Compactly supported cohomology and nearby cycle cohomology of open Shimura varieties of PEL type

Naoki Imai and Yoichi Mieda

ABSTRACT. In this paper, we compare two cohomology groups associated to Shimura varieties of PEL type, which are not proper over the base. One is the compactly supported ℓ-adic cohomology, and the other is the nearby cycle cohomology, namely, the compactly supported cohomology of the nearby cycle complex for the canonical integral model of the Shimura variety over $\mathbb{Z}_p$. We prove that the $G(\mathbb{Q}_p)$-cuspidal part of these cohomology groups are the same, where G denotes the reductive algebraic group naturally attached to the PEL datum. Some applications to unitary Shimura varieties are also given.

1 Introduction

Study of the ℓ-adic cohomology of Shimura varieties is an important theme in number theory and arithmetic algebraic geometry. Let $\mathrm{Sh}_K = \mathrm{Sh}_K(G, X)$ be a Shimura variety associated to a Shimura datum (G, X) and a compact open subgroup K of $G(\mathbb{A}^\infty)$. As is well-known, it is naturally defined over a number field E, called the reflex field, which is canonically attached to (G, X).

First of all, let us consider the case where our Shimura variety is compact; namely, Sh_K is proper over E. For an irreducible algebraic representation ξ of G over $\overline{\mathbb{Q}}_\ell$, the ℓ-adic étale cohomology

$$H^i(\mathrm{Sh}, \mathcal{L}_\xi) = \varinjlim_K H^i(\mathrm{Sh}_K \otimes_E \overline{E}, \mathcal{L}_\xi)$$

of the associated local system $\mathcal{L}_\xi$ is naturally equipped with the action of $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{E}/E)$. By Matsushima's formula, this cohomology group is strongly related to the automorphic spectrum of $G(\mathbb{A})$. If moreover our Shimura variety is so-called PEL type, namely, obtained as a moduli space of abelian varieties with several additional structures, then it has a natural integral model over $\mathcal{O}_{E_v}$, the ring of integers of the completion of E at a finite place v. This integral model, also denoted

by Sh_K, is very useful to investigate the action of the Galois group $\mathrm{Gal}(\overline{E}/E)$ on $H^i(\mathrm{Sh}, \mathcal{L}_\xi)$. If Sh_K is smooth over $\mathcal{O}_{E_v}$, the proper smooth base change theorem says that the ℓ-adic cohomology $H^i(\mathrm{Sh}_K \otimes_E \overline{E}, \mathcal{L}_\xi)$ is canonically isomorphic to the ℓ-adic cohomology of the special fiber $\mathrm{Sh}_{K,v}$ of the integral model. As a result, the $\mathrm{Gal}(\overline{E}/E)$-representation $H^i(\mathrm{Sh}_K \otimes_E \overline{E}, \mathcal{L}_\xi)$ is unramified at v, and the alternating sum of the traces of Frob_v, the Frobenius element at v, on $H^i(\mathrm{Sh}_K \otimes_E \overline{E}, \mathcal{L}_\xi)$ can be computed by the Lefschetz trace formula. Even if Sh_K is not smooth over $\mathcal{O}_{E_v}$, we can investigate the action of $\mathrm{Gal}(\overline{E}_v/E_v)$ on $H^i(\mathrm{Sh}_K \otimes_E \overline{E}, \mathcal{L}_\xi)$ by using the theory of nearby cycle functor. Such a method provides an efficient way to establish a part of the Langlands correspondence; see [HT01], [Far04], [Man04], [Man05], [Shi11] and [Sch10], for instance. The crucial point is that the properness of Sh_K over $\mathcal{O}_{E_v}$ ensures the coincidence of the cohomology of the generic fiber and the nearby cycle cohomology, that is, the cohomology of the nearby cycle complex over the special fiber.

Now we pass to the case where Sh_K is not compact, while is of PEL type. There are many interesting cases satisfying these conditions, including the modular curves and the Siegel modular varieties. For the ℓ-adic cohomology of a Shimura variety over E, we have at least three choices; the ordinary ℓ-adic cohomology, the compactly supported ℓ-adic cohomology and the intersection cohomology. Usually we prefer the intersection cohomology, since it has a close relation to the automorphic spectrum of $G(\mathbb{A})$. However, the theory of minimal compactifications tells us that these three cohomology groups are the same "up to induced representations".

Can we use the natural integral model in this case to investigate the Galois action on the ℓ-adic cohomology? If Sh_K is smooth over $\mathcal{O}_{E_v}$, Fujiwara's trace formula [Fuj97] is a very powerful tool to carry it out. By using it, one can compute the alternating sum of the traces of the action of Frob_v on the intersection cohomology $IH^i(\mathrm{Sh}_K \otimes_E \overline{E}, \mathcal{L}_\xi)$; see [Mor08], [Mor10]. On the other hand, if Sh_K has bad reduction, we cannot use exactly the same method as in the compact case, since the nearby cycle cohomology is different from the ℓ-adic cohomology of the generic fiber, whichever cohomology we choose among the three mentioned above.

The purpose of this paper is to measure the gap between the compactly supported cohomology (of the Shimura variety over E) and the nearby cycle cohomology, that is, the compactly supported cohomology of the nearby cycle complex over the special fiber of the integral model. Our main theorem claims that the gap is non-cuspidal at p, where p is the prime divisible by v. More precise statement is the following:

Theorem 1.1 (Theorem 4.2) *In the kernel and the cokernel of the natural map*

$$\varinjlim_K H^i_c(\mathrm{Sh}_{K,\overline{v}}, R\psi \mathcal{L}_\xi) \longrightarrow \varinjlim_K H^i_c(\mathrm{Sh}_K \otimes_{E_v} \overline{E}_v, \mathcal{L}_\xi),$$

no irreducible supercuspidal representation of $G(\mathbb{Q}_p)$ appears as a subquotient.

Recall that an irreducible smooth representation of $G(\mathbb{Q}_p)$ is said to be supercuspidal if it does not appear as a subquotient of parabolically induced representations from any proper parabolic subgroup.

In particular, for an irreducible admissible representation Π of $G(\mathbb{A}^\infty)$ such that Π_p is supercuspidal, we have an isomorphism of $\mathrm{Gal}(\overline{E}_v/E_v)$-modules

$$\left(\varinjlim_{K} H^i_c(\mathrm{Sh}_{K,\overline{v}}, R\psi\mathcal{L}_\xi)\right)[\Pi] \cong \left(\varinjlim_{K} H^i_c(\mathrm{Sh}_K \otimes_{E_v} \overline{E}_v, \mathcal{L}_\xi)\right)[\Pi]. \qquad (*)$$

Here $(-)[\Pi]$ denotes the Π-isotypic part (for precise definition, see Notation below). As a result, for a representation Π as above, the contribution of it to the compactly supported cohomology of the Shimura variety can be investigated by using the nearby cycle functor. Note that the same thing holds for the ordinary cohomology and the intersection cohomology, since the gaps between the compactly supported cohomology and these two cohomology groups are non-cuspidal, as already explained.

If an irreducible admissible representation Π of $G(\mathbb{A}^\infty)$ has a supercuspidal local component $\Pi_{p'}$ at a prime $p' \neq p$, then it is easy to obtain an isomorphism $(*)$ by using the minimal compactification with hyperspecial level at p. The main difficulty to prove Theorem 1.1 is the absence of a good compactification of our integral Shimura variety with higher level at p. Our main idea to overcome it is to use rigid geometry. Roughly speaking, the gap between the compactly supported cohomology and the nearby cycle cohomology can be measured by the rigid subspace of the generic fiber of our integral model consisting of points corresponding to abelian varieties with bad reduction. We will find a nice stratification on this rigid subspace to conclude that it is "geometrically" induced from proper parabolic subgroups of $G(\mathbb{Q}_p)$. To construct the stratification, we use the theory of arithmetic toroidal compactifications with hyperspecial level at p. Such compactifications were constructed by Faltings and Chai [FC90] for Siegel modular varieties, and by Lan [Lan08] for general Shimura varieties of PEL type. There is also a work by Fujiwara on this topic (*cf.* [FK06, §7]).

It will be worth noting that our method is totally geometric and thus is also valid in the torsion coefficient case. See Theorem 4.13 for an analogue of Theorem 1.1 with the $\overline{\mathbb{F}}_\ell$-coefficients.

We sketch the outline of this paper. In Section 2, we give some preliminary results for constructing a nice stratification. In Section 3, after recalling some notation concerning with Shimura varieties of PEL type, we give a key geometric construction; namely, the construction of a stratification on the rigid subspace that measures the gap of two cohomology groups. We also observe that this stratification behaves well under the Hecke action on the tower of Shimura varieties. In Section 4, we introduce several cohomology groups of the Shimura variety and prove Theorem 1.1. In Section 5, we give some applications of our main result to unitary Shimura varieties. The authors expect that our result has more interesting applications in a symplectic case. We will study it in our future work.

Notation Put $\widehat{\mathbb{Z}} = \prod_{p:\,\mathrm{prime}} \mathbb{Z}_p$ and $\mathbb{A}^\infty = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. For a prime p, put $\widehat{\mathbb{Z}}^p = \prod_{p' \neq p:\,\mathrm{prime}} \mathbb{Z}_{p'}$ and $\mathbb{A}^{\infty,p} = \widehat{\mathbb{Z}}^p \otimes_{\mathbb{Z}} \mathbb{Q}$.

Assume that we are given two locally profinite groups G and Γ and a prime ℓ. For an admissible/continuous representation V of $G \times \Gamma$ over $\overline{\mathbb{Q}}_\ell$ in the sense of [HT01, I.2] and an irreducible admissible representation π of G, put $V[\pi] = \bigoplus_\sigma \sigma^{\oplus m_{\pi \boxtimes \sigma}}$, where σ runs through finite-dimensional irreducible continuous $\overline{\mathbb{Q}}_\ell$-representations of Γ, and $m_{\pi \boxtimes \sigma}$ denotes the coefficient of $[\pi \boxtimes \sigma]$ in the image of V in the Grothendieck group $\mathrm{Groth}_{G \times \Gamma, \ell}(G \times \Gamma)$ considered in [HT01, I.2]. It is a semisimple continuous representation of Γ.

Every sheaf and cohomology are considered in the étale topology.

2 Rigid geometry and semi-abelian schemes

2.1 Notation for adic spaces

Throughout this paper, we will use the framework of adic spaces introduced by Huber (*cf.* [Hub93], [Hub94], [Hub96]). Here we recall some notation briefly.

Let S be a noetherian scheme and S_0 a closed subscheme of S. We denote the formal completion of S along S_0 by $\mathcal{S}$. Put $\mathcal{S}^{\mathrm{rig}} = t(\mathcal{S})_a$, where $t(\mathcal{S})$ is the adic space associated to $\mathcal{S}$ (*cf.* [Hub94, §4]) and $t(\mathcal{S})_a$ denotes the open adic subspace of $t(\mathcal{S})$ consisting of analytic points. It is a quasi-compact analytic adic space.

Let X be a scheme of finite type over S. Put $X_0 = X \times_S S_0$ and denote the formal completion of X along X_0 by $\widehat{X}$. Then we can construct an adic space $\widehat{X}^{\mathrm{rig}}$ in the same way as $\mathcal{S}^{\mathrm{rig}}$. The induced morphism $\widehat{X}^{\mathrm{rig}} \longrightarrow \mathcal{S}^{\mathrm{rig}}$ is of finite type. On the other hand, we can construct another adic space $X \times_S \mathcal{S}^{\mathrm{rig}}$. Indeed, since we have morphisms of locally ringed spaces $(\mathcal{S}^{\mathrm{rig}}, \mathcal{O}_{\mathcal{S}^{\mathrm{rig}}}) \longrightarrow (t(\mathcal{S}), \mathcal{O}_{t(\mathcal{S})}) \longrightarrow (S, \mathcal{O}_S)$ (for the second one, see [Hub94, Remark 4.6 iv)]), we can make the fiber product $X \times_S \mathcal{S}^{\mathrm{rig}}$ in the sense of [Hub94, Proposition 3.8]. For simplicity, we write X^{ad} for $X \times_S \mathcal{S}^{\mathrm{rig}}$, though it depends on (S, S_0). Since the morphism $\mathcal{S}^{\mathrm{rig}} \longrightarrow S$ factors through $S^0 = S \setminus S_0$, we have $X \times_S \mathcal{S}^{\mathrm{rig}} = (X \times_S S^0) \times_{S^0} \mathcal{S}^{\mathrm{rig}}$. In particular, X^{ad} depends only on $X \times_S S^0$. The natural morphism $X^{\mathrm{ad}} \longrightarrow \mathcal{S}^{\mathrm{rig}}$ is locally of finite type, but not necessarily quasi-compact; see the following example.

Example 2.1 Let R be a complete discrete valuation ring and F its fraction field. Consider the case where $S = \mathrm{Spec}\, R$ and S_0 is defined by the maximal ideal of R. Then, for an S-scheme X of finite type, X^{ad} can be regarded as the rigid space over F associated to a scheme $X \times_S \mathrm{Spec}\, F$ over F. For example, $(\mathbb{A}_S^1)^{\mathrm{ad}} = (\mathbb{A}_F^1)^{\mathrm{ad}}$ is the rigid-analytic affine line over F and thus is not quasi-compact. On the other hand, $(\widehat{\mathbb{A}}_S^1)^{\mathrm{rig}}$ is the unit disc "$|z| \le 1$" in $(\mathbb{A}_F^1)^{\mathrm{ad}}$, which is quasi-compact.

Lemma 2.2 *The functors $X \longmapsto \widehat{X}^{\mathrm{rig}}$ and $X \longmapsto X^{\mathrm{ad}}$ commute with fiber products.*

Proof. For the functor $X \longmapsto \widehat{X}^{\mathrm{rig}}$, it can be checked easily (*cf.* [Mie06, Lemma 3.4] and [Mie10b, Proof of Lemma 4.3 v)]). Consider the functor $X \longmapsto X^{\mathrm{ad}}$. Let

$Y \longrightarrow X \longleftarrow Z$ be a diagram of S-schemes of finite type. What we should prove is

$$(Y \times_X Z) \times_S \mathcal{S}^{\mathrm{rig}} \cong (Y \times_S \mathcal{S}^{\mathrm{rig}}) \times_{X \times_S \mathcal{S}^{\mathrm{rig}}} (Z \times_S \mathcal{S}^{\mathrm{rig}}).$$

It is not totally automatic, since $Y \times_X Z$ is not a fiber product in the category of locally ringed spaces. It follows from the fact that morphisms of locally ringed spaces $\mathrm{Spa}(A, A^+) \longrightarrow \mathrm{Spec}\, B$ correspond bijectively to ring homomorphisms $B \longrightarrow A$ (this fact is used implicitly in [Hub94, Remark 4.6 (iv)] to define $t(\mathcal{S}) \longrightarrow S$). ∎

Let us compare $\widehat{X}^{\mathrm{rig}}$ and X^{ad}; by the commutative diagram

$$\begin{array}{ccc} \widehat{X}^{\mathrm{rig}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{S}^{\mathrm{rig}} & \longrightarrow & S \end{array}$$

and the universality of the fiber product $X \times_S \mathcal{S}^{\mathrm{rig}}$, we have a natural morphism $\widehat{X}^{\mathrm{rig}} \longrightarrow X^{\mathrm{ad}}$.

Lemma 2.3   i) If X is separated over S, the natural morphism $\widehat{X}^{\mathrm{rig}} \longrightarrow X^{\mathrm{ad}}$ is an open immersion.

ii) If X is proper over S, the natural morphism $\widehat{X}^{\mathrm{rig}} \longrightarrow X^{\mathrm{ad}}$ is an isomorphism.

Proof. See [Hub94, Remark 4.6 (iv)]. ∎

Remark 2.4 Let $f \colon S' \longrightarrow S$ be a morphism of finite type and $S'_0 = S' \times_S S_0$. We denote by $\mathcal{S}'^{\mathrm{rig}}$ the formal completion of S' along S'_0. Then, all constructions above are compatible with the base change by f. More precisely, for a scheme X of finite type over S, we have $(X \times_S S')^{\wedge \mathrm{rig}} \cong \widehat{X}^{\mathrm{rig}} \times_{\mathcal{S}^{\mathrm{rig}}} \mathcal{S}'^{\mathrm{rig}}$ and $(X \times_S S')^{S'\text{-ad}} \cong X^{\mathrm{ad}} \times_{\mathcal{S}^{\mathrm{rig}}} \mathcal{S}'^{\mathrm{rig}}$. Here $(-)^{S'\text{-ad}}$ denotes the functor $(-)^{\mathrm{ad}}$ for the base (S', S'_0), namely, $(-)^{S'\text{-ad}} = (-) \times_{S'} \mathcal{S}'^{\mathrm{rig}}$.

The following notation is used in Section 3. In the remaining part of this subsection, assume that S is the spectrum of a complete discrete valuation ring and S_0 is the closed point of S. For a scheme of finite type X over S, we have a natural morphism of locally and topologically ringed spaces $(t(\widehat{X}), \mathcal{O}^+_{t(\widehat{X})}) \longrightarrow (\widehat{X}, \mathcal{O}_{\widehat{X}})$ (*cf.* [Hub94, Proposition 4.1]). Note that the underlying continuous map $t(\widehat{X}) \longrightarrow X_0$ is different from the map $t(\widehat{X}) \longrightarrow X$ considered above. We denote the composite $\widehat{X}^{\mathrm{rig}} \hookrightarrow t(\widehat{X}) \longrightarrow X_0$ by $\mathrm{sp}_{\widehat{X}}$, or simply by sp.

Let Y be a closed subscheme of X_0 and $\mathcal{X}$ the formal completion of X along Y. Then we can consider the generic fiber $t(\mathcal{X})_\eta = S^0 \times_S t(\mathcal{X})$ of the adic space $t(\mathcal{X})$. This is so-called the rigid generic fiber of $\mathcal{X}$ due to Raynaud and Berthelot, in the context of adic spaces. If $Y = X_0$, then $t(\mathcal{X})_\eta = \widehat{X}^{\mathrm{rig}}$.

Lemma 2.5 The natural morphism $t(\mathcal{X})_\eta \longrightarrow \widehat{X}^{\mathrm{rig}}$ induced from $\mathcal{X} \longrightarrow \widehat{X}$ is an open immersion. Its image coincides with $\mathrm{sp}^{-1}(Y)^\circ$, where $(-)^\circ$ denotes the interior (in $\widehat{X}^{\mathrm{rig}}$).

Proof. See [Hub98b, Lemma 3.13 i)]. ∎

2.2 Etale sheaves associated to semi-abelian schemes

We continue to use the notation introduced in the beginning of the previous subsection. Let U be an open subscheme of $S^0 = S \setminus S_0$ and p a prime invertible on U. Fix an integer $m > 0$.

Let G be a semi-abelian scheme over S. Namely, G is a separated smooth commutative group scheme over S such that each fiber G_s of G at $s \in S$ is an extension of an abelian variety A_s by a torus T_s. We denote the relative dimension of G over S by d. Assume the following:

- The rank of T_s (called the toric rank of G_s) with $s \in S_0$ is a constant r.
- $G_U = G \times_S U$ is an abelian scheme.

Under the first condition, it is known that G_0 is globally an extension

$$0 \longrightarrow T_0 \longrightarrow G_0 \longrightarrow A_0 \longrightarrow 0,$$

where T_0 is a torus of rank r over S_0 and A_0 is an abelian scheme over S_0 ([FC90, Chapter I, Corollary 2.11]).

Let us consider two group spaces $\widehat{G}^{\mathrm{rig}}[p^m]_{U^{\mathrm{ad}}}$ and $G^{\mathrm{ad}}[p^m]_{U^{\mathrm{ad}}}$ over U^{ad}, where $(-)_{U^{\mathrm{ad}}}$ denotes the restriction to U^{ad}.

Lemma 2.6 *The adic space $G^{\mathrm{ad}}[p^m]_{U^{\mathrm{ad}}}$ is finite étale of degree p^{2dm} over U^{ad}.*

Proof. By Lemma 2.2, we have $G^{\mathrm{ad}}[p^m]_{U^{\mathrm{ad}}} = (G_U[p^m]) \times_U U^{\mathrm{ad}}$. Since $G_U[p^m]$ is finite étale of degree over p^{2dm} over U, $G^{\mathrm{ad}}[p^m]_{U^{\mathrm{ad}}}$ is finite étale of degree p^{2dm} over U^{ad} (*cf.* [Hub96, Corollary 1.7.3 i)]). ■

Lemma 2.7 *The adic space $\widehat{G}^{\mathrm{rig}}[p^m]_{U^{\mathrm{ad}}}$ is finite étale of degree $p^{(2d-r)m}$ over U^{ad}.*

Proof. We may assume that $S = \operatorname{Spec} R$ is affine. Let $I \subset R$ be the defining ideal of S_0. By replacing R by its I-adic completion, we can reduce to the case where R is I-adic complete. Put $S_i = \operatorname{Spec} R/I^{i+1}$ and $G_i = G \times_S S_i$.

By [SGA3, Exposé IX, Théorème 3.6, Théorème 3.6 bis], the exact sequence

$$0 \longrightarrow T_0 \longrightarrow G_0 \longrightarrow A_0 \longrightarrow 0$$

can be lifted canonically to an exact sequence

$$0 \longrightarrow T_i \longrightarrow G_i \longrightarrow A_i \longrightarrow 0$$

over S_i, where T_i is a torus over S_i and A_i is an abelian scheme over S_i (*cf.* [Lan08, §3.3.3]). Let $\widehat{T} = \varinjlim_i T_i$ and $\widehat{A} = \varinjlim_i A_i$ be associated formal groups over $\mathcal{S}$. Then $\widehat{G}$ is an extension of $\widehat{A}$ by $\widehat{T}$.

By taking p^m-torsion points, we get an exact sequence

$$0 \longrightarrow \widehat{T}[p^m] \longrightarrow \widehat{G}[p^m] \longrightarrow \widehat{A}[p^m] \longrightarrow 0$$

of formal groups over $\mathcal{S}$. Since $\widehat{G}^{\mathrm{rig}}[p^m] \cong (\widehat{G}[p^m])^{\mathrm{rig}}$, it suffices to see that $(\widehat{T}[p^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$ (resp. $(\widehat{A}[p^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$) is finite étale of degree p^{rm} (resp. $p^{2(d-r)m}$) over U^{ad}.

First we consider $(\widehat{T}[p^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$. Since $\widehat{T}[p^m] = \varinjlim_i (T_i[p^m])$, it is finite flat over $\mathcal{S} = \mathrm{Spf}\, R$. Therefore there exists a finite flat R-algebra R' such that $\widehat{T}[p^m] = \mathrm{Spf}\, R'$. Moreover, a scheme $T' = \mathrm{Spec}\, R'$ is naturally equipped with a structure of commutative group scheme over $S = \mathrm{Spec}\, R$. Since T' is killed by p^m and p is invertible on U, $T'_U = T' \times_S U$ is a finite étale group scheme over U. By Lemma 2.3 ii), we have $(\widehat{T}[p^m])^{\mathrm{rig}} = (T')^{\wedge \mathrm{rig}} = T'^{\mathrm{ad}} = T' \times_S \mathcal{S}^{\mathrm{rig}}$. Therefore $(\widehat{T}[p^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}} = T'_U \times_U U^{\mathrm{ad}}$ is finite étale over U^{ad} (*cf.* [Hub96, Corollary 1.7.3 i)]). Its degree is clearly p^{rm}.

The same argument also works for $(\widehat{A}[p^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$. ∎

By Lemma 2.6 and Lemma 2.7, we may regard $\widehat{G}^{\mathrm{rig}}[p^m]_{U^{\mathrm{ad}}}$ and $G^{\mathrm{ad}}[p^m]_{U^{\mathrm{ad}}}$ as locally constant constructible sheaves over U^{ad}. Since we have a natural open immersion $\widehat{G}^{\mathrm{rig}} \hookrightarrow G^{\mathrm{ad}}$, $\widehat{G}^{\mathrm{rig}}[p^m]_{U^{\mathrm{ad}}}$ is a subsheaf of $G^{\mathrm{ad}}[p^m]_{U^{\mathrm{ad}}}$.

Remark 2.8 In the setting of Remark 2.4, the construction above is clearly compatible with the base change by $f \colon S' \longrightarrow S$.

In the remaining part of this subsection, consider the case where $S = \mathrm{Spec}\, R$ is the spectrum of a complete discrete valuation ring R, S_0 is the closed point of S and $U = S^0 = S \setminus S_0$. Let $\overline{\eta}$ be a geometric point lying over the unique point of U^{ad}.

As in the proof of Lemma 2.7, $\widehat{G}$ is an extension

$$0 \longrightarrow \widehat{T} \longrightarrow \widehat{G} \longrightarrow \widehat{A} \longrightarrow 0$$

of a formal group $\widehat{A}$ by $\widehat{T}$.

Lemma 2.9 The $\mathbb{Z}/p^m\mathbb{Z}$-modules $\widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$ and $G^{\mathrm{ad}}[p^m]_{\overline{\eta}}$ are free of finite rank. Furthermore, $\widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$ (resp. $T_p \widehat{G}^{\mathrm{rig}}_{\overline{\eta}}$) is a direct summand of $G^{\mathrm{ad}}[p^m]_{\overline{\eta}}$ (resp. $T_p G^{\mathrm{ad}}_{\overline{\eta}}$) as a $\mathbb{Z}_p$-module.

Proof. First $G^{\mathrm{ad}}[p^m]_{\overline{\eta}} \cong G_{\overline{\eta}}[p^m]$ is a free $\mathbb{Z}/p^m\mathbb{Z}$-module of finite rank, since $G_{\overline{\eta}}$ is an abelian variety.

Next we will prove that $\widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$ is a free $\mathbb{Z}/p^m\mathbb{Z}$-module of finite rank. By the exact sequence

$$0 \longrightarrow \widehat{T}^{\mathrm{rig}}[p^m]_{\overline{\eta}} \longrightarrow \widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}} \longrightarrow \widehat{A}^{\mathrm{rig}}[p^m]_{\overline{\eta}} \longrightarrow 0,$$

it suffices to show that $\widehat{T}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$ and $\widehat{A}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$ are free of finite rank. By [SGA3, Exposé X, Théorème 3.2], $\widehat{T}$ can be algebraized into a torus T over S. Therefore, by Lemma 2.2 and Lemma 2.3 ii), we have

$$\widehat{T}^{\mathrm{rig}}[p^m]_{\overline{\eta}} = (T[p^m])^{\wedge \mathrm{rig}}_{\overline{\eta}} = (T[p^m])^{\mathrm{ad}}_{\overline{\eta}} = T_{\overline{\eta}}[p^m],$$

which is obviously free of finite rank. Similar argument works for $\widehat{A}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$, since $\widehat{A}$ is also algebraizable (*cf.* [Lan08, Proposition 3.3.3.6, Remark 3.3.3.9]).

In particular, $\widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$ is an injective $\mathbb{Z}/p^m\mathbb{Z}$-module, and thus is a direct summand of $G^{\mathrm{ad}}[p^m]_{\overline{\eta}}$.

Put $C = T_p G_{\overline{\eta}}^{\mathrm{ad}}/T_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}}$. Then it is easy to see that

$$C \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z} = G^{\mathrm{ad}}[p^m]_{\overline{\eta}}/\widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$$

(use the same argument as above to conclude that $T_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z} \cong \widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$). Therefore $C \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}$ is a finite flat $\mathbb{Z}/p^m\mathbb{Z}$-module, and thus C is a finite flat $\mathbb{Z}_p$-module. Hence C is a free $\mathbb{Z}_p$-module, and thus the inclusion $T_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}} \hookrightarrow T_p G_{\overline{\eta}}^{\mathrm{ad}}$ splits. Namely, $T_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}}$ is a direct summand of $T_p G_{\overline{\eta}}^{\mathrm{ad}}$, as desired. ∎

Remark 2.10 Actually, the extension $0 \longrightarrow \widehat{T} \longrightarrow \widehat{G} \longrightarrow \widehat{A} \longrightarrow 0$ can be algebraized; namely, there exists an exact sequence

$$0 \longrightarrow T \longrightarrow G^{\natural} \longrightarrow A \longrightarrow 0$$

of commutative group schemes over S, where T is a torus over S and A is an abelian scheme over S, such that its formal completion along the special fiber is isomorphic to the extension above (*cf.* [Lan08, Proposition 3.3.3.6, Remark 3.3.3.9]). Such an extension is called the Raynaud extension associated to G.

Our construction above is related to the Raynaud extension in the following way. First, we have a natural isomorphism $\widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}} \xrightarrow{\cong} (G^{\natural})^{\mathrm{ad}}[p^m]_{\overline{\eta}}$, which is induced from an open immersion $\widehat{G}^{\mathrm{rig}} \cong \widehat{(G^{\natural})}^{\mathrm{rig}} \hookrightarrow (G^{\natural})^{\mathrm{ad}}$ (*cf.* Lemma 2.3 i)). Moreover, the image of $\widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}} \hookrightarrow G^{\mathrm{ad}}[p^m]_{\overline{\eta}}$ coincides with the image of the map $G^{\natural}[p^m]_{\overline{\eta}} \longrightarrow G[p^m]_{\overline{\eta}}$ in [Lan08, Corollary 4.5.3.12]. Therefore, the latter statement of Lemma 2.9 also follows from [Lan08, Corollary 4.5.3.12].

Proposition 2.11 *Assume that the fraction field F of R is a finite extension of $\mathbb{Q}_p$. Let λ be a polarization of G_U. Then, alternating bilinear pairings*

$$\langle \ , \ \rangle_\lambda \colon G[p^m]_{\overline{\eta}} \times G[p^m]_{\overline{\eta}} \longrightarrow \mu_{p^m}, \quad \langle \ , \ \rangle_\lambda \colon T_p G_{\overline{\eta}} \times T_p G_{\overline{\eta}} \longrightarrow \mathbb{Z}_p(1)$$

are induced. Then $\widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$ (resp. $T_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}}$) is a coisotropic direct summand of $G^{\mathrm{ad}}[p^m]_{\overline{\eta}}$ (resp. $T_p G_{\overline{\eta}}^{\mathrm{ad}}$) with respect to $\langle \ , \ \rangle_\lambda$. Namely, we have $\widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}}^{\perp} \subset \widehat{G}^{\mathrm{rig}}[p^m]_{\overline{\eta}}$ (resp. $(T_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}})^{\perp} \subset T_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}}$).

Proof. Put $V_p(-) = T_p(-) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then an alternating bilinear pairing

$$\langle \ , \ \rangle_\lambda \colon V_p G_{\overline{\eta}} \times V_p G_{\overline{\eta}} \longrightarrow \mathbb{Q}_p(1)$$

is induced. It suffices to show that $V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}}$ is a coisotropic subspace of $V_p\widehat{G}_{\overline{\eta}}^{\mathrm{ad}}$.

We will prove a more precise result $(V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}})^{\perp} = V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}}$. Since

$$\dim_{\mathbb{Q}_p} V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} + \dim_{\mathbb{Q}_p} V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}} = r + (2d - r) = 2d = \dim_{\mathbb{Q}_p} V_p G_{\overline{\eta}}^{\mathrm{ad}}$$

(*cf.* Lemma 2.7 and its proof), it is sufficient to prove that $V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \subset (V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}})^{\perp}$. Namely, we should prove that the homomorphism $V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}} \longrightarrow \mathbb{Q}_p(1)$ induced by $\langle\ ,\ \rangle_{\lambda}$ is zero.

Since $V_p G_{\overline{\eta}}$ is a semi-stable representation of $\mathrm{Gal}(\overline{F}/F)$, so are $V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}}$ and $V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}}$. Denote the residue field of F by κ_F and put $q = \#\kappa_F$. Consider the action of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}})$ and $D_{\mathrm{st}}(V_p\widehat{A}_{\overline{\eta}}^{\mathrm{rig}})$. Let T and A be as in the proof of Lemma 2.9. Then we have $V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \cong V_p T_{\overline{\eta}}$ and $V_p\widehat{A}_{\overline{\eta}}^{\mathrm{rig}} \cong V_p A_{\overline{\eta}}$. Therefore every eigenvalue of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}})$ is a Weil q^{-2}-number (for the definition of Weil numbers, see [TY07, p. 471]). Similarly, every eigenvalue of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p\widehat{A}_{\overline{\eta}}^{\mathrm{rig}})$ is a Weil q^{-1}-number; this is a consequence of the Weil conjecture for crystalline cohomology of abelian varieties. Therefore, every eigenvalue of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}})$ is either a Weil q^{-4}-number or a Weil q^{-3}-number. On the other hand, every eigenvalue of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(\mathbb{Q}_p(1))$ is equal to q^{-1}, which is a Weil q^{-2}-number. Hence any φ-homomorphism $D_{\mathrm{st}}(V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}}) \longrightarrow D_{\mathrm{st}}(\mathbb{Q}_p(1))$ is zero. Since the functor D_{st} is fully faithful, any $\mathrm{Gal}(\overline{F}/F)$-equivariant homomorphism

$$V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}} \longrightarrow \mathbb{Q}_p(1)$$

is zero. This completes the proof. ∎

3   Shimura varieties of PEL type

3.1   Notation for Shimura varieties of PEL type

In this paper, we are interested in Shimura varieties of PEL type considered in [Kot92, §5] (see also [Lan08, §1.4]). We recall it briefly. Fix a prime p. Consider a 6-tuple $(B, \mathcal{O}_B, *, V, L, \langle\ ,\ \rangle)$, where

- B is a finite-dimensional simple $\mathbb{Q}$-algebra such that $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$,
- $\mathcal{O}_B$ is an order of B whose p-adic completion is a maximal order of $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$,
- $*$ is a positive involution of B (namely, an involution such that $\mathrm{Tr}(bb^*) > 0$ for every $b \in B^{\times}$) which preserves $\mathcal{O}_B$,
- V is a non-zero finite B-module,
- L is a $\mathbb{Z}$-lattice of V preserved by $\mathcal{O}_B$, and
- $\langle\ ,\ \rangle \colon V \times V \longrightarrow \mathbb{Q}$ is a non-degenerate alternating $*$-Hermitian pairing with respect to B-action such that $\langle x, y \rangle \in \mathbb{Z}$ for every $x, y \in L$, and that $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a self-dual lattice of $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

From $(B, *, V, \langle \ , \ \rangle)$, we define a simple $\mathbb{Q}$-algebra $C = \mathrm{End}_B(V)$ with a unique involution $\#$ satisfying $\langle cv, w \rangle = \langle v, c^{\#}w \rangle$ for every $c \in C$ and $v, w \in V$. Moreover we define an algebraic group G over $\mathbb{Q}$ by

$$G(R) = \{ g \in (C \otimes_{\mathbb{Q}} R)^{\times} \mid gg^{\#} \in R^{\times} \}$$

for every $\mathbb{Q}$-algebra R. The condition $gg^{\#} \in R^{\times}$ is equivalent to the existence of $c(g) \in R^{\times}$ such that $\langle gv, gw \rangle = c(g) \langle v, w \rangle$ for every $v, w \in V \otimes_{\mathbb{Q}} R$. By the presence of the lattice L, G can be naturally extended to a group scheme over $\mathbb{Z}$, which is also denoted by the same symbol G. For any ring R, we put $G_R = G \otimes_{\mathbb{Z}} R$.

Consider an $\mathbb{R}$-algebra homomorphism $h \colon \mathbb{C} \longrightarrow C \otimes_{\mathbb{Q}} \mathbb{R}$ preserving involutions (on $\mathbb{C}$, we consider the complex conjugation) such that the symmetric real-valued bilinear form $(v, w) \longmapsto \langle v, h(i)w \rangle$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ is positive definite. Such a 7-tuple $(B, \mathcal{O}_B, *, V, L, \langle \ , \ \rangle, h)$ is said to be an unramified integral PEL datum. Note that the map h induces a homomorphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$, which is also denoted by h.

Let F be the center of B and F^+ be the subfield of F consisting of elements fixed by $*$. The existence of h tells us that $N = [F : F^+](\dim_F C)^{1/2}/2$ is an integer. An unramified integral PEL datum falls into the following three types:

type (A) $[F : F^+] = 2$.

type (C) $[F : F^+] = 1$ and $C \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a product of $M_{2N}(\mathbb{R})$.

type (D) $[F : F^+] = 1$ and $C \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a product of $M_N(\mathbb{H})$.

If we are in the case of type (D), we will exclude the prime $p = 2$.

Using $h \colon \mathbb{C} \longrightarrow C \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow C \otimes_{\mathbb{Q}} \mathbb{C}$, we can decompose the $B \otimes_{\mathbb{Q}} \mathbb{C}$-module $V \otimes_{\mathbb{Q}} \mathbb{C}$ as $V \otimes_{\mathbb{Q}} \mathbb{C} = V_1 \oplus V_2$, where V_1 (resp. V_2) is the subspace of $V \otimes_{\mathbb{Q}} \mathbb{C}$ on which $h(z)$ acts by z (resp. $\bar{z}$) for every $z \in \mathbb{C}$. We denote by E the field of definition of the isomorphism class of the $B \otimes_{\mathbb{Q}} \mathbb{C}$-module V_1, and call it the reflex field. It is a subfield of $\mathbb{C}$ which is finite over $\mathbb{Q}$.

In the sequel, we fix an unramified integral PEL datum $(B, \mathcal{O}_B, *, V, L, \langle \ , \ \rangle, h)$. For a compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, consider the functor Sh_{K^p} from the category of $\mathcal{O}_{E,(p)}$-schemes to the category of sets, that associates S to the set of isomorphism classes of quadruples (A, i, λ, η^p), where

- A is an abelian scheme over S,
- $\lambda \colon A \longrightarrow A^{\vee}$ is a prime-to-p polarization,
- $i \colon \mathcal{O}_B \longrightarrow \mathrm{End}_S(A)$ is an algebra homomorphism such that $\lambda \circ i(b) = i(b^*)^{\vee} \circ \lambda$ for every $b \in \mathcal{O}_B$,
- η^p is a level-K^p structure of (A, i, λ) of type $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p, \langle \ , \ \rangle)$ in the sense of [Lan08, Definition 1.3.7.6],

satisfying the equality of polynomials $\det_{\mathcal{O}_S}(b; \mathrm{Lie}\, A) = \det_E(b; V^1)$ in the sense of [Kot92, §5]. Recall that two quadruples (A, i, λ, η^p) and $(A', i', \lambda', \eta'^p)$ are said to be isomorphic if there exists an isomorphism $f \colon A \longrightarrow A'$ of abelian schemes such that

- $\lambda = f^{\vee} \circ \lambda' \circ f$,

- $f \circ i(b) = i'(b) \circ f$ for every $b \in \mathcal{O}_B$,

- and $f \circ \eta^p = \eta'^p$ in the sense of [Lan08, Definition 1.4.1.4].

If K^p is neat (*cf.* [Lan08, Definition 1.4.1.8]), the functor Sh_{K^p} is represented by a quasi-projective smooth $\mathcal{O}_{E,(p)}$-scheme [Lan08, Corollary 7.2.3.9], which is also denoted by Sh_{K^p}. In this paper, we will call it a Shimura variety of PEL type. The group $G(\mathbb{A}^{p,\infty})$ naturally acts on the tower of schemes $(\mathrm{Sh}_{K^p})_{K^p \subset G(\widehat{\mathbb{Z}}^p)}$ as Hecke correspondences.

Remark 3.1   i) Our definition of Sh_{K^p}, due to [Lan08], is slightly different from that in [Kot92], but they give the same moduli space. See [Lan08, Proposition 1.4.3.4]. We prefer this definition because we can speak about the universal abelian scheme $\mathcal{A}$ over Sh_{K^p}, rather than its isogeny class.

ii) The word "Shimura variety" is an abuse of terminology. Indeed, in some cases $\mathrm{Sh}_{K^p} \otimes_{\mathcal{O}_{E,(p)}} E$ is a finite disjoint union of Shimura varieties in the usual sense.

Assume moreover the following condition due to [Lan08, Condition 1.4.3.10]:

The action of $\mathcal{O}_B$ on L extends to an action of some maximal order $\mathcal{O}'_B$ of B containing $\mathcal{O}_B$.

As explained in the beginning of [Lan08, §5.2.1], this restriction is rather harmless. Under this condition, thanks to [Lan08, Theorem 6.4.1.1], Sh_{K^p} has a toroidal compactification $\mathrm{Sh}_{K^p}^{\mathrm{tor}}$. Furthermore, since K^p is assumed to be neat, we can (and do) take $\mathrm{Sh}_{K^p}^{\mathrm{tor}}$ so that it is a projective scheme over $\mathcal{O}_{E,(p)}$ [Lan08, Proposition 7.3.1.4, Theorem 7.3.3.4]. On $\mathrm{Sh}_{K^p}^{\mathrm{tor}}$, there exists a semi-abelian scheme extending the universal abelian scheme $\mathcal{A}$ on Sh_{K^p}. We denote it by the same symbol $\mathcal{A}$.

In the following, to ease notation, we fix a neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$ and omit the subscript K^p. For an integer $r \geq 0$, let $\mathrm{Sh}_r^{\mathrm{tor}}$ (resp. $\mathrm{Sh}_{\leq r}^{\mathrm{tor}}$, resp. $\mathrm{Sh}_{\geq r}^{\mathrm{tor}}$) be the subset of $\mathrm{Sh}^{\mathrm{tor}}$ consisting of $x \in \mathrm{Sh}^{\mathrm{tor}}$ such that the toric rank of the semi-abelian variety $\mathcal{A}_x$ is equal to (resp. less than or equal to, resp. greater than or equal to) r. By [Lan08, Lemma 3.3.1.4], $\mathrm{Sh}_{\leq r}^{\mathrm{tor}}$ is an open subset of $\mathrm{Sh}^{\mathrm{tor}}$. Therefore $\mathrm{Sh}_{\geq r}^{\mathrm{tor}}$ (resp. $\mathrm{Sh}_r^{\mathrm{tor}}$) is a closed (resp. locally closed) subset of $\mathrm{Sh}^{\mathrm{tor}}$. We regard them as reduced subschemes of $\mathrm{Sh}^{\mathrm{tor}}$. Obviously $\mathrm{Sh}_0^{\mathrm{tor}} = \mathrm{Sh}_{\leq 0}^{\mathrm{tor}}$ coincides with the Shimura variety Sh.

Now choose a place v of E dividing p and write $\mathcal{O}_v$ for the ring of integers of E_v. By taking base change from $\mathcal{O}_{E,(p)}$ to $\mathcal{O}_v$, the schemes $\mathrm{Sh}, \mathrm{Sh}^{\mathrm{tor}}, \dots$ can be regarded as schemes over $\mathcal{O}_v$. We denote them by the same symbols. Moreover, we denote the fibers of them at v by putting the subscripts v. Namely, we set $\mathrm{Sh}_v = \mathrm{Sh} \otimes_{\mathcal{O}_v} \kappa_v$, $\mathrm{Sh}_{v,r}^{\mathrm{tor}} = \mathrm{Sh}_r^{\mathrm{tor}} \otimes_{\mathcal{O}_v} \kappa_v$ and so on, where κ_v denotes the residue field of $\mathcal{O}_v$. Similarly, we denote the generic fibers of them by putting the subscripts η.

Since $\mathrm{Sh}^{\mathrm{tor}}$ is proper over $\mathcal{O}_v$, we may consider the specialization map

$$\mathrm{sp} \colon (\mathrm{Sh}_\eta^{\mathrm{tor}})^{\mathrm{ad}} = (\mathrm{Sh}^{\mathrm{tor}})^{\mathrm{ad}} = (\mathrm{Sh}^{\mathrm{tor}})^{\wedge \mathrm{rig}} \longrightarrow \mathrm{Sh}_v^{\mathrm{tor}}$$

introduced in the end of Section 2.1 (for the second equality, see Lemma 2.3 ii)).

Definition 3.2 For an integer $r \geq 0$, we put

$$\mathrm{Sh}_{]\geq r[} = \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\geq r})^\circ \cap \mathrm{Sh}^{\mathrm{ad}}_\eta, \quad \mathrm{Sh}_{]r[} = \mathrm{Sh}_{]\geq r[} \setminus \mathrm{Sh}_{]\geq r+1[},$$

where $(-)^\circ$ denotes the interior in $(\mathrm{Sh}^{\mathrm{tor}})^{\mathrm{ad}}$. Moreover, we define an open subset $\mathrm{Sh}^\triangle_{]r[}$ of $\mathrm{Sh}_{]r[}$ by

$$\mathrm{Sh}^\triangle_{]r[} = \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,r})^\circ \cap \mathrm{Sh}^{\mathrm{ad}}_\eta = \mathrm{Sh}_{]\geq r[} \cap \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\leq r}).$$

So far, we have only considered level structures which are prime to p. Now we add p^m-level structures on the universal abelian scheme of the generic fiber Sh_η. Let $\mathrm{Sh}_{m,\eta}$ be the scheme over Sh_η classifying principal level-m structures (*cf.* [Lan08, Definition 1.3.6.2]) of the universal object $(\mathcal{A}, i^{\mathrm{univ}}, \lambda^{\mathrm{univ}})$ over Sh_η. We denote the structure morphism $\mathrm{Sh}_{m,\eta} \longrightarrow \mathrm{Sh}_\eta$ by p_m, which is finite and étale.

Let K_m be the compact open subgroup of $G(\mathbb{Q}_p)$ defined as the kernel of $G(\mathbb{Z}_p) \longrightarrow G(\mathbb{Z}/p^m\mathbb{Z})$. Then $\mathrm{Sh}_{m,\eta}$ coincides with the Shimura variety over E_v with $K_m K^p$-level structures. In particular, for an element g of $G(\mathbb{Q}_p)$ and integers $m, m' \geq 0$ satisfying $g^{-1}K_m g \subset K_{m'}$, we can define the Hecke operator $[g]\colon \mathrm{Sh}_{m,\eta} \longrightarrow \mathrm{Sh}_{m',\eta}$, which is finite and étale. For later use, let us recall its description. Let k be an extension field of E_v and x a k-valued point of $\mathrm{Sh}_{m,\eta}$ corresponding to $[(A, i, \lambda, \eta^p, \eta_p)]$. Then $[g](x) = [(A', i', \lambda', \eta'^p, \eta'_p)]$ is given as follows.

First we assume that $L_p \subset gL_p$. By the condition on η_p, it lifts to an isomorphism $\widetilde{\eta}_p\colon L_p \xrightarrow{\cong} T_p A_{\overline{k}}$, where $\overline{k}$ is a separable closure of k. Take an integer $a \geq 0$ such that $p^a gL_p \subset L_p$, and put $A'_{\overline{k}} = A_{\overline{k}}/\widetilde{\eta}_{p,a}(p^a gL_p/p^a L_p)$, where $\widetilde{\eta}_{p,a}\colon L_p/p^a L_p \xrightarrow{\cong} A_{\overline{k}}[p^a]$ is the isomorphism induced by $\widetilde{\eta}_p$. It is easy to see that $A'_{\overline{k}}$ is independent of the choice of a. Moreover, the condition $g^{-1}K_m g \subset K_{m'}(\subset K_0)$ tells us that $A'_{\overline{k}}$ is also independent of the choice of $\widetilde{\eta}_p$ (note that any other choice is $\widetilde{\eta}_p \circ h$ with $h \in K_m$, and that h preserves gL_p). In particular, it descends to an abelian variety A' over k. As $\widetilde{\eta}_{p,a}(p^a gL_p/p^a L_p)$ is stable under $\mathcal{O}_B$, A' carries an $\mathcal{O}_B$-multiplication i' induced from i. A natural p-isogeny $A \longrightarrow A'$ transfers the pair (λ, η^p) on A into a pair (λ', η'^p) on A'. The definition of η'_p is slightly more complicated. By the construction of $A'_{\overline{k}}$, the composite $L_p \xrightarrow{\widetilde{\eta}_p} T_p A_{\overline{k}} \longrightarrow T_p A'_{\overline{k}}$ extends uniquely to an isomorphism $\widetilde{\eta}'_p\colon gL_p \longrightarrow T_p A'_{\overline{k}}$. Taking the mod-$p^{m'}$ reduction of the composite $\widetilde{\eta}'_p \circ g\colon L_p \longrightarrow T_p A'_{\overline{k}}$, we obtain an isomorphism $\eta'_p\colon L_p/p^{m'}L_p \xrightarrow{\cong} A'_{\overline{k}}[p^{m'}]$. By the condition $g^{-1}K_m g \subset K_{m'}$, it is independent of the choice of $\widetilde{\eta}_p$ and thus descends to an isomorphism $\eta'_p\colon L_p/p^{m'}L_p \xrightarrow{\cong} A'[p^{m'}]$, which gives a $p^{m'}$-level structure on A'.

For a general $g \in G(\mathbb{Q}_p)$, we can take an integer b such that $L_p \subset p^b gL_p$. Then we have $[g](x) = [p^b g](x)$, and thus are reduced to the case above.

As in Definition 3.2, we introduce several important subspaces of $\mathrm{Sh}^{\mathrm{ad}}_{m,\eta}$.

Definition 3.3 We put

$$\mathrm{Sh}_{m,]\geq r[} = p_m^{-1}(\mathrm{Sh}_{]\geq r[}), \quad \mathrm{Sh}_{m,]r[} = p_m^{-1}(\mathrm{Sh}_{]r[}), \quad \mathrm{Sh}_{m,]r[}^{\triangle} = p_m^{-1}(\mathrm{Sh}_{]r[}^{\triangle}).$$

Here we write the same symbol p_m for the finite étale morphism $\mathrm{Sh}_{m,\eta}^{\mathrm{ad}} \longrightarrow \mathrm{Sh}_{\eta}^{\mathrm{ad}}$ induced by p_m.

Proposition 3.4 *Let g be an element of $G(\mathbb{Q}_p)$. For integers $m, m' \geq 0$ such that $g^{-1}K_m g \subset K_{m'}$, the Hecke operator $[g]: \mathrm{Sh}_{m,\eta}^{\mathrm{ad}} \longrightarrow \mathrm{Sh}_{m',\eta}^{\mathrm{ad}}$ maps $\mathrm{Sh}_{m,]r[}$ (resp. $\mathrm{Sh}_{m,]\geq r[}$) into $\mathrm{Sh}_{m',]r[}$ (resp. $\mathrm{Sh}_{m',]\geq r[}$).*

A point x of an adic space X which is locally of finite type over $\mathrm{Spa}(E_v, \mathcal{O}_v)$ is said to be classical if the residue field of x is a finite extension of E_v. For a subset Y of X, we denote the set of classical points in Y by $Y(\mathrm{cl})$.

First we consider about classical points in $\mathrm{Sh}_{m,]r[}$.

Lemma 3.5 *Let x be a classical point of $\mathrm{Sh}_{m,]r[}$. Then $[g](x) \in \mathrm{Sh}_{m',\eta}^{\mathrm{ad}}$ lies in $\mathrm{Sh}_{m',]r[}$.*

Proof. First note that $\mathrm{Sh}_{m,]r[}(\mathrm{cl}) = p_m^{-1}(\mathrm{sp}^{-1}(\mathrm{Sh}_{v,r}^{\mathrm{tor}}) \cap \mathrm{Sh}_{\eta}^{\mathrm{ad}})(\mathrm{cl})$. This is an immediate consequence of [Fuj95, Lemma 4.6.4 (c)], whose analogue for adic spaces can be proved similarly.

Let F be the residue field of x and $\mathcal{O}_F$ its ring of integers. Since $\mathrm{Sh}^{\mathrm{tor}}$ is proper over $\mathcal{O}_v$, we have a natural morphism of schemes $\iota_x: \mathrm{Spec}\,\mathcal{O}_F \longrightarrow \mathrm{Sh}^{\mathrm{tor}}$. The image of the generic point η (resp. closed point s) of $\mathrm{Spec}\,\mathcal{O}_F$ under ι_x can be identified with x (resp. $\mathrm{sp}(x)$). Let us denote by $\mathcal{G}$ the pull-back of $\mathcal{A}$ under ι_x. Then, $\mathcal{G}$ is a semi-abelian scheme over $\mathcal{O}_F$ whose fiber at η (resp. s) is the abelian variety $\mathcal{A}_x$ (resp. a semi-abelian variety with toric rank r).

Put $x' = [g](x)$, which is an F-valued point of $\mathrm{Sh}_{m',\eta}^{\mathrm{ad}}$. By the similar construction, we get a semi-abelian scheme $\mathcal{G}'$ over $\mathcal{O}_F$ whose generic fiber $\mathcal{G}'_\eta$ coincides with $\mathcal{A}_{x'}$. It suffices to show that the toric rank of $\mathcal{G}'_s$ is equal to r.

By the description of the Hecke operator $[g]$, $\mathcal{G}'_\eta = \mathcal{A}_{x'}$ is isomorphic to the quotient of $\mathcal{G}_\eta = \mathcal{A}_x$ by some finite étale subgroup scheme C_η. Let C be the closure of C_η in $\mathcal{G}$. Then, [Lan08, Lemma 3.4.3.1] says that the quotient $\mathcal{G}/C$ is represented by a semi-abelian scheme over $\mathcal{O}_F$. Moreover, by [FC90, Chapter I, Proposition 2.7], the isomorphism $(\mathcal{G}/C)_\eta \cong \mathcal{G}'_\eta$ can be extended uniquely to an isomorphism $\mathcal{G}/C \cong \mathcal{G}'$. Therefore $\mathcal{G}'_s$ is a quotient of $\mathcal{G}_s$ by a finite flat group scheme C_s. In particular, we have $\mathrm{rank}_{\mathbb{Z}_p} T_p \mathcal{G}_s = \mathrm{rank}_{\mathbb{Z}_p} T_p \mathcal{G}'_s$ and $\dim \mathcal{G}_s = \dim \mathcal{G}'_s$. This implies that the toric rank of $\mathcal{G}_s$ is the same as that of $\mathcal{G}'_s$. ∎

To deduce Proposition 3.4 from Lemma 3.5, we will use the following easy lemma:

Lemma 3.6 *Let X be an adic space locally of finite type over $\mathrm{Spa}(E_v, \mathcal{O}_v)$.*

  i) *For every non-empty open subset U of X, $U(\mathrm{cl})$ is non-empty.*

  ii) *For every open subset U, $U^-(\mathrm{cl})$ is dense in U^-. Here U^- denotes the closure of U in X.*

13

Proof. For i), we may assume that $U = \mathrm{Spa}(A, A^{\circ})$, where A is an affinoid algebra over E_v and A° is the subring of A consisting of power-bounded elements. Since $A \neq 0$, A has a maximal ideal $\mathfrak{m}$. By [BGR84, Corollary 6.1.2/3], $F = A/\mathfrak{m}$ is a finite extension of E_v. Extend the valuation of E_v to F. Then $v\colon A \longrightarrow A/\mathfrak{m} = F \longrightarrow \mathbb{R}$ gives a classical point of $U = \mathrm{Spa}(A, A^{\circ})$.

ii) is an immediate consequence of i); indeed, $U(\mathrm{cl})$ is dense in U^{-}. ∎

Proof of Proposition 3.4. Denote the complement of $\mathrm{Sh}_{m,]\geq r[}$ in $\mathrm{Sh}^{\mathrm{ad}}_{m,\eta}$ by $(\mathrm{Sh}_{m,]\geq r[})^c$. It coincides with $p_m^{-1}(\mathrm{sp}^{-1}(\mathrm{Sh}_{v,\leq r-1})^{-} \cap \mathrm{Sh}^{\mathrm{ad}}_{\eta})$, which is the closure of the open subset $p_m^{-1}(\mathrm{sp}^{-1}(\mathrm{Sh}_{v,\leq r-1}) \cap \mathrm{Sh}^{\mathrm{ad}}_{\eta})$ of $\mathrm{Sh}^{\mathrm{ad}}_{m,\eta}$ (note that p_m is an open map). Therefore, $(\mathrm{Sh}_{m,]\geq r[})^c(\mathrm{cl})$ is dense in $(\mathrm{Sh}_{m,]\geq r[})^c$ by Lemma 3.6. Similarly, $([g]^{-1}(\mathrm{Sh}_{m',]\geq r[})^c)(\mathrm{cl})$ is dense in $[g]^{-1}(\mathrm{Sh}_{m',]\geq r[})^c$.

On the other hand, Lemma 3.5 says that $(\mathrm{Sh}_{m,]\geq r[})^c(\mathrm{cl}) = ([g]^{-1}(\mathrm{Sh}_{m',]\geq r[})^c)(\mathrm{cl})$. From it we can deduce $(\mathrm{Sh}_{m,]\geq r[})^c = [g]^{-1}(\mathrm{Sh}_{m',]\geq r[})^c$, as both sides are closed in $\mathrm{Sh}^{\mathrm{ad}}_{m,\eta}$. Thus we have $\mathrm{Sh}_{m,]\geq r[} = [g]^{-1}(\mathrm{Sh}_{m',]\geq r[})$ and $\mathrm{Sh}_{m,]r[} = [g]^{-1}(\mathrm{Sh}_{m',]r[})$. ∎
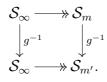
3.2 The set $\mathcal{S}_m$

This subsection provides some basic preliminaries for the next subsection. We denote by $\mathcal{S}_{\infty}$ the set of $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-submodules $I \subset L_p$ such that

– I is a non-trivial direct summand of L_p as an $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module, and

– I is coisotropic, namely, $I^{\perp} \subset I$.

For an element $I \in \mathcal{S}_{\infty}$ and an integer $m \geq 1$, we have an $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-submodule $I/p^m I \subset L_p/p^m L_p$. We denote the set consisting of such submodules by $\mathcal{S}_m$.

Proposition 3.7   i) *The set $\mathcal{S}_{\infty}$ can be identified with the set $\mathcal{S}_{\mathbb{Q}_p}$ consisting of non-trivial coisotropic $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$-submodules of V_p. In particular, $\mathcal{S}_{\infty}$ is naturally equipped with an action of $G(\mathbb{Q}_p)$.*

ii) *There exist finitely many proper parabolic subgroups $P_1, \ldots, P_k$ of $G_{\mathbb{Q}_p}$ such that $\mathcal{S}_{\infty}$ is isomorphic to $\coprod_{i=1}^k G(\mathbb{Q}_p)/P_i(\mathbb{Q}_p)$ as $G(\mathbb{Q}_p)$-sets.*

iii) *We have a natural $G(\mathbb{Z}_p)$-equivariant bijection $K_m \backslash \mathcal{S}_{\infty} \xrightarrow{\cong} \mathcal{S}_m$.*

iv) *Let g be an element of $G(\mathbb{Q}_p)$. For integers $m, m' \geq 1$ with $g^{-1}K_m g \subset K_{m'}$, we have a unique map $g^{-1}\colon \mathcal{S}_m \longrightarrow \mathcal{S}_{m'}$ that makes the following diagram commutative:*

$$
\begin{array}{ccc}
\mathcal{S}_{\infty} & \twoheadrightarrow & \mathcal{S}_m \\
\downarrow{\scriptstyle g^{-1}} & & \downarrow{\scriptstyle g^{-1}} \\
\mathcal{S}_{\infty} & \twoheadrightarrow & \mathcal{S}_{m'}.
\end{array}
$$

Proof of Proposition 3.7 i). First let I be a $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$-submodule of V_p. As $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order of $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$, it is hereditary [Rei75, Theorem 21.4]. Therefore the

$\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module $L_p/I \cap L_p$ is projective, since it is finitely generated and torsion-free as a $\mathbb{Z}_p$-module (*cf.* [Rei75, Corollary 21.5]). Therefore the canonical surjection $L_p \longrightarrow L_p/I \cap L_p$ splits, and $I \cap L_p$ is a direct summand of L_p as an $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module.

By the observation above, we can define the following two maps

$$\mathcal{S}_{\mathbb{Q}_p} \longrightarrow \mathcal{S}_{\infty}; \ I \longmapsto I \cap L_p, \qquad \mathcal{S}_{\infty} \longrightarrow \mathcal{S}_{\mathbb{Q}_p}; \ I \longmapsto I \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

It is easy to see that these are the inverse of each other. ∎

Remark 3.8 By the proof above, we know that L_p is a projective $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module.

For proving the remaining part, we need some preparation; the proof of Proposition 3.7 is completed in the end of this subsection.

First let us introduce more notation. Let Λ be one of $\mathbb{Z}/p^m\mathbb{Z}$, $\mathbb{Z}_p$ or $\mathbb{Q}_p$, and put $\mathcal{O}_{\Lambda} = \mathcal{O}_B \otimes_{\mathbb{Z}} \Lambda$. For the center F of B, we have canonical decompositions $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{\tau} F_{\tau}$ and $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{\tau} \mathcal{O}_{\tau}$, where τ runs through places of F lying over p. These decompositions are extended to the decomposition $\mathcal{O}_{\Lambda} = \prod_{\tau} \mathcal{O}_{\tau,\Lambda}$.

By the assumption, $\mathcal{O}_{\tau,\Lambda}$ is isomorphic to a matrix algebra $M_{d_{\tau}}(\mathcal{O}_{\tau} \otimes_{\mathbb{Z}_p} \Lambda)$ over $\mathcal{O}_{\tau} \otimes_{\mathbb{Z}_p} \Lambda$. As in [Lan08, §1.1.3], let us denote by $M_{\tau,\Lambda}$ the $\mathcal{O}_{\tau,\Lambda}$-module $(\mathcal{O}_{\tau} \otimes_{\mathbb{Z}_p} \Lambda)^{d_{\tau}}$. By [Lan08, Lemma 1.1.3.4], every finitely generated projective $\mathcal{O}_{\Lambda}$-module M is isomorphic to $\bigoplus_{\tau} M_{\tau,\Lambda}^{m_{\tau}}$, for uniquely determined integers $m_{\tau} \geq 0$. We call $\underline{m} = (m_{\tau})$ the multi-rank of M. Put $|\underline{m}| = \sum_{\tau} m_{\tau}$.

Similarly, we can define the multi-rank of a finitely generated projective $B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$-module. For a finitely generated projective $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$-module M, the multi-rank of the $B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$-module $M \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ is not equal to the multi-rank of M in general.

Lemma 3.9 *For $I \in \mathcal{S}_{\mathbb{Q}_p}$, the stabilizer P_I of $I \subset V_p$ in $G_{\mathbb{Q}_p}$ is a proper parabolic subgroup of $G_{\mathbb{Q}_p}$.*

Proof. Let $\underline{m}$ be the multi-rank of the $B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$-module $I \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ and X the moduli space classifying coisotropic $B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$-submodules of $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$ with multi-rank $\underline{m}$. Since X is a closed subscheme of a Grassmannian, it is projective over $\overline{\mathbb{Q}}_p$.

Let us prove that $G(\overline{\mathbb{Q}}_p)$ acts on $X(\overline{\mathbb{Q}}_p)$ transitively. Take an arbitrary coisotropic $B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$-submodule I' of $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$ with multi-rank $\underline{m}$. It suffices to find $g \in G(\overline{\mathbb{Q}}_p)$ such that $I' = g(I \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)$. Put $J = (I \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)^{\perp}$ and $J' = I'^{\perp}$. These are totally isotropic $B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$-submodules of $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$ with the same multi-rank. It is sufficient to find $g \in G(\overline{\mathbb{Q}}_p)$ such that $J' = gJ$. Fix a $B \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$-isomorphism $f \colon J \xrightarrow{\cong} J'$. By [Lan08, Lemma 1.2.4.4], we can find orthogonal decompositions

$$V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p = (J \oplus J^{\vee}) \oplus (J \oplus J^{\vee})^{\perp} = (J' \oplus J'^{\vee}) \oplus (J' \oplus J'^{\vee})^{\perp}.$$

Since $(J \oplus J^{\vee})^{\perp}$ and $(J' \oplus J'^{\vee})^{\perp}$ have the same multi-rank, by [Lan08, Corollary 1.2.3.10], there exists a $B \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$-isomorphism $h \colon (J \oplus J^{\vee})^{\perp} \xrightarrow{\cong} (J' \oplus J'^{\vee})^{\perp}$ preserving the alternating pairings. Then the composite

$$V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p = (J \oplus J^{\vee}) \oplus (J \oplus J^{\vee})^{\perp} \xrightarrow{(f \oplus (f^{\vee})^{-1}) \oplus h} (J' \oplus J'^{\vee}) \oplus (J' \oplus J'^{\vee})^{\perp} = V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$$

gives an element g of $G(\overline{\mathbb{Q}}_p)$ satisfying $gJ = J'$.

Now we know that X_{red} is a homogeneous space for $G_{\overline{\mathbb{Q}}_p}$ in the sense of [Spr98, 2.3.1]. Furthermore, we can easily see that the $G_{\overline{\mathbb{Q}}_p}$-space X_{red} satisfies the conditions in [Spr98, Corollary 5.5.4]. Therefore, by [Spr98, Theorem 5.5.5], X_{red} is a quotient of $G_{\overline{\mathbb{Q}}_p}$ by $P_{I,\overline{\mathbb{Q}}_p} = P_I \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$. Hence $G_{\overline{\mathbb{Q}}_p}/P_{I,\overline{\mathbb{Q}}_p}$ is projective over $\overline{\mathbb{Q}}_p$, that is, $P_{I,\overline{\mathbb{Q}}_p}$ is a parabolic subgroup of $G_{\overline{\mathbb{Q}}_p}$. Thus we can conclude that P_I is a parabolic subgroup of $G_{\mathbb{Q}_p}$.

Finally we will observe $P_I \neq G_{\mathbb{Q}_p}$. Put $J = (I \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)^{\perp}$ and consider the orthogonal decomposition $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p = (J \oplus J^{\vee}) \oplus (J \oplus J^{\vee})^{\perp}$ as above. It is easy to see that $(J^{\vee})^{\perp}$ gives a $\overline{\mathbb{Q}}_p$-valued point of X distinct from $I \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$. Therefore $G_{\overline{\mathbb{Q}}_p}/P_{I,\overline{\mathbb{Q}}_p} = X_{\mathrm{red}}$ has more than one closed point, and thus $P_I \neq G_{\mathbb{Q}_p}$. ∎

Lemma 3.10 *Let $(M, \langle\ ,\ \rangle)$ be a finitely generated self-dual symplectic projective $\mathcal{O}_{\mathbb{Z}_p}$-module in the sense of [Lan08, Definition 1.1.4.7], and $N \subset M$ a totally isotropic direct summand as an $\mathcal{O}_{\mathbb{Z}_p}$-module. Put $M_m = M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}$ and $N_m = N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}$ for simplicity.*

By [Lan08, Lemma 1.2.4.4], we can find an embedding $N_m^{\vee} \hookrightarrow M_m$ with totally isotropic image such that M_m is decomposed into the orthogonal direct sum:

$$M_m = (N_m \oplus N_m^{\vee}) \oplus (N_m \oplus N_m^{\vee})^{\perp}.$$

Let us fix such an embedding. Then, we can extend it to an embedding $N^{\vee} \hookrightarrow M$ with totally isotropic image, such that M is decomposed into the orthogonal direct sum:

$$M = (N \oplus N^{\vee}) \oplus (N \oplus N^{\vee})^{\perp}.$$

Proof. We use the similar induction argument as in the proof of [Lan08, Proposition 1.2.4.6]. If $N \neq 0$, we can find a direct sum decomposition $N = N' \oplus N''$ such that the multi-rank $\underline{n}'$ of N' satisfies $|\underline{n}'| = 1$ (in the proof of [Lan08, Proposition 1.2.4.6], such a decomposition is written as $M_1 \cong M_{1,0} \oplus M_1'$). Let τ be a unique place of F lying over p with $n'_{\tau} = 1$. Then $N' \cong M_{\tau,\mathbb{Z}_p}$ is generated by one element x. Moreover, $N'^{\vee} \cong M_{\tau \circ *,\mathbb{Z}_p}$ is also generated by a single element. Take a generator $\overline{y}$ of $N_m'^{\vee}$ and regard it as an element of M_m by the fixed embedding $N_m'^{\vee} \subset N_m^{\vee} \hookrightarrow M_m$.

Let $(\!|\ ,\ |\!) \colon M \times M \longrightarrow \mathcal{O}_{\mathbb{Z}_p}$ be the skew-Hermitian pairing associated to $\langle\ ,\ \rangle$ (*cf.* [Lan08, Lemma 1.1.4.5]). We have $(\!|x, x|\!) = 0$, since N is totally isotropic. Choose a lift $y \in M$ of $\overline{y}$. Then we have $(\!|y, y|\!) \in p^m\mathcal{O}_{\mathbb{Z}_p}$, since $N_m^{\vee}$ is totally isotropic. As in the proof of [Lan08, Proposition 1.2.4.6], we may assume that $(\!|x, y|\!) = 1_{\tau \circ *} \in \mathcal{O}_{\tau \circ *,\mathbb{Z}_p}$. Moreover, replacing y by $1_{\tau \circ *}y$, we may assume that $y = 1_{\tau \circ *}y$.

If $\tau \neq \tau \circ *$, put $y' = y$. Then $(\!|y', y'|\!) = 1_{\tau \circ *}(\!|y', y'|\!)1_{\tau} = 0$ and $(\!|x, y'|\!) = 1_{\tau \circ *}$. Consider the case $\tau = \tau \circ *$. Since $(\!|y, y|\!) \in p^m\mathcal{O}_{\mathbb{Z}_p}$, we can find $b \in p^m\mathcal{O}_{\mathbb{Z}_p}$ such that $b - b^* = (\!|y, y|\!)$ by [Lan08, Proposition 1.2.2.1]. Put $y' = y + bx$. Then we have $(\!|y', y'|\!) = 0$ and $(\!|x, y'|\!) = 1_{\tau \circ *}$.

Denote by $N'^{\sim}$ the $\mathcal{O}_{\tau \circ *,\mathbb{Z}_p}$-span of y' in M. The pairing $\langle\ ,\ \rangle$ gives a natural homomorphism $N'^{\sim} \longrightarrow N'^{\vee}$, whose mod-$p^m$ reduction is the identity (we regard

$N_m'^\vee$ as a submodule of M by the fixed embedding $N_m'^\vee \subset N_m^\vee \hookrightarrow M_m$). Therefore it is an isomorphism and gives an embedding $N'^\vee \cong N'^\sim \hookrightarrow M$ extending the fixed embedding $N_m'^\vee \subset N_m^\vee \hookrightarrow M_m$. By [Lan08, Lemma 1.2.3.9], we have an orthogonal decomposition

$$M = (N' \oplus N'^\vee) \oplus (N' \oplus N'^\vee)^\perp.$$

Since $N \subset M$ is totally isotropic, the $N'^\vee$-component of every element of N'' is zero. Therefore, if we write N''' for the image of N'' under the projection $M \longrightarrow (N' \oplus N'^\vee)^\perp$, we have $N = N' \oplus N'''$. As the image of the embedding $N_m''^\vee \subset N_m^\vee \hookrightarrow M_m$ is contained in $(N_m' \oplus N_m'^\vee)^\perp$, N_m''' coincides with N_m''. Hence we can apply the induction hypothesis to $N''' \subset (N' \oplus N'^\vee)^\perp$ and $N_m'''^\vee = N_m''^\vee \hookrightarrow (N_m' \oplus N_m'^\vee)^\perp$. This concludes the proof. ∎

Lemma 3.11 *Let $(M, \langle\ ,\ \rangle)$ and $(N, \langle\ ,\ \rangle)$ be finitely generated self-dual symplectic projective $\mathcal{O}_{\mathbb{Z}_p}$-modules. Assume that for an integer $m \geq 1$ we are given an $\mathcal{O}_{\mathbb{Z}/p^m\mathbb{Z}}$-isomorphism $h\colon M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\cong} N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}$ which preserves the alternating pairings up to $(\mathbb{Z}/p^m\mathbb{Z})^\times$-multiplication. Then there exists an $\mathcal{O}_{\mathbb{Z}_p}$-isomorphism $\widetilde{h}\colon M \xrightarrow{\cong} N$ preserving the alternating pairings up to $\mathbb{Z}_p^\times$-multiplication, such that $\widetilde{h} \bmod p^m = h$.*

Proof. For a $\mathbb{Z}_p$-algebra R, let $H(R)$ (resp. $X(R)$) be the group (resp. set) consisting of $\mathcal{O}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} R$-automorphisms of $N \otimes_{\mathbb{Z}_p} R$ (resp. $\mathcal{O}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} R$-isomorphisms $M \otimes_{\mathbb{Z}_p} R \xrightarrow{\cong} N \otimes_{\mathbb{Z}_p} R$) preserving the alternating pairings up to $R^\times$-multiplication. The functor $R \longmapsto H(R)$ (resp. $R \longmapsto X(R)$) is obviously represented by a group scheme (resp. scheme) over $\mathbb{Z}_p$. The group scheme H naturally acts on X.

Since $M \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}$ is isomorphic to $N \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}$, M and N are isomorphic as $\mathcal{O}_{\mathbb{Z}_p}$-modules (*cf.* [Lan08, Lemma 1.1.3.1]). Therefore, by [Lan08, Corollary 1.2.3.10], X is an H-torsor (with respect to the étale topology on $\operatorname{Spec}\mathbb{Z}_p$). On the other hand, by [Lan08, Corollary 1.2.3.12], H is smooth over $\mathbb{Z}_p$. Thus we can conclude that X is smooth over $\mathbb{Z}_p$. In particular, the natural map $X(\mathbb{Z}_p) \longrightarrow X(\mathbb{Z}/p^m\mathbb{Z})$ is surjective. Hence we can find $\widetilde{h} \in X(\mathbb{Z}_p)$ which is mapped to $h \in X(\mathbb{Z}/p^m\mathbb{Z})$ under the map above, as desired. ∎

Proof of Proposition 3.7 ii), iii), iv). First we prove iii). Since the natural surjection $\mathcal{S}_\infty \longrightarrow \mathcal{S}_m$ is $G(\mathbb{Z}_p)$-equivariant and K_m acts trivially on $\mathcal{S}_m$, a $G(\mathbb{Z}_p)$-equivariant surjection $K_m \backslash \mathcal{S}_\infty \longrightarrow \mathcal{S}_m$ is induced. We should prove its injectivity. Let I and I' be two elements of $\mathcal{S}_\infty$ such that $I_m = I'_m$ (as in the proof of Lemma 3.10, we write I_m for $I \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}$). Put $N = I^\perp$ and $N' = I'^\perp$. They are totally isotropic direct summands of L_p. It suffices to find an element $g \in K_m$ such that $gN = N'$.

By [Lan08, Lemma 1.2.4.4], Remark 3.8 and Lemma 3.10, we can find embeddings $i\colon N^\vee \hookrightarrow L_p$ and $i'\colon N'^\vee \hookrightarrow L_p$ with totally isotropic images such that

$i \bmod p^m = i' \bmod p^m$. These give the following orthogonal decompositions:

$$L_p = (N \oplus N^\vee) \oplus (N \oplus N^\vee)^\perp = (N' \oplus N'^\vee) \oplus (N' \oplus N'^\vee)^\perp.$$

Since we have

$$(N \oplus N^\vee)^\perp \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z} = (N_m \oplus N_m^\vee)^\perp$$
$$= (N_m' \oplus N_m'^\vee)^\perp = (N' \oplus N'^\vee)^\perp \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z},$$

Lemma 3.11 enables us to find an $\mathcal{O}_{\mathbb{Z}_p}$-isomorphism $h \colon (N \oplus N^\vee)^\perp \xrightarrow{\cong} (N' \oplus N'^\vee)^\perp$ preserving the alternating pairings up to multiplication by $a \in \mathbb{Z}_p^\times$ such that $h \bmod p^m = \mathrm{id}$. On the other hand, it is easy to find an $\mathcal{O}_{\mathbb{Z}_p}$-isomorphism $f \colon N \xrightarrow{\cong} N'$ such that $f \bmod p^m = \mathrm{id}$ (*cf.* proof of Lemma 3.11). The composite

$$L_p = (N \oplus N^\vee) \oplus (N \oplus N^\vee)^\perp \xrightarrow{(f \oplus a(f^\vee)^{-1}) \oplus h} (N' \oplus N'^\vee) \oplus (N' \oplus N'^\vee)^\perp = L_p$$

gives an element of K_m satisfying $gN = N'$. This conclude the proof of iii).

Next we prove ii). By Proposition 3.7 i), we identify the $G(\mathbb{Q}_p)$-set $\mathcal{S}_\infty$ with $\mathcal{S}_{\mathbb{Q}_p}$. By iii), $K_1 \backslash \mathcal{S}_\infty$ is a finite set. In particular, $G(\mathbb{Q}_p) \backslash \mathcal{S}_\infty = G(\mathbb{Q}_p) \backslash \mathcal{S}_{\mathbb{Q}_p}$ is also a finite set. Let $I_1, \ldots, I_k \in \mathcal{S}_{\mathbb{Q}_p}$ be a system of representatives of $G(\mathbb{Q}_p) \backslash \mathcal{S}_{\mathbb{Q}_p}$. Let P_i be the stabilizer of $I_i \subset V_p$ in $G_{\mathbb{Q}_p}$. Then we have an obvious isomorphism of $G(\mathbb{Q}_p)$-sets $\coprod_{i=1}^k G(\mathbb{Q}_p)/P_i(\mathbb{Q}_p) \cong \mathcal{S}_{\mathbb{Q}_p}$. Since P_i is a proper parabolic subgroup of $G_{\mathbb{Q}_p}$ by Lemma 3.9, this completes the proof of ii).

The remaining part iv) is an immediate consequence of iii). Indeed, we can define $g^{-1} \colon K_m \backslash \mathcal{S}_\infty \longrightarrow K_m \backslash \mathcal{S}_\infty$ by $K_m I \longmapsto K_{m'} g^{-1} I$, which is well-defined as $g^{-1} K_m g \subset K_{m'}$. ∎

3.3 Partition of $\mathrm{Sh}_{m,]r[}$

Fix integers $m, r \geq 1$. In this subsection, we give a partition of $\mathrm{Sh}_{m,]r[}$ indexed by $\mathcal{S}_m$. First, let us apply the construction introduced in Section 2.2 to the case where $S = \mathrm{Sh}_{\leq r}^{\mathrm{tor}}$, $S_0 = \mathrm{Sh}_{v,r}^{\mathrm{tor}}$, $U = \mathrm{Sh}_\eta$ and $G = \mathcal{A}|_{\mathrm{Sh}_{\leq r}^{\mathrm{tor}}}$. By Lemma 2.5, $U^{\mathrm{ad}} = U \times_S t(\mathcal{S})_a = \mathrm{Sh}_{]r[}^\triangle$ in this case. Therefore, we have two locally constant constructible sheaves

$$\widehat{\mathcal{A}}^{\mathrm{rig}}[p^m]_{\mathrm{Sh}_{]r[}^\triangle} \subset \mathcal{A}^{\mathrm{ad}}[p^m]_{\mathrm{Sh}_{]r[}^\triangle}.$$

By taking pull-back under $p_m \colon \mathrm{Sh}_{m,]r[}^\triangle = p_m^{-1}(\mathrm{Sh}_{]r[}^\triangle) \longrightarrow \mathrm{Sh}_{]r[}^\triangle$ and using the universal level structure $p_m^* \mathcal{A}[p^m] \cong L_p/p^m L_p$ over $\mathrm{Sh}_{m,]r[}^\triangle$, we get a locally constant subsheaf of $L_p/p^m L_p$ corresponding to $\widehat{\mathcal{A}}^{\mathrm{rig}}[p^m]_{\mathrm{Sh}_{]r[}^\triangle}$, which we denote by $\mathcal{F}_{m,r}$.

Lemma 3.12 *For every $x \in \mathrm{Sh}_{m,]r[}^\triangle$, the stalk $\mathcal{F}_{m,r,\overline{x}}$ is an element of $\mathcal{S}_m$.*

Proof. Since $\mathrm{Sh}^{\triangle}_{m,]r[}(\mathrm{cl})$ is dense in $\mathrm{Sh}^{\triangle}_{m,]r[}$ (Lemma 3.6), we may assume that x is a classical point.

By definition, the level structure $\mathcal{A}[p^m]_{\overline{x}} \cong L_p/p^m L_p$ lifts to an isomorphism $T_p \mathcal{A}_{\overline{x}} \cong L_p$ of $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-modules which preserves alternating bilinear pairings up to multiplication by $\mathbb{Z}_p^{\times}$; here $T_p \mathcal{A}_{\overline{x}}$ is equipped with the alternating bilinear pairing $T_p \mathcal{A}_{\overline{x}} \times T_p \mathcal{A}_{\overline{x}} \longrightarrow \mathbb{Z}_p(1)$ induced from the universal polarization. Therefore, it suffices to show that $T_p \widehat{\mathcal{A}}^{\mathrm{rig}}_{\overline{x}}$ is a non-trivial coisotropic direct summand of $T_p \mathcal{A}^{\mathrm{ad}}_{\overline{x}}$ as an $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module. First note that $T_p \widehat{\mathcal{A}}^{\mathrm{rig}}_{\overline{x}}$ is an $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-submodule of $T_p \mathcal{A}^{\mathrm{ad}}_{\overline{x}}$, since the semi-abelian scheme $\mathcal{A}$ over $\mathrm{Sh}^{\mathrm{tor}}$ is endowed with an $\mathcal{O}_B$-multiplication, which is an extension of the universal $\mathcal{O}_B$-multiplication on $\mathcal{A}|_{\mathrm{Sh}}$.

By Lemma 2.9, $T_p \widehat{\mathcal{A}}^{\mathrm{rig}}_{\overline{x}}$ is a direct summand of $T_p \mathcal{A}^{\mathrm{ad}}_{\overline{x}}$ as a $\mathbb{Z}_p$-module. As we assume that $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order of $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$, it is hereditary. Therefore the quotient $T_p \mathcal{A}^{\mathrm{ad}}_{\overline{x}} / T_p \widehat{\mathcal{A}}^{\mathrm{rig}}_{\overline{x}}$ is a projective $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module, and thus $T_p \widehat{\mathcal{A}}^{\mathrm{rig}}_{\overline{x}}$ is a direct summand of $T_p \mathcal{A}^{\mathrm{ad}}_{\overline{x}}$ as an $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module.

By Lemma 2.7, we have $\mathrm{rank}_{\mathbb{Z}_p} T_p \widehat{\mathcal{A}}^{\mathrm{rig}}_{\overline{x}} = \dim_{\mathbb{Q}_p} V_p - r$. Since $1 \leq r \leq 1/2 \dim_{\mathbb{Q}_p} V_p$, $T_p \widehat{\mathcal{A}}^{\mathrm{rig}}_{\overline{x}}$ is a non-trivial submodule of $T_p \mathcal{A}^{\mathrm{ad}}_{\overline{x}}$. Finally, Proposition 2.11 tells us that $(T_p \widehat{\mathcal{A}}^{\mathrm{rig}}_{\overline{x}})^{\perp} \subset T_p \widehat{\mathcal{A}}^{\mathrm{rig}}_{\overline{x}}$. ∎

Definition 3.13 For $I \in \mathcal{S}_m$, let $\mathrm{Sh}^{\triangle}_{m,]r[,I}$ be the subset of $\mathrm{Sh}^{\triangle}_{m,]r[}$ consisting of $x \in \mathrm{Sh}^{\triangle}_{m,]r[}$ such that $\mathcal{F}_{m,r,\overline{x}} = I$. It is open and closed in $\mathrm{Sh}^{\triangle}_{m,]r[}$. Denote the closure of $\mathrm{Sh}^{\triangle}_{m,]r[,I}$ in $\mathrm{Sh}_{m,]r[}$ by $\mathrm{Sh}_{m,]r[,I}$.

Proposition 3.14 *We have* $\mathrm{Sh}_{m,]r[} = \coprod_{I \in \mathcal{S}_m} \mathrm{Sh}_{m,]r[,I}$ *as topological spaces.*

Proof. First note that

$$\mathrm{Sh}^{\triangle}_{m,]r[} = \mathrm{Sh}_{m,]\geq r[} \cap p_m^{-1} \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\leq r}), \quad \mathrm{Sh}_{m,]r[} = \mathrm{Sh}_{m,]\geq r[} \cap \big(p_m^{-1} \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\leq r})\big)^{-}.$$

Since $p_m^{-1} \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\leq r})$ is a quasi-compact open subset of the spectral space $(\mathrm{Sh}^{\mathrm{tor}}_{m,\eta})^{\mathrm{ad}}$, its closure $(p_m^{-1} \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\leq r}))^{-}$ consists of specializations of points in $p_m^{-1} \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\leq r})$ (*cf.* [Hoc69, Corollary of Theorem 1]). Thus every point in $\mathrm{Sh}_{m,]r[}$ is a specialization of a point in $\mathrm{Sh}^{\triangle}_{m,]r[}$. Similarly, every point in $\mathrm{Sh}_{m,]r[,I}$ is a specialization of a point in $\mathrm{Sh}^{\triangle}_{m,]r[,I}$; note that $\mathrm{Sh}^{\triangle}_{m,]r[,I} = \mathrm{Sh}_{m,]r[,I} \cap p_m^{-1} \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\leq r}) \hookrightarrow \mathrm{Sh}_{m,]r[,I}$ is a quasi-compact open immersion between locally spectral spaces.

Now let x be a point of $\mathrm{Sh}_{m,]r[}$. Then there exists a generalization $y \in \mathrm{Sh}^{\triangle}_{m,]r[}$ of x. Since $\mathrm{Sh}^{\triangle}_{m,]r[}$ is a disjoint union of $\mathrm{Sh}^{\triangle}_{m,]r[,I}$, there exists $I \in \mathcal{S}_m$ with $y \in \mathrm{Sh}^{\triangle}_{m,]r[,I}$. Then x lies in $\mathrm{Sh}_{m,]r[,I}$.

Next we will prove that $\mathrm{Sh}_{m,]r[,I}$ and $\mathrm{Sh}_{m,]r[,I'}$ are disjoint for $I, I' \in \mathcal{S}_m$ with $I \neq I'$. Assume that we can take a point x in $\mathrm{Sh}_{m,]r[,I} \cap \mathrm{Sh}_{m,]r[,I'}$. Then there exist points $y \in \mathrm{Sh}^{\triangle}_{m,]r[,I}$ and $y' \in \mathrm{Sh}^{\triangle}_{m,]r[,I'}$ both of which are generalizations of x. By [Hub96, Lemma 1.1.10 i)], y is a specialization of y' or y' is a specialization of y. Since $\mathrm{Sh}^{\triangle}_{m,]r[,I}$ is stable under specializations and generalizations in $\mathrm{Sh}^{\triangle}_{m,]r[}$, y lies in $\mathrm{Sh}^{\triangle}_{m,]r[,I} \cap \mathrm{Sh}^{\triangle}_{m,]r[,I'}$. This is contradiction.

We have obtained $\mathrm{Sh}_{m,]r[} = \coprod_{I \in \mathcal{S}_m} \mathrm{Sh}_{m,]r[,I}$ as sets. Since $\mathrm{Sh}_{m,]r[,I}$ is closed in $\mathrm{Sh}_{m,]r[}$ and $\mathcal{S}_m$ is a finite set, $\mathrm{Sh}_{m,]r[,I}$ is also open in $\mathrm{Sh}_{m,]r[}$. Therefore we have an equality of topological spaces $\mathrm{Sh}_{m,]r[} = \coprod_{I \in \mathcal{S}_m} \mathrm{Sh}_{m,]r[,I}$, as desired. ∎

Proposition 3.15 *Let g be an element of $G(\mathbb{Q}_p)$. For integers $m, m' \geq 1$ such that $g^{-1}K_m g \subset K_{m'}$, the Hecke operator $[g]\colon \mathrm{Sh}_{m,\eta}^{\mathrm{ad}} \longrightarrow \mathrm{Sh}_{m',\eta}^{\mathrm{ad}}$ maps $\mathrm{Sh}_{m,]r[,I}$ into $\mathrm{Sh}_{m',]r[,g^{-1}I}$ (for the definition of $g^{-1}I$, see Proposition 3.7 iv)).*

Proof. By Lemma 3.6, $\mathrm{Sh}_{m,]r[,I}^{\triangle}(\mathrm{cl})$ is dense in $\mathrm{Sh}_{m,]r[,I}^{\triangle}$. Therefore $\mathrm{Sh}_{m,]r[,I}^{\triangle}(\mathrm{cl})$ is dense in $\mathrm{Sh}_{m,]r[,I}$. By this fact and Proposition 3.4, it suffices to show that $[g]$ maps a classical point $x \in \mathrm{Sh}_{m,]r[,I}^{\triangle}$ to a point in $\mathrm{Sh}_{m',]r[,g^{-1}I}^{\triangle}$. Note that $\mathrm{Sh}_{m,]r[}^{\triangle}(\mathrm{cl}) = p_m^{-1}(\mathrm{sp}^{-1}(\mathrm{Sh}_{v,r}^{\mathrm{tor}}) \cap \mathrm{Sh}_{\eta}^{\mathrm{ad}})(\mathrm{cl}) = \mathrm{Sh}_{m,]r[}(\mathrm{cl})$ (*cf.* [Fuj95, Lemma 4.6.4 (c)]). Hence $[g](x)$ lies in $\mathrm{Sh}_{m',]r[}^{\triangle}$ by Proposition 3.4.

We can reduce the problem to the case where $L_p \subset gL_p$. We use the same notation as in the proof of Lemma 3.5. Then we have the following commutative diagram of adic spaces over $\mathrm{Spa}(F, \mathcal{O}_F)$:

$$
\begin{array}{ccc}
\widehat{\mathcal{G}}^{\mathrm{rig}}[p^m] & \hookrightarrow & \mathcal{G}^{\mathrm{ad}}[p^m] \\
\downarrow & & \downarrow \\
\widehat{\mathcal{G}}'^{\mathrm{rig}}[p^m] & \hookrightarrow & \mathcal{G}'^{\mathrm{ad}}[p^m].
\end{array}
$$

This induces the commutative diagram

$$
\begin{array}{ccccc}
T_p\widehat{\mathcal{G}}_{\overline{\eta}}^{\mathrm{rig}} & \hookrightarrow & T_p\mathcal{G}_{\overline{\eta}}^{\mathrm{ad}} & \xleftarrow[\cong]{\widetilde{\eta}_p} & L_p \\
\downarrow & & \downarrow & & \uparrow \\
T_p\widehat{\mathcal{G}}_{\overline{\eta}}'^{\mathrm{rig}} & \hookrightarrow & T_p\mathcal{G}_{\overline{\eta}}'^{\mathrm{ad}} & \xleftarrow[\cong]{\widetilde{\eta}_p'} & gL_p & \xleftarrow[\cong]{g} & L_p.
\end{array}
$$

Here $\widetilde{\eta}_p$ and $\widetilde{\eta}_p'$ are the same as in the description of the Hecke operator $[g]$ (page 12). By definition, $\mathcal{F}_{m',r,[g](x)}$ is the mod-$p^{m'}$ reduction of $g^{-1}\widetilde{\eta}_p'^{-1}(T_p\widehat{\mathcal{G}}_{\overline{\eta}}'^{\mathrm{rig}})$. Therefore, the diagram above tells us that $\mathcal{F}_{m',r,[g](x)} = g^{-1}\mathcal{F}_{m,r,x} = g^{-1}I$, as desired. ∎

4 Cohomology of Shimura varieties

4.1 Compactly supported cohomology and nearby cycle cohomology

As in Section 3.1, let us fix a neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$. Fix a prime ℓ which is different from p. Let ξ be an algebraic representation of $G_{\mathbb{Q}_\ell}$ on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. It naturally defines a smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\xi$ (or $\mathcal{L}_{\xi,m,K^p}$, if we need to indicate m and K^p) on $\mathrm{Sh}_{m,\eta}$ (*cf.* [HT01, III.2]). Moreover,

$\mathcal{L}_\xi$ is equivariant with respect to the Hecke action. For example, let g be an element of $G(\mathbb{Q}_p)$ satisfying $g^{-1}K_m g \subset K_{m'}$ for integers $m, m' \geq 0$. Then we have a natural isomorphism $[g]^* \mathcal{L}_{\xi,m',K^p} \xrightarrow{\cong} \mathcal{L}_{\xi,m,K^p}$ of smooth sheaves over $\mathrm{Sh}_{m,K^p,\eta}$.

Consider the compactly supported cohomology

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi) = \varinjlim_m H^i_c(\mathrm{Sh}_{m,K^p,\eta} \otimes_{E_v} \overline{E}_v, \mathcal{L}_\xi).$$

The group $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{E}_v/E_v)$ naturally acts on it. By this action, $H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi)$ becomes an admissible/continuous representation of $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{E}_v/E_v)$ in the sense of [HT01, I.2].

On the other hand, we may also consider the nearby cycle cohomology defined as follows:

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi) = \varinjlim_m H^i_c(\mathrm{Sh}_{K^p,\overline{v}}, R\psi(p_{m*}\mathcal{L}_\xi)),$$

where we put $\mathrm{Sh}_{K^p,\overline{v}} = \mathrm{Sh}_{K^p,v} \otimes_{\kappa_v} \overline{\kappa}_v$. Obviously the group $\mathrm{Gal}(\overline{E}_v/E_v)$ acts on it. The following lemma gives an action of $G(\mathbb{Q}_p)$ on $H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi)$.

Lemma 4.1 *We have a natural action of $G(\mathbb{Q}_p)$ on $H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi)$. By this action, $H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi)$ becomes an admissible/continuous $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{E}_v/E_v)$-representation.*

Proof. To ease notation, we omit the subscript K^p.

As in [Man05, §6], we can construct a tower $(\mathrm{Sh}_m)_{m \geq 0}$ of schemes over $\mathcal{O}_v$ with finite transition maps such that Sh_m gives an integral model of $\mathrm{Sh}_{m,\eta}$ and $\mathrm{Sh}_0 = \mathrm{Sh}$. In this situation, we have

$$H^i_c(\mathrm{Sh}_{\overline{v}}, R\psi(p_{m*}\mathcal{L}_\xi)) = H^i_c(\mathrm{Sh}_{m,\overline{v}}, R\psi\mathcal{L}_\xi),$$

where $\mathrm{Sh}_{m,\overline{v}} = \mathrm{Sh}_m \otimes_{\mathcal{O}_v} \overline{\kappa}_v$.

Put $G^+(\mathbb{Q}_p) = \{g \in G(\mathbb{Q}_p) \mid g^{-1}L_p \subset L_p\}$. For $g \in G^+(\mathbb{Q}_p)$, let $e(g)$ be the minimal non-negative integer such that $gL_p \subset p^{-e(g)}L_p$. Then we can construct a tower $(\mathrm{Sh}_{m,g})_{m \geq e(g)}$ of schemes over $\mathcal{O}_v$ and two morphisms

$$\mathrm{pr}\colon \mathrm{Sh}_{m,g} \longrightarrow \mathrm{Sh}_m, \quad [g]\colon \mathrm{Sh}_{m,g} \longrightarrow \mathrm{Sh}_{m-e(g)}$$

which are compatible with the transition maps. It is known that these are proper morphisms, pr induces an isomorphism on the generic fibers, and $[g]$ induces the Hecke action of g on the generic fibers [Man05, Proposition 16, Proposition 17]. In particular, we have a canonical cohomological correspondence (*cf.* [SGA5, Exposé III], [Fuj97])

$$c_g\colon [g]^*_\eta \mathcal{L}_{\xi,m-e(g)} \xrightarrow{\cong} \mathrm{pr}^*_\eta \mathcal{L}_{\xi,m} = R\,\mathrm{pr}^!_\eta \mathcal{L}_{\xi,m}.$$

Let

$$R\psi(c_g)\colon [g]^*_{\overline{v}} R\psi\mathcal{L}_{\xi,m-e(g)} \longrightarrow R\,\mathrm{pr}^!_{\overline{v}} R\psi\mathcal{L}_{\xi,m}$$

be the specialization of c_g (*cf.* [Fuj97, §1.5], [IM10, §6]). Since $[g]_{\overline{v}}$ is proper, this induces a homomorphism

$$H_c^i(\mathrm{Sh}_{m-e(g),\overline{v}}, R\psi\mathcal{L}_\xi) \xrightarrow{H_c^i(R\psi(c_g))} H_c^i(\mathrm{Sh}_{m,\overline{v}}, R\psi\mathcal{L}_\xi).$$

Taking the inductive limit, we get

$$\gamma_g \colon H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi) \longrightarrow H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi).$$

From an obvious relation $c_{gg'} = c_g \circ g^* c_{g'}$ for $g, g' \in G^+(\mathbb{Q}_p)$, we deduce $\gamma_{gg'} = \gamma_g \circ \gamma_{g'}$ (*cf.* [IM10, Corollary 6.3]). On the other hand, by [Man05, Proposition 16 (3), Proposition 17 (3)], $\gamma_{p^{-1}}$ is the identity. Since $G(\mathbb{Q}_p)$ is generated by $G^+(\mathbb{Q}_p)$ and p as a monoid, we can extend γ_g to the whole $G(\mathbb{Q}_p)$. By [Man05, Proposition 16 (4)], the restriction of this action to $K_0 = G(\mathbb{Z}_p)$ coincides with the inductive limit of the natural action of K_0 on $H_c^i(\mathrm{Sh}_{m,\overline{v}}, R\psi\mathcal{L}_\xi)$. In particular, it is a smooth action. Furthermore, for integers $m' \geq m \geq 1$, we have

$$H_c^i(\mathrm{Sh}_{m',\overline{v}}, R\psi\mathcal{L}_\xi)^{K_m/K_{m'}} = H_c^i(\mathrm{Sh}_{m,\overline{v}}, R\psi\mathcal{L}_\xi)$$

since $K_m/K_{m'}$ is a p-group (*cf.* [Mie10a, Proposition 2.5]). Taking inductive limit, we obtain

$$H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi)^{K_m} = H_c^i(\mathrm{Sh}_{m,\overline{v}}, R\psi\mathcal{L}_\xi).$$

This implies that $H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi)$ is an admissible/continuous representation of $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{E}_v/E_v)$. ∎

Now we can state our main theorem in this article.

Theorem 4.2 *The kernel and cokernel of the canonical homomorphism*

$$H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi) \longrightarrow H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi)$$

(cf. [SGA7, Exposé XIII, (2.1.7.3)]) are non-cuspidal, namely, they have no supercuspidal subquotient of $G(\mathbb{Q}_p)$. In particular, for an irreducible supercuspidal representation π of $G(\mathbb{Q}_p)$, we have an isomorphism

$$H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi)[\pi] \cong H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi)[\pi].$$

Remark 4.3 Let $H^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi) = \varinjlim_m H^i(\mathrm{Sh}_{m,K^p,\eta} \otimes_{E_v} \overline{E}_v, \mathcal{L}_\xi)$ be the ordinary cohomology of our Shimura variety. This is also an admissible/continuous representation of $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{E}_v/E_v)$. By using the minimal compactification of $\mathrm{Sh}_{m,K^p,\eta}$ and its natural stratification (*cf.* [Pin92, §3.7]), it is easy to see that the kernel and the cokernel of the canonical homomorphism

$$H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi) \longrightarrow H^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi)$$

are non-cuspidal as $G(\mathbb{Q}_p)$-representations (in fact, we can use the similar argument as in the next subsection). Therefore, to prove Theorem 4.2, it suffices to show that the kernel and the cokernel of the composite

$$H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi) \longrightarrow H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi) \longrightarrow H^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi)$$

are non-cuspidal.

Remark 4.4 Let $IH^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi) = \varinjlim_m H^i(\mathrm{Sh}^{\min}_{m,K^p,\eta} \otimes_{E_v} \overline{E}_v, j_{!*}\mathcal{L}_\xi)$ be the intersection cohomology of our Shimura variety, where $j\colon \mathrm{Sh}_{m,K^p,\eta} \hookrightarrow \mathrm{Sh}^{\min}_{m,K^p,\eta}$ denotes the minimal compactification of $\mathrm{Sh}_{m,K^p,\eta}$. Then, as in the previous remark, it is easy to see that the kernel and cokernel of the canonical homomorphism

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi) \longrightarrow IH^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi)$$

is non-cuspidal. Therefore, by Theorem 4.2, we have an isomorphism

$$IH^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi)[\pi] \cong H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi)[\pi]$$

for an irreducible supercuspidal representation π of $G(\mathbb{Q}_p)$.

Corollary 4.5 *Put*

$$H^i_c(\mathrm{Sh}_{\infty,\overline{\eta}}, \mathcal{L}_\xi) = \varinjlim_{K^p} H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi), \quad H^i_c(\mathrm{Sh}_{\infty,\overline{v}}, R\psi\mathcal{L}_\xi) = \varinjlim_{K^p} H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi).$$

These are $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{E}_v/E_v)$-representations.

Let Π be an irreducible admissible representation of $G(\mathbb{A}^\infty)$ such that Π_p is a supercuspidal representation of $G(\mathbb{Q}_p)$. Then, Π does not appear as a subquotient of the kernel or the cokernel of the canonical homomorphism $H^i_c(\mathrm{Sh}_{\infty,\overline{v}}, R\psi\mathcal{L}_\xi) \longrightarrow H^i_c(\mathrm{Sh}_{\infty,\overline{\eta}}, \mathcal{L}_\xi)$. In particular, we have an isomorphism of $\mathrm{Gal}(\overline{E}_v/E_v)$-representations

$$H^i_c(\mathrm{Sh}_{\infty,\overline{\eta}}, \mathcal{L}_\xi)[\Pi] \cong H^i_c(\mathrm{Sh}_{\infty,\overline{v}}, R\psi\mathcal{L}_\xi)[\Pi].$$

Proof. Take a neat compact open subgroup $K^p \subset G(\widehat{\mathbb{Z}}^p)$ such that $\Pi^{K^p} \neq 0$. If Π appears as a subquotient of the kernel or the cokernel of $H^i_c(\mathrm{Sh}_{\infty,\overline{v}}, R\psi\mathcal{L}_\xi) \longrightarrow H^i_c(\mathrm{Sh}_{\infty,\overline{\eta}}, \mathcal{L}_\xi)$, then Π_p appears as a subquotient of the kernel or the cokernel of $H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi) \longrightarrow H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \mathcal{L}_\xi)$. This contradicts to Theorem 4.2. ∎

Remark 4.6 If an irreducible admissible representation Π of $G(\mathbb{A}^\infty)$ is supercuspidal at a prime $p' \neq p$, we can prove the similar result as above. This case is due to Tetsushi Ito and the second author. Its proof is easier, since we may use minimal compactifications over $\mathbb{Z}_p$.

4.2 Proof of main theorem

The starting point of our proof of Theorem 4.2 is the following rigid-geometric interpretation of the nearby cycle cohomology.

Lemma 4.7 *There exist natural isomorphisms*

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi) \cong \varinjlim_m H^i_c\big(\mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,0})_{\overline{\eta}}, p_{m*}\mathcal{L}^{\mathrm{ad}}_\xi\big) \cong \varinjlim_m H^i_{(\mathrm{Sh}_{m,]\geq 1[})^c_{\overline{\eta}}}(\mathrm{Sh}^{\mathrm{ad}}_{m,\overline{\eta}}, \mathcal{L}^{\mathrm{ad}}_\xi).$$

23

Here the subscript $\overline{\eta}$ denotes the base change from E_v to $\overline{E}_v$, $(\mathrm{Sh}_{m,]\geq 1[})^c$ denotes the complement of $\mathrm{Sh}_{m,]\geq 1[}$ in $\mathrm{Sh}_{m,\eta}^{\mathrm{ad}}$, and $\mathcal{L}_\xi^{\mathrm{ad}}$ denotes the ℓ-adic sheaf on $\mathrm{Sh}_{m,\eta}^{\mathrm{ad}}$ induced from $\mathcal{L}_\xi$. As in Section 3.1, we omit the subscript K^p.

By Proposition 3.4, $\varinjlim_m H^i_{(\mathrm{Sh}_{m,]\geq 1[})^c_{\overline{\eta}}}(\mathrm{Sh}_{m,\overline{\eta}}^{\mathrm{ad}}, \mathcal{L}_\xi^{\mathrm{ad}})$ is naturally equipped with an action of $G(\mathbb{Q}_p)$. Under this action, the isomorphism

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi) \cong \varinjlim_m H^i_{(\mathrm{Sh}_{m,]\geq 1[})^c_{\overline{\eta}}}(\mathrm{Sh}_{m,\overline{\eta}}^{\mathrm{ad}}, \mathcal{L}_\xi^{\mathrm{ad}})$$

is $G(\mathbb{Q}_p)$-equivariant.

Remark 4.8 Before proving it, we should clarify the definition of the ℓ-adic cohomology of rigid spaces appearing in the lemma. We define them simply by taking projective limit of the cohomology with torsion coefficients and tensoring $\overline{\mathbb{Q}}_\ell$. Note that this definition is, at least a priori, different from the compactly supported ℓ-adic cohomology introduced in [Hub98a]. However our naive definition also works well, since all the cohomology groups we encounter are known to be finitely generated.

Proof of Lemma 4.7. Let $\mathcal{F}$ be an arbitrary ℓ^n-torsion sheaf on $\mathrm{Sh}_{m,\overline{\eta}}$. We shall observe that there exist natural isomorphisms

$$H^i_c(\mathrm{Sh}_{\overline{v}}, R\psi p_{m*}\mathcal{F}) \cong H^i_c\big(\mathrm{sp}^{-1}(\mathrm{Sh}_{v,0}^{\mathrm{tor}})_{\overline{\eta}}, p_{m*}\mathcal{F}^{\mathrm{ad}}\big) \cong H^i_{(\mathrm{Sh}_{m,]\geq 1[})^c_{\overline{\eta}}}(\mathrm{Sh}_{m,\overline{\eta}}^{\mathrm{ad}}, \mathcal{F}^{\mathrm{ad}}).$$

By [Hub96, Theorem 5.7.6, Theorem 3.7.2], we obtain the first isomorphism as the composite of

$$H^i_c(\mathrm{Sh}_{\overline{v}}, R\psi p_{m*}\mathcal{F}) \cong H^i_c\big((\mathrm{Sh}^{\wedge\mathrm{rig}})_{\overline{\eta}}, (p_{m*}\mathcal{F})^{\mathrm{ad}}\big) = H^i_c\big(\mathrm{sp}^{-1}(\mathrm{Sh}_{v,0}^{\mathrm{tor}})_{\overline{\eta}}, p_{m*}\mathcal{F}^{\mathrm{ad}}\big).$$

To construct the second isomorphism, we prove the following:

- $(\mathrm{Sh}_{]\geq 1[})^c$ contains $\mathrm{sp}^{-1}(\mathrm{Sh}_{v,0}^{\mathrm{tor}})$, and
- $(\mathrm{Sh}_{]\geq 1[})^c$ is closed in $(\mathrm{Sh}_\eta^{\mathrm{tor}})^{\mathrm{ad}}$.

The former is clear from the definition. For the latter, note that $(\mathrm{Sh}_\eta^{\mathrm{tor}})^{\mathrm{ad}} \setminus \mathrm{Sh}_\eta^{\mathrm{ad}} = (\mathrm{Sh}_{\geq 1,\eta}^{\mathrm{tor}})^{\mathrm{ad}}$ is stable under generalizations [Hub96, Lemma 1.1.10 v)]. By this fact, the closure of the quasi-compact open subset $\mathrm{sp}^{-1}(\mathrm{Sh}_{v,0}^{\mathrm{tor}})$ in $(\mathrm{Sh}_\eta^{\mathrm{tor}})^{\mathrm{ad}}$ coincides with the closure of it in $\mathrm{Sh}_\eta^{\mathrm{ad}}$, that is, $(\mathrm{Sh}_{]\geq 1[})^c$ (*cf.* [Hoc69, Corollary of Theorem 1]). Therefore $(\mathrm{Sh}_{]\geq 1[})^c$ is closed in $(\mathrm{Sh}_\eta^{\mathrm{tor}})^{\mathrm{ad}}$.

By these two facts and the properness of $(\mathrm{Sh}_\eta^{\mathrm{tor}})^{\mathrm{ad}}$, we have a natural homomorphism

$$H^i_c\big(\mathrm{sp}^{-1}(\mathrm{Sh}_{v,0}^{\mathrm{tor}})_{\overline{\eta}}, p_{m*}\mathcal{F}^{\mathrm{ad}}\big) \longrightarrow H^i_{(\mathrm{Sh}_{]\geq 1[})^c_{\overline{\eta}}}\big((\mathrm{Sh}^{\mathrm{tor}})_{\overline{\eta}}^{\mathrm{ad}}, j_! p_{m*}\mathcal{F}^{\mathrm{ad}}\big)$$
$$= H^i_{(\mathrm{Sh}_{]\geq 1[})^c_{\overline{\eta}}}(\mathrm{Sh}_{\overline{\eta}}^{\mathrm{ad}}, p_{m*}\mathcal{F}^{\mathrm{ad}}) = H^i_{(\mathrm{Sh}_{m,]\geq 1[})^c_{\overline{\eta}}}(\mathrm{Sh}_{m,\overline{\eta}}^{\mathrm{ad}}, \mathcal{F}^{\mathrm{ad}}),$$

where $j\colon \mathrm{Sh}_\eta \hookrightarrow \mathrm{Sh}_\eta^{\mathrm{tor}}$ denotes the canonical open immersion. Let us prove that this is an isomorphism. Put $\mathcal{G} = p_{m*}\mathcal{F}$. Consider the following distinguished triangle:

$$R\Gamma_c(\mathrm{Sh}_{\overline{v}}, R\psi\mathcal{G}) \longrightarrow R\Gamma(\mathrm{Sh}_{\overline{v}}^{\mathrm{tor}}, R\psi j_!\mathcal{G}) \longrightarrow R\Gamma\big(\mathrm{Sh}_{\overline{v},\geq 1}^{\mathrm{tor}}, (R\psi j_!\mathcal{G})|_{\mathrm{Sh}_{\overline{v},\geq 1}^{\mathrm{tor}}}\big) \xrightarrow{+1}.$$

By [Ber96, Theorem 3.1] and Lemma 2.5, we have isomorphisms

$$R\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{\overline{v}}, R\psi j_! \mathcal{G}) \cong R\Gamma\big((\mathrm{Sh}^{\mathrm{tor}})^{\mathrm{ad}}_{\overline{\eta}}, j_! \mathcal{G}^{\mathrm{ad}}\big),$$

$$R\Gamma\big(\mathrm{Sh}^{\mathrm{tor}}_{\overline{v},\geq 1}, (R\psi j_! \mathcal{G})|_{\mathrm{Sh}^{\mathrm{tor}}_{\overline{v},\geq 1}}\big) \cong R\Gamma\big(\mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\geq 1})^{\circ}_{\overline{\eta}}, j_! \mathcal{G}^{\mathrm{ad}}\big).$$

Therefore, the natural morphism

$$R\Gamma_c(\mathrm{Sh}_{\overline{v}}, R\psi \mathcal{G}) \cong R\Gamma_c\big(\mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,0})_{\overline{\eta}}, \mathcal{G}^{\mathrm{ad}}\big) \longrightarrow H^i_{(\mathrm{Sh}_{]\geq 1[})^c_{\overline{\eta}}}\big((\mathrm{Sh}^{\mathrm{tor}})^{\mathrm{ad}}_{\overline{\eta}}, j_! \mathcal{G}^{\mathrm{ad}}\big)$$

is an isomorphism (note that $(\mathrm{Sh}^{\mathrm{tor}}_{\eta})^{\mathrm{ad}} \setminus \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,\geq 1})^{\circ} = (\mathrm{Sh}_{]\geq 1[})^c$). This completes a proof of the second isomorphism.

By taking projective limit, we obtain natural isomorphisms

$$H^i_c(\mathrm{Sh}_{\overline{v}}, R\psi p_{m*}\mathcal{L}_{\xi}) \cong H^i_c\big(\mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{tor}}_{v,0})_{\overline{\eta}}, p_{m*}\mathcal{L}^{\mathrm{ad}}_{\xi}\big) \cong H^i_{(\mathrm{Sh}_{m,]\geq 1[})^c_{\overline{\eta}}}(\mathrm{Sh}^{\mathrm{ad}}_{m,\overline{\eta}}, \mathcal{L}^{\mathrm{ad}}_{\xi})$$

in the lemma. It is straightforward to see that the isomorphism

$$\varinjlim_m H^i_c(\mathrm{Sh}_{\overline{v}}, R\psi p_{m*}\mathcal{L}_{\xi}) \cong \varinjlim_m H^i_{(\mathrm{Sh}_{m,]\geq 1[})^c_{\overline{\eta}}}(\mathrm{Sh}^{\mathrm{ad}}_{m,\overline{\eta}}, \mathcal{L}^{\mathrm{ad}}_{\xi})$$

is $G(\mathbb{Q}_p)$-equivariant. ∎

Corollary 4.9 *We have the following long exact sequence of $G(\mathbb{Q}_p)$-modules:*

$$\cdots \longrightarrow H^i_c(\mathrm{Sh}_{\infty, K^p, \overline{v}}, R\psi \mathcal{L}_{\xi}) \longrightarrow H^i(\mathrm{Sh}_{\infty, K^p, \overline{\eta}}, \mathcal{L}_{\xi}) \longrightarrow \varinjlim_m H^i(\mathrm{Sh}_{m,]\geq 1[, \overline{\eta}}, \mathcal{L}^{\mathrm{ad}}_{\xi})$$

$$\longrightarrow H^{i+1}_c(\mathrm{Sh}_{\infty, K^p, \overline{v}}, R\psi \mathcal{L}_{\xi}) \longrightarrow \cdots,$$

where the action of $G(\mathbb{Q}_p)$ on $\varinjlim_m H^i(\mathrm{Sh}_{m,]\geq 1[, \overline{\eta}}, \mathcal{L}^{\mathrm{ad}}_{\xi})$ comes from Proposition 3.4.

Proof. Write $\mathcal{L}_{\xi} = (\mathcal{F}_n)_n \otimes \overline{\mathbb{Q}}_{\ell}$, where $\mathcal{F}_n$ is a locally constant constructible ℓ^n-torsion sheaf. By the proof of Lemma 4.7, we have a natural isomorphism

$$H^i_c(\mathrm{Sh}_{K^p, \overline{v}}, R\psi p_{m*}\mathcal{F}_n) \cong H^i_{(\mathrm{Sh}_{m,]\geq 1[})^c_{\overline{\eta}}}(\mathrm{Sh}^{\mathrm{ad}}_{m,\overline{\eta}}, \mathcal{F}^{\mathrm{ad}}_n).$$

On the other hand, by the comparison isomorphism [Hub96, Theorem 3.8.1], we have

$$H^i(\mathrm{Sh}_{m, K^p, \overline{\eta}}, \mathcal{F}_n) \cong H^i(\mathrm{Sh}^{\mathrm{ad}}_{m,\overline{\eta}}, \mathcal{F}^{\mathrm{ad}}_n).$$

Therefore we have the following long exact sequence:

$$\cdots \longrightarrow H^i_c(\mathrm{Sh}_{K^p, \overline{v}}, R\psi p_{m*}\mathcal{F}_n) \longrightarrow H^i(\mathrm{Sh}_{m, K^p, \overline{\eta}}, \mathcal{F}_n) \longrightarrow H^i(\mathrm{Sh}_{m,]\geq 1[, \overline{\eta}}, \mathcal{F}^{\mathrm{ad}}_n)$$

$$\longrightarrow H^{i+1}_c(\mathrm{Sh}_{K^p, \overline{v}}, R\psi p_{m*}\mathcal{F}_n) \longrightarrow \cdots.$$

On the other hand, clearly $H^i_c(\mathrm{Sh}_{K^p, \overline{v}}, R\psi p_{m*}\mathcal{F}_n)$ and $H^i(\mathrm{Sh}_{m, K^p, \overline{\eta}}, \mathcal{F}_n)$ are finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-modules for every i, and thus $H^i_c(\mathrm{Sh}_{m,]\geq 1[, \overline{\eta}}, \mathcal{F}^{\mathrm{ad}}_n)$ is also finitely generated. Therefore the above exact sequence of projective systems satisfies the Mittag-Leffler condition, and the projective limit preserves exactness. By taking projective limit with respect to n, tensoring $\overline{\mathbb{Q}}_{\ell}$, and taking inductive limit with respect to m, we obtain the desired exact sequence. ∎

Lemma 4.10 *Let $\mathcal{F}$ be a constructible ℓ^n-torsion sheaf on $\mathrm{Sh}_{m,\overline{\eta}}$. Then, for every $r \geq 1$, $H^i(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{F}^{\mathrm{ad}})$ is a finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-module.*

Proof. Let $j \colon \mathrm{Sh}_\eta \hookrightarrow \mathrm{Sh}_\eta^{\mathrm{tor}}$ be the canonical open immersion. Then, by [Hub96, Theorem 3.7.2, Theorem 3.8.1], [Ber96, Theorem 3.1] and Lemma 2.5, we have

$$
\begin{aligned}
H^i(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{F}^{\mathrm{ad}}) &= H^i(\mathrm{Sh}_{]\geq r[,\overline{\eta}}, p_{m*}\mathcal{F}^{\mathrm{ad}}) = H^i\big(\mathrm{sp}^{-1}(\mathrm{Sh}_{v,\geq r}^{\mathrm{tor}})_{\overline{\eta}}^{\circ}, Rj_* p_{m*}\mathcal{F}^{\mathrm{ad}}\big) \\
&\cong H^i\big(\mathrm{sp}^{-1}(\mathrm{Sh}_{v,\geq r}^{\mathrm{tor}})_{\overline{\eta}}^{\circ}, (Rj_* p_{m*}\mathcal{F})^{\mathrm{ad}}\big) \\
&\cong H^i\big(\mathrm{Sh}_{\overline{v},\geq r}^{\mathrm{tor}}, (R\psi Rj_* p_{m*}\mathcal{F})|_{\mathrm{Sh}_{\overline{v},\geq r}^{\mathrm{tor}}}\big).
\end{aligned}
$$

This is a finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-module, as $(R\psi Rj_* p_{m*}\mathcal{F})|_{\mathrm{Sh}_{\overline{v},\geq r}^{\mathrm{tor}}}$ is a constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-complex on $\mathrm{Sh}_{\overline{v},\geq r}^{\mathrm{tor}}$. ∎

Corollary 4.11 *Let $r \geq 1$ be an integer. Put $V_{\geq r}^i = \varinjlim_m H^i(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}})$ and $V_r^i = \varinjlim_m H^i_{\mathrm{Sh}_{m,]r[,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}})$.*

  i) *The group $G(\mathbb{Q}_p)$ naturally acts on $V_{\geq r}^i$ and V_r^i, and these are admissible $G(\mathbb{Q}_p)$-representations.*
  ii) *We have the following long exact sequence of $G(\mathbb{Q}_p)$-modules:*

$$
\cdots \longrightarrow V_r^i \longrightarrow V_{\geq r}^i \longrightarrow V_{\geq r+1}^i \longrightarrow V_r^{i+1} \longrightarrow \cdots .
$$

Proof. First let us construct the long exact sequence in ii). As in the proof of Corollary 4.9, write $\mathcal{L}_\xi = (\mathcal{F}_n)_n \otimes \overline{\mathbb{Q}}_\ell$. Obviously we have the following long exact sequence:

$$
\begin{aligned}
\cdots \longrightarrow H^i_{\mathrm{Sh}_{m,]r[,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{F}_n^{\mathrm{ad}}) &\longrightarrow H^i(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{F}_n^{\mathrm{ad}}) \longrightarrow H^i(\mathrm{Sh}_{m,]\geq r+1[,\overline{\eta}}, \mathcal{F}_n^{\mathrm{ad}}) \\
&\longrightarrow H^{i+1}_{\mathrm{Sh}_{m,]r[,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{F}_n^{\mathrm{ad}}) \longrightarrow \cdots .
\end{aligned}
$$

Lemma 4.10 tells us that every term of this long exact sequence is a finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-module. Therefore, as in the proof of 4.9, we obtain the long exact sequence

$$
\cdots \longrightarrow V_r^i \longrightarrow V_{\geq r}^i \longrightarrow V_{\geq r+1}^i \longrightarrow V_r^{i+1} \longrightarrow \cdots
$$

by taking projective limit, tensoring $\overline{\mathbb{Q}}_\ell$ and taking inductive limit.

By Proposition 3.4, the group $G(\mathbb{Q}_p)$ acts on $V_{\geq r}^i$ and V_r^i. Clearly they are smooth $G(\mathbb{Q}_p)$-representations, and the long exact sequence above is $G(\mathbb{Q}_p)$-equivariant. Moreover, as in the proof of Lemma 4.1, we can easily see that for an integer $m \geq 1$,

$$
(V_{\geq r}^i)^{K_m} = H^i(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}}), \quad (V_r^i)^{K_m} = H^i_{\mathrm{Sh}_{m,]r[,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}}).
$$

Since $H^i(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{F}_n^{\mathrm{ad}})$ and $H^i_{\mathrm{Sh}_{m,]r[,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{F}_n^{\mathrm{ad}})$ are finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-modules for every $n \geq 1$, $H^i(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}})$ and $H^i_{\mathrm{Sh}_{m,]r[,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}})$ are finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces. Therefore we conclude that $V_{\geq r}^i$ and V_r^i are admissible $G(\mathbb{Q}_p)$-representations. ∎

Proposition 4.12 *Let $P_1, \ldots, P_k$ be the parabolic subgroups of $G_{\mathbb{Q}_p}$ as in Proposition 3.7 ii). For an integer j with $1 \leq j \leq k$, let $I(j) \in \mathcal{S}_\infty$ be the image of $[\mathrm{id}] \in G(\mathbb{Q}_p)/P_j(\mathbb{Q}_p) \subset \coprod_{i=1}^k G(\mathbb{Q}_p)/P_i(\mathbb{Q}_p)$ under the isomorphism $\coprod_{i=1}^k G(\mathbb{Q}_p)/P_i(\mathbb{Q}_p) \cong \mathcal{S}_\infty$ constructed in Proposition 3.7 ii). For $m \geq 1$, denote by $I(j)_m$ the image of $I(j)$ under the natural map $\mathcal{S}_\infty \longrightarrow \mathcal{S}_m$. Put $W_{r,j}^i = \varinjlim_m H^i_{\mathrm{Sh}_{m,]r[,I(j)_m,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}})$.*

   i) *For every j with $1 \leq j \leq k$, we have a natural smooth action of $P_j(\mathbb{Q}_p)$ on $W_{r,j}^i$.*

   ii) *We have a natural $G(\mathbb{Q}_p)$-equivariant isomorphism $V_r^i \cong \bigoplus_j \mathrm{Ind}_{P_j(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W_{r,j}^i$.*

   iii) *The $G(\mathbb{Q}_p)$-representation V_r^i is non-cuspidal.*

Proof. First note that $\mathrm{Sh}_{m+1,]r[,I(j)_{m+1}}$ is open and closed in $p_{m+1,m}^{-1}(\mathrm{Sh}_{m,]r[,I(j)_m})$, where $p_{m+1,m}\colon \mathrm{Sh}_{m+1,\eta} \longrightarrow \mathrm{Sh}_{m,\eta}$ denotes the natural transition map; it follows from Proposition 3.14 and Proposition 3.15. Therefore the transition map

$$H^i_{\mathrm{Sh}_{m,]r[,I(j)_m,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}}) \longrightarrow H^i_{\mathrm{Sh}_{m+1,]r[,I(j)_{m+1},\overline{\eta}}}(\mathrm{Sh}_{m+1,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}})$$

of $W_{r,j}^i$ is induced.

Now i) is clear from Proposition 3.15. Let us prove ii). We follow the proof of [IM10, Proposition 5.20]. By the Frobenius reciprocity, we have a homomorphism of $G(\mathbb{Q}_p)$-modules

$$\bigoplus_{j=1}^k \mathrm{Ind}_{P_j(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W_{r,j}^i \longrightarrow V_r^i.$$

We shall observe that this is bijective. For an integer $m \geq 1$, we have

$$H^i_{\mathrm{Sh}_{m,]r[,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}}) \overset{(1)}{=} \bigoplus_{I \in \mathcal{S}_m} H^i_{\mathrm{Sh}_{m,]r[,I,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}})$$

$$\overset{(2)}{=} \bigoplus_{j=1}^k \bigoplus_{g \in K_m \backslash G(\mathbb{Q}_p)/P_j(\mathbb{Q}_p)} H^i_{\mathrm{Sh}_{m,]r[,g^{-1}I(j)_m,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}})$$

$$\overset{(3)}{=} \bigoplus_{j=1}^k \bigoplus_{g \in K_m \backslash K_0/P_j(\mathbb{Q}_p) \cap K_0} H^i_{\mathrm{Sh}_{m,]r[,g^{-1}I(j)_m,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}})$$

$$\overset{(4)}{\cong} \bigoplus_{j=1}^k \mathrm{Ind}_{(P_j(\mathbb{Q}_p) \cap K_0)/(P_j(\mathbb{Q}_p) \cap K_m)}^{K_0/K_m} H^i_{\mathrm{Sh}_{m,]r[,I(j)_m,\overline{\eta}}}(\mathrm{Sh}_{m,]\geq r[,\overline{\eta}}, \mathcal{L}_\xi^{\mathrm{ad}}).$$

Here (1) follows from Proposition 3.14, (2) from Proposition 3.7, (3) from the Iwasawa decomposition $G(\mathbb{Q}_p) = P_j(\mathbb{Q}_p)K_0$. The isomorphism (4) is a consequence of Proposition 3.15 and [Boy99, Lemme 13.2]. By taking the inductive limit, we obtain K_0-isomorphisms

$$V_r^i \cong \bigoplus_{j=1}^k \mathrm{Ind}_{P_j(\mathbb{Q}_p) \cap K_0}^{K_0} W_{r,j}^i \overset{\cong}{\longleftarrow} \mathrm{Ind}_{P_j(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} W_{r,j}^i$$

(the second map is an isomorphism by the Iwasawa decomposition). By the proof of [Boy99, Lemme 13.2], it is easy to see that the first isomorphism above is nothing but the K_0-homomorphism obtained by the Frobenius reciprocity for $P_j(\mathbb{Q}_p) \cap K_0 \subset K_0$. Therefore the composite of the two isomorphisms above coincides with the $G(\mathbb{Q}_p)$-homomorphism introduced at the beginning of our proof of ii). Thus we conclude the proof of ii).

Finally consider iii). By ii), we have only to prove that the unipotent radical of $P_j(\mathbb{Q}_p)$ acts trivially on $W^i_{r,j}$. By [Boy99, Lemme 13.2.3], it suffices to prove that $W^i_{r,j}$ is an admissible $P_j(\mathbb{Q}_p)$-representation. For an integer $m \geq 1$, $(W^i_{r,j})^{P_j(\mathbb{Q}_p) \cap K_m}$ is a subspace of $(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P_j(\mathbb{Q}_p)} W^i_{r,j})^{K_m}$. By ii) and Corollary 4.11 i), $(\mathrm{Ind}^{G(\mathbb{Q}_p)}_{P_j(\mathbb{Q}_p)} W^i_{r,j})^{K_m} \subset (\bigoplus_{j=1}^k \mathrm{Ind}^{G(\mathbb{Q}_p)}_{P_j(\mathbb{Q}_p)} W^i_{r,j})^{K_m} \cong (V^i_r)^{K_m}$ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Hence $W^i_{r,j}$ is an admissible $P_j(\mathbb{Q}_p)$-representation, as desired. ∎

Proof of Theorem 4.2. By Remark 4.3 and Corollary 4.9, it suffices to show that $V^i_{\geq 1} = \varinjlim_m H^i(\mathrm{Sh}_{m,]\geq 1[,\overline{\eta}}, \mathcal{L}^{\mathrm{ad}}_\xi)$ is non-cuspidal for every i. By Corollary 4.11 and Proposition 4.12, the non-cuspidality of $V^i_{\geq r}$ is equivalent to that of $V^i_{\geq r+1}$. If r is large enough, $V^i_{\geq r} = 0$. Hence we can conclude that $V^i_{\geq r}$ is non-cuspidal for every $r \geq 1$, by the descending induction on r. ∎

4.3 Torsion coefficients

Since our method of proving Theorem 4.2 is totally geometric, we may also obtain an analogous result for ℓ-torsion coefficients. For simplicity, we will only consider a constant coefficient $\overline{\mathbb{F}}_\ell$. As in Section 4.1, put

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \overline{\mathbb{F}}_\ell) = \varinjlim_m H^i_c(\mathrm{Sh}_{m,K^p,\eta} \otimes_{E_v} \overline{E}_v, \overline{\mathbb{F}}_\ell),$$

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{F}}_\ell) = \varinjlim_m H^i_c(\mathrm{Sh}_{K^p,\overline{v}}, R\psi(p_{m*}\overline{\mathbb{F}}_\ell)).$$

They are naturally endowed with actions of $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{E}_v/E_v)$. They are admissible/continuous $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{E}_v/E_v)$-representations; note that if $m \geq 1$ we have

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \overline{\mathbb{F}}_\ell)^{K_m} = H^i_c(\mathrm{Sh}_{m,K^p,\overline{\eta}}, \overline{\mathbb{F}}_\ell),$$

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{F}}_\ell)^{K_m} = H^i_c(\mathrm{Sh}_{K^p,\overline{v}}, R\psi(p_{m*}\overline{\mathbb{F}}_\ell)),$$

since K_m is a pro-p group (*cf.* [Mie10a, Proposition 2.5]).

The following theorem can be proven in exactly the same way as Theorem 4.2.

Theorem 4.13 *The kernel and the cokernel of the canonical homomorphism*

$$H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{F}}_\ell) \longrightarrow H^i_c(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \overline{\mathbb{F}}_\ell)$$

have no supercuspidal subquotient of $G(\mathbb{Q}_p)$. For the definition of supercuspidal representations over $\overline{\mathbb{F}}_\ell$, see [Vig96, II.2.5].

5   Applications

In this section, we give a very simple application of our main result. Proofs in this section are rather sketchy, since the technique is more or less well-known. More detailed and general study will be given elsewhere.

Here we consider the Shimura variety for $GU(1, n-1)$ over $\mathbb{Q}$. Let F be an imaginary quadratic extention of $\mathbb{Q}$ and $\mathrm{Spl}_{F/\mathbb{Q}}$ the set of rational primes over which $F/\mathbb{Q}$ splits. Fix an integer $n \geq 2$ and a prime $p \in \mathrm{Spl}_{F/\mathbb{Q}}$. Consider the integral PEL datum $(B, \mathcal{O}_B, *, V, L, \langle \ , \ \rangle, h)$ as follows:

- $B = F$, $\mathcal{O}_B = \mathcal{O}_F$ and $*$ is the unique non-trivial element of $\mathrm{Gal}(F/\mathbb{Q})$.
- $V = F^n$ and $L = \mathcal{O}_F^n$.
- $\langle \ , \ \rangle \colon V \times V \longrightarrow \mathbb{Q}$ be an alternating pairing satisfying the following conditions:
  - $\langle x, y \rangle \in \mathbb{Z}$ for every $x, y \in L$,
  - $\langle bx, y \rangle = \langle x, b^* y \rangle$ for every $x, y \in V$ and $b \in F$, and
  - $G_{\mathbb{R}} \cong GU(1, n-1)$ (for the definition of G, see Section 3.1).
- $h \colon \mathbb{C} \longrightarrow \mathrm{End}_F(V) \otimes_{\mathbb{Q}} \mathbb{R} \cong M_n(\mathbb{C})$ is given by $z \longmapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} I_{n-1} \end{pmatrix}$.

Put $\Sigma = \{ p \in \mathrm{Spl}_{F/\mathbb{Q}} \mid L_p = L_p^{\perp} \}$. Our integral Shimura datum is unramified at every $p \in \Sigma$. Moreover, for such p, $G_{\mathbb{Q}_p}$ is isomorphic to $\mathrm{GL}_n(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$ (*cf.* [Far04, §1.2.3]). In this case, the reflex field E is equal to F. To a neat compact open subgroup K of $G(\widehat{\mathbb{Z}})$, we can attach the Shimura variety Sh_K, which is not proper over $\mathrm{Spec}\, F$. If $K = K_0 K^p$ for some compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, we have $\mathrm{Sh}_K = \mathrm{Sh}_{K^p} \otimes_{\mathcal{O}_{F,(p)}} F$, where Sh_{K^p} is the moduli space introduced in Section 3.1.

Let us fix a prime number ℓ. Put

$$H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell) = \varinjlim_K H_c^i(\mathrm{Sh}_K \otimes_F \overline{F}, \overline{\mathbb{Q}}_\ell).$$

It is an admissible/continuous $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{F}/F)$-representation over $\overline{\mathbb{Q}}_\ell$.

Theorem 5.1 *Let Π be an irreducible admissible representation of $G(\mathbb{A}^\infty)$ over $\overline{\mathbb{Q}}_\ell$. Assume that there exists a prime $p \in \Sigma$ such that Π_p is a supercuspidal representation of $G(\mathbb{Q}_p)$. Then $H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)[\Pi] = 0$ unless $i = n - 1$.*

Remark 5.2 For proper Shimura varieties, an analogous result is known ([Clo91], [HT01, Corollary IV.2.7]). It would be possible to give an "automorphic" proof of Theorem 5.1 by using results in [Mor10]. However, the authors think that our proof, consisting of purely local arguments, is simpler and has some importance.

Proof. Let ℓ' be another prime number and fix an isomorphism of fields $\iota \colon \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_{\ell'}$. Then ι induces an isomorphism $H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)[\Pi] \cong H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_{\ell'})[\iota\Pi]$, where $\iota\Pi$ is the representation of $G(\mathbb{A}^\infty)$ over $\overline{\mathbb{Q}}_{\ell'}$ induced by Π and ι. It is easy to observe that Π_p is supercuspidal if and only if $(\iota\Pi)_p$ is supercuspidal. Therefore, we can change our

ℓ freely, and thus we can assume that there exists a prime $p \in \Sigma \setminus \{\ell\}$ such that Π_p is supercuspidal. Fix such p and take a place v of F lying over p. Then, for an integer $m \geq 0$ and a neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, $\mathrm{Sh}_{K_m K^p} \otimes_F F_v$ is isomorphic to $\mathrm{Sh}_{m,K^p,\eta}$ introduced in Section 3.1. Therefore we have an isomorphism

$$H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell) \cong \varinjlim_{m,K^p} H_c^i(\mathrm{Sh}_{m,K^p,\overline{\eta}}, \overline{\mathbb{Q}}_\ell) = \varinjlim_{K^p} H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \overline{\mathbb{Q}}_\ell).$$

Thus it suffices to show that $H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{\eta}}, \overline{\mathbb{Q}}_\ell)[\pi] = 0$ for a supercuspidal representation π of $G(\mathbb{Q}_p)$, a neat compact open subgroup K^p, and an integer $i \neq n-1$. By Theorem 4.2, it is equivalent to showing that $H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell)[\pi] = 0$.

For an integer $h \geq 0$, let $\mathrm{Sh}_{K^p,v}^{[h]}$ be the reduced closed subscheme of $\mathrm{Sh}_{K^p,v}$ consisting of points x such that the étale rank of $\mathcal{A}_x[v^\infty]$ is less than or equal to h (cf. [HT01, p. 111]); recall that $\mathcal{A}$ denotes the universal abelian scheme over Sh_{K^p}. Put $\mathrm{Sh}_{K^p,v}^{(h)} = \mathrm{Sh}_{K^p,v}^{[h]} \setminus \mathrm{Sh}_{K^p,v}^{[h-1]}$. Our proof of the theorem is divided into the subsequent two lemmas. ∎

Lemma 5.3 *For every supercuspidal representation π of $G(\mathbb{Q}_p)$, we have*

$$H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell)[\pi] = \left(\varinjlim_m H_c^i\big(\mathrm{Sh}_{K^p,\overline{v}}^{[0]}, (R\psi p_{m*}\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{K^p,\overline{v}}^{[0]}}\big)\right)[\pi].$$

Proof. First recall that our Shimura variety $\mathrm{Sh}_{m,K^p,\eta}$ has a good integral model over $\mathcal{O}_v$. For an integer $m \geq 0$, consider the functor from the category of $\mathcal{O}_v$-schemes to the category of sets, that associates S to the set of isomorphism classes of 6-tuples $(A, i, \lambda, \eta^p, \eta_v, \eta_{p,0})$, where

- $[(A, i, \lambda, \eta^p)] \in \mathrm{Sh}_{K^p}(S)$,
- $\eta_v \colon L \otimes_{\mathbb{Z}} (v^{-m}\mathcal{O}_v/\mathcal{O}_v) \longrightarrow A[v^m]$ be a Drinfeld v^m-level structure (cf. [HT01, II.2]), and
- $\eta_{p,0} \colon p^{-m}\mathbb{Z}/\mathbb{Z} \longrightarrow \mu_{p^m,S}$ be a Drinfeld p^m-level structure.

Then it is easy to see that this functor is represented by a scheme Sh_{m,K^p} which is finite over Sh_{K^p}. Moreover the generic fiber of Sh_{m,K^p} can be naturally identified with $\mathrm{Sh}_{m,K^p,\eta}$ (cf. the moduli problem $\mathfrak{X}'_U$ introduced in [HT01, p. 92]). As in [HT01, III.4], we can extend the Hecke action of $G(\mathbb{Q}_p)$ on $(\mathrm{Sh}_{m,K^p,\eta})_{m \geq 0}$ to the tower $(\mathrm{Sh}_{m,K^p})_{m \geq 0}$. We have a $G(\mathbb{Q}_p)$-equivariant isomorphism

$$H_c^i(\mathrm{Sh}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell) \cong \varinjlim_m H_c^i\big(\mathrm{Sh}_{m,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell\big).$$

Let us denote by $\mathrm{Sh}_{m,K^p,v}^{[h]}$ (resp. $\mathrm{Sh}_{m,K^p,v}^{(h)}$) the inverse image of $\mathrm{Sh}_{K^p,v}^{[h]}$ (resp. $\mathrm{Sh}_{K^p,v}^{(h)}$) under $\mathrm{Sh}_{m,K^p} \longrightarrow \mathrm{Sh}_{K^p}$. For an integer $h \geq 0$, it is easy to observe that

$$\varinjlim_m H_c^i\big(\mathrm{Sh}_{K^p,\overline{v}}^{[h]}, (R\psi p_{m*}\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{K^p,\overline{v}}^{[h]}}\big) \cong \varinjlim_m H_c^i\big(\mathrm{Sh}_{m,K^p,\overline{v}}^{[h]}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{m,K^p,\overline{v}}^{[h]}}\big),$$

$$\varinjlim_m H_c^i\big(\mathrm{Sh}_{K^p,\overline{v}}^{(h)}, (R\psi p_{m*}\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{K^p,\overline{v}}^{(h)}}\big) \cong \varinjlim_m H_c^i\big(\mathrm{Sh}_{m,K^p,\overline{v}}^{(h)}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{m,K^p,\overline{v}}^{(h)}}\big)$$

and that they are admissible $G(\mathbb{Q}_p)$-representations. Moreover, by considering the kernel of the universal Drinfeld v^m-level structure η_v^{univ}, we can decompose $\mathrm{Sh}_{m,K^p,v}^{(h)}$ into finitely many open and closed subsets indexed by the set consisting of direct summands of $L \otimes_{\mathbb{Z}} (v^{-m}\mathcal{O}_v/\mathcal{O}_v)$ with rank $n - h$ (cf. [Boy99, Définition 10.4.1, Proposition 10.4.2] and [IM10, Definition 5.1, Lemma 5.3]). Using this partition, we can prove that the $G(\mathbb{Q}_p)$-representation

$$\varinjlim_m H_c^i\big(\mathrm{Sh}_{m,K^p,\overline{v}}^{(h)}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{m,K^p,\overline{v}}^{(h)}}\big)$$

is parabolically induced from a proper parabolic subgroup of $G(\mathbb{Q}_p)$. Therefore, by the same argument as in the proof of Theorem 4.2, we can conclude that the kernel and the cokernel of

$$\varinjlim_m H_c^i(\mathrm{Sh}_{m,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell) \longrightarrow \varinjlim_m H_c^i\big(\mathrm{Sh}_{m,K^p,\overline{v}}^{[0]}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{m,K^p,\overline{v}}^{[0]}}\big)$$

are non-cuspidal. This completes the proof of the lemma. ∎

Lemma 5.4 *Let π be a supercuspidal representation of $G(\mathbb{Q}_p)$. If $i \neq n - 1$, we have*
$$\Big(\varinjlim_m H_c^i\big(\mathrm{Sh}_{K^p,\overline{v}}^{[0]}, (R\psi p_{m*}\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{K^p,\overline{v}}^{[0]}}\big)\Big)[\pi] = 0.$$

Proof. Let $\mu_h \colon \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{\mathbb{C}}$ be the homomorphism of algebraic groups over $\mathbb{C}$ defined as the composite of

$$\mathbb{G}_{m,\mathbb{C}} \xrightarrow{z \longmapsto (z,1)} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \overset{(*)}{\cong} (\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{h_{\mathbb{C}}} G_{\mathbb{C}},$$

where $(*)$ is given by $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\cong} \mathbb{C} \times \mathbb{C}; \ a \otimes b \longmapsto (ab, a\overline{b})$. Fix an isomorphism of fields $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ and denote by $\mu \colon \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \longrightarrow G_{\overline{\mathbb{Q}}_p}$ the induced cocharacter of $G_{\overline{\mathbb{Q}}_p}$. Let b be a unique basic element of $B(G_{\mathbb{Q}_p}, \mu)$ (for the definition of $B(G, \mu)$, we refer to [Far04, §2.1.1]), and denote by $\mathcal{M}$ the Rapoport-Zink space associated to the local unramified PEL datum $(F \otimes_{\mathbb{Q}} \mathbb{Q}_p, \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p, *, V_p, L_p, \langle\ ,\ \rangle, b, \mu)$ (cf. [Far04, §2.3.5]). The Rapoport-Zink space $\mathcal{M}$ is equipped with an action of the group $J(\mathbb{Q}_p)$, where J denotes the algebraic group over $\mathbb{Q}_p$ associated to b (cf. [RZ96, Proposition 1.12]). By [Far04, §2.3.7.1], $\mathcal{M}$ is isomorphic to $\mathcal{M}_{\mathrm{LT}} \times \mathbb{Q}_p^{\times}/\mathbb{Z}_p^{\times}$, where $\mathcal{M}_{\mathrm{LT}}$ is the Lubin-Tate space for $\mathrm{GL}_n(\mathbb{Q}_p)$. Furthermore, $J(\mathbb{Q}_p)$ is isomorphic to $D^{\times} \times \mathbb{Q}_p^{\times}$, where D denotes the central division algebra over $\mathbb{Q}_p$ with invariant $1/n$. The action of $J(\mathbb{Q}_p)$ on $\mathcal{M}$ is identified with the well-known action of $D^{\times} \times \mathbb{Q}_p^{\times}$ on $\mathcal{M}_{\mathrm{LT}} \times \mathbb{Q}_p^{\times}/\mathbb{Z}_p^{\times}$.

By the p-adic uniformization theorem of Rapoport-Zink ([RZ96, Theorem 6.30], [Far04, Corollaire 3.1.9]), we have an isomorphism

$$\coprod_{\ker^1(\mathbb{Q},G)} I(\mathbb{Q})\backslash \mathcal{M} \times G(\mathbb{A}^{\infty,p})/K^p \cong \mathrm{Sh}_{K^p}^{\wedge},$$

where I is an algebraic group over $\mathbb{Q}$ satisfying $I(\mathbb{A}^\infty) \cong J(\mathbb{Q}_p) \times G(\mathbb{A}^{\infty,p})$ and $\mathrm{Sh}_{K^p}^\wedge$ denotes the formal completion of $\mathrm{Sh}_{K^p} \otimes_{\mathcal{O}_v} W(\overline{\mathbb{F}}_p)$ along $\mathrm{Sh}_{K^p,\overline{v}}^{[0]}$, the basic locus of $\mathrm{Sh}_{K^p,\overline{v}}$. By this isomorphism, we know that $\mathrm{Sh}_{K^p,\overline{v}}^{[0]}$, which coincides with $\mathrm{Sh}_{K^p}^\wedge$ as topological spaces, consists of finitely many closed points; indeed, the left hand side of the isomorphism above is a finite disjoint sum of formal schemes of the form $\Gamma \backslash \mathcal{M}$, where $\Gamma \subset J(\mathbb{Q}_p)$ is a discrete cocompact subgroup (*cf.* [Far04, Lemme 3.1.7]). Therefore, by [Ber96, Theorem 3.1], we have an isomorphism

$$H_c^i\big(\mathrm{Sh}_{K^p,\overline{v}}^{[0]}, (R\psi p_{m*}\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{K^p,\overline{v}}^{[0]}}\big) = H^i\big(\mathrm{Sh}_{K^p,\overline{v}}^{[0]}, (R\psi p_{m*}\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{K^p,\overline{v}}^{[0]}}\big)$$
$$\cong H^i\big(\mathrm{Sh}_{m,K^p,\overline{\eta}}(b), \overline{\mathbb{Q}}_\ell\big),$$

where $\mathrm{Sh}_{m,K^p}(b) = p_m^{-1}(\mathrm{sp}^{-1}(\mathrm{Sh}_{K^p,v}^{[0]})^\circ)$.

Now use the Hochschild-Serre spectral sequence [Far04, Théorème 4.5.12]

$$E_2^{r,s} = \varinjlim_m \mathrm{Ext}_{J(\mathbb{Q}_p)\text{-smooth}}^r\big(H_c^{2(n-1)-s}(\mathcal{M}_{K_m}, \overline{\mathbb{Q}}_\ell)(n-1), \mathcal{A}(I)_{\mathbf{1}}^{K^p}\big)$$
$$\Longrightarrow \varinjlim_m H^{r+s}(\mathrm{Sh}_{m,K^p,\overline{\eta}}(b), \overline{\mathbb{Q}}_\ell),$$

where $\mathcal{A}(I)_{\mathbf{1}}$ is the space of automorphic forms on $I(\mathbb{A}^\infty)$ (see [Far04, Définition 4.5.8] for detail). Since $J(\mathbb{Q}_p) = D^\times \times \mathbb{Q}_p^\times$ is anisotropic modulo center, it is easy to see that $E_2^{r,s} = 0$ unless $r = 0$. If $r = 0$, we have

$$E_2^{0,s} = \varinjlim_m \mathrm{Hom}_{J(\mathbb{Q}_p)}\big(H_c^{2(n-1)-s}(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)(n-1), \mathcal{A}(I)_{\mathbf{1}}^{K^p}\big)^{K_m},$$

where we put $H_c^i(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell) = \varinjlim_m H_c^i(\mathcal{M}_{K_m}, \overline{\mathbb{Q}}_\ell)$.

By [Mie10a], the $G(\mathbb{Q}_p)$-representation $H_c^{2(n-1)-s}(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)(n-1)$ has non-zero supercuspidal part only if $s = n - 1$. Indeed, for an irreducible supercuspidal representation $\pi = \pi_1 \otimes \chi$ of $G(\mathbb{Q}_p)$, where π_1 is an irreducible supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ and χ is a character of $\mathrm{GL}_1(\mathbb{Q}_p)$, we have

$$H_c^i(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)[\pi] = H_c^i(\mathcal{M}_{\mathrm{LT},\infty}, \overline{\mathbb{Q}}_\ell)[\pi_1] \otimes \chi,$$

as we see in [Far04, p. 168]. Therefore $E_2^{0,s}$ has a supercuspidal subquotient only if $s = n - 1$.

Hence we can conclude that

$$\varinjlim_m H_c^i\big(\mathrm{Sh}_{K^p,\overline{v}}^{[0]}, (R\psi p_{m*}\overline{\mathbb{Q}}_\ell)|_{\mathrm{Sh}_{K^p,\overline{v}}^{[0]}}\big) \cong \varinjlim_m H^i(\mathrm{Sh}_{m,K^p,\overline{\eta}}(b), \overline{\mathbb{Q}}_\ell)$$

has non-zero supercuspidal part only if $i = n - 1$. ∎

We also have a similar result for the torsion coefficient case. For a neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, put

$$H_c^i(\mathrm{Sh}_{K^p}, \overline{\mathbb{F}}_\ell) = \varinjlim_m H_c^i(\mathrm{Sh}_{K_m K^p} \otimes_F \overline{F}, \overline{\mathbb{F}}_\ell).$$

It is an admissible/continuous $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{F}/F)$-representation over $\overline{\mathbb{F}}_\ell$.

Theorem 5.5 *Let p be a prime in $\Sigma \setminus \{\ell\}$ and π an irreducible supercuspidal $\overline{\mathbb{F}}_\ell$-representation of $G(\mathbb{Q}_p)$. Then, for every neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, we have $H_c^i(\mathrm{Sh}_{K^p}, \overline{\mathbb{F}}_\ell)[\pi] = 0$ unless $i = n - 1$.*

Remark 5.6
   i) Theorem 5.5 for proper Shimura varieties is due to S. W. Shin. His method, using Mantovan's formula, is slightly different from ours.

  ii) Using the result in [Dat10], it is possible to describe the action of $W_{\mathbb{Q}_p}$ on $H_c^{n-1}(\mathrm{Sh}_{K^p}, \overline{\mathbb{F}}_\ell)[\pi]$ by means of the mod-ℓ local Langlands correspondence. Such study has also been carried out by S. W. Shin when the Shimura variety is proper.

Proof. Almost all arguments in the proof of Theorem 5.1 work well. The only one point which should be modified is about the vanishing of the supercuspidal part of $E_2^{r,s}$ for $(r,s) \neq (0, n-1)$ in the proof of Lemma 5.4; note that an irreducible $\overline{\mathbb{F}}_\ell$-representation of $J(\mathbb{Q}_p)$, being supercuspidal, is not necessarily injective in the category of smooth $\overline{\mathbb{F}}_\ell$-representations of $J(\mathbb{Q}_p)$ with the fixed central character. For this point, we can use the same argument as that by Shin, in which he uses the vanishing of the supercuspidal part $H_c^i(\mathcal{M}_{\mathrm{LT},\infty}, \overline{\mathbb{F}}_\ell)_{\mathrm{sc}}$ for $i \neq n-1$ (*cf.* [Dat10, proof of Proposition 3.1.1, Remarque 3.1.5]) and the projectivity of the $D^\times$-representation $H_c^{n-1}(\mathcal{M}_{\mathrm{LT},\infty}, \overline{\mathbb{F}}_\ell)_{\mathrm{sc}}$ (*cf.* [Dat10, §3.2.2, Remarque iii)]). ∎

References

[Ber96]   V. G. Berkovich, *Vanishing cycles for formal schemes. II*, Invent. Math. **125** (1996), no. 2, 367–390.

[BGR84]   S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften, vol. 261, Springer-Verlag, Berlin, 1984.

[Boy99]   P. Boyer, *Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale*, Invent. Math. **138** (1999), no. 3, 573–629.

[Clo91]   L. Clozel, *Représentations galoisiennes associées aux représentations automorphes autoduales de* GL(n), Inst. Hautes Études Sci. Publ. Math. (1991), no. 73, 97–145.

[Dat10]   J.-F. Dat, *Théorie de Lubin-Tate non-abélienne ℓ-entière*, preprint, arXiv:1011.1887, 2010.

[Far04]   L. Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, Astérisque (2004), no. 291, 1–199, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales.

[FC90]   G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 22, Springer-Verlag, Berlin, 1990.

[FK06] K. Fujiwara and F. Kato, *Rigid geometry and applications*, Moduli spaces and arithmetic geometry, Adv. Stud. Pure Math., vol. 45, Math. Soc. Japan, Tokyo, 2006, pp. 327–386.

[Fuj95] K. Fujiwara, *Theory of tubular neighborhood in étale topology*, Duke Math. J. **80** (1995), no. 1, 15–57.

[Fuj97] _____, *Rigid geometry, Lefschetz-Verdier trace formula and Deligne's conjecture*, Invent. Math. **127** (1997), no. 3, 489–533.

[Hoc69] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969), 43–60.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[Hub93] R. Huber, *Continuous valuations*, Math. Z. **212** (1993), no. 3, 455–477.

[Hub94] _____, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551.

[Hub96] _____, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Hub98a] _____, *A comparison theorem for l-adic cohomology*, Compositio Math. **112** (1998), no. 2, 217–235.

[Hub98b] _____, *A finiteness result for direct image sheaves on the étale site of rigid analytic varieties*, J. Algebraic Geom. **7** (1998), no. 2, 359–403.

[IM10] T. Ito and Y. Mieda, *Cuspidal representations in the ℓ-adic cohomology of the Rapoport-Zink space for* GSp(4), preprint, arXiv:1005.5619, 2010.

[Kot92] R. E. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444.

[Lan08] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, preprint, 2008.

[Man04] E. Mantovan, *On certain unitary group Shimura varieties*, Astérisque (2004), no. 291, 201–331, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales.

[Man05] _____, *On the cohomology of certain PEL-type Shimura varieties*, Duke Math. J. **129** (2005), no. 3, 573–610.

[Mie06] Y. Mieda, *On the action of the Weil group on the l-adic cohomology of rigid spaces over local fields*, Int. Math. Res. Not. (2006), Art. ID 16429, 1–11.

[Mie10a] _____, *Non-cuspidality outside the middle degree of ℓ-adic cohomology of the Lubin-Tate tower*, Adv. Math. **225** (2010), no. 4, 2287–2297.

[Mie10b] _____, *Variants of formal nearby cycles*, preprint, arXiv:1005.5616, 2010.

[Mor08] S. Morel, *Complexes pondérés sur les compactifications de Baily-Borel: le cas des variétés de Siegel*, J. Amer. Math. Soc. **21** (2008), no. 1, 23–61 (electronic).

[Mor10] ———, *On the cohomology of certain noncompact Shimura varieties*, Annals of Mathematics Studies, vol. 173, Princeton University Press, Princeton, NJ, 2010, With an appendix by Robert Kottwitz.

[Pin92] Richard Pink, *On l-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification*, Math. Ann. **292** (1992), no. 2, 197–240.

[Rei75] I. Reiner, *Maximal orders*, Academic Press, London-New York, 1975, London Mathematical Society Monographs, No. 5.

[RZ96] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.

[Sch10] P. Scholze, *The Local Langlands correspondence for* GL_n *over p-adic fields*, preprint, arXiv:1010.1540, 2010.

[Shi11] S. W. Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), no. 3, 1645–1741.

[Spr98] T. A. Springer, *Linear algebraic groups*, second ed., Progress in Mathematics, vol. 9, Birkhäuser Boston Inc., Boston, MA, 1998.

[TY07] R. Taylor and T. Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493.

[Vig96] M.-F. Vignéras, *Représentations l-modulaires d'un groupe réductif p-adique avec l ≠ p*, Progress in Mathematics, vol. 137, Birkhäuser Boston Inc., Boston, MA, 1996.

[SGA3] *Schémas en groupes (SGA3)*, Lecture Notes in Mathematics, Vol. 151, 152, 153, Springer-Verlag, Berlin, 1970.

[SGA5] *Cohomologie l-adique et fonctions L (SGA5)*, Lecture Notes in Mathematics, Vol. 589, Springer-Verlag, Berlin, 1977.

[SGA7] *Groupes de monodromie en géométrie algébrique (SGA7)*, Lecture Notes in Mathematics, Vol. 288, 340, Springer-Verlag, Berlin, 1972.

Naoki Imai
Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606–8502, Japan
E-mail address: `naoki@kurims.kyoto-u.ac.jp`

Yoichi Mieda
Faculty of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka, 819–0395, Japan
E-mail address: `mieda@math.kyushu-u.ac.jp`