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# Motion of phase boundaries by surface diffusion

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# 1 Introduction

There are several models to describe motion of interphase boundaries in coarsening processes. The Cahn-Hilliard equation is one of typical models describing such evolutions from macroscopic point of view. It is known that as a singular limit of the Cahn-Hilliard equation there are several evolution equations for phase boundaries. These interface models are geometric in the sense that evolution law is completely determined by geometry of phase boundaries. If the surface diffusion is dominated, the interface model comes to be a quasilinear diffusion equation for position of phase boundaries. However, this equation is not second order but fourth order so there are several phenomena which are different from those of second order model like the mean curvature flow equation which is a model describing motion of antiphase boundaries. The authors analyse such fourth order models from rigorous mathematical point of view. The authors for example proved: (i) a curve that moves by surface diffusion may pinch in a finite time, (ii) a curve that moves by surface diffusion may not preserve its convexity in a finite time. These phenomena do not happen for the curve shortening equation. Several other behaviours of solutions will be presented in the lecture.

We study motion by surface diffusion which was first derived by Mullins [7].

Let $\Gamma_t \subset \mathbf{R}^2$ be a closed evolving curve depending on time $t$ with initial data $\Gamma_t|_{t=0} = \Gamma_0$. The governing equation for evolving curves by surface diffusion is of the form

$$V = -\kappa_{ss}. \tag{1}$$

Here $V$ denotes the outward normal velocity and $\kappa$ denotes the outward curvature; $s$ denotes the arclength parameter of $\Gamma_t$. There are several derivations of this equation other than Mullins [7]. See for example Cahn and Taylor [1] and Cahn, Elliott and Novick-Cohen [2]. In the latter paper, (1) is obtained as some formal limit of Cahn-Hilliard equations. A typical feature of $\Gamma_t$ moved by (1) is that the area enclosed by $\Gamma_t$ is preserved. Related equations to (1) are well explained in Elliott and Garcke [3] and Cahn and Taylor [1]. For physical background of these equations, see [1, 3] and references cited there.

In [3] local existence of solution for (1) was proved without uniqueness as well as for other equations. They proved that if initial data is close to a circle, then $\Gamma_t$ exists globally in time and it converges to a circle with the same area enclosed by $\Gamma_0$ as $t$ tends to infinity. They also conjectured that $\Gamma_t$ moved by (1) may pinch for some simple smooth initial data. After this work was completed, we were informed of a recent work of Escher, Mayer and Simonett [4] on unique local existence of solutions of (1). They proved the unique existence of local-in-time solutions even for higher dimensional version of (1) in small Hölder spaces by appealing abstract semigroup theory. They also studied the large time behavior of solutions of the higher dimensional version of (1) if initial data are close to a sphere. These results are regarded as a natural extension of the results of [3] to higher dimensional setting. Moreover, they showed numerical evidence of existence of pinchings for various closed curves.

In this paper we present a rigorous mathematical result proving Elliott and Garcke's conjecture without detailed proof. The detailed proof is given in our paper [6]. Let us explain our idea. We consider a smooth closed simple curve $\Gamma_0$ which is symmetric with respect to $x$-axis and $y$-axis. We assume that $\Gamma_0$ is of the form

$$\Gamma_0 = \{(x, y); y = \pm u_0(x)\},$$

where $u_0(x)$ is even and $u_0(x)$ takes the only local minimum at $x = 0$. If $\Gamma_t$ is represented by $y = u(t, x)$, then (1) becomes a fourth order equation of $u(t, x)$. If we linearize (1) around $u = 0$, we obtain

$$u_t = -u_{xxxx}.$$

If we consider the Cauchy problem for this equation with $u_0(x) \geq 0$ and $u_0(x) = x^4 + \delta$ for small $\delta > 0$ near $x = 0$, then $u(t,0)$ would be negative in a short time. In other words, the comparison principle does not hold. It is easy to guess this phenomenon since $u(t,x) = x^4 - 4!t + \delta$ solves $u_t = -u_{xxxx}$. For a good choice of $u_0(x)$, $u(t,0)$ becomes negative in short time during the period that solution $\Gamma_t$: $y = u(t,x)$ of (1) exists as smooth curves. Since $\Gamma_t$ is represented by $y = u(t,x)$, and symmetric with respect to $y = 0$, this means that $\Gamma_t$ pinches in short time even if $\Gamma_0$ is simple. This is a rough idea of our proof.

## 2 Pinching of evolving closed curves

We summarize here a parametrization of (1) by following Elliott and Garcke [4].

Let $M^0$ be a fixed reference $C^\infty$ (or at least $C^5$) closed curve with arclength $2L$. For $\mathbf{T} = \mathbf{R}/(2L\mathbf{Z})$, let

$$X^0 : \quad \begin{array}{l} \mathbf{T} \to M^0, \\ \eta \mapsto X^0(\eta) \end{array}$$

be an arclength parametrization of $M^0$. By definition, $X^0$ is a function on $\mathbf{T}$ or equivalently $2L$-periodic function. Then, $\tau^0(\eta) = X^0_\eta(\eta)$ is the unit tangent vector of $M^0$ and the Frenet formula gives

$$\begin{aligned} \tau^0_\eta(\eta) &= \kappa^0(\eta) n^0(\eta), \\ n^0_\eta(\eta) &= -\kappa^0(\eta) \tau^0(\eta), \end{aligned}$$

where $n^0(\eta)$ is the unit normal vector and $\kappa^0(\eta)$ is the curvature of $M^0$ with the sign convention that the curvature of a circle is negative.

Let $\Gamma_t \subset \mathbf{R}^2$ be a closed curve moved by surface diffusion law with respect to time $t \geq 0$ starting from initial closed curve $\Gamma_0$. For small $T > 0$ we expect that $\Gamma_t$ is parametrized by

$$X : \begin{array}{ccc} [0,T) \times \mathbf{T} & \to & \Gamma_t, \\ (t,\eta) & \mapsto & X(t,\eta), \end{array}$$

$$X(t,\eta) = X^0(\eta) + d(t,\eta) n^0(\eta)$$

with some $d(t,\eta)$ defined on $[0,T) \times \mathbf{T}$. If $\Gamma_0$ is embedded and $\Gamma_t$ is close to $\Gamma_0$, then $d(t,\eta)$ is the distance function from $M^0$. By this parametrization, (1) is equivalent to

$$\frac{1 - d\kappa^0}{J} d_t = -\frac{1}{J}\partial_\eta \left(\frac{1}{J}\partial_\eta \kappa\right),$$

where $J = |X_\eta| = \partial s / \partial \eta$ is the Jacobian and $\kappa(t,\eta)$ is the curvature of $\Gamma_t$ in the direction of $n^0$. Their explicit forms are

$$J = J(\eta, \alpha_0, \alpha_1)|_{(\alpha_0,\alpha_1)=(d,d_\eta)} = (d_\eta^2 + (1 - d\kappa^0)^2)^{1/2},$$

$$\kappa = \frac{1}{J^3}\{(1 - d\kappa^0) d_{\eta\eta} + 2\kappa^0 d_\eta^2 + \kappa^0_\eta d d_\eta + \kappa^0 (1 - d\kappa^0)^2\}.$$

Thus, the equation (1) for $d(t,\eta)$ with initial data $\Gamma_t|_{t=0} = \Gamma_0$ is of the form:

$$\left\{ \begin{array}{l} d_t + J^{-4} d_{\eta\eta\eta\eta} + P d_{\eta\eta\eta} + Q = 0, \ \ 0 < t < T, \ \ \eta \in \mathbf{T}, \\ d(0,\eta) = d_0(\eta), \ \ \eta \in \mathbf{T}, \end{array} \right. \tag{2}$$

where $P$ and $Q$ are polynomials with arguments $(1-\kappa^0 d)^{-1}$, $J^{-1}$, $\kappa^0$, $\kappa^0_\eta$, $\kappa^0_{\eta\eta}$, $\kappa^0_{\eta\eta\eta}$, $d$, $d_\eta$ and $d_{\eta\eta}$. We note that $\kappa^0$ toghther with its derivatives $\kappa^0_\eta$, $\kappa^0_{\eta\eta}$, $\kappa^0_{\eta\eta\eta}$ is continuous and bounded on $\mathbf{T}$ since $M^0$ is at least $C^5$. We show that there is an evolving closed curve which pinches in finite time, even if initial curve is simple.

Let us explain our idea of the proof. Let $M^0 = \{X^0(\eta); \eta \in \mathbf{T} = \mathbf{R}/(2L\mathbf{Z})\}$ be a dumbbell like curve symmetric with respect to both $x$-axis and $y$-axis and its neck is so thin so that it is just a segment on the $x$-axis. It is normalized by setting $X^0(0) = X^0(L) =$ the origin $(0,0)$. Let $\Gamma_0 = \{X^0(\eta) + d_0(\eta)n^0(\eta); \eta \in \mathbf{T}\}$ with $d_0(\eta) > 0$ be symmetric with respect to both $x$-axis and $y$-axis and assume that $d_0(\eta)$ takes its global isolated minimum at $\eta = 0$ and $L$. Then, by symmetry of the equation (2), the solution $\Gamma_t = \{X^0(\eta) + d(t,\eta)n^0(\eta); \eta \in \mathbf{T}\}$ stays symmetric with respect to both $x$-axis and $y$-axis. In particular, $d_\eta(t,0) = 0$ and $d_{\eta\eta\eta}(t,0) = 0$. Thus if $d(t,\eta)$ solves (2), then

$$d_t(0,0) = -\partial_\eta^4 d(0,0) + 3(\partial_\eta^2 d(0,0))^3.$$

Thus, by the fundamental theorem of calculus,

$$\begin{aligned} d(t,0) &= d(0,0) + d_t(0,0)t + \int_0^t \int_0^\tau d_{ss}(s,0)\,ds\,d\tau \\ &\le d(0,0) + (-\partial_\eta^4 d(0,0) + 3(\partial_\eta^2 d(0,0))^3)t + t^2 \cdot \sup_{t\in[0,\bar{t}],\eta\in\mathbf{T}} |d_{tt}(t,\eta)|, \end{aligned} \tag{3}$$

where $\bar{t}$ is taken so that $d(t,\eta)$ exists for $[0,\bar{t}]$. Roughly speaking, if $d(0,0)$ is sufficiently small and $-\partial_\eta^4 d(0,0) + 3(\partial_\eta^2 d(0,0))^3 < 0$, then $d(t,0)$ may be nagative for $t$ between two roots of the quadratic equation of $t$: the R.H.S. of (3)$=0$, which will imply a pinching of $\Gamma_t$.

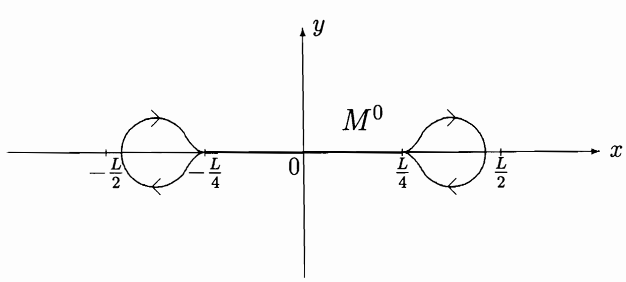
We shall state our result rigorously in the following. To do this, we define a special $(C^\infty)$ reference curve $M^0$. This is parametrized by

$$X^0(\eta) = (X_1^0(\eta), X_2^0(\eta)) \quad \text{for } \eta \in \mathbf{T} = \mathbf{R}/(2L\mathbf{Z})$$

satisfying

$$\begin{cases} X_1^0(\eta) = -X_1^0(-\eta), & 0 \le \eta \le L, \\ X_2^0(\eta) = X_2^0(-\eta), & 0 \le \eta \le L, \\ X^0(\eta) = (\eta, 0), & 0 \le \eta \le L/4, \\ (X_1^0)_\eta(\eta) > 0, & 0 \le \eta \le L/2, \\ X_1^0(L/2+\eta) = X_1^0(L/2-\eta), & 0 \le \eta \le L/2, \\ X_2^0(\eta) > 0, & L/4 < \eta < L/2, \\ X_2^0(L/2+\eta) = -X_2^0(L/2-\eta), & 0 \le \eta \le L/2, \end{cases}$$

where $\eta$ is an arclength parameter.



We define a set of functions in $\mathbf{T}$ depending on positive parameters $N$ and $\varepsilon$:

$$
\begin{aligned}
D_0(N,\varepsilon) \quad = \quad & \{d_0 : \text{smooth}; d_0(-\eta) = d_0(\eta) = d_0(L-\eta), \quad d_0(\eta) > 0 \quad (\forall \eta \in \mathbf{T}), \\
& \|d_0\|_{H^9(\mathbf{T})} \leq N, \quad d_0(0) \leq \varepsilon, \quad d_0^{(4)}(0) - 3d_0''(0)^3 > 0, \\
& d_0(\eta) \text{ attains its global minimum at } \eta = 0\}.
\end{aligned}
$$

Here $\|d_0\|_{H^9(\mathbf{T})}$ denotes the sum of $L^2$-norms of derivatives of $d_0$ up to order 9. Note that closed curves $\Gamma_0$ parametrized by $X(0,\eta) = X^0(\eta) + d_0(\eta)n^0(\eta)$ with $d_0 \in D_0(N,\varepsilon)$ are simple in $\mathbf{R}^2$. A typical result is:

**Theorem 1** *(Pinching of evolving closed curves). For any $N > 0$ depending on $M^0$, there is an $\varepsilon_0 > 0$; for any $\varepsilon \in (0,\varepsilon_0)$ and any $d_0 \in D_0(N,\varepsilon)$, there are $t_0 \in (0, T_1(N))$ (where $T_1(N)$ is an existing time of the solution of (2)) and $t_1 (> t_0)$ such that for initial simple closed curve $\Gamma_0$ with parametrization*

$$\Gamma_0 = \{X(0,\eta) = X^0(\eta) + d_0(\eta)n^0(\eta); \eta \in \mathbf{T}\},$$

*the solution curve $\Gamma_t$ with parametrization*

$$\Gamma_t = \{X(t,\eta) = X^0(\eta) + d(t,\eta)n^0(\eta); \eta \in \mathbf{T}\}, \quad t \in [0, T_1(N)],$$

*where $d \in D_{T_1(N)}(N)$ is the unique solution of (2), ceases to be simple for at least $t_0 < t < \min(t_1, T_1(N))$.*

This result looks stronger than the one presented in [6] in the sense that $d_0$ is taken arbitrary but clearly the proof in [6] yields this result.

# 3 Nonpreserving of convexity of closed curves

We conclude this paper by stating our recent study for nonpreserving of convexity of evolving closed curves driven by surface diffusion. This phenomenon is markedly different from that of second order model like the mean curvature flow equation. In fact, the mean curvature flow equation preserves convexity of evolving closed curves as long as they exist (see Gage and Hamilton [5]).

Our idea of the proof of this phenomenon is roughly stated as follows. Let $\Sigma$ be the set of all simple convex closed curves which are symmetric with respect to the $y$-axis. We can construct a family of mappings $\{S^{\varepsilon,\delta}\}_{0<\varepsilon,\delta\ll 1}$ such that each $S^{\varepsilon,\delta}$ maps from $\Sigma$ to a set of closed curves which are also symmetric with respect to the $y$-axis. Then, for any $\Gamma_0 \in \Sigma$, we can choose a suitable $\varepsilon$, $\delta$ such that $S^{\varepsilon,\delta}$ maps $\Gamma_0$ to another simple convex closed curve $\Gamma_0^{\varepsilon,\delta}$ and weakens the convexity of $\Gamma_0$ locally. It can be also shown that the solution curve $\Gamma_t^{\varepsilon,\delta}$ of (1) starting from $\Gamma_0^{\varepsilon,\delta} = S^{\varepsilon,\delta}(\Gamma_0)$ exists in a time interval uniformly in $0 < \varepsilon \ll 1$ and it is simple and closed. After similar computations as in (3), we can show that this $\Gamma_t^{\varepsilon,\delta}$ loses its convexity in the above time interval.

We summarize our result in the following.

**Theorem 2** *(Nonpreserving of convexity). Let $\Gamma_0 \in \Sigma$. Then, there is a $\delta_0 > 0$; for any $\delta \in (0,\delta_0)$, there is an $\varepsilon_0^\delta > 0$ such that for any $\varepsilon \in (0,\varepsilon_0^\delta)$ there are $t_0^{\varepsilon,\delta}$ and $t_1^{\varepsilon,\delta}$ with $0 < t_0^{\varepsilon,\delta} < t_1^{\varepsilon,\delta}$ such that the solution curve $\Gamma_t^{\varepsilon,\delta}$ of (1) starting from $\Gamma_0^{\varepsilon,\delta} = S^{\varepsilon,\delta}(\Gamma_0)$ loses its convexity for at least $t_0^{\varepsilon,\delta} \leq t \leq \min(T_0^{*,\delta}, t_1^{\varepsilon,\delta})$, where $T_0^{*,\delta} > 0$ is an existing time of $\Gamma_t^{\varepsilon,\delta}$.*

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