Local Saito-Kurokawa A-packets and ℓ-adic cohomology of Rapoport-Zink tower for GSp(4): announcement

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1 Introduction

This is an announcement of a recent joint work of Tetsushi Ito and the author on the ℓ-adic cohomology of the Rapoport-Zink tower for GSp_4. The Rapoport-Zink tower for GSp_4 is a p-adic local counterpart of the Siegel threefold. Its ℓ-adic cohomology H^i_{RZ} is naturally equipped with actions of three groups; the Weil group of $\mathbb{Q}_p$, $\mathrm{GSp}_4(\mathbb{Q}_p)$ and a non-trivial inner form $J(\mathbb{Q}_p)$ of $\mathrm{GSp}_4(\mathbb{Q}_p)$. These actions are expected to be strongly related with the local Langlands correspondence, but they are not fully understood yet. In this work, we focus on a certain class of non-tempered local A-packets of $J(\mathbb{Q}_p)$, called the local Saito-Kurokawa A-packets. We determine how these A-packets and the associated L-packets contribute to the $\mathrm{GSp}_4(\mathbb{Q}_p)$-supercuspidal part of H^i_{RZ}. See Theorem 3.1 for the precise statement.

The outline of this article is as follows. In Section 2, we give a brief review of the local Langlands correspondence. We also recall the Lubin-Tate tower, which is essential to prove the local Langlands correspondence for GL_n. In Section 3, we introduce the Rapoport-Zink tower for GSp_4, which is a GSp_4-version of the Lubin-Tate tower. After that, we state our main theorem and explain the ideas of the proof.

2 Local Langlands correspondence

Throughout this article, we fix a prime number p. In this section, we briefly recall the local Langlands correspondence. Let G be a connected reductive group over $\mathbb{Q}_p$. We assume that G is an inner form of a split group for simplicity. We write $\Pi(G)$ for the set of the isomorphism classes of irreducible smooth representations (over $\mathbb{C}$) of $G(\mathbb{Q}_p)$, and $\Phi(G)$ for the set of the $\widehat{G}$-conjugacy classes of L-parameters $W_{\mathbb{Q}_p}\times \mathrm{SL}_2(\mathbb{C})\to \widehat{G}$. Here $W_{\mathbb{Q}_p}$ denotes the Weil group of $\mathbb{Q}_p$, and $\widehat{G}$ denotes the dual group of G over $\mathbb{C}$. The local Langlands correspondence for G is a conjectural map $\mathrm{LLC}\colon \Pi(G)\to\Phi(G)$ with finite fibers. The fiber Π^G_ϕ of $\phi\in\Phi(G)$ is called the L-packet of ϕ. The map LLC is expected to be surjective when G is split.

If $G=\mathrm{GL}_n$, then $\widehat{G}$ equals $\mathrm{GL}_n(\mathbb{C})$, and an L-parameter $W_{\mathbb{Q}_p}\times \mathrm{SL}_2(\mathbb{C})\to\widehat{G}$ is identified with an n-dimensional semisimple representation of $W_{\mathbb{Q}_p}\times\mathrm{SL}_2(\mathbb{C})$. The local Langlands correspondence for GL_n has been proved by Harris-Taylor [HT01] (see also [Hen00] and [Sch13]). In this case, every L-packet is a singleton; in other words, the map $\mathrm{LLC}\colon \Pi(\mathrm{GL}_n) \to \Phi(\mathrm{GL}_n)$ is bijective. Let us briefly recall the construction of $\mathrm{LLC}(\pi)$ for a supercuspidal $\pi \in \Pi(\mathrm{GL}_n)$. It is given by using the Lubin-Tate tower $\{M_K\}_{K\subset \mathrm{GL}_n(\mathbb{Z}_p)}$, which is a projective system of rigid spaces over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ indexed by compact open subgroups of $\mathrm{GL}_n(\mathbb{Z}_p)$. Here are basic geometric properties of the Lubin-Tate tower:

- $M_{\mathrm{GL}_n(\mathbb{Z}_p)} = \coprod_{\mathbb{Z}}((n-1)\text{-dimensional open unit disk over } \widehat{\mathbb{Q}}_p^{\mathrm{ur}})$.
- $M_K/M_{\mathrm{GL}_n(\mathbb{Z}_p)}$ is a finite étale covering. In particular, each M_K is an $(n-1)$-dimensional smooth rigid space over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$. If K is an open normal subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$, then $M_K/M_{\mathrm{GL}_n(\mathbb{Z}_p)}$ is a Galois covering with Galois group $\mathrm{GL}_n(\mathbb{Z}_p)/K$.

The group $\mathrm{GL}_n(\mathbb{Q}_p)$ acts on the projective system $\{M_K\}_{K\subset \mathrm{GL}_n(\mathbb{Z}_p)}$; it is a local analogue of the Hecke action. The group $D^\times$ also acts on the tower, where D is the central division algebra over $\mathbb{Q}_p$ with invariant $1/n$. Now we fix a prime number ℓ and an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. We put $H^i_{\mathrm{LT}} = \varinjlim_K H^i_c(M_K \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$. It is equipped with an action of $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$. Roughly speaking, the L-parameter $\mathrm{LLC}(\pi)$ for a supercuspidal $\pi \in \Pi(\mathrm{GL}_n)$ is constructed by using the irreducible decomposition of H^{n-1}_{LT}.

Theorem 2.1 ([Car86], [HT01], [Boy09]) *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. We put $\rho = JL(\pi)$, where JL denotes the Jacquet-Langlands correspondence between $\mathrm{GL}_n(\mathbb{Q}_p)$ and $D^\times$. Then $\mathrm{LLC}(\pi)$ is a unique irreducible n-dimensional representation of $W_{\mathbb{Q}_p}$ (which is regarded as a representation of $W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C})$ by the first projection) satisfying the following:*

$$\mathrm{Hom}_{D^\times}(H^{n-1}_{\mathrm{LT}}, \rho)^{\mathrm{sm}} \cong \pi \boxtimes \mathrm{LLC}(\pi)\Big(\frac{n-1}{2}\Big).$$

Here $(-)^{\mathrm{sm}}$ denotes the smooth part with respect to the $\mathrm{GL}_n(\mathbb{Q}_p)$-action, and $(\frac{n-1}{2})$ denotes the Tate twist.

Remark 2.2 If $i \neq n-1$, we have $\mathrm{Hom}_{D^\times}(H^i_{\mathrm{LT}}, \rho)^{\mathrm{sm}} = 0$. See [Boy09].

The key of the proof of Theorem 2.1 is to relate $\{M_K\}_{K\subset \mathrm{GL}_n(\mathbb{Z}_p)}$ to a certain Shimura variety. Let us explain it in the case $n = 2$. In the following we write $\mathbb{A}$ for the ring of adeles of $\mathbb{Q}$. For a compact open subgroup $K' \subset \mathrm{GL}_2(\mathbb{A}^\infty)$, let $\mathrm{Sh}_{K'}$ denote the modular curve over $\mathbb{Q}$ with level K'. We write $\mathrm{Sh}^{\mathrm{an}}_{K',\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$ for the rigid space over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ associated with $\mathrm{Sh}_{K',\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} = \mathrm{Sh}_{K'} \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_p^{\mathrm{ur}}$. We fix a sufficiently small compact open subgroup K^p of $\mathrm{GL}_2(\mathbb{A}^{\infty,p})$. We write $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\widehat{\mathbb{Z}}_p^{\mathrm{ur}}}$ for the integral modular curve over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ with level $\mathrm{GL}_2(\mathbb{Z}_p)K^p$. The supersingular locus of its mod p fiber $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\overline{\mathbb{F}}_p}$ is denoted by $\mathrm{Sh}^{\mathrm{ss}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\overline{\mathbb{F}}_p}$. We have the specialization map $\mathrm{sp}\colon \mathrm{Sh}^{\mathrm{an}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \to \mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\overline{\mathbb{F}}_p}$. Let $\mathrm{Sh}^{\text{ss-red}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$ be the rigid analytic open subset of $\mathrm{Sh}^{\mathrm{an}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$ obtained as the inverse image of $\mathrm{Sh}^{\mathrm{ss}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\overline{\mathbb{F}}_p}$ (strictly speaking, we are in fact working in the framework of adic spaces, so we need to take the interior of the inverse image). The open subset $\mathrm{Sh}^{\text{ss-red}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$ is called the supersingular reduction locus, since its classical point corresponds to an elliptic curve with good supersingular reduction. Finally, for a compact open subgroup K of $\mathrm{GL}_2(\mathbb{Z}_p)$, let $\mathrm{Sh}^{\text{ss-red}}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$ be the inverse image of $\mathrm{Sh}^{\text{ss-red}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$ in $\mathrm{Sh}^{\mathrm{an}}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$. Then the following holds:

Proposition 2.3 (p-adic uniformization) *We have an isomorphism*

$$\mathrm{Sh}^{\text{ss-red}}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \cong \widetilde{D}^\times \backslash (M_K \times \mathrm{GL}_2(\mathbb{A}^{\infty,p})/K^p),$$

where $\widetilde{D}$ is the quaternion division algebra over $\mathbb{Q}$ which ramifies exactly at ∞ and p.

In this work, we use the local Langlands correspondence for $G = \mathrm{GSp}_4$ and its non-trivial inner form J. Both of the dual groups $\widehat{G}$ and $\widehat{J}$ are equal to $\mathrm{GSp}_4(\mathbb{C})$. The local Langlands correspondence for G and J are due to Gan-Takeda [GT11] and Gan-Tantono [GT14], respectively. Unlike the GL_n-case, no geometry is needed in the proofs of them. They used the local theta lifting to reduce the local Langlands correspondence for G and J to that for GL_2 and GL_4. However, the author is still interested in how the local Langlands correspondence for these groups interacts with geometry.

Let $\phi\colon W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GSp}_4(\mathbb{C})$ be an element of $\Phi(G) = \Phi(J)$. The corresponding L-packets Π^G_ϕ and Π^J_ϕ are not necessarily singletons. We are particularly interested in the case where Π^G_ϕ contains a supercuspidal representation. Such L-parameters are classified as follows:

Proposition 2.4 *Let $r\colon \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C})$ denote the natural embedding. If Π^G_ϕ contains a supercuspidal representation, then one of the following holds:*

(i) *There exists a 4-dimensional irreducible representation ϕ_0 of $W_{\mathbb{Q}_p}$ such that $r \circ \phi = \phi_0 \boxtimes \mathbf{1}$, where $\mathbf{1}$ denotes the trivial representation of $\mathrm{SL}_2(\mathbb{C})$. In this case, each of Π^G_ϕ and Π^J_ϕ consists of one supercuspidal representation.*

(ii) *There exist distinct 2-dimensional irreducible representations ϕ_0 and ϕ_1 of $W_{\mathbb{Q}_p}$ such that $r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\phi_1 \boxtimes \mathbf{1})$. In this case, each of Π^G_ϕ and Π^J_ϕ consists of two supercuspidal representations.*

(iii) *There exist a 2-dimensional irreducible representation ϕ_0 of $W_{\mathbb{Q}_p}$ and a character χ of $W_{\mathbb{Q}_p}$ such that $r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\chi \boxtimes \mathbf{Std})$, where $\mathbf{Std}$ denotes the standard representation of $\mathrm{SL}_2(\mathbb{C})$. In this case, each of Π^G_ϕ and Π^J_ϕ consists of one supercuspidal representation and one non-supercuspidal discrete series representation.*

(iv) *There exist distinct characters χ_0, χ_1 of $W_{\mathbb{Q}_p}$ such that $r \circ \phi = (\chi_0 \boxtimes \mathbf{Std}) \oplus (\chi_1 \boxtimes \mathbf{Std})$. In this case, Π^G_ϕ consists of one supercuspidal representation and one non-supercuspidal discrete series representation, and Π^J_ϕ consists of two non-supercuspidal discrete series representations.*

In this article we focus on the case (iii). We write π_{sc} (resp. π_{disc}) for the supercuspidal (resp. non-supercuspidal) representation belonging to Π^G_ϕ. Similarly, we write ρ_{sc} (resp. ρ_{disc}) for the supercuspidal (resp. non-supercuspidal) representation belonging to Π^J_ϕ.

We also need to consider the A-parameter ψ obtained as the composite of

$$W_{\mathbb{Q}_p}\times \mathrm{SL}_2(\mathbb{C})\times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\text{swap } \mathrm{SL}_2 \text{ factors}} W_{\mathbb{Q}_p}\times \mathrm{SL}_2(\mathbb{C})\times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\phi\boxtimes\mathbf{1}} \mathrm{GSp}_4(\mathbb{C}).$$

Let Π^G_ψ (resp. Π^J_ψ) be the local A-packet attached to ψ. We should clarify what Π^G_ψ and Π^J_ψ mean, since local A-packets for J has not been fully constructed yet (see [GT19] for the construction of local A-packets for G). Recall that our ϕ satisfies $r\circ\phi = (\phi_0\boxtimes\mathbf{1})\oplus(\chi\boxtimes\mathbf{Std})$. This implies that $\det\phi_0=\chi^2$. Therefore, the A-parameter $\psi'=\psi\otimes\chi^{-1}$ factors through $\mathrm{Sp}_4(\mathbb{C})\subset\mathrm{GSp}_4(\mathbb{C})$. Since $\mathrm{Sp}_4(\mathbb{C})=\widehat{\mathrm{SO}_5}$, ψ' can be regarded as an A-parameter for both $G^{\mathrm{ad}}=\mathrm{SO}_5(\mathbb{Q}_p)$ and J^{ad}. Local A-packets for $\mathrm{SO}_5(\mathbb{Q}_p)$ was fully constructed by Arthur [Art13]. In particular we have the local A-packet $\Pi^{\mathrm{SO}_5}_{\psi'}$, which can be regarded as a subset of $\Pi(G)$. We put $\Pi^G_\psi=\{\pi'\otimes(\chi\circ\mathrm{sim})\mid \pi'\in\Pi^{\mathrm{SO}_5}_{\psi'}\}$, where $\mathrm{sim}\colon G(\mathbb{Q}_p)\to\mathbb{Q}_p^\times$ denotes the similitude character and χ is regarded as a character $\mathbb{Q}_p^\times\to\mathbb{C}^\times$ by the local class field theory $W^{\mathrm{ab}}_{\mathbb{Q}_p}\cong\mathbb{Q}_p^\times$. As for J^{ad}, the local A-packet $\Pi^{J^{\mathrm{ad}}}_{\psi'}$ for the particular A-parameter ψ' was constructed in [Gan08]. Therefore we get the local A-packet Π^J_ψ in the same way as above.

We call Π^G_ψ and Π^J_ψ the local Saito-Kurokawa A-packets. The structure of them are as follows:

- Π^G_ψ consists of π_{sc} and a non-tempered representation π_{nt}.
- Π^J_ψ consists of a supercuspidal representation ρ'_{sc} and a non-tempered representation ρ_{nt}. As a consequence of our main theorem, ρ'_{sc} turns out to be equal to ρ_{sc} (see Remark 3.2 (ii)).

3 Main Theorem

We continue to write G for GSp_4 and J for its unique non-trivial inner form over $\mathbb{Q}_p$. To state our main theorem, we introduce the (basic) Rapoport-Zink tower for GSp_4, which is the GSp_4-version of the Lubin-Tate tower. It is a projective system of rigid spaces over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ indexed by compact open subgroups of $G(\mathbb{Z}_p)$. Here are basic geometric properties of the Rapoport-Zink tower for GSp_4:

- $M_{G(\mathbb{Z}_p)}$ is a 3-dimensional smooth rigid space over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ (unlike the Lubin-Tate case, we do not have an elementary expression of it).
- $M_K/M_{G(\mathbb{Z}_p)}$ is a finite étale covering. In particular, each M_K is a 3-dimensional smooth rigid space over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$. If K is an open normal subgroup of $G(\mathbb{Z}_p)$, then $M_K/M_{G(\mathbb{Z}_p)}$ is a Galois covering with Galois group $G(\mathbb{Z}_p)/K$.

As in the Lubin-Tate case, the tower $\{M_K\}_{K\subset G(\mathbb{Z}_p)}$ is equipped with an action of $G(\mathbb{Q}_p)\times J(\mathbb{Q}_p)$. We put $H^i_{\mathrm{RZ}}=\varinjlim_K H^i_c(M_K\otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}\mathbb{C}_p,\overline{\mathbb{Q}}_\ell)$, which is a representation of

$G(\mathbb{Q}_p)\times J(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}$. For an irreducible smooth representation ρ of $J(\mathbb{Q}_p)$, we put $H^{i,j}_{\mathrm{RZ}}[\rho] := (\mathrm{Ext}^j_{J(\mathbb{Q}_p)}(H^i_{\mathrm{RZ}},\rho)^{\mathcal{D}_c\text{-sm}})_{\mathrm{sc}}$, where $(-)_{\mathrm{sc}}$ denotes the $G(\mathbb{Q}_p)$-supercuspidal part. For the definition of $(-)^{\mathcal{D}_c\text{-sm}}$, see [Mie14, Notation]. Note that $H^{i,j}_{\mathrm{RZ}}[\rho]$ is a representation of $G(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}$. Since the split semisimple rank of J is 1, we have $H^{i,j}_{\mathrm{RZ}}[\rho]=0$ for $j\geq 2$.

Let $\phi\in\Phi(G)$ be an L-parameter satisfying Proposition 2.4 (iii); namely, there exist a 2-dimensional irreducible representation ϕ_0 of $W_{\mathbb{Q}_p}$ and a character χ of $W_{\mathbb{Q}_p}$ such that $r\circ\phi=(\phi_0\boxtimes\mathbf{1})\oplus(\chi\boxtimes\mathbf{Std})$. We use the same notation as in the previous section. We are interested in how Π^G_ϕ, Π^J_ϕ, Π^G_ψ and Π^J_ψ contribute to H^i_{RZ}. Now we can state our main theorem:

Theorem 3.1 (joint work with Tetsushi Ito) *We have the following:*

(i) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}]\cong\begin{cases}\pi_{\mathrm{sc}}\boxtimes\phi_0(\frac{3}{2}) & i=3,\\ 0 & i\neq 3,\end{cases}\quad H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}]=0,$

$H^{i,0}_{\mathrm{RZ}}[\rho'_{\mathrm{sc}}]\cong\begin{cases}\pi_{\mathrm{sc}}\boxtimes\phi_0(\frac{3}{2}) & i=3,\\ 0 & i\neq 3,\end{cases}\quad H^{i,1}_{\mathrm{RZ}}[\rho'_{\mathrm{sc}}]=0.$

(ii) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}]\cong\begin{cases}\pi_{\mathrm{sc}}\boxtimes\chi(1) & i=3,\\ 0 & i\neq 3,\end{cases}\quad H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}]\cong\begin{cases}\pi_{\mathrm{sc}}\boxtimes\chi(2) & i=4,\\ 0 & i\neq 4.\end{cases}$

(iii) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}]\cong\begin{cases}\pi_{\mathrm{sc}}\boxtimes\chi(2) & i=4,\\ 0 & i\neq 4,\end{cases}\quad H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}]\cong\begin{cases}\pi_{\mathrm{sc}}\boxtimes\chi(1) & i=3,\\ 0 & i\neq 3.\end{cases}$

Here are very rough summary of the main theorem:

- A piece of the local Langlands correspondence for G and J appears in H^3_{RZ}. This is similar to the Kottwitz conjecture (see [Rap95]).
- The non-tempered local A-packet Π^J_ψ contributes to H^4_{RZ}.
- There exists a supercuspidal representation of $G(\mathbb{Q}_p)$ appearing outside the middle degree. In fact, it happens only when its L-parameter has non-trivial $\mathrm{SL}_2(\mathbb{C})$-part (see Remark 3.2 (iv)).

Remark 3.2 (i) By working in a suitable derived category, we may also consider the derived version $H^*_{\mathrm{RZ}}[\rho] := (\mathrm{Ext}^*_{J(\mathbb{Q}_p)}(R\Gamma_{\mathrm{RZ}},\rho)^{\mathcal{D}_c\text{-sm}})_{\mathrm{sc}}$ of $H^{i,j}_{\mathrm{RZ}}[\rho]$. We can recover ϕ and ψ from the $W_{\mathbb{Q}_p}$-action and the Lefschetz operator on $H^*_{\mathrm{RZ}}[\rho_{\mathrm{disc}}]$ and $H^*_{\mathrm{RZ}}[\rho_{\mathrm{nt}}]$, respectively (*cf.* [Dat12] in the GL_n case).

(ii) By using Theorem 3.1, we can prove that the semisimple L-parameters attached to π_{sc}, ρ_{sc} and ρ'_{sc} by Fargues-Scholze [FS] are equal to $\phi|_{W_{\mathbb{Q}_p}}$. This implies that $\rho_{\mathrm{sc}}\cong\rho'_{\mathrm{sc}}$.

(iii) By using recent results of Fargues-Scholze [FS], we can improve the theorem above. We will explain it elsewhere.

(iv) For the L-packets of type (i) and (ii) in Proposition 2.4, we can obtain similar results as Theorem 3.1 (i). On the other hand, up to now we cannot treat the

L-packets of type (iv) in Proposition 2.4. The reason is that the theory of local A-packets for J (or J^{ad}) is not available in this case.

The proof of Theorem 3.1 is given by combination of local and global methods. First we recall some results obtained from local geometry.

Theorem 3.3 ([IM]) *Unless* $2 \leq i \leq 4$, $H^i_{\mathrm{RZ,sc}} = 0$.

Here 2 (resp. 4) appears in the statement since it is equal to $\dim M_{G(\mathbb{Z}_p)} - \dim \mathcal{M}_{\mathrm{red}}$ (resp. $\dim M_{G(\mathbb{Z}_p)} + \dim \mathcal{M}_{\mathrm{red}}$), where $\mathcal{M}$ is the natural formal model of $M_{G(\mathbb{Z}_p)}$. The equality $\dim \mathcal{M}_{\mathrm{red}} = 1$ is related to the fact that the supersingular locus of the Siegel threefold is 1-dimensional. The method of the proof of Theorem 3.3 is similar to the author's proof of $H^i_{\mathrm{LT,sc}} = 0$ for $i \neq n-1$ (see [Mie10]), but it is much more complicated, mainly because connected components of $\mathcal{M}$ are not affine (even not quasi-compact).

Theorem 3.4 *The representation* $H^2_{\mathrm{RZ,sc}}$ *of* $J(\mathbb{Q}_p)$ *does not contain non-supercuspidal subquotient.*

This is a consequence of Theorem 3.3 and the fact that $H^2_{\mathrm{RZ},G(\mathbb{Q}_p)\text{-sc},J(\mathbb{Q}_p)\text{-non-sc}}$ and $H^5_{\mathrm{RZ},G(\mathbb{Q}_p)\text{-sc},J(\mathbb{Q}_p)\text{-non-sc}}$ are related by the Zelevinsky involution (see [Mie]).

Theorem 3.5 ([Mie20]) *Assume that the central character of* π_{sc} *is trivial on* $p^{\mathbb{Z}} \subset \mathrm{GSp}_4(\mathbb{Q}_p)$ *(we can always twist* π_{sc} *by a character so that it satisfies this condition). Then, the representation* $(\varinjlim_K H^i_c((M_K/p^{\mathbb{Z}}) \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell))[\pi_{\mathrm{sc}}^\vee]$ *of* $J(\mathbb{Q}_p)$ *has finite length.*

This was proved by using the duality isomorphism between the Rapoport-Zink tower for G and that for J due to [KW] and [CFS].

Next we discuss the global aspect. As in the Lubin-Tate case, we use the relation between the Rapoport-Zink tower $\{M_K\}_{K\subset G(\mathbb{Z}_p)}$ and the Siegel threefold. For a compact open subgroup $K' \subset G(\mathbb{A}^\infty)$, let $\mathrm{Sh}_{K'}$ denote the Siegel threefold over $\mathbb{Q}$ with level K'. We put $H^i_c(\mathrm{Sh}) = \varinjlim_{K'} H^i_c(\mathrm{Sh}_{K'} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$, which is a representation of $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This representation is rather understood by using the global Langlands correspondence for GSp_4 (see [Tay93] and [Wei09]).

Let us fix a sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}^{\infty,p})$. As in Section 2, for a compact open subgroup $K \subset G(\mathbb{Q}_p)$ we can define a rigid analytic open subset $\mathrm{Sh}^{\text{ss-red}}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$ of $\mathrm{Sh}^{\mathrm{an}}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$, which is called the supersingular reduction locus. The following is an analogue of Proposition 2.3:

Proposition 3.6 (p-adic uniformization, [RZ96]) *We have an isomorphism*

$$\mathrm{Sh}^{\text{ss-red}}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \cong \widetilde{J}(\mathbb{Q})\backslash(M_K \times G(\mathbb{A}^{\infty,p})/K^p),$$

where $\widetilde{J}$ *is a suitable inner form of* GSp_4 *over* $\mathbb{Q}$ *such that* $\widetilde{J} \otimes_{\mathbb{Q}} \mathbb{R}$ *is anisotropic modulo center,* $\widetilde{J} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \cong G \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ *and* $\widetilde{J} \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong J$.

We put $H^i(\mathrm{Sh}^{\text{ss-red}}_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}) = \varinjlim_{K,K^p} H^i(\mathrm{Sh}^{\text{ss-red}}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$, which is a representation of $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$. By Proposition 3.6, we have the Hochschild-Serre spectral sequence

$$E_2^{r,s} = \mathrm{Ext}^r_{J(\mathbb{Q}_p)}(H^{6-s}_{\mathrm{RZ}}(3), \mathcal{A}(\widetilde{J})_{\mathbf{1}})_{\mathrm{sc}} \Rightarrow H^{r+s}(\mathrm{Sh}^{\text{ss-red}}_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}})_{\mathrm{sc}},$$

which is due to [Far04]. Here $\mathcal{A}(\widetilde{J})_{\mathbf{1}}$ is the space of automorphic forms on $\widetilde{J}(\mathbb{A})$ which are trivial on $\widetilde{J}(\mathbb{R})$. By Boyer's trick and a result in [IM20] or [LS18], we have $H^{r+s}(\mathrm{Sh}^{\text{ss-red}}_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}})_{\mathrm{sc}} \cong H_c^{r+s}(\mathrm{Sh})_{\mathrm{sc}}$. Therefore we obtain:

Proposition 3.7 *We have a spectral sequence*

$$E_2^{r,s} = \mathrm{Ext}^r_{J(\mathbb{Q}_p)}(H^{6-s}_{\mathrm{RZ}}(3), \mathcal{A}(\widetilde{J})_{\mathbf{1}})_{\mathrm{sc}} \Rightarrow H_c^{r+s}(\mathrm{Sh})_{\mathrm{sc}}.$$

Now we are ready to sketch the proof of Theorem 3.1. The point is that we begin with $H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}]$. By using Gan's result [Gan08], we can choose

- a cuspidal automorphic representation Π of $G(\mathbb{A})$
- and a cuspidal automorphic representation Σ of $\widetilde{J}(\mathbb{A})$

such that

- $\Pi_p \cong \pi_{\mathrm{sc}}$ and Π^∞ contributes to $H_c^2(\mathrm{Sh})$ and $H_c^4(\mathrm{Sh})$.
- if Π' is an automorphic representation of $G(\mathbb{A})$ such that $\Pi'_v \cong \Pi_v$ for all places $v \neq p, \infty$ and Π'_p is supercuspidal, then $\Pi = \Pi'$. It is a kind of the strong multiplicity one theorem.
- $\Sigma_p \cong \rho_{\mathrm{nt}}$ and $\Sigma_\infty \cong \mathbf{1}$.
- if Σ' is an automorphic representation of $\widetilde{J}(\mathbb{A})$ such that $\Sigma'_v \cong \Sigma_v$ for all places $v \neq p$, then $\Sigma = \Sigma'$. It is a kind of the strong multiplicity one theorem.
- $\Pi^{\infty,p} = \Sigma^{\infty,p}$; recall that we have $G(\mathbb{A}^{\infty,p}) = \widetilde{J}(\mathbb{A}^{\infty,p})$.

By taking the $\Pi^{\infty,p}$-isotypic part of the spectral sequence in Proposition 3.7, we get a short exact sequence

$$0 \to H^{i+1,1}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \to \pi_{\mathrm{sc}} \boxtimes H_c^{6-i}(\mathrm{Sh})[\Pi^\infty](3) \to H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \to 0.$$

By assumption, $H_c^{6-i}(\mathrm{Sh})[\Pi^\infty](3) \neq 0$ only if $i = 2, 4$. On the other hand, by Theorems 3.3 and 3.4, we have $H^{5,1}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] = H^{2,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] = 0$. Hence we conclude

$$H^{4,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \cong \pi_{\mathrm{sc}} \boxtimes H_c^2(\mathrm{Sh})[\Pi^\infty](3), \quad H^{3,1}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \cong \pi_{\mathrm{sc}} \boxtimes H_c^4(\mathrm{Sh})[\Pi^\infty](3).$$

Next we investigate $H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}]$. We choose Π and Σ similarly as above, but so that Π^∞ contributes to $H_c^3(\mathrm{Sh})$. Then we get a short exact sequence

$$0 \to H^{4,1}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}] \to \pi_{\mathrm{sc}} \boxtimes H_c^3(\mathrm{Sh})[\Pi^\infty](3) \to H^{3,0}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}] \to 0.$$

Since $H_c^3(\mathrm{Sh})[\Pi^\infty](3)$ is 2-dimensional indecomposable as a $W_{\mathbb{Q}_p}$-representation, it suffices to determine $\dim H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}][\pi_{\mathrm{sc}}]$. This is done by using the following facts:

– $[\rho_{\mathrm{nt}}] + [\rho_{\mathrm{disc}}] = [\text{induced}]$ in the Grothendieck group of finite length representations of $J(\mathbb{Q}_p)$.
– $\sum_{i=0}^{\infty}(-1)^i \dim \mathrm{Ext}^i_{J(\mathbb{Q}_p)/p^{\mathbb{Z}}}(V, \text{induced}) = 0$ for every $J(\mathbb{Q}_p)/p^{\mathbb{Z}}$-representation V of finite length ([SS97]).

To apply the second fact, we need the finiteness result in Theorem 3.5.

We can treat $H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}]$ and $H^{i,j}_{\mathrm{RZ}}[\rho'_{\mathrm{sc}}]$ in the same way. These cases are the simplest because $H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}] = H^{i,1}_{\mathrm{RZ}}[\rho'_{\mathrm{sc}}] = 0$.

References

[Art13] J. Arthur, *The endoscopic classification of representations: Orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013.

[Boy09] P. Boyer, *Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples*, Invent. Math. **177** (2009), no. 2, 239–280.

[Car86] H. Carayol, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 409–468.

[CFS] M. Chen, L. Fargues, and X. Shen, *On the structure of some p-adic period domains*, arXiv:1710.06935.

[Dat12] J.-F. Dat, *Opérateur de Lefschetz sur les tours de Drinfeld et Lubin-Tate*, Compos. Math. **148** (2012), no. 2, 507–530.

[Far04] L. Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, Astérisque (2004), no. 291, 1–199, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales.

[FS] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459.

[Gan08] W. T. Gan, *The Saito-Kurokawa space of* PGSp_4 *and its transfer to inner forms*, Eisenstein series and applications, Progr. Math., vol. 258, Birkhäuser Boston, Boston, MA, 2008, pp. 87–123.

[GT11] W. T. Gan and S. Takeda, *The local Langlands conjecture for* $\mathrm{GSp}(4)$, Ann. of Math. (2) **173** (2011), no. 3, 1841–1882.

[GT14] W. T. Gan and W. Tantono, *The local Langlands conjecture for* $\mathrm{GSp}(4)$*, II: The case of inner forms*, Amer. J. Math. **136** (2014), no. 3, 761–805.

[GT19] T. Gee and O. Taïbi, *Arthur's multiplicity formula for* $\mathbf{GSp}_4$ *and restriction to* $\mathbf{Sp}_4$, J. Éc. polytech. Math. **6** (2019), 469–535.

[Hen00] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. **139** (2000), no. 2, 439–455.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton

University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[IM] T. Ito and Y. Mieda, *Cuspidal representations in the ℓ-adic cohomology of the Rapoport-Zink space for* $\mathrm{GSp}(4)$, preprint, arXiv:1005.5619.

[IM20] N. Imai and Y. Mieda, *Potentially good reduction loci of Shimura varieties*, Tunis. J. Math. **2** (2020), no. 2, 399–454.

[KW] T. Kaletha and J. Weinstein, *On the Kottwitz conjecture for local Shimura varieties*, arXiv:1709.06651.

[LS18] K.-W. Lan and B. Stroh, *Nearby cycles of automorphic étale sheaves*, Compos. Math. **154** (2018), no. 1, 80–119.

[Mie] Y. Mieda, *Zelevinsky involution and ℓ-adic cohomology of the Rapoport-Zink tower*, preprint, arXiv:1401.5135.

[Mie10] ______, *Non-cuspidality outside the middle degree of ℓ-adic cohomology of the Lubin-Tate tower*, Adv. Math. **225** (2010), no. 4, 2287–2297.

[Mie14] ______, *Geometric approach to the local Jacquet-Langlands correspondence*, Amer. J. Math. **136** (2014), no. 4, 1067–1091.

[Mie20] ______, *On irreducible components of Rapoport-Zink spaces*, Int. Math. Res. Not. IMRN (2020), no. 8, 2361–2407.

[Rap95] M. Rapoport, *Non-Archimedean period domains*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 423–434.

[RZ96] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.

[Sch13] P. Scholze, *The Local Langlands correspondence for* GL_n *over p-adic fields*, Invent. Math. **192** (2013), no. 3, 663–715.

[SS97] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 85, 97–191.

[Tay93] R. Taylor, *On the l-adic cohomology of Siegel threefolds*, Invent. Math. **114** (1993), no. 2, 289–310.

[Wei09] R. Weissauer, *Endoscopy for* $\mathrm{GSp}(4)$ *and the cohomology of Siegel modular threefolds*, Lecture Notes in Mathematics, vol. 1968, Springer-Verlag, Berlin, 2009.

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