Toward generalization of the non-abelian Lubin-Tate theory

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Outline

- The Lubin-Tate theory =
  Geometric realization of the maximal abelian extension of a $p$-adic field (≡ local class field theory)
  Use a 1-dimensional formal group with height 1

- The non-abelian Lubin-Tate theory =
  Geometric realization of the local Langlands correspondence for $GL_n$
  Use the universal deformation space of a 1-dimensional formal group with height $n$ (Lubin-Tate space)

- Generalization of the non-abelian Lubin-Tate theory =
  Geometric realization of the local Langlands correspondence for $p$-adic reductive groups
  Use Rapoport-Zink spaces
Notation on Galois groups

- $p$: prime number
- $\Gamma = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$: the absolute Galois group

$$1 \longrightarrow I \longrightarrow \Gamma \xrightarrow{(*)} \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow 1$$

- $\text{Frob} \in \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$: the geometric Frobenius element ($x \mapsto x^{1/p}$)
- $W \subset \Gamma$: the inverse image of $\text{Frob}^\mathbb{Z} \subset \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ under $(*)$
  (the Weil group for $\mathbb{Q}_p$)
The local Langlands correspondence for $GL_n$

**Theorem (LLC for $GL_n$, Harris-Taylor, Henniart)**

There is a natural bijection between

- irreducible smooth representations of $GL_n(\mathbb{Q}_p)$, and
- Frobenius-semisimple Weil-Deligne representations
  \[ \phi: W \times SL_2(\mathbb{C}) \rightarrow GL_n(\mathbb{C}) \]

- A representation $(\pi, V)$ of a topological group $H$ is said to be smooth if for every $x \in V$, $\text{Stab}_H(x) = \{ h \in H \mid hx = x \}$ is open in $H$.
- A Weil-Deligne representation $\phi$ is said to be Frobenius-semisimple if for every $w \in W$, $\phi(w)$ is semisimple.
- For a prime number $\ell \neq p$, Weil-Deligne representations are in bijection with continuous $\ell$-adic representations $W \rightarrow GL_n(\mathbb{Q}_\ell)$.
  (Grothendieck’s monodromy theorem)
  Denote by $WD(\sigma)$ the Weil-Deligne representation corresponding to a continuous $\ell$-adic representation $\sigma$. 
Example of LLC for $GL_n$

**Example ($n = 1$)**

$\text{Art}: \mathbb{Q}_p^\times \xrightarrow{\cong} W^{ab}$: isomorphism of local class field theory

$\chi: GL_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$ corresponds to $W \longrightarrow W^{ab} \xrightarrow{\chi \circ \text{Art}^{-1}} \mathbb{C}^\times$.

**Example (general $n$)**

- trivial rep. of $GL_n(\mathbb{Q}_p) \leftrightarrow n$-dim. trivial rep. of $W \times SL_2(\mathbb{C})$
- Steinberg rep. $\text{St}_n \leftrightarrow \phi$: irred., $\phi|_W = 1$ (write $\text{Sp}_n$ for such $\phi$)
  - $\text{St}_n$: irred. quotient of $\text{Ind}_{B}^{GL_n(\mathbb{Q}_p)} 1$ ($B$: upper triangular matrices)
- supercuspidal rep. $\leftarrow \phi$: irred., $\phi|_{SL_2(\mathbb{C})} = 1$

Supercuspidal representation is a representation which does not appear in a subquotient of any proper parabolic induction.
G: connected reductive group over $\mathbb{Q}_p$. For simplicity assume $G$ is split.

$\hat{G}$: the dual group of $G$ (the algebraic group over $\mathbb{C}$ obtained by changing roots and coroots of $G$)

LLC for $G$ is not bijective in general!

**Conjecture (LLC for $G$)**

1. There is a natural surjection with finite fibers
   - from isom. classes of irred. smooth rep. of $G(\mathbb{Q}_p)$ to
   - $\hat{G}(\mathbb{C})$-conjugacy classes of $L$-parameters $\phi: W \times SL_2(\mathbb{C}) \rightarrow \hat{G}(\mathbb{C})$

   The fiber $\Pi^G_\phi$ of $\phi$ is called $L$-packet.

2. Put $S_\phi = \pi_0(Cent_{\hat{G}(\mathbb{C})}(\phi))$.
   Then there is a natural bijection $\text{Irr}(S_\phi) \cong \Pi^G_\phi$.

- Known for some smaller groups (e.g. $SL_2$, $U(3)$)
- Recent progress for classical groups (cf. Arthur’s book)
LLC for $G = \text{GSp}_4$

$G = \text{GSp}_4 \hookrightarrow \hat{G} = \text{GSpin}_5 = \text{GSp}_4$

In this case, there is a candidate for LLC by Gan-Takeda.

### Classification of $L$-packets

- **$r$:** $\text{GSp}_4 \hookrightarrow \text{GL}_4$: natural embedding
- **$\phi$:** $W \times \text{SL}_2(\mathbb{C}) \longrightarrow \text{GSp}_4(\mathbb{C})$: $L$-parameter $\sim r \circ \phi$: Weil-Deligne rep.

Assume that $\Pi^G_\phi$ contains a supercuspidal rep. of $G(\mathbb{Q}_p)$

1. $(r \circ \phi)|_{\text{SL}_2(\mathbb{C})} = 1$, $r \circ \phi$: irred. $\sim \Pi^G_\phi = \{\pi\}$
2. $(r \circ \phi)|_{\text{SL}_2(\mathbb{C})} = 1$, $r \circ \phi = \varphi_1 \oplus \varphi_2$ ($\varphi_i$: irred., $\varphi_1 \ncong \varphi_2$, $\dim \varphi_i = 2$) $\sim \Pi^G_\phi = \{\pi_1, \pi_2\}$, $\pi_1$, $\pi_2$ are supercuspidal
3. $r \circ \phi = \varphi \oplus (\chi \otimes \text{Sp}_2)$, $\varphi|_{\text{SL}_2(\mathbb{C})} = 1$, $\varphi$: irred. $\sim \Pi^G_\phi = \{\pi_1, \pi_2\}$, $\pi_1$ is supercuspidal, $\pi_2$ is not supercuspidal
4. $r \circ \phi = (\chi_1 \otimes \text{Sp}_2) \oplus (\chi_2 \otimes \text{Sp}_2)$ ($\chi_1 \neq \chi_2$) $\sim \Pi^G_\phi = \{\pi_1, \pi_2\}$, $\pi_1$ is supercuspidal, $\pi_2$ is not supercuspidal
LLC for $GU(2, D)$

$D$: quaternion div. alg. over $\mathbb{Q}_p$, $J = GU(2, D)$: inner form of $GSp_4(\mathbb{Q}_p)$
In this case, there is a candidate for LLC by Gan-Tantono.

$\phi: \mathcal{W} \times SL_2(\mathbb{C}) \to GSp_4(\mathbb{C}) \sim \Pi^J_{\phi}$: $L$-packet

$\Pi^J_{\phi}$ can be empty. $\Pi^J_{\phi} \neq \emptyset$ if $\phi$ is discrete.

Classification of $L$-packets

$\phi: \mathcal{W} \times SL_2(\mathbb{C}) \to GSp_4(\mathbb{C})$: $L$-parameter.
Assume that $\Pi^G_{\phi}$ contains a supercuspidal rep.
Then, for previous cases (I)–(IV), we have

(I) $\Pi^J_{\phi} = \{\rho\}$, $\rho$ is supercuspidal

(II) $\Pi^J_{\phi} = \{\rho_1, \rho_2\}$, $\rho_1$, $\rho_2$ are supercuspidal

(III) $\Pi^J_{\phi} = \{\rho_1, \rho_2\}$, $\rho_1$ is supercuspidal, $\rho_2$ is not supercuspidal

(IV) $\Pi^J_{\phi} = \{\rho_1, \rho_2\}$, $\rho_1$, $\rho_2$ are not supercuspidal
What we want to know

- Geometric reason why $L$-packets naturally appear.
- Characterization of $L$-packets of type (I)–(IV) from geometric viewpoint.

These are new problems which does not appear in the case of $GL_n$. 
Rapoport-Zink space for GSp\(_{2n}\)

- \(\mathbb{X}\): an \(n\)-dimensional \(p\)-divisible group over \(\overline{\mathbb{F}}_p\) with slope \(1/2\) (e.g. \(E\): a supersingular elliptic curve over \(\overline{\mathbb{F}}_p\), \(\mathbb{X} = E[p^\infty]^{\oplus n}\))

- \(\lambda_0: \mathbb{X} \xrightarrow{\cong} \mathbb{X}^\vee\): a polarization \((\lambda_0^\vee = -\lambda_0)\)

- \(\text{Nilp}\): the category of \(\hat{\mathbb{Z}}_p^{ur}\)-algebras \(A\) in which \(p\) is unipotent

**Definition (Rapoport-Zink space)**

Define a functor \(\mathcal{M}: \text{Nilp} \to \text{Set}\) as follows:

\[
\mathcal{M}(A) = \{(X, \lambda, \rho)\}/ \cong
\]

- \(X\): \(p\)-divisible group over \(A\), \(\lambda\): polarization of \(X\)

- \(\rho: X \otimes_{\overline{\mathbb{F}}_p} A/pA \to X \otimes_A A/pA\): quasi-isogeny \((= p^{-m} \circ \text{isogeny})\)

- \(\rho^{-1} \circ (\lambda \mod p) \circ \rho \in \mathbb{Q}_p^\times \cdot \lambda_0\)

\(\mathcal{M}\) is represented by a formal scheme over \(\hat{\mathbb{Z}}_p^{ur}\) called the Rapoport-Zink space.
Rapoport-Zink space for $\text{GSp}_{2n}$

$\mathcal{M}$ is a very large formal scheme.
- Countably many connected components (corresponding to $\text{deg } \rho$)
- Each connected component is not quasi-compact. If $n = 2$, $\mathcal{M}^{\text{red}}$ is a chain of infinitely many $\mathbb{P}^1$’s.

**Group action on $\mathcal{M}$**

$J = \text{QIsog}(\overline{X}, \lambda_0)$: the group of self-quasi-isogenies $\rightsquigarrow J = GU(n, D)$

$\mathcal{M} \curvearrowright J$ (right action): $(X, \lambda, \rho) \cdot h = (X, \lambda, \rho \circ h)$

**Relation to Shimura varieties (p-adic uniformization)**

$\text{Sh}$: Shimura variety for $\text{GSp}_{2n}$ (the moduli space of principally polarized $n$-dimensional abelian varieties)

$\text{Sh}^{\text{ss}} \subset \text{Sh}_{\mathbb{F}_p}$: supersingular locus, $\text{Sh}|_{\text{Sh}^{\text{ss}}}$: formal completion along $\text{Sh}^{\text{ss}}$

$\rightsquigarrow \text{Sh}|_{\text{Sh}^{\text{ss}}}^{\wedge}$ is uniformized by $\mathcal{M}$: $\text{Sh}|_{\text{Sh}^{\text{ss}}}^{\wedge} = \bigsqcup_{i=1}^{k} \mathcal{M}/\Gamma_i$ ($\Gamma_i \subset J$)
Rapoport-Zink tower for $\text{GSp}_{2n}$

$M = \text{rigid generic fiber of } \mathcal{M}$ (rigid space over $\hat{\mathbb{Q}}_p^{ur}$)

**Rapoport-Zink tower**

$\{M_K\}_K$: proj. system of étale coverings of $M$ (the Rapoport-Zink tower)

- $K$ runs through compact open subgroups of $\text{GSp}_{2n}(\mathbb{Z}_p)$
- Defined by using $K$-level str. of the universal polarized $p$-div. group
- $M_{\text{GSp}_{2n}(\mathbb{Z}_p)} = M$
- If $K$ is a congruence subgp $K_m = \text{Ker}(\text{GSp}_{2n}(\mathbb{Z}_p) \to \text{GSp}_{2n}(\mathbb{Z}/p^m\mathbb{Z}))$, $K_m$-level str. = trivialization of $p^m$-torsion points (preserving pol.)

**Group actions on Rapoport-Zink tower**

- $J$ acts on $M_K$ (preserve levels)
- $G = \text{GSp}_{2n}(\mathbb{Q}_p)$ acts on the tower $\{M_K\}_K$ (doesn’t preserve levels)
  
  $g \in G \leadsto M_K \longrightarrow M_{g^{-1}Kg}$ (Hecke action)
Relation to Shimura varieties

$\text{Sh}^{[ss]} \subset \text{Sh}_{\mathbb{Q}_p}^{\text{ur}}$: locus consisting of abelian varieties with supersingular reduction (rigid locally closed subset)

$\text{Sh}_{K,\mathbb{Q}_p}^{\text{ur}}$: Shimura variety with $K$-level structure, $\text{Sh}_{K}^{[ss]} \subset \text{Sh}_{K,\mathbb{Q}_p}^{\text{ur}}$: inverse image of $\text{Sh}^{[ss]}$

$\sim \text{Sh}_{K}^{[ss]}$ is uniformized by $M_K$

$l$-adic étale cohomology of the Rapoport-Zink tower

$l$: prime number different from $p$

$H^i_{\text{RZ}} := \lim_{\longrightarrow_K} H^i_c(M_K \otimes_{\mathbb{Q}_p^{\text{ur}}} \mathbb{Q}_p^{\text{ac}} \otimes \mathbb{Q}_l):$ representation of $W \times G \times J$

Goal: describe $H^i_{\text{RZ}}$ via LLC for $G$ and $J$
The case of $GL_n$

Change the definition of the Rapoport-Zink space as follows:

- $X$: 1-dim. $p$-div. group (⇔ formal group) over $\overline{F}_p$ with slope $1/n$
- forget all "polarizations"

$\leadsto$ the Lubin-Tate space, the Lubin-Tate tower, cohomology $H^i_{LT}$

- $G = GL_n(\mathbb{Q}_p)$, $J = D_n^\times$
  
  ($D_n$ is the central division algebra over $\mathbb{Q}_p$ with $\text{inv } D_n = 1/n$)

Theorem (non-abelian Lubin-Tate theory, Harris-Taylor)

$\pi$: supercuspidal rep. of $GL_n(\mathbb{Q}_p)$, $\phi$: Weil-Deligne rep. s.t. $\Pi^{GL_n}_\phi = \{\pi\}$

$\Pi_{\phi}^{D_n^\times} = \{\rho\}$ ($\pi \leftrightarrow \rho$: Jacquet-Langlands correspondence)

$\sigma: W \longrightarrow GL_n(\overline{\mathbb{Q}_\ell})$: $\ell$-adic rep. s.t. $WD(\sigma) = \phi$

$$\text{Hom}_{GL_n(\mathbb{Q}_p)}(H^i_{LT}, \pi) = \begin{cases} 
\sigma\left(\frac{1-n}{2}\right) \otimes \rho & (i = n - 1) \\
0 & (i \neq n - 1)
\end{cases}$$
The case of $\text{GSp}_4$

(joint work with Tetsushi Ito)

- $G = \text{GSp}_4(\mathbb{Q}_p)$, $J = \text{GU}(2, D)$
- $\phi: \mathcal{W} \times \text{SL}_2(\mathbb{C}) \to \text{GSp}_4(\mathbb{C})$: $L$-parameter
  Assume that $\Pi^G_\phi$ contains a supercuspidal rep.
- For $\rho \in \Pi^J_\phi$, put $H^i_{\text{RZ}}[\rho] := \text{Hom}_J(H^i_{\text{RZ}}, \rho)^{G\text{-sm}}$ (rep. of $\mathcal{W} \times G$)
- $H^i_{\text{RZ}}[\rho]_{\text{cusp}}$: the supercuspidal part of $H^i_{\text{RZ}}[\rho]$

Classification of $L$-parameters (again)

- $r: \text{GSp}_4 \hookrightarrow \text{GL}_4$: natural embedding
  - (I) $(r \circ \phi)|_{\text{SL}_2(\mathbb{C})} = 1$, $r \circ \phi$: irred.
  - (II) $(r \circ \phi)|_{\text{SL}_2(\mathbb{C})} = 1$, $r \circ \phi = \varphi_1 \oplus \varphi_2$ ($\varphi_i$: irred., $\varphi_1 \not\cong \varphi_2$, $\dim \varphi_i = 2$)
  - (III) $r \circ \phi = \varphi \oplus (\chi \otimes \text{Sp}_2)$, $\varphi|_{\text{SL}_2(\mathbb{C})} = 1$, $\varphi$: irred.
  - (IV) $r \circ \phi = (\chi_1 \otimes \text{Sp}_2) \oplus (\chi_2 \otimes \text{Sp}_2)$ ($\chi_1 \neq \chi_2$)
Consider the case where $\phi|_{\text{SL}_2(\mathbb{C})} = 1$ (i.e. type (I) or (II))

**Main theorem A**

$\phi$: type (I) or (II), $\rho \in \Pi^J_{\phi}$: supercuspidal rep.

1. If $i \neq 3$, then $H^i_{\text{RZ}}[\rho]_{\text{cusp}} = 0$.
2. $H^3_{\text{RZ}}[\rho]_{\text{cusp}} = \bigoplus_{\pi \in \Pi^G_{\phi}} \sigma_{\pi} \otimes \pi$  
   $\sigma_{\pi}$: irred. $\ell$-adic rep. of $W$ s.t. $\bigoplus_{\pi \in \Pi^G_{\phi}} \text{WD}(\sigma_{\pi}) = r \circ \phi$

- If $\phi$ is type (II), we can determine which of $\varphi_1$ or $\varphi_2$ is equal to $\text{WD}(\sigma_{\pi})$.
- The above theorem is a precise version of Kottwitz’s conjecture for $\sum_i (-1)^i H^i_{\text{RZ}}$. 
The case of $GSp_4$ (when $\phi|_{SL_2(\mathbb{C})} \neq 1$)

Consider the case where $\phi|_{SL_2(\mathbb{C})} \neq 1$ (i.e. type (III) or (IV))

**Main theorem $A'$**

$\phi$: type (III), i.e. $r \circ \phi = \phi \oplus (\chi \otimes Sp_2)$

$\rho \in \Pi^J_\phi$: supercuspidal rep. (unique)

1. If $i \neq 3$, then $H^i_{RZ}[\rho]_{cusp} = 0$.
2. $H^3_{RZ}[\rho]_{cusp} = \sigma_\pi \otimes \pi$
   $\pi \in \Pi^G_\pi$: supercuspidal rep. (unique), $WD(\sigma_\pi) = \varphi$

**Main theorem $B$**

$\phi$: type (III) or (IV), $\pi \in \Pi^G_\phi$: supercuspidal rep. (unique)

Then $\pi$ appears as a subquotient of $H^4_{RZ}$. 
The case of GSp$_4$ (when $\phi|_{\text{SL}_2(\mathbb{C})} \neq 1$)

Our more precise expectation is the following (in progress)

**Expectation**

- $\phi$: type (III) or (IV), $\chi \otimes \text{Sp}_2 \subset r \circ \phi$
- $\pi \in \Pi^G_\phi$: supercuspidal rep. (unique), $\rho \in \Pi^J_\phi$: non-supercuspidal rep.
  - $\chi \otimes \pi^\vee \otimes \rho$ occurs in $H^3_{\text{RZ}}$.
  - $\chi \otimes \pi^\vee \otimes \text{Zel}(\rho)^\vee$ occurs in $H^4_{\text{RZ}}$.
    - (Zel: Zelevinsky involution, $\text{Zel}(\rho)^\vee$: non-tempered rep.)

- We hope $H^2_{\text{RZ}} = 0$.
- If $\Pi^J_\phi = \{\rho, \rho'\}$, then $\{\rho', \text{Zel}(\rho)^\vee\}$ is a non-tempered $A$-packet of $J$. 
Outline of proof

We explain the outline of our proof of main theorem B.

Main theorem B (again)

\( \phi \): type (III) or (IV), \( \pi \in \Pi^G_\phi \): supercuspidal rep.

Then \( \pi \) appears as a subquotient of \( H^4_{RZ} \).

Relate \( H^i_{RZ} \) to the cohomology of the Shimura variety.

\( \text{Sh}^{[\text{ss}]}_K \) is uniformized by \( M_K \leadsto \) the Hochschild-Serre spectral sequence.

Hochschild-Serre spectral sequence (Harris, Fargues)

\[
E_2^{i,j} = \text{Ext}^j_{J-\text{sm}}(H^6_{RZ} - j, A)(-3) \Longrightarrow \lim_{K} H^{i+j}(\text{Sh}^{[\text{ss}]}_K, \overline{\mathbb{Q}}_\ell)
\]

\( A \): space of automorphic forms on \( \text{GSp}_4(\mathbb{A}^{\infty,P}) \times J \)

\[
H^i(\text{Sh}^{[\text{ss}]}_\infty) := \lim_{K} H^{i+j}(\text{Sh}^{[\text{ss}]}_K, \overline{\mathbb{Q}}_\ell)
\]
Outline of proof

$\text{Sh}^{[ss]}_K$: supersingular reduction locus
$\text{Sh}^{[\text{good}]}_K$: good reduction locus

$\text{Sh}^{[ss]}_K \subset \text{Sh}^{[\text{good}]}_K \subset \text{Sh}_K$

$\sim \quad \text{IH}^i(\text{Sh}_\infty)_{\text{cusp}} \overset{(1)}{\to} \text{H}^i(\text{Sh}_\infty)_{\text{cusp}} \overset{(2)}{\to} \text{H}^i(\text{Sh}^{[\text{good}]}_\infty)_{\text{cusp}} \overset{(3)}{\to} \text{H}^i(\text{Sh}^{[ss]}_\infty)_{\text{cusp}}$

(1) Use the minimal compactification over $\mathbb{C}$. True for every Shimura variety.

(2) Joint work with Naoki Imai. True for fairly general Shimura varieties.

(3) Boyer’s trick. Only valid for $\text{GSp}_4$. (proof is not applicable to $\text{GSp}_{2n}$ with $n \geq 3$)
Outline of proof

Another key is:

Theorem (non-cuspidality)

If \( i \neq 2, 3, 4 \), no supercuspidal rep. appears as a subquotient of \( H^i_{RZ} \).

- Since \( \dim M = 3 \), \( H^i_{RZ} = 0 \) unless \( 0 \leq i \leq 6 \).
- \( 2 = 3 - 1 = \dim M - \dim M^{\text{red}}, \quad 4 = 3 + 1 = \dim M + \dim M^{\text{red}} \)
- This theorem is proved by a purely local method.
- The method is similar to our purely local proof of non-cuspidality for the Lubin-Tate tower. However, we encounter many new difficulties since \( M \) is very large.
- For a proof, we introduce variants of formal nearby cycle functor. (in order to capture “invisible boundaries”)
Outline of proof

Main theorem B (again)

$\phi$: type (III) or (IV), $\pi \in \Pi^G_\phi$: supercuspidal rep. (unique)

Then $\pi$ appears as a subquotient of $H^4_{RZ}$.

Proof of main theorem B

Take a cuspidal automorphic rep. $\Pi$ of $\text{GSp}_4(\mathbb{A})$ s.t.

- $\Pi$ occurs in $IH^2(\text{Sh}_\infty)$ (a condition on $\Pi_\infty$)
- $\Pi_p \cong \pi^\vee$

(Need the assumption that $\phi$ is type (III) or (IV))

By the Hochschild-Serre spectral sequence, there are $i, j$ with $i + j = 2$ s.t.
$\Pi_p$ contributes to $\text{Ext}^i_{J-\text{sm}}(H^{6-j}_{RZ}, \mathcal{A})$.

By the non-cuspidality, we have $6 - j \neq 5, 6$.

So $\Pi_p$ appears in $\text{Hom}_J(H^4_{RZ}, \mathcal{A})$, and thus $\pi$ appears in $H^4_{RZ}$. 
Outline of proof

Our proof of main theorems $A, A'$ is similar. Need to globalize $\pi$ and $\rho$ carefully.

Take a cuspidal automorphic rep. $\Pi$ of $\text{GSp}_4(\mathbb{A})$ s.t.

- $\Pi$ occurs in $IH^3(\text{Sh}_\infty)$ (a condition on $\Pi_\infty$)
- $\Pi_p \cong \pi$
- $\Pi$ satisfies the strong multiplicity one theorem at $p$. Namely, an autom. rep. $\Pi'$ of $\text{GSp}_4(\mathbb{A})$ with $\Pi^p = (\Pi')^p$ coincides with $\Pi$.

(cf. Arthur’s multiplicity conjecture)

Globalize $\rho$ to $\Pi^J$ so that similar conditions and $\Pi^{\infty,p} = (\Pi^J)^{\infty,p}$ are satisfied.

Take “$\Pi^{\infty,p}$-parts” of the Hochschild-Serre spectral sequence.
Local method

Global method cannot answer our question why $L$-packets naturally appear. Another purely local method is usage of Lefschetz trace formula for open rigid (or adic) spaces.

**Theorem (Lefschetz trace formula for open rigid spaces)**

$X$: quasi-compact smooth adic space over $\hat{Q}_p^{ac}$

$X \subset \overline{X}$: compactification

$f: X \rightarrow X$: proper morphism, assumed to be extended to $\overline{f}: \overline{X} \rightarrow \overline{X}$

Assume that for every $x \in \overline{X} \setminus X$, $x$ and $\overline{f}(x)$ can be separated by closed constructible subsets of $\overline{X}$. Then,

$$\sum_i (-1)^i \text{Tr}(f^*; H^i_c(X, \overline{Q}_\ell)) = \# \text{Fix}(f)$$

Apply this formula to quasi-compact open subsets of $M_K$. 
Local method

**Theorem**

\[ \phi: W \times \text{SL}_2(\mathbb{C}) \longrightarrow \text{GSp}_4(\mathbb{C}): \text{L-parameter of type (I) or (II)} \]

- Assume the character relation between \( \Pi^G_\phi \) and \( \Pi^J_\phi \).
  
  (It is expected to be proved by the stable trace formula. It has been proved in some cases, e.g. \( \phi \) is a TRSELP.)

- Assume that \( H_{RZ}^i[\rho] \) is a finitely generated \( G \)-module for \( \rho \in \Pi^J_\phi \).

Then, for every elliptic regular semisimple element \( g \in G \),

\[
\sum_{\rho \in \Pi^J_\phi} \theta_{H_{RZ}[\rho]}(g) = 4 \sum_{\pi \in \Pi^G_\phi} \theta_{\pi}(g)
\]

- \( H_{RZ}[\rho] = \sum_i (-1)^i H_{RZ}^i[\rho] \)

- \( \theta_{H_{RZ}[\rho]}, \theta_{\pi} \): distribution characters. They are locally constant functions over the set of regular semisimple elements of \( G \).
Local method

- For $GL_n$, there are preceding works by Faltings and Strauch.
- By this method, we get no information on the action of $W$.
- To count fixed points in $M_K$, we use the $p$-adic period map (the map attaching the Hodge filtration of the Dieudonné module to a $p$-divisible group)
  $\rightsquigarrow$ by $p$-adic Hodge theory, stable conjugacy classes and their transfers naturally appear.
  (these are closely related to $L$-packets)
- With more precise study, we expect to show

$$\theta_{H_RZ}[\rho](g) = \frac{4}{\#\prod_\phi^G} \sum_{\pi \in \prod_\phi^G} \theta_\pi(g)$$
Comments on other groups

- Similar methods are applicable to $GU(2, 1)$, and $GL_4$ with slope $1/2$.
- Boyer’s trick is valid for $GU(n - 1, 1)$ (Mantovan, Shen)
  $\leadsto$ similar results for the alternating sum of $H^i_{RZ}$ might be possible.
  Proof of the non-cuspidality doesn’t work in this case.
- It seems important to consider the case of $GU(2, D)$ (the “dual tower” for $GSp_4$) simultaneously (cf. Faltings isomorphism)