_0 or V_∞ could be zero. Note also that since G is p-adic, any irreducible representation is admissible. So in particular, V_∞ does not contain any irreducible subrepresentation. However, it may have many irreducible quotients and subquotients. Thus, the space V_∞ is in general not G-semisimple. Hence, the structure of the compactly induced representation $\mathrm{c}-\mathrm{Ind}_L^G\sigma$ heavily depends on the supercuspidal supports of its irreducible subquotients.
We go back to our previous notations. Let $0 \leqslant \theta \leqslant \theta_{\max}$, let ρ be a smooth irreducible representation of J_θ and let χ be a character of $\mathrm{Z}(J(\mathbb{Q}_p))$ agreeing with the central character of ρ on $\mathrm{Z}(J(\mathbb{Q}_p)) \cap J_\theta$. Since the group $\mathrm{Z}(J(\mathbb{Q}_p))J_\theta$ contains the center and is compact modulo the center, we have a canonical decomposition

$$(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi \simeq V_{\rho,\chi,0} \oplus V_{\rho,\chi,\infty}.$$

In order to describe the spaces $V_{\rho,\chi,0}$ and $V_{\rho,\chi,\infty}$, we determine the supercuspidal supports of the irreducible subquotients of $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho$ through type theory, with the assumption that ρ is inflated from $\mathcal{J}_\theta$. For our purpose, it will be enough to analyze only the case $\theta = \theta_{\max}$. In this case, $\dim V^1_{\theta_{\max}}$ is equal to 0 or 1 so that $\mathrm{GU}(V^1_{\theta_{\max}}) = \{1\}$ or $\mathbb{F}_{p^2}^\times$ has no proper parabolic subgroup. In particular, if ρ is a cuspidal representation of $\mathrm{GU}(V^0_{\theta_{\max}})$, then its inflation to the reductive quotient

$$\mathcal{J}_{\theta_{\max}} \simeq \mathrm{G}(\mathrm{U}(V^0_{\theta_{\max}}) \times \mathrm{U}(V^1_{\theta_{\max}}))$$

is also cuspidal.

In the following paragraphs, we recall a few general facts from type theory. For more details, we refer to [BK98] and [Mor99]. Let G be the group of F-rational points of a reductive connected group $\mathbf{G}$ over a p-adic field F. A parabolic subgroup P (resp. Levi complement L) of G is defined as the group of F-rational points of an F-rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ (resp. an F-rational Levi complement $\mathbf{L} \subset \mathbf{G}$). Every parabolic subgroup P admits a Levi decomposition $P = LU$ where U is the unipotent radical of P. We denote by $X^{\mathrm{un}}(G)$ the set of **unramified characters** of G, ie. the continuous characters of G which are trivial $G^\circ := \bigcap_\psi \mathrm{Ker}|\psi|_F$ where ψ runs over all the F-rational algebraic characters of G and $|\cdot|_F$ is the normalized valuation on F. We consider pairs (L,τ) where L is a Levi complement of G and τ is a supercuspidal representation of L. Two pairs (L,τ) and (L',τ') are said to be **inertially equivalent** if for some $g \in G$ and $\chi \in X^{\mathrm{un}}(G)$ we have $L' = L^g$ and $\tau' \simeq \tau^g \otimes \chi$ where τ^g is the representation of L^g defined by $\tau^g(l) := \tau(g^{-1}lg)$. This is an equivalence relation, and we denote by $[L,\tau]_G$ or $[L,\tau]$ the inertial equivalence class of (L,τ) in G. The set of all inertial equivalence classes is denoted $\mathrm{IC}(G)$. If P is a parabolic subgroup of G, we write ι_P^G for the normalised parabolic induction functor. Any smooth irreducible representation π of G is isomorphic to a subquotient of some parabolically induced representation $\iota_P^G(\tau)$, where $P = LU$ for some Levi complement L and τ is a supercuspidal representation of L. We denote by $\ell(\pi) \in \mathrm{IC}(G)$ the inertial equivalence class $[L,\tau]$. This is uniquely determined by π and it is called the **inertial support** of π.

Let $\mathfrak{s} \in \mathrm{IC}(G)$. We denote by $\mathrm{Rep}^{\mathfrak{s}}(G)$ the full subcategory of $\mathrm{Rep}(G)$ whose objects are the smooth representations of G all of whose irreducible subquotients have inertial support $\mathfrak{s}$. This definition corresponds to the one given in [BD84] Proposition-Définition 2.8. If $\mathfrak{S} \subset \mathrm{IC}(G)$, we write $\mathrm{Rep}^{\mathfrak{S}}(G)$ for the direct product of the categories $\mathrm{Rep}^{\mathfrak{s}}(G)$ where $\mathfrak{s}$ runs over $\mathfrak{S}$. The following statement is Proposition 2.10 of loc. cit.

Theorem 4.19. *The category* $\mathrm{Rep}(G)$ *decomposes as the direct product of the subcategories* $\mathrm{Rep}^{\mathfrak{s}}(G)$ *where* $\mathfrak{s}$ *runs over* $\mathrm{IC}(G)$*. Moreover, if* $\mathfrak{S} \subset \mathrm{IC}(G)$ *then the category* $\mathrm{Rep}^{\mathfrak{S}}(G)$ *is stable under direct sums and subquotients.*

Type theory was then introduced in [BK98] in order to describe the categories $\mathrm{Rep}^{\mathfrak{s}}(G)$ which are called the **Bernstein blocks**. Let $\mathfrak{S}$ be a subset of $\mathrm{IC}(G)$. A **$\mathfrak{S}$-type** in G is a pair (K, ρ) where K is an open compact subgroup of G and ρ is a smooth irreducible representation of K, such that for every smooth irreducible representation π of G we have

$$\pi_{|K} \text{ contains } \rho \iff \ell(\pi) \in \mathfrak{S}.$$

When $\mathfrak{S}$ is a singleton $\{\mathfrak{s}\}$, we call it an $\mathfrak{s}$-type instead.

Remark 4.20. By Frobenius reciprocity, the condition that $\pi_{|K}$ contains ρ is equivalent to π being isomorphic to an irreducible quotient of $\mathrm{c} - \mathrm{Ind}_K^G \rho$. In fact, we can say a little bit more. Let K be an open compact subgroup of G and let ρ be an irreducible smooth representation of K. Let $\mathrm{Rep}_\rho(G)$ denote the full subcategory of $\mathrm{Rep}(G)$ whose objects are those representations which are generated by their ρ-isotypic component. If (K, ρ) is an $\mathfrak{S}$-type, then [BK98] Theorem 4.3 establishes the equality of categories $\mathrm{Rep}_\rho(G) = \mathrm{Rep}^{\mathfrak{S}}(G)$. By definition of compact induction, the representation $\mathrm{c} - \mathrm{Ind}_K^G \rho$ is generated by its ρ-isotypic vectors. Therefore any irreducible subquotient of $\mathrm{c} - \mathrm{Ind}_K^G \rho$ has inertial support in $\mathfrak{S}$.

An important class of types are those of depth zero, and they are the only ones we shall encounter. First, we recall the following result. If K is a parahoric subgroup of G, we denote by $\mathcal{K}$ its maximal reductive quotient. It is a finite group of Lie type over the residue field of F. The following statement is [Mor99] Proposition 4.1

Proposition 4.21. *Let K be a maximal parahoric subgroup of G and let ρ be an irreducible cuspidal representation of $\mathcal{K}$, seen as a representation of K by inflation. Let π be an irreducible smooth representation of G and assume that $\pi_{|K}$ contains ρ. Then π is supercuspidal and there exists an irreducible smooth representation $\tilde{\rho}$ of the normalizer $\mathrm{N}_G(K)$ such that $\tilde{\rho}_{|K}$ contains ρ and $\pi \simeq \mathrm{c} - \mathrm{Ind}_{N_G(K)}^G \tilde{\rho}$.*

Such representations π are called **depth-0 supercupidal representations** of G. More generally, a smooth irreducible representation π of G is said to be of **depth-0** if it contains a non-zero vector that is fixed by the pro-unipotent radical of some parahoric subgroup of G. A **depth-0 type** in G is a pair (K, ρ) where K is a parahoric subgroup of G and ρ is an irreducible cuspidal representation of $\mathcal{K}$, inflated to K. The name is justified by [Mor99] Theorem 4.8.

Theorem 4.22. *Let (K, ρ) be a depth-0 type. Then there exists a (unique) finite set $\mathfrak{S} \subset \mathrm{IC}(G)$ such that (K, ρ) is an $\mathfrak{S}$-type of G.*

Let K be a parahoric subgroup of G. Using the Bruhat-Tits building of G, one may canonically associate a Levi complement L of G such that $K_L := L \cap K$

is a maximal parahoric subgroup of L, whose maximal reductive quotient $\mathcal{K}_L$ is naturally identified with $\mathcal{K}$. This is precisely described in [Mor99] paragraph 2.1. Moreover, we have $L = G$ if and only if K is a maximal parahoric subgroup of G. Now, let (K, ρ) be a depth-0 type of G and denote by $\mathfrak{S}$ the finite subset of $\mathrm{IC}(G)$ such that it is an $\mathfrak{S}$-type of G. Since ρ is a cuspidal representation of $\mathcal{K} \simeq \mathcal{K}_L$, we may inflate it to K_L. Then, the pair (K_L, ρ) is a depth-0 type of L. We say that (K, ρ) is a G-**cover** of (K_L, ρ). By the previous theorem, there is a finite set $\mathfrak{S}_L \subset \mathrm{IC}(L)$ such that (K_L, ρ) is an $\mathfrak{S}_L$-type of L. Then the proof of Theorem 4.8 in [Mor99] shows that we have the relation

$$\mathfrak{S} = \left\{ [M, \tau]_G \,\middle|\, [M, \tau]_L \in \mathfrak{S}_L \right\}.$$

In this set, M is some Levi complement of L, therefore it may also be seen as a Levi complement in G. Thus, an inertial equivalence class $[M, \tau]_L$ in L gives rise to a class $[M, \tau]_G$ in G. Since K_L is maximal in L, in virtue of the proposition above any element of $\mathfrak{S}_L$ has the form $[L, \pi]_L$ for some supercuspidal representation π of L. In particular, every smooth irreducible representation of G containing the type (K, ρ) has a conjugate of L as cuspidal support. We deduce the following corollary.

Corollary 4.23. *Let (K, ρ) be a depth-0 type in G and assume that K is not a maximal parahoric subgroup. Then no smooth irreducible representation π of G containing the type (K, ρ) is supercuspidal.*

Thus, up to replacing G with a Levi complement, the study of any depth-0 type (K, ρ) can be reduced to the case where K is a maximal parahoric subgroup. Let us assume that it is the case, and let $\mathfrak{S}$ be the associated finite subset of $\mathrm{IC}(G)$. While $\mathfrak{S}$ is in general not a singleton, it becomes one once we modify the pair (K, ρ) a little bit according to [Mor99] Theorem Variant 4.7. Let $\widehat{K}$ be the maximal open compact subgroup of $\mathrm{N}_G(K)$. We have $K \subset \widehat{K}$ but in general this inclusion may be strict. Let $\tilde{\rho}$ be a smooth irreducible representation of $\mathrm{N}_G(K)$ such that $\tilde{\rho}_{|K}$ contains ρ. Let $\widehat{\rho}$ be any irreducible component of the restriction $\tilde{\rho}_{|\widehat{K}}$. Eventually, let $\pi := \mathrm{c} - \mathrm{Ind}_{\mathrm{N}_G(K)}^G \tilde{\rho}$ be the associated depth-0 supercuspidal representation of G.

Theorem 4.24. *The pair $(\widehat{K}, \widehat{\rho})$ is a $[G, \pi]$-type.*

The conclusion does not depend on the choice of $\widehat{\rho}$ as an irreducible component of $\tilde{\rho}_{|\widehat{K}}$. Any one of them affords a type for the same singleton $\mathfrak{s} = [G, \pi]$. Let us now consider a parahoric subgroup K along with an irreducible representation ρ of its maximal reductive quotient $\mathcal{K} = K/K^+$, where K^+ is the pro-unipotent radical of K. Assume that ρ is not cuspidal. Thus, there exists a proper parabolic subgroup

$\mathcal{P} \subset \mathcal{K}$ with Levi complement $\mathcal{L}$, and a cuspidal irreducible representation τ of $\mathcal{L}$, such that ρ is an irreducible component of the Harish-Chandra induction $\iota_{\mathcal{P}}^{\mathcal{K}}\tau$. The preimage of $\mathcal{P}$ via the quotient map $K \twoheadrightarrow \mathcal{K}$ is a parahoric subgroup $K' \subsetneq K$, whose maximal reductive quotient $\mathcal{K}' := K'/K'^{+}$ is naturally identified with $\mathcal{L}$. We have $K^{+} \subset K'^{+} \subset K'$ and the intermediate quotient K'^{+}/K^{+} is identified with the unipotent radical $\mathcal{N}$ of $\mathcal{P} \simeq K'/K^{+}$. Consider ρ as an irreducible representation of K inflated from $\mathcal{K}$. The invariants $\rho^{K'^{+}}$ form a representation of K' which coincides with the inflation of the Harish-Chandra restriction of ρ (as a representation of $\mathcal{K}$) to $\mathcal{L}$. Thus, $\rho^{K'^{+}}$ contains the inflation of τ to a representation of K'. In other words, we have a K'-equivariant map

$$\tau \longrightarrow \rho_{|K'}.$$

By Frobenius reciprocity, it gives a map

$$\mathrm{c}-\mathrm{Ind}_{K'}^{K}\tau \longrightarrow \rho,$$

which is surjective by irreducibility of ρ. Applying the functor $\mathrm{c}-\mathrm{Ind}_{K}^{G} : \mathrm{Rep}(K) \to \mathrm{Rep}(G)$, which is exact, and using transitivity of compact induction, we deduce the existence of a natural surjection

$$\mathrm{c}-\mathrm{Ind}_{K'}^{G}\tau \twoheadrightarrow \mathrm{c}-\mathrm{Ind}_{K}^{G}\rho.$$

Now, (K',τ) is a depth-0 type in G. Let $\mathfrak{S} \subset \mathrm{IC}(G)$ be the subset such that (K',τ) is an $\mathfrak{S}$-type, and let L be the (proper) Levi complement of G associated to K' as in the previous paragraph. By Remark **??**, it follows that any irreducible subquotient of $\mathrm{c}-\mathrm{Ind}_{K}^{G}\rho$ has inertial support in $\mathfrak{S}$. Since all elements of $\mathfrak{S}$ are of the form $[L,\pi]$ for some supercuspidal representation π of L, we reach the following conclusion.

Proposition 4.25. *Let K be a parahoric subgroup of G and let ρ be a non cuspidal irreducible representation of its maximal reductive quotient $\mathcal{K}$. Then no irreducible subquotient of* $\mathrm{c}-\mathrm{Ind}_{K}^{G}\rho$ *is supercuspidal.*

We go back to the context of the unitary similitude group $J(\mathbb{Q}_p)$. We may now determine the inertial support of any irreducible subquotient of a representation of the form $\mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J}\rho$ with ρ inflated from a unipotent representation of $\mathrm{GU}(V_{\theta_{\max}}^{0})$. In particular, all the terms $E_1^{0,b}$ are of this form according to Corollary 4.12. More precisely, let λ be a partition of $2\theta_{\max}+1$ and let Δ_t be its 2-core (see Section 2). Thus $2\theta_{\max}+1 = \frac{t(t+1)}{2} + 2e$ for some $e \geqslant 0$. The integer $\frac{t(t+1)}{2}$ is odd, so it can be written as $2f+1$ for some $f \geqslant 0$, and we have $\theta_{\max} = f+e$. Recall the basis of $\mathbf{V}$ that we fixed in Section 1.1. The images of the vectors $e_{\pm i}$ for

$1 \leqslant i \leqslant \theta_{\max}$ and of e_0^{an} in $V^0_{\theta_{\max}} = \Lambda_{\theta_{\max}}/p\Lambda_{\theta_{\max}}$ define a basis of $V^0_{\theta_{\max}}$, allowing us to identify $\mathrm{GU}(V^0_{\theta_{\max}})$ with the matrix group $\mathrm{GU}_{2\theta_{\max}+1}(\mathbb{F}_p)$. The cuspidal support of ρ_λ is (L_t, ρ_t) according to Section 2. Let P_t be the standard parabolic subgroup with Levi complement L_t. By direct computation, one may check that the preimage of P_t in $J_{\theta_{\max}}$ is the parahoric subgroup $J_{f,\dots,\theta_{\max}} := J_f \cap J_{f+1} \cap \ldots \cap J_{\theta_{\max}}$. Let L_f be the Levi complement of $J(\mathbb{Q}_p)$ that is associated to the parahoric subgroup $J_{f,\dots,\theta_{\max}}$. Let $\mathbf{V}^f$ be the subspace of $\mathbf{V}$ generated by $\mathbf{V}^{\mathrm{an}}$ and by the vectors $e_{\pm 1}, \dots, e_{\pm f}$. It is equipped with the restriction of the hermitian form of $\mathbf{V}$. Then $L_f \simeq \mathrm{G}(\mathrm{U}(\mathbf{V}^f) \times \mathrm{U}_1(\mathbb{Q}_p)^e)$.
The group $L_f \cap J_{f,\dots,\theta_{\max}}$ is a maximal parahoric subgroup of L_f, and ρ_t can be inflated to it. In particular, the pair $(L_f \cap J_{f,\dots,\theta_{\max}}, \rho_t)$ is a level-0 type in L_f. Since we work with unitary groups over an unramified quadratic extension, $L_f \cap J_{f,\dots,\theta_{\max}}$ is also a maximal compact subgroup of L_f. In particular, $(L_f \cap J_{f,\dots,\theta_{\max}}, \rho_t)$ is a type for a singleton of the form $[L_f, \tau_f]_{L_f}$. Then τ_f has the form

$$\tau_f = \mathrm{c} - \mathrm{Ind}_{\mathrm{N}_{L_f}(L_f \cap J_{f,\dots,\theta_{\max}})}^{L_f} \widetilde{\rho}_t,$$

where $\widetilde{\rho}_t$ is some smooth irreducible representation of $\mathrm{N}_{L_f}(L_f \cap J_{f,\dots,\theta_{\max}})$ containing ρ_t upon restriction. It follows that if we inflate ρ_t to $J_{f,\dots,\theta_{\max}}$ then $(J_{f,\dots,\theta_{\max}}, \rho_t)$ is a $[L_f, \tau_f]$-type in $J(\mathbb{Q}_p)$. Moreover the compactly induced representation $\mathrm{c} - \mathrm{Ind}_{J_{\theta_{\max}}}^J \rho_\lambda$ is a quotient of $\mathrm{c} - \mathrm{Ind}_{J_{f,\dots,\theta_{\max}}}^J \rho_t$. In particular, we reach the following conclusion.

Proposition 4.26. *Let λ be a partition of $2\theta_{\max}+1$ with 2-core Δ_t. Write $\frac{t(t+1)}{2} = 2f+1$ for some $f \geqslant 0$. Any irreducible subquotient of $\mathrm{c} - \mathrm{Ind}_{J_{\theta_{\max}}}^J \rho_\lambda$ has inertial support $[L_f, \tau_f]$.*

In particular, if $f < \theta_{\max}$ then none of these irreducible subquotients are supercuspidal.

Let us keep the notations of the previous paragraph. Since unipotent representations of finite groups of Lie type have trivial central characters, if χ is an unramified character of $\mathrm{Z}(J(\mathbb{Q}_p))$ then $\chi_{\mathrm{Z}(J(\mathbb{Q}_p)) \cap J_{\theta_{\max}}}$ coincides with the central character of ρ_λ inflated to $J_{\theta_{\max}}$. As in Theorem 4.18, we have

$$\left(\mathrm{c} - \mathrm{Ind}_{J_{\theta_{\max}}}^J \rho_\lambda\right)_\chi \simeq V_{\rho_\lambda,\chi,0} \oplus V_{\rho_\lambda,\chi,\infty}.$$

If $f < \theta_{\max}$, then no irreducible supercuspidal representation can occur. Thus $V_{\rho_\lambda,\chi,0} = 0$.
On the other hand, assume now that $f = \theta_{\max}$ so that $L_f = J$ and ρ_λ is equal to the cuspidal representation $\rho_{\Delta_{\theta_{\max}}}$. As seen in Proposition 1.14, we have $\mathrm{N}_J(J_{\theta_{\max}}) = \mathrm{Z}(J(\mathbb{Q}_p))J_{\theta_{\max}}$ unless $n = 2$ (thus $\theta_{\max} = 0$) in which case $J_0 = J^\circ$ and $\mathrm{Z}(J(\mathbb{Q}_p))J_0$

is of index 2 in $\mathrm{N}_J(J_0) = J$. A representative of the non-trivial coset is given by g_0 as defined in Section 1.1. If $n \neq 2$, define

$$\tau_{\theta_{\max},\chi} := \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J(\mathbb{Q}_p))J_{\theta_{\max}}}^{J} \chi \otimes \rho_\lambda.$$

Then $\tau_{\theta_{\max},\chi}$ is an irreducible supercuspidal representation of $J(\mathbb{Q}_p)$, and we have

$$\left(\mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_\lambda\right)_\chi \simeq \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J(\mathbb{Q}_p))J_{\theta_{\max}}}^{J} \chi \otimes \rho_\lambda = \tau_{\theta_{\max},\chi}.$$

Thus $V_{\rho_\lambda,\chi,\infty} = 0$ and $V_{\rho_\lambda,\chi\infty} = \tau_{\theta_{\max},\chi}$ in this case.
When $n = 2$, $\rho_\lambda = \rho_{\Delta_0} = \mathbf{1}$ is the trivial representation of $J_0 = J^\circ$. Let $\chi_0 : J \to \overline{\mathbb{Q}_\ell}^\times$ be the unique non-trivial character of $J(\mathbb{Q}_p)$ which is trivial on $\mathrm{Z}(J(\mathbb{Q}_p))J_0$. Then $\left(\mathrm{c}-\mathrm{Ind}_{J_0}^{J}\mathbf{1}\right)_\chi$ is the sum of an unramified character $\tau_{0,\chi}$ of $J(\mathbb{Q}_p)$ whose central character is χ, and of the character $\chi_0\tau_{0,\chi}$. Both characters are supercuspidal, and they are the only unramified characters of $J(\mathbb{Q}_p)$ with central character χ.

According to Proposition 4.4 and Corollary 4.12, the terms $E_1^{0,b}$ are a sum of representations of the form

$$\mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_\lambda,$$

with λ a partition of $2\theta_{\max}+1$ having 2-core Δ_0 if b is even, and Δ_1 if b is odd. Moreover, we have

$$E_2^{0,2(n-1-\theta_{\max})} \simeq \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \mathbf{1}, \qquad E_2^{0,2(n-1-\theta_{\max})+1} \simeq \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_{(2\theta_{\max},1)}.$$

In particular, summing up the discussion of the previous paragraph, we have reached the following statement.

Proposition 4.27. *Let χ be an unramified character of $\mathrm{Z}(J(\mathbb{Q}_p))$.*

- *Assume that $n \geqslant 3$. The representation $(E_2^{0,2(n-1-\theta_{\max})})_\chi$ contains no non-zero admissible subrepresentation, and it is not $J(\mathbb{Q}_p)$-semisimple. Moreover, any irreducible subquotient has inertial support $[L_0,\tau_0]$. If $n \geqslant 5$, then the same statement holds for $(E_2^{0,2(n-1-\theta_{\max})+1})_\chi$ with the inertial support being $[L_1,\tau_1]$.*
- *For $n = 1,2,3,4$, let $b = 0,2,3,5$ respectively. Then $\theta_{\max} = 0$ when $1,2$ and $\theta_{\max} = 1$ when $n = 3,4$. Let χ be an unramified character of $\mathrm{Z}(J(\mathbb{Q}_p))$. The representation $\tau_{\theta_{\max},\chi}$ is irreducible supercuspidal, and we have*

$$(E_2^{0,b})_\chi \simeq \begin{cases} \tau_{\theta_{\max},\chi} & \text{if } n = 1,3,4, \\ \tau_{\theta_{\max},\chi} \oplus \chi_0\tau_{\theta_{\max},\chi} & \text{if } n = 2. \end{cases}$$

In particular, we deduce the following important corollary.

Corollary 4.28. *Let χ be an unramified character of $\mathrm{Z}(J(\mathbb{Q}_p))$. If $n \geqslant 3$ then $\mathrm{H}_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}},\overline{\mathbb{Q}_\ell})_\chi$ is not $J(\mathbb{Q}_p)$-admissible. If $n \geqslant 5$ then the same holds for $\mathrm{H}_c^{2(n-1-\theta_{\max})+1}(\mathcal{M}^{\mathrm{an}},\overline{\mathbb{Q}_\ell})_\chi$.*

4.3 The case $n = 3, 4$

Let us focus on the case $\theta_{\max} = 1$, that is $n = 3$ or 4. Recall that $N(\Lambda_0)$ denotes the set of lattices $\Lambda \in \mathcal{L}_0$ with type $t(\Lambda) = t_{\max} = 3$ containing Λ_0. It has cardinality $\nu(1,2) = p+1$ when $n = 3$ and $\nu(2,3) = p^3+1$ when $n = 4$. In particular, we may locate the non zero terms $E_{1,\mathrm{alt}}^{a,b}$ of the alternating Čech spectral sequence as follows.

$$E_{1,\mathrm{alt}}^{a,b} \neq 0 \iff \begin{cases} (a,b) \in \{(0,2); (0,3); (-k,4) \,|\, 0 \leqslant k \leqslant p\} & \text{if } n = 3, \\ (a,b) \in \{(0,4); (0,5); (-k,6) \,|\, 0 \leqslant k \leqslant p^3\} & \text{if } n = 4. \end{cases}$$

In Figure 1 below, we draw the shape of the first page $E_{1,\mathrm{alt}}$ for $n = 3$. The case of $n = 4$ is similar, except that two more 0 rows should be added at the bottom. To alleviate the notations, we write φ_{-a} for the differential $\varphi^{a,2(n-1)}$.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\varphi_4} & E_{1,\mathrm{alt}}^{-3,4} & \xrightarrow{\varphi_3} & E_{1,\mathrm{alt}}^{-2,4} & \xrightarrow{\varphi_2} & E_{1,\mathrm{alt}}^{-1,4} & \xrightarrow{\varphi_1} & \mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1} \\
 & & & & & & & & \\
 & & & & & & & & \mathrm{c}-\mathrm{Ind}_{J_1}^{J}\rho_{\Delta_2} \\
 & & & & & & & & \\
 & & & & & & & & \mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1} \\
 & & & & & & & & \\
 & & & & & & & & 0 \\
 & & & & & & & & \\
 & & & & & & & & 0
\end{array}$$

Figure 1: The first page $E_{1,\mathrm{alt}}$ of the alternating Čech spectral sequence when $n = 3$.

Let $i \in \mathbb{Z}$ such that ni is even. For $\Lambda, \Lambda' \in \mathcal{L}_i$, we define the distance $d(\Lambda, \Lambda')$ as the smallest integer $d \geqslant 0$ such that there exists a sequence $\Lambda = \Lambda^0, \ldots, \Lambda^d = \Lambda'$ of lattices of $\mathcal{L}_i$ with $\{\Lambda^j, \Lambda^{j+1}\}$ being an edge for all $0 \leqslant j \leqslant d-1$. This definition makes sense for any n. When $\theta_{\max} = 1$, any lattice $\Lambda \in \mathcal{L}_i$ has type 1 or 3, and two lattices forming an edge can not have the same type. Therefore, the value of $t(\Lambda^j)$ alternates between 1 and 3. In particular, if $t(\Lambda) = t(\Lambda')$ then $d(\Lambda, \Lambda')$ is even. According to [Vol10] Proposition 3.7, the simplicial complex $\mathcal{L}_i$ is in fact a tree. We will use this to prove the following proposition.

Proposition 4.29. *Assume that $n = 3$ or 4. We have $E_2^{-1,2(n-1)} = 0$.*

For now, $n \geqslant 3$ is still any integer. By Proposition 4.6, we may use the alternating Čech spectral sequence to show that $E_2^{-1,2(n-1)} = \mathrm{Ker}(\varphi_1)/\mathrm{Im}(\varphi_2)$ vanishes. The term $E_1^{a,2(n-1)}$ is the $\overline{\mathbb{Q}_\ell}$-vector space generated by the set I_{-a+1}, and $E_{1,\mathrm{alt}}^{a,2(n-1)}$ is the subspace consisting of all the vectors $v = \sum_{\gamma \in I_{-a+1}} \lambda_\gamma \gamma$ such that for all $\sigma \in \mathfrak{S}_{-a+1}$ we have $\lambda_{\sigma(\gamma)} = \mathrm{sgn}(\sigma)\lambda_\gamma$. Here the λ_γ's are scalars which are almost all zero. To prove the proposition, let us look at the differential φ_2. It acts on the basis vectors in the following way.

$$\left.\begin{matrix}(\Lambda,\Lambda,\Lambda)\\(\Lambda,\Lambda,\Lambda')\\(\Lambda',\Lambda,\Lambda)\end{matrix}\right\} \mapsto (\Lambda,\Lambda), \qquad \forall \Lambda,\Lambda' \in \mathcal{L}^{(1)} \text{ such that } U_\Lambda \cap U_{\Lambda'} \neq \varnothing,$$

$$(\Lambda,\Lambda',\Lambda) \mapsto (\Lambda',\Lambda) - (\Lambda,\Lambda) + (\Lambda,\Lambda'), \qquad \forall \Lambda,\Lambda' \in \mathcal{L}^{(1)} \text{ such that } U_\Lambda \cap U_{\Lambda'} \neq \varnothing,$$

$$(\Lambda,\Lambda',\Lambda'') \mapsto (\Lambda',\Lambda'') - (\Lambda,\Lambda'') + (\Lambda,\Lambda'), \quad \forall \Lambda,\Lambda',\Lambda'' \in \mathcal{L}^{(1)} \text{ such that } U_\Lambda \cap U_{\Lambda'} \cap U_{\Lambda''} \neq \varnothing.$$

We note that for a collection of lattices $\Lambda^1,\ldots,\Lambda^s \in \mathcal{L}_i^{(1)}$, the condition $U_{\Lambda^1} \cap \ldots \cap U_{\Lambda^s} \neq \varnothing$ is equivalent to $d(\Lambda^j,\Lambda^{j'}) = 2$ for all $1 \leqslant j \neq j' \leqslant s$. Towards a contradiction, we assume that $\mathrm{Im}(\varphi_2) \subsetneq \mathrm{Ker}(\varphi_1)$. Let $v \in \mathrm{Ker}(\varphi_1)\backslash\mathrm{Im}(\varphi_2)$. Since $v \in E_{1,\mathrm{alt}}^{-1,2(n-1)}$, it decomposes under the form

$$v = \sum_{j=1}^{r} \lambda_j(\gamma_j - \tau(\gamma_j)),$$

where $r \geqslant 1$, the γ_j's are of the form (Λ,Λ') with $d(\Lambda,\Lambda') = 2$, the scalars λ_j's are non zero and $\tau \in \mathfrak{S}_2$ is the transposition. We may assume that r is minimal among all the vectors in the complement $\mathrm{Ker}(\varphi_1)\backslash\mathrm{Im}(\varphi_2)$. In particular, there exists a single $i \in \mathbb{Z}$ such that ni is even, and for all $1 \leqslant j \leqslant r$ the lattices in γ_j belong to $\mathcal{L}_i^{(1)}$. We may further assume $i = 0$ without loss of generality. We say that an element $\gamma \in I_2$ occurs in v if $\gamma = \gamma_j$ or $\tau(\gamma_j)$ for some $1 \leqslant j \leqslant r$. Similarly, we say that a lattice $\Lambda \in \mathcal{L}_0^{(1)}$ occurs in v if it is a constituent of some γ_j.

Lemma 4.30. *Let $\gamma = (\Lambda,\Lambda') \in I_2$ be an element occuring in v. Then there exists $\Lambda'' \in \mathcal{L}_0^{(1)}$ such that $(\Lambda,\Lambda'') \in I_2$ occurs in v and $d(\Lambda',\Lambda'') = 4$.*

Proof. Let us write $(\Lambda,\Lambda^j) \in I_2, 1 \leqslant j \leqslant s$ for the various elements occuring in v whose first component is Λ. Up to reordering the γ_j's and swapping them with $\tau(\gamma_j)$ if necessary, we may assume that $(\Lambda,\Lambda^j) = \gamma_j$ for all $1 \leqslant j \leqslant s$, and that $\Lambda^1 = \Lambda'$. The coordinate of $\varphi_1(v)$ along the basis vector $(\Lambda) \in I_1$ is equal to $-2\sum_{j=1}^{s} \lambda_j$. Since $\varphi_1(v) = 0$, this sum is zero. Since $\lambda_1 \neq 0$ by hypothesis, we have in particular $s \geqslant 2$. For all $2 \leqslant j \leqslant s$, we have $2 \leqslant d(\Lambda',\Lambda^j) \leqslant 4$ by the triangular inequality. Towards a contradiction, assume that $d(\Lambda',\Lambda^j) = 2$ for all

$2 \leqslant j \leqslant s$. In particular, $\delta_j := (\Lambda^j, \Lambda, \Lambda') \in I_3$ for all $2 \leqslant j \leqslant s$. Consider the vector

$$w := \frac{1}{3} \sum_{j=2}^{s} \sum_{\sigma \in \mathfrak{S}_6} \operatorname{sgn}(\sigma) \lambda_j \sigma(\delta_j) \in E_{1,\mathrm{alt}}^{-2,2(n-1)}.$$

Then we compute

$$\begin{aligned}
\varphi_2(w) &= -\lambda_1((\Lambda, \Lambda') - (\Lambda', \Lambda)) - \sum_{j=2}^{s} \lambda_j((\Lambda, \Lambda^j) - (\Lambda^j, \Lambda)) + \sum_{j=2}^{s} \lambda_j((\Lambda', \Lambda^j) - (\Lambda^j, \Lambda')) \\
&= -\sum_{j=1}^{s} \lambda_j(\gamma_j - \tau(\gamma_j)) + \sum_{j=2}^{s} \lambda_j((\Lambda', \Lambda^j) - (\Lambda^j, \Lambda')).
\end{aligned}$$

In particular, we get

$$v + \varphi_2(w) = \sum_{j=2}^{s} \lambda_j((\Lambda^j, \Lambda') - (\Lambda', \Lambda^j)) + \sum_{j=s+1}^{r} \lambda_j(\gamma_j - \tau(\gamma_j)) \in \operatorname{Ker}(\varphi_1) \backslash \operatorname{Im}(\varphi_2),$$

which contradicts the minimality of r. $\square$

From now on, let us assume that $n = 3$ or 4, so that $\mathcal{L}_0$ is a tree. To conclude the proof of the proposition, let us pick $\Lambda = \Lambda^0 \in \mathcal{L}_0^{(1)}$ which occurs in v, say in a pair $(\Lambda, \Lambda') \in I_2$. Write $\Lambda^1 := \Lambda'$. By induction, we build a sequence $(\Lambda^k)_{k \geqslant 0}$ of lattices in $\mathcal{L}_0^{(1)}$ such that for all k, the pair $(\Lambda^k, \Lambda^{k+1})$ occurs in v and we have $d(\Lambda^0, \Lambda^k) = 2k$. It follows that the Λ^k's are pairwise distinct, and it leads to a contradiction since only a finite number of such lattices can occur in v.
Let us assume that $\Lambda^0, \ldots, \Lambda^k$ are already built for some $k \geqslant 1$. Since $(\Lambda^{k-1}, \Lambda^k)$ occurs in v, so does $(\Lambda^k, \Lambda^{k-1})$. By the Lemma applied to latter pair, there exists $\Lambda^{k+1} \in \mathcal{L}_0^{(1)}$ such that the pair $(\Lambda^k, \Lambda^{k+1}) \in I_2$ occurs in v and $d(\Lambda^{k-1}, \Lambda^{k+1}) = 4$. By the triangular inequality, we have

$$\begin{aligned}
d(\Lambda^0, \Lambda^{k+1}) &\leqslant d(\Lambda^0, \Lambda^k) + d(\Lambda^k, \Lambda^{k+1}) = 2k + 2 = 2(k+1), \\
d(\Lambda^0, \Lambda^{k+1}) &\geqslant |d(\Lambda^0, \Lambda^k) - d(\Lambda^k, \Lambda^{k+1})| = 2(k-1).
\end{aligned}$$

Thus $d(\Lambda^0, \Lambda^{k+1}) = 2(k-1), 2k$ or $2(k+1)$. We prove that it must be equal to the latter.

Assume that $d(\Lambda^0, \Lambda^{k+1}) = 2(k-1)$. There exists a path $\Lambda^0 = L^0, \ldots, L^{2(k-1)} = \Lambda^{k+1}$. We obtain a cycle

$$\begin{tikzcd}
 & \Lambda^0 \cap \Lambda^1 \arrow[r, dash] & \Lambda^1 \arrow[r, dash] & \ldots \arrow[r, dash] & \Lambda^{k-1} \arrow[r, dash] & \Lambda^{k-1} \cap \Lambda^k \arrow[dr, dash] & \\
\Lambda^0 \arrow[ur, dash] \arrow[dr, dash] & & & & & & \Lambda^k \\
 & L^1 \arrow[r, dash] & L^2 \arrow[r, dash] & \ldots \arrow[r, dash] & L^{2(k-1)} = \Lambda^{k+1} \arrow[r, dash] & \Lambda^k \cap \Lambda^{k+1} \arrow[ur, dash] &
\end{tikzcd}$$

Since $\mathcal{L}_0$ is a tree, this cycle must be trivial, ie. the lower and upper paths, which are of the same length, are the same. In particular, we have $\Lambda^{k-1} = \Lambda^{k+1}$, which is absurd since $d(\Lambda^{k-1}, \Lambda^{k+1}) = 4$.

Assume that $d(\Lambda^0, \Lambda^{k+1}) = 2k$. There exists a path $\Lambda^0 = L_0, \dots, L^{2k} = \Lambda^{k+1}$. We obtain a cycle

$$\begin{tikzcd}
& \Lambda^0 \cap \Lambda^1 \arrow[r, dash] & \Lambda^1 \arrow[r, dash] & \dots \arrow[r, dash] & \Lambda^{k-1} \cap \Lambda^k \arrow[r, dash] & \Lambda^k \arrow[dr, dash] & \\
\Lambda^0 \arrow[ur, dash] \arrow[dr, dash] & & & & & & \Lambda^k \cap \Lambda^{k+1} \\
& L^1 \arrow[r, dash] & L^2 \arrow[r, dash] & \dots \arrow[r, dash] & L^{2k-1} \arrow[r, dash] & L^{2k} = \Lambda^{k+1} \arrow[ur, dash] &
\end{tikzcd}$$

Since $\mathcal{L}_0$ is a tree, this cycle must be trivial, ie. the lower and upper paths, which are of the same length, are the same. In particular, we have $\Lambda^k = \Lambda^{k+1}$, which is absurd since $d(\Lambda^k, \Lambda^{k+1}) = 2$.

Thus, we have $d(\Lambda^0, \Lambda^{k+1}) = 2(k+1)$ so that Λ^{k+1} meets all the requirements. It concludes the proof of Proposition 4.29.

In particular, we obtain the following statement.

Theorem 4.31. *Assume that $n = 3$ or 4. We have*

$$\mathrm{H}_c^{2(n-1)-1}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c} - \mathrm{Ind}_{J_1}^{J} \rho_{\Delta_2},$$

with the rational Frobenius τ acting like multiplication by $-p^{2(n-1)-1}$.

5 The cohomology of the supersingular locus of the Shimura variety for $n = 3, 4$

5.1 The Hochschild-Serre spectral sequence induced by p-adic uniformization

In this section, $n \geqslant 1$ is still any integer. We recover the notations of Section 3. Let $\xi : \mathbb{G} \to W_\xi$ be a finite-dimensional irreducible algebraic $\overline{\mathbb{Q}_\ell}$-representation of $\mathbb{G}$. Such representations have been classified in [HT01] Chapter III.2. We think of $\mathbb{V}_{\overline{\mathbb{Q}_\ell}} := \mathbb{V} \otimes \overline{\mathbb{Q}_\ell}$ as a representation of $\mathbb{G}$, whose dual is denoted by $\mathbb{V}_0$. Using the alternating form $\langle \cdot, \cdot \rangle$, we have an isomorphism $\mathbb{V}_0 \simeq \mathbb{V}_{\overline{\mathbb{Q}_\ell}} \otimes c^{-1}$, where c is the multiplier character of G. Then, W_ξ can be described as follows.

Proposition 5.1. *There exists unique integers $t(\xi), m(\xi) \geqslant 0$ and an idempotent $\epsilon(\xi) \in \mathrm{End}(\mathbb{V}_0^{\otimes m(\xi)})$ such that*

$$W_\xi \simeq c^{t(\xi)} \otimes \epsilon(\xi)(\mathbb{V}_0^{\otimes m(\xi)}).$$

The weight $w(\xi)$ is defined by

$$w(\xi) := m(\xi) - 2t(\xi) \in \mathbb{Z}.$$

To any ξ as above, we can associate a local system $\mathcal{L}_\xi$ which is defined on the tower $(\mathrm{S}_{K^p})_{K^p}$ of Shimura varieties. We denote by $\overline{\mathcal{L}_\xi}$ its restriction to the special fiber $\overline{\mathrm{S}}_{K^p}$. Let A_{K^p} be the universal abelian scheme over S_{K^p}. We write $\pi^m_{K^p} : A^m_{K^p} \to \mathrm{S}_{K^p}$ for the structure morphism of the m-fold product of A_{K^p} with itself over S_{K^p}. If $m = 0$ it is just the identity on S_{K^p}. According to [HT01] Chapter III.2, we have an isomorphism

$$\mathcal{L}_\xi \simeq \epsilon(\xi)\epsilon_{m(\xi)} \left(\mathrm{R}^{m(\xi)}(\pi^{m(\xi)}_{K^p})_*\overline{\mathbb{Q}_\ell}(t(\xi))\right),$$

where $\epsilon_{m(\xi)}$ is some idempotent. In particular, if ξ is the trivial representation of $\mathbb{G}$ then $\mathcal{L}_\xi = \overline{\mathbb{Q}_\ell}$.

We fix an irreducible algebraic representation $\xi : \mathbb{G} \to W_\xi$ as above. We associate the space $\mathcal{A}_\xi$ of **automorphic forms of I of type ξ at infinity**. Explicitly, it is given by

$$\mathcal{A}_\xi = \left\{ f : I(\mathbb{A}_f) \to W_\xi \;\middle|\; \begin{array}{l} f \text{ is } I(\mathbb{A}_f)\text{-smooth by right translations} \\ \text{and } \forall \gamma \in I(\mathbb{Q}), f(\gamma\,\cdot) = \xi(\gamma)f(\cdot) \end{array} \right\}.$$

Notation. Let $\mathrm{Sh}^{\mathrm{an}}_{K_0K^p} := (\mathrm{S}_{K^p}\otimes_{\mathbb{Z}_{p^2}}\mathbb{Q}_{p^2})^{\mathrm{an}}$ denote the analytification of the generic fiber of S_{K^p}, on which the analytified local system $\mathcal{L}^{\mathrm{an}}_\xi$ is defined. Let $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss,an}} \subset \mathrm{Sh}^{\mathrm{an}}_{K_0K^p}$ denote the analytical tube of the supersingular locus, or in other words the generic fiber of the formal scheme $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss}}$. We write $\mathrm{H}^\bullet((\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss,an}}, \mathcal{L}^{\mathrm{an}}_\xi)$ for the cohomology of $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss,an}} \otimes \mathbb{C}_p$ with coefficients in $\mathcal{L}^{\mathrm{an}}_\xi$.

In [Far04] Théorème 4.5.12, Fargues builds a spectral sequence associated to the p-adic uniformization theorem in order to compute the cohomology of $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss,an}}$.

Theorem 5.2. *There is a W-equivariant spectral sequence*

$$F_2^{a,b}(K^p) : \mathrm{Ext}^a_J\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(n-1), \mathcal{A}_\xi^{K^p}\right) \implies \mathrm{H}^{a+b}((\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss,an}}, \mathcal{L}^{\mathrm{an}}_\xi).$$

These spectral sequences are compatible as the open compact subgroup K^p varies in $\mathbb{G}(\mathbb{A}_f^p)$.

The W-action on $F_2^{a,b}(K^p)$ is inherited from the cohomology group $\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(n-1)$. By the compatibility with variation of the level K^p, we may take the limit and obtain a $\mathbb{G}(\mathbb{A}_f^p) \times W$-equivariant spectral sequence $F_2^{a,b} := \varinjlim_{K^p} F_2^{a,b}(K^p)$. Since $\theta_{\max}$ is the semisimple rank of $J(\mathbb{Q}_p)$, the terms $F_2^{a,b}(K^p)$ are zero for $a > \theta_{\max}$ according to [Far04] Lemme 4.4.12. Therefore, the non-zero terms $F_2^{a,b}$ are located in the finite strip delimited by $0 \leqslant a \leqslant \theta_{\max}$ and $0 \leqslant b \leqslant 2(n-1)$. Let us look at the abutment of the sequence. Since S_{K^p} is smooth, Berkovich's comparison theorem, cf [Ber96] Corollary 3.6, gives an isomorphism

$$\mathrm{H}^{a+b}(\overline{\mathrm{S}}_{K^p}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \mathrm{H}^{a+b}((\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss,an}}, \mathcal{L}_\xi^{\mathrm{an}}).$$

Since $\overline{\mathrm{S}}_{K^p}^{\mathrm{ss}}$ has dimension $\theta_{\max}$, the cohomology $\mathrm{H}^\bullet((\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss,an}}, \mathcal{L}_\xi^{\mathrm{an}})$ is concentrated in degrees 0 to $2\theta_{\max}$.

Let $\mathcal{A}(I)$ denote the set of all automorphic representations of I counted with multiplicities. We write $\check{\xi}$ for the dual of ξ. We also define

$$\mathcal{A}_\xi(I) := \{\Pi \in \mathcal{A}(I) \mid \Pi_\infty = \check{\xi}\}.$$

According to [Far04] Section 4.6, we have an identification

$$\mathcal{A}_\xi^{K_p} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \Pi_p \otimes (\Pi^p)^{K_p}.$$

It yields, for every a and b, an isomorphism

$$F_2^{a,b}(K^p) \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^a \left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(n-1), \Pi_p\right) \otimes (\Pi^p)^{K_p}.$$

Taking the limit over K^p, we deduce that

$$F_2^{a,b} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^a \left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(n-1), \Pi_p\right) \otimes \Pi^p.$$

The spectral sequence defined by the terms $F_2^{a,b}$ computes $\mathrm{H}^{a+b}((\widehat{\mathrm{S}})^{\mathrm{ss,an}}, \mathcal{L}_\xi^{\mathrm{an}}) := \varinjlim_{K^p} \mathrm{H}^{a+b}((\widehat{\mathrm{S}}_{K^p})^{\mathrm{ss,an}}, \mathcal{L}_\xi^{\mathrm{an}})$. It is isomorphic to $\mathrm{H}^{a+b}(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) := \varinjlim_{K^p} \mathrm{H}^{a+b}(\overline{\mathrm{S}}_{K^p}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$.

Recall from Corollary 4.5 that we have a decomposition

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{b \leqslant b' \leqslant 2(n-1)} E_2^{b-b',b'},$$

and $E_2^{b-b',b'}$ corresponds to the eigenspace of τ associated to the eigenvalue $(-p)^{b'}$. Accordingly, we have a decomposition

$$F_2^{a,b} \simeq \bigoplus_{\substack{2(n-1)-b \leqslant \\ b' \leqslant 2(n-1)}} \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^a \left(E_2^{2(n-1)-b-b',b'}(n-1), \Pi_p\right) \otimes \Pi^p.$$

For $\Pi \in \mathcal{A}_\xi(I)$, we denote by ω_Π the central character. We define

$$\delta_{\Pi_p} := \omega_{\Pi_p}(p^{-1} \cdot \mathrm{id}) p^{-w(\xi)} \in \overline{\mathbb{Q}_\ell}^\times.$$

Let ι be any isomorphism $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$, and write $|\cdot|_\iota := |\iota(\cdot)|$. The center of $I(\mathbb{Q})$ is identified with $\mathbb{E}^\times$, and the element $p^{-1} \cdot \mathrm{id} \in \mathrm{Z}(J(\mathbb{Q}_p))$ is the image of $p^{-1} \in \mathbb{E}^\times \simeq \mathrm{Z}(I(\mathbb{Q})) \hookrightarrow \mathrm{Z}(J(\mathbb{Q}_p))$. We have $\omega_\Pi(p^{-1}) = 1$. Moreover, for any finite place $q \neq p$, the element p^{-1} lies inside the maximal compact subgroup of $\mathrm{Z}(I(\mathbb{Q}_q))$, so $|\omega_{\Pi_q}(p^{-1})|_\iota = 1$. Besides $\Pi_\infty = \check{\xi}$, so we have

$$|\omega_{\Pi_p}(p^{-1} \cdot \mathrm{id})|_\iota = |\omega_{\check{\xi}}(p^{-1})|_\iota^{-1} = |\omega_\xi(p^{-1})|_\iota = |p^{w(\xi)}|_\iota = p^{w(\xi)}.$$

The last equality comes from the isomorphism $W_\xi \simeq c^{t(\xi)} \otimes \epsilon(\xi)(\mathbb{V}_0^{\otimes m(\xi)})$, see Proposition 5.1. In particular $|\delta_{\Pi_p}|_\iota = 1$ for any isomorphism ι.

Proposition 5.3. *The W-action on $\mathrm{Ext}_J^a(E_2^{2(n-1)-b-b',b'}(n-1), \Pi_p)$ is trivial on the inertia I, and the Frobenius element* Frob *acts like multiplication by* $(-1)^{-b'}\delta_{\Pi_p} p^{-b'+2(n-1)+w(\xi)}$.

Proof. Let us write $X := E_2^{2(n-1)-b-b',b'}(n-1)$. By convention, the action of Frob on a space $\mathrm{Ext}_J^a(X, \Pi_p)$ is induced by functoriality of Ext applied to $\mathrm{Frob}^{-1} : X \to X$. Let us consider a projective resolution of X in the category of smooth representations of $J(\mathbb{Q}_p)$

$$\cdots \xrightarrow{u_3} P_2 \xrightarrow{u_2} P_1 \xrightarrow{u_1} P_0 \xrightarrow{u_0} X \longrightarrow 0.$$

Since Frob^{-1} commutes with the action of $J(\mathbb{Q}_p)$, we can choose a lift $\mathcal{F} = (\mathcal{F}_i)_{i \geqslant 0}$ of Frob^{-1} to a morphism of chain complexes.

$$\begin{tikzcd}
\cdots \arrow[r, "u_3"] & P_2 \arrow[r, "u_2"] \arrow[d, "\mathcal{F}_2"] & P_1 \arrow[r, "u_1"] \arrow[d, "\mathcal{F}_1"] & P_0 \arrow[r, "u_0"] \arrow[d, "\mathcal{F}_0"] & X \arrow[r] \arrow[d, "\mathrm{Frob}^{-1}"] & 0 \\
\cdots \arrow[r, "u_3"] & P_2 \arrow[r, "u_2"] & P_1 \arrow[r, "u_1"] & P_0 \arrow[r, "u_0"] & X \arrow[r] & 0
\end{tikzcd}$$

After applying $\mathrm{Hom}_J(\cdot, \Pi_p)$ and forgetting about the first term, we obtain a morphism $\mathcal{F}^*$ of chain complexes.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_J(P_0, \Pi_p) \arrow[r] \arrow[d, "\mathcal{F}_0^*"] & \mathrm{Hom}_J(P_1, \Pi_p) \arrow[r] \arrow[d, "\mathcal{F}_1^*"] & \mathrm{Hom}_J(P_2, \Pi_p) \arrow[r] \arrow[d, "\mathcal{F}_2^*"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_J(P_0, \Pi_p) \arrow[r] & \mathrm{Hom}_J(P_1, \Pi_p) \arrow[r] & \mathrm{Hom}_J(P_2, \Pi_p) \arrow[r] & \cdots
\end{tikzcd}$$

Here $\mathcal{F}_i^* f(v) := f(\mathcal{F}_i(v))$. It induces morphisms on the cohomology

$$\mathcal{F}_i^* : \mathrm{Ext}_J^i(X, \Pi_p) \to \mathrm{Ext}_J^i(X, \Pi_p),$$

which do not depend on the choice of the lift $\mathcal{F}$. Recall that Frob is the composition of φ and $p \cdot \mathrm{id} \in J(\mathbb{Q}_p)$. Since φ is multiplication by the scalar $(-1)^{b'} p^{b'-2(n-1)}$ on X, we may choose the lift $\mathcal{F}_i := (-1)^{b'} p^{-b'+2(n-1)}(p^{-1} \cdot \mathrm{id})$ for all i.

Consider an element of $\mathrm{Ext}^i_J(X, \Pi_p)$ represented by a morphism $f : P_i \to \Pi_p$. For any $v \in P_i$ we have

$$\mathcal{F}_i^* f(v) = f(\mathcal{F}_i(v)) = (-1)^{-b'} p^{-b'+2(n-1)} f((p^{-1}\cdot\mathrm{id})\cdot v) = (-1)^{-b'} p^{-b'+2(n-1)} \omega_{\Pi_p}(p^{-1}\cdot\mathrm{id}) f(v).$$

It follows that Frob acts on $\mathrm{Ext}^i_J(X, \Pi_p)$ via multiplication by the scalar $(-1)^{-b'} \delta_{\Pi_p} p^{-b'+2(n-1)+w(\xi)}$. □

In general, the Hochschild-Serre spectral sequence has many differentials between non-zero terms. However, focusing on the diagonal defined by $a + b = 0$, it is possible to compute $\mathrm{H}^0(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$. Recall that $X^{\mathrm{un}}(J)$ denotes the set of unramified characters of $J(\mathbb{Q}_p)$, ie. the characters which are trivial on J°. If $x \in \overline{\mathbb{Q}_\ell}^\times$ is any non-zero scalar, we denote by $\overline{\mathbb{Q}_\ell}[x]$ the 1-dimensional representation of W where the inertia I acts trivially and Frob acts like multiplication by x.

Proposition 5.4. *We have an isomorphism of $\mathbb{G}(\mathbb{A}_f^p) \times W$-representations*

$$\mathrm{H}^0(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}].$$

Proof. The only non-zero term $F_2^{a,b}$ on the diagonal $a + b = 0$ is $F_2^{0,0}$. Since there is no non-zero arrow pointing at nor coming from this term, it is untouched in all the successive pages of the sequence. Therefore we have an isomorphism

$$F_2^{0,0} \simeq \mathrm{H}^0(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}).$$

Using Proposition 4.14, we also have isomorphisms

$$\begin{aligned} F_2^{0,0} &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J\left(\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(n-1), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J\left((\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1})(n-1), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J^\circ}\left(\mathbf{1}(n-1), \Pi_{p|J^\circ}\right) \otimes \Pi^p. \end{aligned}$$

Thus, only the automorphic representations $\Pi \in \mathcal{A}_\xi(I)$ with $\Pi_p^{J^\circ} \neq 0$ contribute to the sum. Consider such a Π. The irreducible representation Π_p is generated by a J°-invariant vector. Since J° is normal in $J(\mathbb{Q}_p)$, the whole representation Π_p is trivial on J°. Thus, it is an irreducible representation of $J/J^\circ \simeq \mathbb{Z}$. Therefore, it is an unramified character. Moreover the W-representation $V_\Pi^0 := \mathrm{Hom}_{J^\circ}(\mathbf{1}(n-1), \Pi_p)$ is 1-dimensional and the Frobenius action was described in Proposition 5.3. □

5.2 The case $n = 3, 4$

In this section, we assume that $\theta_{\max} = 1$, ie. $n = 3$ or 4. Let ξ be an irreducible finite dimensional algebraic representation of $\mathbb{G}$. The semisimple rank of $J(\mathbb{Q}_p)$ is 1, therefore the terms $F_2^{a,b}$ are zero for $a > 1$. In particular, the spectral sequence already degenerates on the second page. Since it computes the cohomology of the supersingular locus $\overline{\mathrm{S}}^{\mathrm{ss}}$ which is 1-dimensional, we also have $F_2^{0,b} = 0$ for $b \geqslant 3$, and $F_2^{1,b} = 0$ for $b \geqslant 2$. In Figure 2, we draw the second page F_2 and we write between brackets the *complex modulus* of the possible eigenvalues of Frob on each term under any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$, as computed in Proposition 5.3.

Remark 5.5. The fact that no eigenvalue of complex modulus $p^{w(\xi)}$ appears in $F_2^{0,1}$ nor in $F_2^{1,1}$ follows from Proposition 4.29, where we proved that $E_2^{-1,2(n-1)} = 0$.

$$\begin{array}{cc}
F_2^{0,2}[p^{w(\xi)+2}, p^{w(\xi)}] & 0 \\
& \\
F_2^{0,1}[p^{w(\xi)+1}] & F_2^{1,1}[p^{w(\xi)+1}] \\
& \\
F_2^{0,0}[p^{w(\xi)}] & F_2^{1,0}[p^{w(\xi)}]
\end{array}$$

Figure 2: The second page F_2 with the complex modulus of possible eigenvalues of Frob on each term.

Proposition 5.6. *We have $F_2^{1,1} = 0$ and the eigenspaces of* Frob *on $F_2^{0,2}$ attached to any eigenvalue of complex modulus $p^{w(\xi)}$ are zero.*

Proof. By the machinery of spectral sequences, there is a $\mathbb{G}(\mathbb{A}_f^p) \times W$-subspace of $\mathrm{H}^2(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ isomorphic to $F_2^{1,1}$, and the quotient by this subspace is isomorphic to $F_2^{0,2}$. We prove that all eigenvalues of Frob on $\mathrm{H}^2(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ have complex modulus $p^{w(\xi)+2}$. The proposition then readily follows.
We need the Ekedahl-Oort stratification on the supersingular locus of the Shimura variety. Let $K^p \subset G(\mathbb{A}_f^p)$ be small enough. In [VW11] Sections 3.3 and 6.3, the authors define the Ekedahl-Oort stratification on $\mathcal{M}_{\mathrm{red}}$ and on $\overline{\mathrm{S}}^{\mathrm{ss}}_{K^p}$ respectively, and they are compatible via the p-adic uniformization isomorphism. For $n = 3$ or 4, the stratification on the supersingular locus take the following form

$$\overline{\mathrm{S}}^{\mathrm{ss}}_{K^p} = \overline{\mathrm{S}}^{\mathrm{ss}}_{K^p}[1] \sqcup \overline{\mathrm{S}}^{\mathrm{ss}}_{K^p}[3].$$

The stratum $\overline{\mathrm{S}}^{\mathrm{ss}}_{K^p}[1]$ is closed and 0-dimensional, whereas the other stratum $\overline{\mathrm{S}}^{\mathrm{ss}}_{K^p}[3]$ is open, dense and 1-dimensional. In particular, we have a Frobenius equivariant isomorphism between the cohomology groups of highest degree

$$\mathrm{H}^2(\overline{\mathrm{S}}^{\mathrm{ss}}_{K^p} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \mathrm{H}^2_c(\overline{\mathrm{S}}_{K^p}[3] \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}).$$

According the [VW11] Section 5.3, the closed Bruhat-Tits strata $\mathcal{M}_\Lambda$ and $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ also admit an Ekedahl-Oort stratification of a similar form, and we have a decomposition

$$\overline{\mathrm{S}}^{\mathrm{ss}}_{K^p}[3] = \bigsqcup_{\substack{1\leqslant k\leqslant s\\ [\Lambda]\in\Gamma_k\backslash\mathcal{L}^{(1)}}} \overline{\mathrm{S}}_{K^p,\Lambda,k}[3],$$

into a finite disjoint union of open and closed subvarieties (we used the notations of Section 3). As a consequence, we have the following Frobenius equivariant isomorphisms

$$\mathrm{H}^2_c(\overline{\mathrm{S}}_{K^p}[3]\otimes\mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{1\leqslant k\leqslant s\\ [\Lambda]\in\Gamma_k\backslash\mathcal{L}^{(1)}}} \mathrm{H}^2_c(\overline{\mathrm{S}}_{K^p,\Lambda,k}[3]\otimes\mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{1\leqslant k\leqslant s\\ [\Lambda]\in\Gamma_k\backslash\mathcal{L}^{(1)}}} \mathrm{H}^2(\overline{\mathrm{S}}_{K^p,\Lambda,k}\otimes\mathbb{F}, \overline{\mathcal{L}_\xi})$$

where the last isomorphism follows from the stratification on the closed Bruhat-Tits strata $\overline{\mathrm{S}}_{K^p,\Lambda,k}$. Now, recall that the local system $\mathcal{L}_\xi$ is given by

$$\mathcal{L}_\xi \simeq \epsilon(\xi)\epsilon_{m(\xi)}\left(\mathrm{R}^{m(\xi)}(\pi^{m(\xi)}_{K^p})_*\overline{\mathbb{Q}_\ell}(t(\xi))\right).$$

It implies that $\overline{\mathcal{L}_\xi}$ is pure of weight $w(\xi)$. Since the variety $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ is smooth and projective, it follows that all the eigenvalues of Frob on the cohomology group $\mathrm{H}^2(\overline{\mathrm{S}}_{K^p,\Lambda,k}\otimes\mathbb{F}, \overline{\mathcal{L}_\xi})$ have complex modulus $p^{w(\xi)+2}$ under any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$. The result follows by taking the limit over K^p. □

In this paragraph, let us compute the term

$$\begin{aligned} F_2^{1,0} &\simeq \bigoplus_{\Pi\in\mathcal{A}_\xi(I)} \mathrm{Ext}^1_J\left(\mathrm{H}^{2(n-1)}_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(n-1), \Pi_p\right)\otimes\Pi^p \\ &\simeq \bigoplus_{\Pi\in\mathcal{A}_\xi(I)} \mathrm{Ext}^1_J\left(\mathrm{c}-\mathrm{Ind}^J_{J^\circ}\mathbf{1}(n-1), \Pi_p\right)\otimes\Pi^p. \end{aligned}$$

Let St_J denote the Steinberg representation of $J(\mathbb{Q}_p)$.

Proposition 5.7. *Let π be an irreducible smooth representation of $J(\mathbb{Q}_p)$. Then*

$$\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}^J_{J^\circ}\mathbf{1}, \pi) = \begin{cases} \overline{\mathbb{Q}_\ell} & \textit{if } \exists\chi\in X^{\mathrm{un}}(J), \pi\simeq\chi\cdot\mathrm{St}_J, \\ 0 & \textit{otherwise.}\end{cases}$$

In order to prove this proposition, we need a few general facts about restriction of smooth representations to normal subgroups. Let G be a locally profinite group and let H be a closed normal subgroup. If (σ, W) is a representation of H, for $g \in G$ we define the representation (σ^g, W) by $\sigma^g : h \mapsto \sigma(g^{-1}hg)$. The representation σ is irreducible if and only if σ^g is for any (or for all) $g \in G$.

Lemma 5.8. *Assume that* $\mathrm{Z}(G)H$ *has finite index in* G*.*

(1) *Let* π *be a smooth irreducible admissible representation of* G*. There exists a smooth irreducible representation* σ *of* H*, an integer* $r \geqslant 1$ *and* $g_1, \ldots, g_r \in G$ *such that*

$$\pi_{|H} \simeq \sigma^{g_1} \oplus \ldots \oplus \sigma^{g_r}.$$

Moreover $r \leqslant [\mathrm{Z}(G)H : G]$*, and for any* $g \in G$ *there exists some* $1 \leqslant i \leqslant r$ *such that* $\sigma^g \simeq \sigma^{g_i}$*.*

(2) *Assume furthermore that* G/H *is abelian. Let* π_1 *and* π_2 *be two smooth admissible irreducible representations of* G*. The three following statements are equivalent.*
   - $(\pi_1)_{|H} \simeq (\pi_2)_{|H}$.
   - *There exists a smooth character* χ *of* G *which is trivial on* H *such that* $\pi_2 \simeq \chi \cdot \pi_1$.
   - $\mathrm{Hom}_H(\pi_1, \pi_2) \neq 0$.

(3) *Assume that* G/H *is abelian and that* $[\mathrm{Z}(G)H : G] = 2$*. Let* $g_0 \in G \backslash \mathrm{Z}(G)H$ *and let* π *be a smooth admissible irreducible representation of* G*. If there exists an irreducible representation* σ *of* H *such that* $\pi_{|H} \simeq \sigma \oplus \sigma^{g_0}$*, then* $\sigma \not\simeq \sigma^{g_0}$*.*

Proof. For (1) and (2), we refer to [Ren09] VI.3.2 Proposition. The result there is stated in the context of a p-adic group G with normal subgroup $H = {}^0G$ such that $G/{}^0G \simeq \mathbb{Z}^d$ for some $d \geqslant 0$, but the same arguments work as verbatim in the generality of the lemma. Admissibility of the representations involved is assumed only in order to apply Schur's lemma, insuring for instance the existence of central characters of smooth irreducible representations. In particular, if G/K is at most countable for any open compact subgroup K of G, then it is not necessary to assume admissibility.

Let us prove (3). Assume towards a contradiction that $\pi_{|H} \simeq \sigma \oplus \sigma^{g_0}$ and that $\sigma \simeq \sigma^{g_0}$. We build a smooth admissible irreducible representation Π of G such that $\Pi_{|H} = \sigma$, which results in a contradiction in regards to (2) since $\mathrm{Hom}_H(\Pi, \pi) \neq 0$ but $\Pi_{|H} \not\simeq \pi_{|H}$. Let χ be the central character of π. Then $\chi_{|\mathrm{Z}(G) \cap H}$ coincides with the central character of σ. Let W denote the underlying vector space of σ. By hypothesis, there exists a linear automorphism $f : W \to W$ such that for every $h \in H$ and $w \in W$,

$$f(\sigma(g_0^{-1}hg_0) \cdot w) = \sigma(h) \circ f(w).$$

Let us write $g_0^2 = z_0h_0$ for some $z_0 \in \mathrm{Z}(G)$ and $h_0 \in H$. We define $\varphi := f^2 \circ \sigma(h_0)^{-1}$. Then for all $h \in H$ and $w \in W$, we have

$$\begin{aligned}\varphi(\sigma(h)\cdot w) = f^2(\sigma(h_0^{-1}h)\cdot w) &= f^2(\sigma(h_0^{-1}hh_0)\sigma(h_0^{-1})\cdot w)\\ &= f^2(\sigma(g_0^{-2}hg_0^2)\sigma(h_0^{-1})\cdot w)\\ &= \sigma(h)\circ f^2(\sigma(h_0)^{-1}\cdot w)\\ &= \sigma(h)\circ\varphi(w).\end{aligned}$$

Thus $\varphi : \sigma \xrightarrow{\sim} \sigma$. By Schur's lemma we have $\varphi = \lambda \cdot \mathrm{id}$ for some $\lambda \in \overline{\mathbb{Q}_\ell}$. Up to replacing f by $(\chi(z_0)\lambda^{-1})^{1/2}f$, we may assume that $\varphi = \chi(z_0)\cdot\mathrm{id}$, ie. $f^2 = \chi(z_0)\sigma(h_0)$.
We build a G-representation Π on W which extends σ. Let $g \in G$ and define

$$\Pi(g) = \begin{cases}\chi(z)\sigma(h) & \text{if } g = zh \in Z(G)H,\\ \chi(z)f\circ\sigma(h) & \text{if } g = g_0zh \in g_0Z(G)H.\end{cases}$$

Then one may check that Π is a well defined group morphism $G \to \mathrm{GL}(W)$. The fact that it is smooth irreducible and admissible follows from $\Pi_{|H} \simeq \sigma$ by construction, and it concludes the proof. □

We may now move on to the proof of Proposition 5.7.

Proof. First, let us compute $\mathrm{Ext}^1_{J^\circ}(\mathbf{1},\sigma)$ for any irreducible representation σ of J° with trivial central character. Let $J^1 = \mathrm{U}(\mathbf{V})$ denote the unitary group of $\mathbf{V}$ (recall that $J = \mathrm{GU}(\mathbf{V})$ is the group of unitary similitudes). Then $J^1(\mathbb{Q}_p)$ is a normal subgroup both of J° and of $J(\mathbb{Q}_p)$. Moreover, $J^\circ/J^1(\mathbb{Q}_p)$ is isomorphic to the image of the multiplier $c_{|J^\circ} : J^\circ \to \mathbb{Z}_p^\times$, in particular it is compact and abelian. Thus, we have

$$\mathrm{Ext}^1_{J^\circ}(\mathbf{1},\sigma) \simeq \mathrm{Ext}^1_{J^1}(\mathbf{1},\sigma_{|J^1(\mathbb{Q}_p)})^{J^\circ/J^1(\mathbb{Q}_p)}.$$

Since σ has trivial central character, the J°-action on $\mathrm{Ext}^1_{J^1}(\mathbf{1},\sigma_{|J^1(\mathbb{Q}_p)})$ is actually trivial on $\mathrm{Z}(J^\circ)J^1(\mathbb{Q}_p)$. Since $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ is unramified, we actually have $\mathrm{Z}(J^\circ)J^1(\mathbb{Q}_p) = J^\circ$. Hence, J° acts trivially on $\mathrm{Ext}^1_{J^1}(\mathbf{1},\sigma_{|J^1(\mathbb{Q}_p)})$.
Since J^1 is an algebraic group, we may use Theorem 2 of [NP20], a generalization of a duality theorem of Schneider and Stühler, to finish the computation. Namely, we have

$$\mathrm{Ext}^1_{J^1}(\mathbf{1},\sigma_{|J^1(\mathbb{Q}_p)}) \simeq \mathrm{Hom}_{J^1}(\sigma_{|J^1(\mathbb{Q}_p)}, D(\mathbf{1}))^\vee,$$

where D denotes the Aubert-Zelevinsky involution in $J^1(\mathbb{Q}_p)$. We note that $D(\mathbf{1}) = \mathrm{St}_{J^1}$ is the Steinberg representation of $J^1(\mathbb{Q}_p)$. Let St_{J° denote the representation of $J^\circ = \mathrm{Z}(J^\circ)J^1(\mathbb{Q}_p)$ obtained by letting the center act trivially on St_{J^1}.

We have proved that for any irreducible representation σ of J° with trivial central character, we have

$$\mathrm{Ext}^1_{J^\circ}(\mathbf{1},\sigma) \simeq \mathrm{Hom}_{J^1}(\sigma_{|J^1(\mathbb{Q}_p)}, \mathrm{St}_{J^1})^\vee \simeq \begin{cases} \overline{\mathbb{Q}_\ell} & \text{if } \sigma \simeq \mathrm{St}_{J^\circ}, \\ 0 & \text{else.} \end{cases}$$

Now, let π be an irreducible representation of $J(\mathbb{Q}_p)$. By Frobenius reciprocity we have

$$\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1},\pi) \simeq \mathrm{Ext}^1_{J^\circ}(\mathbf{1},\pi_{|J^\circ}).$$

By functoriality of Ext, we have $\mathrm{Ext}^1_{J^\circ}(\mathbf{1},\pi_{|J^\circ}) = 0$ if the central character of π is not unramified. Thus, let us now assume that the central character is unramified. By the above, $\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1},\pi)$ is non zero if and only if $\pi_{|J^\circ}$ contains St_{J°. Besides, as will be proved in Lemma 5.9, we have $(\mathrm{St}_J)_{|J^\circ} = \mathrm{St}_{J^\circ}$. Thus, Lemma 5.8 (2) implies that $\pi_{|J^\circ}$ contains St_{J° if and only if $\pi \simeq \chi\cdot\mathrm{St}_J$ for some unramified character $\chi \in X^{\mathrm{un}}(J)$. Since $\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1},\chi\cdot\mathrm{St}_J) \simeq \overline{\mathbb{Q}_\ell}$, we are done. □

Lemma 5.9. *We have* $(\mathrm{St}_J)_{|J^\circ} \simeq \mathrm{St}_{J^\circ}$.

Proof. Since the Steinberg representation St_J has trivial central character, it is enough to prove that $(\mathrm{St}_J)_{|J^1(\mathbb{Q}_p)} \simeq \mathrm{St}_{J^1}$. The Steinberg representation St_J (resp. St_{J^1}) can be characterized as the unique irreducible representation ρ of $J(\mathbb{Q}_p)$ (resp. of $J^1(\mathbb{Q}_p)$) such that $\mathrm{Ext}^2_J(\mathbf{1},\rho) \neq 0$ (resp. $\mathrm{Ext}^1_{J^1}(\mathbf{1},\rho) \neq 0$). The gap between the degrees of the Ext groups for $J(\mathbb{Q}_p)$ and for $J^1(\mathbb{Q}_p)$ is explained by the non-compactness of the center of $J(\mathbb{Q}_p)$. By [NP20] Proposition 3.4 we have

$$\mathrm{Ext}^2_J(\mathbf{1},\mathrm{St}_J) \simeq \mathrm{Ext}^1_{J,\mathbf{1}}(\mathbf{1},\mathrm{St}_J) \oplus \mathrm{Ext}^2_{J,\mathbf{1}}(\mathbf{1},\mathrm{St}_J),$$

where the Ext groups on the right-hand side are taken in the category of smooth representations of $J(\mathbb{Q}_p)$ on which the center acts trivially. Equivalently, this is the category of smooth representations of $J(\mathbb{Q}_p)/\mathrm{Z}(J(\mathbb{Q}_p))$. Consider the normal subgroup $\mathrm{Z}(J(\mathbb{Q}_p))J^1(\mathbb{Q}_p)/\mathrm{Z}(J(\mathbb{Q}_p)) \simeq J^1(\mathbb{Q}_p)/\mathrm{Z}(J(\mathbb{Q}_p))\cap J^1(\mathbb{Q}_p) = J^1(\mathbb{Q}_p)/\mathrm{Z}(J^1(\mathbb{Q}_p))$. The quotient group is isomorphic to $J(\mathbb{Q}_p)/\mathrm{Z}(J(\mathbb{Q}_p))J^1(\mathbb{Q}_p)$, which is trivial if n is odd and $\mathbb{Z}/2\mathbb{Z}$ is n is even. Thus, we have

$$\begin{aligned}
\mathrm{Ext}^\bullet_{J,\mathbf{1}}(\mathbf{1},\mathrm{St}_J) &\simeq \mathrm{Ext}^\bullet_{J/\mathrm{Z}(J)}(\mathbf{1},\mathrm{St}_J) \\
&\simeq \mathrm{Ext}^\bullet_{J^1/\mathrm{Z}(J^1)}(\mathbf{1},(\mathrm{St}_J)_{|J^1(\mathbb{Q}_p)})^{J(\mathbb{Q}_p)/\mathrm{Z}(J(\mathbb{Q}_p))J^1(\mathbb{Q}_p)} \\
&\simeq \mathrm{Ext}^\bullet_{J^1,\mathbf{1}}(\mathbf{1},(\mathrm{St}_J)_{|J^1(\mathbb{Q}_p)})^{J(\mathbb{Q}_p)/\mathrm{Z}(J(\mathbb{Q}_p))J^1(\mathbb{Q}_p)} \\
&\simeq \mathrm{Ext}^\bullet_{J^1}(\mathbf{1},(\mathrm{St}_J)_{|J^1(\mathbb{Q}_p)})^{J(\mathbb{Q}_p)/\mathrm{Z}(J(\mathbb{Q}_p))J^1(\mathbb{Q}_p)},
\end{aligned}$$

the last line following from the same Proposition 3.4 as above, but applied to $J^1(\mathbb{Q}_p)$. In [Far04] Lemme 4.4.12, it is explained that $\mathrm{Ext}^i_{J^1}(\pi_1,\pi_2)$ vanishes for any smooth representations π_1, π_2 of $J^1(\mathbb{Q}_p)$ as soon as i is greater than the semisimple rank of $J(\mathbb{Q}_p)$, that is 1 in our case. Hence, $\mathrm{Ext}^2_{J,\mathbf{1}}(\mathbf{1}, \mathrm{St}_J) = 0$ and we have

$$\mathrm{Ext}^2_J(\mathbf{1}, \mathrm{St}_J) \simeq \mathrm{Ext}^1_{J,\mathbf{1}}(\mathbf{1}, \mathrm{St}_J) \simeq \mathrm{Ext}^1_{J^1}(\mathbf{1}, (\mathrm{St}_J)_{|J^1(\mathbb{Q}_p)})^{J(\mathbb{Q}_p)/\mathrm{Z}(J(\mathbb{Q}_p))J^1(\mathbb{Q}_p)}.$$

In particular, the right-hand side is non zero, which proves that $(\mathrm{St}_J)_{|J^1(\mathbb{Q}_p)}$ contains St_{J^1}. It remains that to justify that $(\mathrm{St}_J)_{|J^1(\mathbb{Q}_p)}$ is irreducible. If n is odd so that $\mathrm{Z}(J(\mathbb{Q}_p))J^1(\mathbb{Q}_p) = J(\mathbb{Q}_p)$, it is automatic. If n is even, in virtue of point (3) of Lemma 5.8, it remains to justify that for any $g \in J(\mathbb{Q}_p)$ we have $\mathrm{St}_{J^1}^g \simeq \mathrm{St}_{J^1}$. This follows from the following computation

$$\mathrm{Ext}^1_{J^1}(\mathbf{1}, \mathrm{St}^g_{J^1}) = \mathrm{Ext}^1_{J^1}(\mathbf{1}^{g^{-1}}, \mathrm{St}_{J^1}) = \mathrm{Ext}^1_{J^1}(\mathbf{1}, \mathrm{St}_{J^1}) \neq 0.$$

□

We may now compute the cohomology of the supersingular locus. Recall the supercuspidal representation τ_1 of the Levi complement $M_1 \subset J(\mathbb{Q}_p)$ that we defined in Section 4.2. When $n = 3$ or 4, we actually have $M_1 = J(\mathbb{Q}_p)$ and

$$\tau_1 = \mathrm{c}-\mathrm{Ind}^J_{\mathrm{N}_J(J_1)} \widetilde{\rho_{\Delta_2}}$$

is a supercuspidal representation of $J(\mathbb{Q}_p)$, where $\mathrm{N}_J(J_1) = \mathrm{Z}(J(\mathbb{Q}_p))J_1$ (see Proposition 1.14) and $\widetilde{\rho_{\Delta_2}}$ is the inflation of ρ_{Δ_2} to $\mathrm{N}_J(J_1)$.

Theorem 5.10. *Assume that $n = 3$ or 4. There are $\mathbb{G}(\mathbb{A}_f^p) \times W$-equivariant isomorphisms*

$$\mathrm{H}^0(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

$$\mathrm{H}^1(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi\cdot \mathrm{St}_J}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi\cdot \tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}],$$

$$\mathrm{H}^2(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

Proof. The statement regarding $\mathrm{H}^0(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ was already proved in Proposition 5.4. Let us prove the statement regarding $\mathrm{H}^2(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ first. By Proposition 5.6, we have

$$\mathrm{H}^2(\overline{\mathrm{S}}^{\mathrm{ss}} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq F_2^{0,2} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J \left(E_2^{0,2(n-2)}(n-1), \Pi_p\right) \otimes \Pi^p.$$

The term $E_2^{0,2(n-2)}$ is isomorphic to $\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\,\mathbf{1}$. Therefore, by Frobenius reciprocity we have

$$\mathrm{Hom}_J\left(E_2^{0,b}(n-1),\Pi_p\right)\simeq \mathrm{Hom}_{J_1}\left(\mathbf{1}(n-1),\Pi_p\right).$$

Hence, only the automorphic representations $\Pi\in\mathcal{A}_\xi(I)$ with $\Pi_p^{J_1}\neq 0$ contribute to $F_2^{0,2}$. Such a representation Π_p is said to be J_1**-spherical**. Since J_1 is a special maximal compact subgroup of $J(\mathbb{Q}_p)$, according to [Mın11] 2.1, we have $\dim(\pi^{J_1})=1$ for every smooth irreducible J_1-spherical representation π of $J(\mathbb{Q}_p)$. The result follows using Proposition 5.3 to describe the eigenvalues of Frob.

We now prove the statement regarding $\mathrm{H}^1(\overline{\mathrm{S}}^{\mathrm{ss}}\otimes\mathbb{F},\overline{\mathcal{L}_\xi})$. By the Hochschild-Serre spectral sequence, there exists a $G(\mathbb{A}_f^p)\times W$-subspace V' of this cohomology group such that

$$V'\simeq F_2^{1,0}\quad\text{and}\quad \mathrm{H}^1(\overline{\mathrm{S}}^{\mathrm{ss}}\otimes\mathbb{F},\overline{\mathcal{L}_\xi})/V'\simeq F_2^{0,1}.$$

We have

$$\begin{aligned}F_2^{1,0}&\simeq\bigoplus_{\Pi\in\mathcal{A}_\xi(I)}\mathrm{Ext}^1_J\left(\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}},\overline{\mathbb{Q}_\ell})(n-1),\Pi_p\right)\otimes\Pi^p\\&\simeq\bigoplus_{\Pi\in\mathcal{A}_\xi(I)}\mathrm{Ext}^1_J\left(\mathrm{c}-\mathrm{Ind}_{J^\circ}^{J}\,\mathbf{1}(n-1),\Pi_p\right)\otimes\Pi^p\\&\simeq\bigoplus_{\substack{\Pi\in\mathcal{A}_\xi(I)\\ \exists\chi\in X^{\mathrm{un}}(J),\\ \Pi_p=\chi\cdot\mathrm{St}_J}}\Pi^p\otimes\overline{\mathbb{Q}_\ell}[\delta_{\Pi_p}p^{w(\xi)}],\end{aligned}$$

according to Proposition 5.7, and with the eigenvalues of Frob being given by Proposition 5.3. On the other hand, we have

$$F_2^{0,1}\simeq\bigoplus_{\Pi\in\mathcal{A}_\xi(I)}\mathrm{Hom}_J\left(E_2^{0,2(n-1)-1}(n-1),\Pi_p\right)\otimes\Pi^p.$$

By Proposition 5.3, Frob acts on a summand of $F_2^{0,1}$ by the scalar $-\delta_{\Pi_p}p^{w(\xi)+1}$. Since $\mathrm{Frob}_{|V'}$ has no eigenvalue of complex modulus $p^{w(\xi)+1}$, the quotient actually splits so that $F_2^{0,1}$ is naturally a subspace of $\mathrm{H}^1(\overline{\mathrm{S}}^{\mathrm{ss}}\otimes\mathbb{F},\overline{\mathcal{L}_\xi})$. It remains to compute it. We have

$$E_2^{0,2(n-1)-1}\simeq\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\,\rho_{\Delta_2}.$$

Hence, we have an isomorphism

$$\begin{aligned}F_2^{0,1}&\simeq\bigoplus_{\Pi\in\mathcal{A}_\xi(I)}\mathrm{Hom}_J\left(\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\,\rho_{\Delta_2}(n-1),\Pi_p\right)\otimes\Pi^p\\&\simeq\bigoplus_{\Pi\in\mathcal{A}_\xi(I)}\mathrm{Hom}_{J_1}\left(\rho_{\Delta_2}(n-1),\Pi_{p|J_1}\right)\otimes\Pi^p.\end{aligned}$$

It follows that only the automorphic representations $\Pi \in \mathcal{A}_\xi(I)$ whose p-component Π_p contains the supercuspidal representation ρ_{Δ_2} when restricted to J_1, contribute to the sum. According to Proposition 4.21, such Π_p are precisely those of the form $\chi \cdot \tau_1$ for some $\chi \in X^{\mathrm{un}}(J)$. By the Mackey formula we have

$$\begin{aligned}\mathrm{Hom}_J\left(\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\rho_{\Delta_2}, \chi\cdot\tau_1\right) &\simeq \mathrm{Hom}_{J_1}\left(\rho_{\Delta_2}, \tau_{1|J_1}\right)\\ &\simeq \mathrm{Hom}_{J_1}\left(\rho_{\Delta_2}, (\mathrm{c}-\mathrm{Ind}_{\mathrm{N}_J(J_1)}^{J}\widetilde{\rho_{\Delta_2}})_{|J_1}\right)\\ &\simeq \bigoplus_{h\in J_1\backslash J(\mathbb{Q}_p)/\mathrm{N}_J(J_1)} \mathrm{Hom}_{J_1\cap {}^h\mathrm{N}_J(J_1)}(\rho_{\Delta_2}, {}^h\widetilde{\rho_{\Delta_2}}),\end{aligned}$$

where in the last formula we omitted to write the restrictions to $J_1 \cap {}^h\mathrm{N}_J(J_1)$. We used the fact that $\chi_{|J_1}$ is trivial. Since $\widetilde{\rho_{\Delta_2}}$ is just the inflation of ρ_{Δ_2} from J_1 to $\mathrm{N}_J(J_1) = \mathrm{Z}(J(\mathbb{Q}_p))J_1$, we have a bijection

$$\mathrm{Hom}_{J_1\cap {}^h\mathrm{N}_J(J_1)}(\rho_{\Delta_2}, {}^h\widetilde{\rho_{\Delta_2}}) \simeq \mathrm{Hom}_{\mathrm{N}_J(J_1)\cap {}^h\mathrm{N}_J(J_1)}(\widetilde{\rho_{\Delta_2}}, {}^h\widetilde{\rho_{\Delta_2}}).$$

Now, $\mathrm{N}_J(J_1)$ contains the center, is compact modulo the center, and $\tau_1 = \mathrm{c}-\mathrm{Ind}_{\mathrm{N}_J(J_1)}^{J}\widetilde{\rho_{\Delta_2}}$ is supercuspidal. It follows that an element $h \in J(\mathbb{Q}_p)$ intertwines $\widetilde{\rho_{\Delta_2}}$ if and only if $h \in \mathrm{N}_J(J_1)$ (see for instance [BH06] 11.4 Theorem along with Remarks 1 and 2). Therefore, only the trivial double coset contributes to the sum and we have

$$\mathrm{Hom}_J\left(\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\rho_{\Delta_2}, \chi\cdot\tau_1\right) \simeq \mathrm{Hom}_{J_1}(\rho_{\Delta_2}, \rho_{\Delta_2}) \simeq \overline{\mathbb{Q}_\ell}.$$

To sum up, we have

$$F_2^{0,1} \simeq \bigoplus_{\substack{\Pi\in\mathcal{A}_\xi(I)\\ \exists\chi\in X^{\mathrm{un}}(J),\\ \Pi_p=\chi\cdot\tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}].$$

It concludes the proof.

□

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